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Dealing with uncertainty and imprecision by means of fuzzy numbers

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Abstract

The problem of the combination of imprecision and uncertainty combination from the approximate reasoning point of view is addressed. An imprecise and uncertain information can be represented as a fuzzy quantity together with a certainty value. In order to simplify the use of such information, it is necessary to combine the imprecision and uncertainty of the fuzzy number. In this paper we propose a method for combining them based on the use of information measures. The first step consists in truncating the fuzzy number by the certainty value. Since non-normalized fuzzy numbers are difficult to use, we transform the truncated fuzzy number into a normalized fuzzy number which contains the same amount of information. To formalize this process, we develop a theoretical context for the information measures on fuzzy values. We study the fuzzy numbers transformation and its properties, and give an approximate reasoning interpretation to the approach.

Keywords: Uncertainty; Imprecision; Fuzzy number; Implication function; Information function

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1. Introduction

The co-existence of imprecision and uncertainty within a concrete datum appears in many applications. For example, in the study of optimization methods in fuzzy graphs [4] or in the framework of uncertain fuzzy databases [1]. In these two cases, the given solutions give rise to a series of inconveniences derived, mainly, from the use of non-normalized or non-trapezoidal fuzzy sets, respectively.

In this paper we investigate the problems associated with the combination of imprecision and uncertainty from the approximate reasoning point of view. To do that, we use the general transformation function $\mathscr{T}$ introduced in [14] that will allow us to relate our results to other approaches of the literature and that will open new ways for the treatment of this problem.

This type of information can be expressed, in general, by an imprecise value A (represented, for example, by a trapezoidal fuzzy number) together with a certainty level α associated with such value. The situation can be formulated as a conditional expression in the following terms.

If the datum is totally true then its value is A. Since we have a certainty level $\alpha < 1$, the generalized modus ponens could be formulated as:

If the certainty level is 1, *then the value is* A.

If the certainty level is $\alpha < 1$, *then the value is* A'.

This situation is equivalent to the generic case:

$$\begin{array}{l} \text{if } \alpha = 1 \text{ then } A \\ \quad \alpha' \\ \hline \qquad\qquad\qquad A' \end{array}$$

Therefore, a natural way to solve the problem is to consider that the datum we are handling is A' defined as: $\mu_{A'}(x) = I(\alpha, \mu_A(x))$ where I is a material implication function which reflects the interpretation given to the compatibility degree.

There exist in the literature two main ways of dealing with imprecise and uncertain data and can be interpreted as follows:

1. *To Truncate*: If the datum is (α, A), then A' is defined by the membership function $\mu_{A'}(x) = \min(\alpha, \mu_A(x))$ which directly implies that we are using Mamdani's implication in our reasoning.
2. *To Expand*: If we assume that α is a necessity, then A' is given by the membership function $\mu_{A'}(x) = \max(1 - \alpha, \mu_A(x))$, which corresponds to Kleene–Dienes' implication as foundation of our reasoning.

These two approaches correspond to the disjunctive and conjunctive representation of the inference rule, respectively.

From our point of view, we understand that the use of these implication functions [2,3,15] for information representation (which is our final objective) could induce an error, since datum A' we are using will be evaluated in terms of compatibility with other data and, in these cases:

1. Mamdani's implication results in a decrease of the compatibility to level α in any case. This result seems to be reasonable as α is the certainty degree but it obliges us to work with non-normalized fuzzy numbers.
2. Kleene–Dienes' implication imposes that any datum would be compatible with A' at least at level $1 - \alpha$. This result may not be suitable for some applications, since it assigns the same possibility to all the points of the underlying domain independently from the distance to the support set of A. Let us think that, for example, if A is *very heavy* with certainty α, the values close to 0 will have the same possibility than those close to the support of very heavy.

Our main objective is to find a transformation function that, based on different criteria, ensures us a suitable change. The intuitive ideas used to find such a transformation function are:

- To truncate A at level α (we obtain A^{α}).
- To normalize A^{α} (we obtain $A^{\mathscr{T}}$).

If we assume the translation of uncertainty into imprecision, then imprecision of $A^{\mathscr{T}}$ must be larger than A imprecision but $A^{\mathscr{T}}$ will never be defined on the whole domain. The idea is to increase such imprecision around the support set of value A. The transformation used is according to equitative distribution of imprecision on the support of A which is valid when no more information is provided, i.e., imprecision is distributed according to a metric which takes into account the nearness to the original information. Following these ideas, when we have the information that *X is black* with certainty α, we will never give a positive possibility to colour white but to colours near enough to black depending on value α.

This way of reasoning has not been used yet and will permit us to ensure that the information amount provided by an uncertain imprecise value A is the same as the information provided by its transformation $A^{\mathscr{T}}$, which is fully true and normalized.

The paper is organized as follows. In Section 2, the preliminary concepts and the notation used are introduced. In Section 3, the axiomatic definition of an information function on fuzzy numbers and its properties are given. Based on this information function, a transformation for fuzzy numbers is introduced in Section 4. This transformation ensures that the amount of information before and after its application remains equal. In Section 5, we are going to prove that the transformation function defined is an implication function. To do it, we are checking that all the conditions an implication function must hold, are also held by our transformation function. Finally, in Section 6, the main conclusions of this work are summarized.

2. Preliminary concepts

A fuzzy value is a fuzzy representation about the real value of a property (attribute) when it is not precisely known.

In this paper, according to Goguen's Fuzzification Principle [10], we will call every fuzzy set of the real line *fuzzy quantity*. A *fuzzy number* is a particular case of a fuzzy quantity with the following properties.

Definition 1. The fuzzy quantity A with membership function $\mu_A(x)$ is a *fuzzy number* iff:
1. $\forall\alpha \in [0,1],\ A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geqslant \alpha\}$ (α-cuts of A) is a convex set.
2. $\mu_A(x)$ is an upper-semicontinuous function.
3. The support set of A $\text{Supp}(A) = \{x \in \mathbb{R} \mid \mu_A(x) > 0\}$ is a bounded set of $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

The given definition is based on the definition given by Dubois and Prade [7] but we do not require either normalization or that the modal interval is a singleton.

We will use $\tilde{\mathbb{R}}$ to denote the set of fuzzy numbers, and $h(A)$ to denote the height of the fuzzy number A. For the sake of simplicity, we will use capital letters at the beginning of the alphabet to represent fuzzy numbers.

The interval $[a_\alpha, b_\alpha]$ (see Fig. 1) is called the α-cut of A. So then, fuzzy numbers are fuzzy quantities whose α-cuts are closed and bounded intervals: $A_\alpha = [a_\alpha, b_\alpha]$ with $\alpha \in (0,1]$. The set $\text{Supp}(A) = \{x \in \mathbb{R} \mid \mu_A(x) > 0\}$ is called the *support set of A*.

If there is, at least, one point x verifying $\mu_A(x) = 1$ we say that A is a *normalized* fuzzy number.

Sometimes, a trapezoidal shape is used to represent fuzzy numbers. This representation is very useful as the fuzzy number is completely characterized by

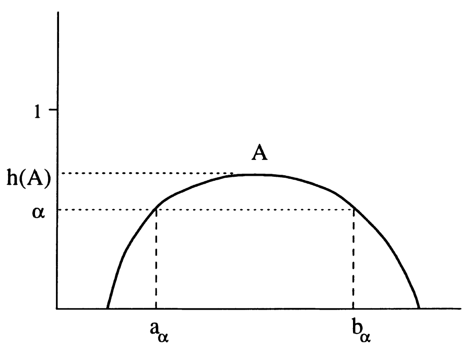


Fig. 1. Fuzzy number.

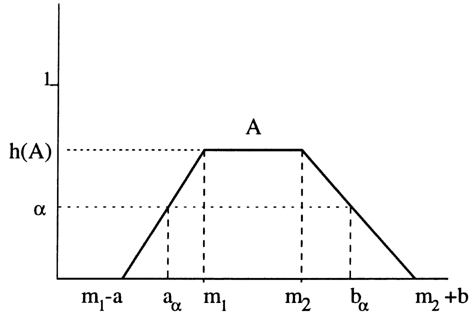


Fig. 2. Trapezoidal fuzzy number.

four parameters (m_1, m_2, a, b) and the height $h(A)$ as shown in Fig. 2. Other parametrical representation for fuzzy numbers can be found in [5]. We will call *modal set* all values in the interval $[m_1, m_2]$, i.e., the set $\{x \in \text{Supp}(A) \mid \forall y \in \mathbb{R},\ \mu_A(x) \geqslant \mu_A(y)\}$. The values a and b are called left and right *spreads*, respectively.

When a fuzzy number is not normalized, this situation can be interpreted as a lack of confidence in the information provided by such numbers [6,11]. In fact, the height of the fuzzy number could be considered as a certainty degree of the represented value. On the other hand, if we assume these considerations, normalized fuzzy numbers represent imprecise quantities on which we have complete certainty. As we will see along this paper, this uncertainty can be translated, using some suitable transformations, into imprecision, taking into account that the less the uncertainty (or the more the certainty) about a fuzzy number, the more is the imprecision of such a number. This transformation will be done in such a way that the amount of information of the fuzzy number will be constant before and after the modification.

3. An information measure on fuzzy values

As pointed out in Section 2, we are going to translate uncertainty into imprecision and vice versa under certain conditions. The most important of these conditions is that the amount of information provided by the fuzzy number remains equal before and after the transformation. So then, the first step is to define an information function for fuzzy numbers.

We propose an axiomatic definition of information, partially inspired in the theory of generalized information given by Kampé de Fériet [13] and that can be related to the precision indexes [8] and the specificity concept, introduced by Yager in [16].

Definition 2. Let $\mathscr{D} \subseteq \tilde{\mathbb{R}} \mid \mathbb{R} \subseteq \mathscr{D}$; we say that the application I defined as

$$I : \mathscr{D} \to [0, 1]$$

is an *information* on $\mathscr{D}$ if it verifies:
1. $I(A) = 1 \quad \forall A \in \mathbb{R}$,
2. $\forall A, B \in \mathscr{D} \mid h(A) = h(B)$ and $A \subseteq B \Rightarrow I(B) \leqslant I(A)$.
The first condition means that real numbers are totally informative and, the second one, that considering two fuzzy numbers with the same height, if one of them is contained in the other one, then it is obvious that the first one, which is more precise, is also more informative.

The given definition of information is very similar to the definition of the precision index, in fact, when applied to normalized fuzzy numbers, both of them coincide. This coincidence is very reasonable because, when there is no uncertainty, information is equivalent to precision. In this way, the information function is a generalization of the precision indexes.

Definition 3. Let $A \in \mathscr{D}$ be a fuzzy number. We say that A has the *maximum information* or that A is totally informative with respect to I iff $I(A) = 1$.

Obviously, real numbers are totally informative with respect to any information measure I, that is, $\forall r \in \mathbb{R},\ I(r) = 1$.

The information about fuzzy numbers depends on different factors, in particular, on imprecision and certainty. We focus on general types of information related only to this two factors.

To compute a measure of the imprecision contained in a fuzzy number, we will consider a measure of the imprecision of its α-cuts, which are closed intervals on which the following function is defined:

$$\forall A \in \tilde{\mathbb{R}}, \quad f_A(\alpha) = \begin{cases} b_\alpha - a_\alpha & \text{if } \alpha \leqslant h(A), \\ 0 & \text{otherwise.} \end{cases}$$

From this imprecision function on the α-cuts, we define the total imprecision of a fuzzy value as a combination of the imprecision in every level α. When $\alpha = 0$, we will consider that $f_A(0)$ is the length of the support set.

Definition 4. The *imprecision* of a fuzzy number is defined as follows:

$$f : \tilde{\mathbb{R}} \to \mathbb{R}_0^+,$$

$$\forall A \in \tilde{\mathbb{R}}, \quad f(A) = \int_0^h (A) f_A(\alpha)\, \mathrm{d}\alpha.$$

The imprecision function f coincides with the area below the membership function of the fuzzy value, as shown in Fig. 3.

Obviously, it is held that $\forall A, B \in \tilde{\mathbb{R}} \mid A \subseteq B \Rightarrow f_A(\alpha) \leqslant f_B(\alpha)\ \forall \alpha \in [0, h]$ when $h(A) = h(B) = h$ and, therefore $f(A) \leqslant f(B)$.

Related to the height (certainty) and the imprecision of a fuzzy value, we define the following general type of function on $\tilde{\mathbb{R}}$:

$$I_{\mathscr{F}} : \tilde{\mathbb{R}} \to [0, 1],$$

$$I_{\mathscr{F}}(A) = \mathscr{F}(h(A), f(A)).$$

The following result guarantees that, for certain types of $\mathscr{F}$ functions, $I_{\mathscr{F}}$ is a information function on $\mathbb{R}$.

Proposition 1. *Let* $\mathscr{F} : (0, 1] \times \mathbb{R}_0^+ \to [0, 1]$ *such that*
1. $\mathscr{F}(1, 0) = 1$,
2. $\forall y, z \in \mathbb{R}_0^+ \mid y \leqslant z \Rightarrow \mathscr{F}(x, z) \leqslant \mathscr{F}(x, y) \quad \forall x \in (0, 1]$.

then, $I_{\mathscr{F}}$ *is an information function on* $\tilde{\mathbb{R}}$.

Proof. Let $A \in \mathbb{R}$. Then obviously $h(A) = 1$ and $f(A) = 0$. Then, $I_{\mathscr{F}}(A) = \mathscr{F}(1, 0) = 1$.

Let $A, B \in \tilde{\mathbb{R}} \mid A \subseteq B$ and $h(A) = h(B)$. Then $f(A) \leqslant f(B)$ and it is verified that

$$\mathscr{F}(h(B), f(B)) = I_{\mathscr{F}}(B) \leqslant \mathscr{F}(h(A), f(A)) = I_{\mathscr{F}}(A)$$

and therefore, $I_{\mathscr{F}}$ is an information on $\tilde{\mathbb{R}}$. □

When $\mathscr{F}$ verifies the previous conditions, we will call function $I_{\mathscr{F}}$ an $\mathscr{F}$-information. In this way, associated with a class of functions, we can build some particular types of information on $\tilde{\mathbb{R}}$ with the property of not depending

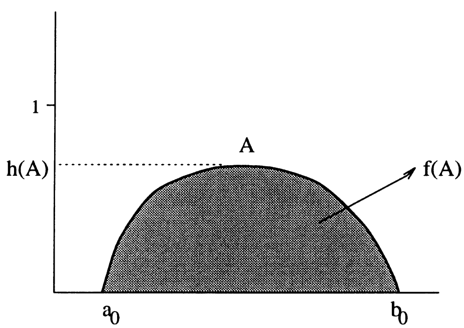


Fig. 3.

on the position the fuzzy value has on $\mathbb{R}$, as shown in the following proposition.

Proposition 2. *Let $\mathscr{F}$ be a function verifying conditions established in Proposition* 1, $A \in \tilde{\mathbb{R}}$ *and* $t \in \mathbb{R}$. *Then,*

$$I_{\mathscr{F}}(A) = I_{\mathscr{F}}(A \oplus t).$$

Proof. If $A \in \tilde{\mathbb{R}}$ and $t \in \mathbb{R}$ then $A \oplus t \in \tilde{\mathbb{R}}$ and $\mu_{A\oplus t}(z) = \mu_A(z-t)\ \forall z \in \mathbb{R}$. Besides, $h(A \oplus t) = \sup_{z\in\mathbb{R}}\{\mu_{A\oplus t}(z)\} = \sup_{z\in\mathbb{R}}\{\mu_A(z-t)\} = \sup_{z\in\mathbb{R}}\{\mu_A(z)\} = h(A)$ and $(A \oplus t)_\alpha = A_\alpha + t$, resulting that $f_A(\alpha) = f_{A\oplus t}(\alpha)$ and $f(A) = f(A \oplus t)$; therefore the result is immediate. □

There are many ways to build information functions but, for our purpose, we are defining information associated with a particular function. This $\mathscr{F}$-information will permit, subsequently, the definition of transformations that keep constant the amount of information a fuzzy number provides.

Let us consider the function

$$\mathscr{F} : (0,1] \times \mathbb{R}_0^+ \to [0,1],$$

$$\mathscr{F}(x,y) = \frac{x}{k \cdot y + 1}, \quad k \in \mathbb{R}^+,$$

that trivially verifies the conditions established in Proposition 1. Hence, we can define the following $\mathscr{F}$-information.

Definition 5. We define the function

$$I_{\mathscr{F}} : \tilde{\mathbb{R}} \to [0,1],$$

$$\forall A \in \tilde{\mathbb{R}}, \quad I_{\mathscr{F}}(A) = \frac{h(A)}{k \cdot f(A) + 1},$$

where $h(A)$ is the fuzzy number height, $f(A)$ is the imprecision associated with A and $k \neq 0$ a parameter which depends on the domain scale (in Section 5, this parameter is widely explained).

Evidently, by Proposition 1, $I_{\mathscr{F}}$ is an information function and, it trivially follows that

$$\forall A \in \tilde{\mathbb{R}}, \quad 0 \leqslant I_{\mathscr{F}}(A) \leqslant h(A) \leqslant 1.$$

As can be immediately deduced from its definition, information $I_{\mathscr{F}}$ is always bounded by the fuzzy number height. Therefore, fuzzy numbers with maximum information with respect to $I_{\mathscr{F}}$ must also have maximum height ($h(A) = 1$) and, consequently, minimum imprecision ($f(A) = 0$).

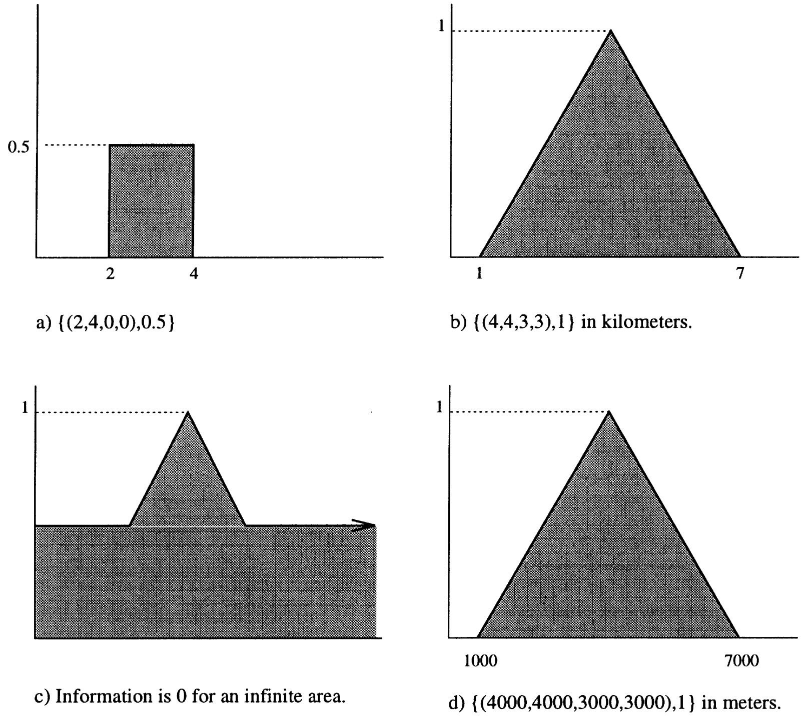


Fig. 4. Fuzzy values examples.

The fuzzy numbers shown in Figs. 4(a) and (b) provide the same information as

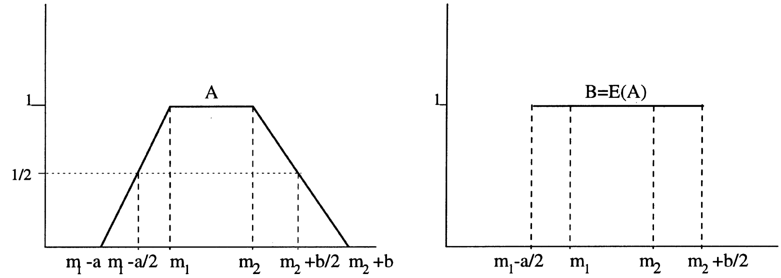
$$\frac{0.5}{1+1} = \frac{1}{3+1} = 0.25$$

assuming $k = 1$.

The fuzzy numbers shown in Figs. 4(b) and (d) are the same fuzzy numbers expressed in different domain scales. As the information provided by both numbers should be the same, the k parameter must be adapted to the scale changes considering a base or reference scale where k is set to 1 (in this case kilometers)

$$\frac{1}{3+1} = \frac{1}{\frac{1}{1000} \cdot 3000 + 1},$$

so then, if the base scale is kilometers and the current scale in meters, k parameter must be set to $1/1000$.



a) Normalized trapezoidal fuzzy number. b) Interval with the same information amount.

Fig. 5. Normalized trapezoidal fuzzy number and the corresponding interval.

The case of Fig. 4(c) is not really a fuzzy number representation if we strictly follow Definition 1, but it is very illustrative to see how these types of fuzzy quantities with infinite support, provide information 0, as $f(A) = \infty$.

We could also wonder which interval has the same information amount as a concrete normalized trapezoidal fuzzy number. The answer to this question is the following – Let us suppose our normalized trapezoidal fuzzy number is the general one represented in Fig. 5(a). One of the possible intervals (the one centered in the modal set) with the same information as the fuzzy number is represented in Fig. 5(b) expressed by $B = [m_1 - \frac{a}{2}, m_2 + \frac{b}{2}]$. It can be easily proved that $I(A) = I(B)$ and that $B = E(A)$, i.e. the mean value of A in the sense of Dubois and Prade [9,12].

Once we have an information function on fuzzy numbers, we want to use it to define transformations which preserve such information function value. The idea is to find an *equivalent* representation of the considered fuzzy number in such a way that we change uncertainty by imprecision keeping constant the relationship between them defined by the information function.

4. Fuzzy numbers transformations

4.1. Basic model

The aim of the transformations we are proposing in this section is, basically, to be able to modify the height of a fuzzy number but keeping the information contained in it. The reason for doing this is that, in most applications, it is very convenient that fuzzy numbers are normalized (simplicity, better understanding for users, etc...). Given a fuzzy number, a transformation on it will give another fuzzy number with the same information amount but different height. So then, to define transformations, we will request that the information function remains fixed, i.e., we will modify certainty and imprecision but keeping constant the relation between both numbers, which is defined by the information function.

The definition of transformation will be obtained from the condition of equality in the information but, as a first step, we must establish what we understand for transformation of a fuzzy number on a subset of $\tilde{\mathbb{R}}$.

Definition 6. Let us consider $\alpha \in (0,1]$ and the class of fuzzy numbers $\mathscr{D} \subseteq \tilde{\mathbb{R}}$. We say that

$$\mathscr{T}_\alpha : \mathscr{D} \to \tilde{\mathbb{R}}$$

is a *transformation* for an information function I on $\mathscr{D}$, if it verifies that:
1. $\mathscr{T}_\alpha(A) \in \mathscr{D}$,
2. $h(\mathscr{T}_\alpha(A)) = \alpha$,
3. $I(\mathscr{T}_\alpha(A)) = I(A)\ \forall A \in \mathscr{D}$.

In this way, for a height level α, $\mathscr{T}_\alpha(A)$ need not exist but, if it does, it must verify the conditions above.

Definition 7. Given the transformation $\mathscr{T}_\alpha$, we say that $A \in \mathscr{D}$ is *transformable* for $\alpha = (0,1]$ if there exists $\mathscr{T}_\alpha(A)$.

We will denote $H(A) = \{\alpha \in (0,1] \mid \exists \mathscr{T}_\alpha(A)\}$ the set of levels, where A is transformable.

Though most of the results obtained here can be generalized for any type of fuzzy number, we will focus on trapezoidal ones for the sake of simplicity in the transformation function. We will note by τ the class of trapezoidal fuzzy numbers on $\mathbb{R}$.

Given a fuzzy number $A \in \tau$, we are looking for the conditions that another fuzzy number B, with fixed height $\alpha \in (0,1]$, must hold to have the same information amount as A.

Proposition 3. *Let $A, B \in \tau$ be two fuzzy numbers with heights $h(A) = \alpha_A$ and $h(B) = \alpha_B$, respectively. Then,*

$$I_{\mathscr{F}}(A) = I_{\mathscr{F}}(B) \Longleftrightarrow f_B(0) + f_B(\alpha_B) = f_A(0) + f_A(\alpha_A) + \frac{2}{k}\Delta(\alpha_A, \alpha_B),$$

where

$$\Delta(\alpha_A, \alpha_B) = \frac{\alpha_B - \alpha_A}{\alpha_A \cdot \alpha_B}.$$

Proof. It is immediate for trapezoidal fuzzy numbers, taking into account that $f(A) = ((f_A(0) + f_A(\alpha_A))/2) \cdot \alpha_A$. $\square$

That is, the sum of base imprecision and modal imprecision must be modified by the value $(2/k) \cdot \Delta(\alpha_A, \alpha_B)$ for A can be transformed into a fuzzy number B of fixed height. Besides, if we pretend to put up the height of A ($\alpha_A < \alpha_B$), then $\Delta(\alpha_A, \alpha_B)$ is positive and the sum of base imprecision and modal imprecision of B must augment; on the other hand, to put down the height ($\alpha_B < \alpha_A$), since $\Delta(\alpha_A, \alpha_B)$ is negative, imprecision must be decreased. When the height is fixed, it is obvious that imprecision remains equal.

So then, the relation between uncertainty and imprecision is the following:

- An increase of certainty means an increase of imprecision.
- A decrease of imprecision means a decrease of certainty.

Proposition 3 permits us to define a transformation assuming that:

1. Modal imprecision is preserved.
2. The increase/decrease of imprecision is equally distributed in the right and left sides of the fuzzy number independently from its shape.

Definition 8. Let $A \in \tau$ be a fuzzy number such that

$$A = \{(m_1, m_2, a, b), \alpha_A\},$$

where m_1, m_2, a and b are shown in Fig. 2 and α_A is the height of A.

Let $\alpha \in (0, 1]$. We will denote $\Delta(\alpha_A, \alpha) = \Delta$ and define

$$\mathscr{T}_\alpha(A) = \left\{ \left(m_1, m_2, a + \frac{\Delta}{k}, b + \frac{\Delta}{k} \right), \alpha \right\}$$

for those α in which the transformation makes sense.

Proposition 4. *$\mathscr{T}_\alpha$ is a transformation for trapezoidal fuzzy numbers.*

Proof. Let us assume that there exists $\mathscr{T}_\alpha(A)$ for $\alpha \in (0, 1]$. Then, obviously $\mathscr{T}_\alpha(A) \in \tau$ and $h(\mathscr{T}_\alpha(A)) = \alpha$.

On the other hand,

$$f_{\mathscr{T}_\alpha(A)}(0) + f_{\mathscr{T}_\alpha(A)}(\alpha) = f_A(0) + f_A(\alpha_A) + \frac{2}{k}\Delta.$$

By Proposition 3, $I_{\mathscr{F}}(A) = I_{\mathscr{F}}(\mathscr{T}_\alpha(A))$ and using Definition 6, $\mathscr{T}_\alpha$ is an information on τ. $\square$

Definition 9. Let $A = \{(m_1, m_2, a, b), \alpha_A\}$ be a trapezoidal fuzzy number. We define the lowest limit of the transformation as

$$l(A) = \max\left\{ \frac{\alpha_A}{k \cdot a \cdot \alpha_A + 1}, \frac{\alpha_A}{k \cdot b \cdot \alpha_A + 1} \right\}.$$

It can be proved immediately that $l(A)$ is a number in the interval $(0, 1]$ and it is less or equal than the height of A.

Proposition 5. *$A \in \tau$ is transformable* $\iff \alpha \geqslant l(A)$.

Proof. $A \in \tau$ is transformable $\iff \exists \mathscr{T}_\alpha(A)$ and the existence of $\mathscr{T}_\alpha(A)$ means that the spreads of A are positive or null, as it is the only possible restriction to build it. Therefore,

$$\left.\begin{array}{l} a + \frac{\Delta}{k} \geqslant 0 \\ b + \frac{\Delta}{k} \geqslant 0 \end{array}\right\} \iff \alpha \geqslant l(A). \qquad \square$$

Following this result, the transformation domain is $H(A) = [l(A), 1]$, where $A \in \tau$. Since the lowest limit of the transformation is always less or equal than the height of A, it is always possible to make a transformation for putting up the height of a fuzzy number but, on the contrary, there is a minimum level from which transformations are not possible. In Fig. 6 we have represented graphically the behavior of $\mathscr{T}_\alpha$ when the height is decreased and, therefore, imprecision is also decreased. On the other hand, in Fig. 7 it is shown how an increment of height produces an increment of imprecision. This result agrees with the following assertion: "*Imprecision and uncertainty can be considered as two antagonistic points of view about the same reality, which is human imperfection... and if the contents of a proposition is made more precise, then uncer-*

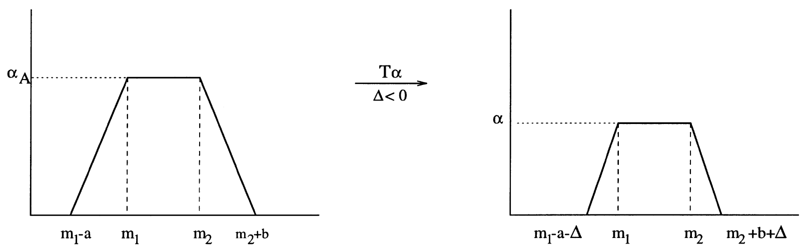


Fig. 6. Transformation that decreases imprecision.

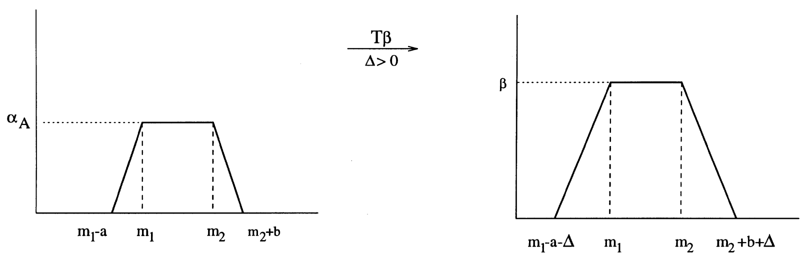


Fig. 7. Transformation that augments the certainty.

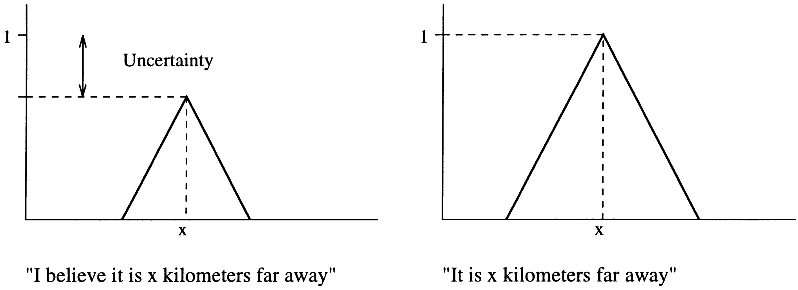


Fig. 8. An increase of certainty produces an increase of imprecision.

tainty will have to be augmented" [6], which is a way to enunciate the principle of incompatibility between certainty and precision, established by Zadeh in [17].

Considering that $f(A)$ is a measure for the imprecision of the fuzzy number A and that $1 - h(A)$ is a measure of its uncertainty, this principle can be enunciated as:

- If $f(A)$ decreases, then $h(A)$ decreases.
- If $h(A)$ increases, then $f(A)$ increases.

Function $I_{\mathscr{F}}$ fixes the constant relationship between imprecision and uncertainty and is associated with the concept we represent using a fuzzy number. On the other hand, as transformations to put up the height are always possible, we can always normalize ($\alpha = 1$) the fuzzy numbers we are working with. Normalization means a loss of uncertainty, i.e., the security on the validity of the fuzzy representation. In Fig. 8 it is shown how the fuzzy number *x kilometers* with certainty degree less than 1 is transformed into a *bigger* fuzzy number with certainty 1.

Note. We can see how, contrary to the model of expanding imprecision over the whole domain, our model assumes implicitly that imprecision must be distributed depending on the nearness to the original concept.

Proposition 6 ($\mathscr{T}_\alpha$ Properties). *Let $A \in \tau$ and $\alpha, \beta \in H(A)$. Then the following properties are verified*:

1. $\mathscr{T}_{h(A)}(A) = A$,
2. $\mathscr{T}_\alpha(\mathscr{T}_\beta(A)) = \mathscr{T}_\alpha(A)$,
3. $\mathscr{T}_{h(A)}(\mathscr{T}_\alpha(A)) = A$.

Proof. Let us consider $A = \{(m_1, m_2, a, b), h(A)\}$, then

1. Since $\Delta(h(A), h(A)) = 0$, then $\mathscr{T}_{h(A)}(A) = A$,
2. $\mathscr{T}_\beta(A) = \{(m_1, m_2, a + \Delta(h(A), \beta), b + \Delta(h(A), \beta)), \beta\}$,

$$\mathscr{T}_\alpha(\mathscr{T}_\beta(A)) = \left\{ \left(m_1, m_2, a + \frac{\Delta(h(A),\beta) + \Delta(\beta,\alpha)}{k}, \right. \right.$$
$$\left. \left. b + \frac{\Delta(h(A),\beta) + \Delta(\beta,\alpha)}{k} \right), \alpha \right\}$$

and, since $\Delta(h(A),\beta) + \Delta(\beta,\alpha) = \Delta(h(A),\alpha)$ then $\mathscr{T}_\alpha(\mathscr{T}_\beta(A)) = \mathscr{T}_\alpha(A)$,
3. It trivially follows from (1) and (2).

4.2. The k parameter and its experimental computation

As we pointed out at the end of Section 3, k parameter is adjusted depending on the domain scale taking into account that there is a pre-fixed base scale, for which k parameter is set to 1. The idea is that identical fuzzy numbers, though expressed in a different scale, must provide exactly the same information amount and that this information must be the same before and after a transformation is applied.

In the next sub-sections we are going to illustrate with some examples the use of k parameter in the case that the base scale is used ($k = 1$) and in the case when it is not.

4.2.1. Fuzzy values in the same domain scale: k = 1

Let us suppose we are given the fuzzy number $A = \{(3,4,1,1), 0.5\}$ for the concept 'I believe it is few kilometers far away' where the believe has been quantified by 0.5, and we want A to be normalized, that is, $\alpha = 1$ for the proposition become 'It is few kilometers far away'. Since the information amount before and after the transformation must be the same,

$$I_{\mathscr{F}}(A) = I_{\mathscr{F}}(\mathscr{T}(A))$$

and, by Proposition 3, $\Delta(0.5,1) = (1-0.5)/(1 \times 0.5)$. Therefore, the transformation of A is

$$\mathscr{T}(A) = \left\{ \left(3, 4, 1 + \frac{1}{k} \cdot \frac{(1-0.5)}{(1 \times 0.5)}, 1 + \frac{1}{k} \cdot \frac{(1-0.5)}{(1 \times 0.5)} \right), 1 \right\} = \{(3,4,2,2),1\}$$

considering $k = 1$, since the transformation is from kilometers into kilometers.

4.2.2. Fuzzy values in different domain scale

In the previous case, $k = 1$ as we were considering a fixed scale, but what would happen if we were given the same information in two different scales? In this case, k is set to the number of units of the base scale contained in a unit of the scale we are using. For example, if the base scale is *kilometers* and the scale in use is *meters*, $k = 1/1000$, i.e. the number of kilometers contained in a meter. If the base scale is *centimeters* and the scale in use is *meters*, then $k = 100$.

Let us see the information function behavior with an example. In this case, we must establish a base scale as a reference, for example, kilometers. In this situation, let us suppose we want to normalize the following fuzzy number given in kilometers and meters,

$$A_{\mathrm{km}} = \{(3,4,1,1),0.5\} \text{ in kilometers,}$$
$$A_{\mathrm{m}} = \{(3000,4000,1000,1000),0.5\} \text{ in meters.}$$

A_{km} transformation gives the result obtained above: $\mathscr{T}(A_{\mathrm{km}}) = \{(3,4,2,2),1\}$.

For A_{m} transformation, k parameter must be set to $1/1000$ (the relation between the scale we are using and the base scale) and the result is $\mathscr{T}(A_{\mathrm{m}}) = \{(3000,4000,2000,2000),1\}$ which is the same result obtained for $\mathscr{T}(A_{\mathrm{km}})$ but represented in the corresponding scale.

If we consider now that $A_{\mathrm{km}} = \{(3000,4000,1000,1000),0.5\}$ in kilometers, k parameter will be 1 and the transformation will be $\mathscr{T}(A_{\mathrm{km}}) = \{(3000,4000,1001,1001),1\}$.

As we can see, the transformation function $\mathscr{T}$ is correct with respect to the change to different domain scales.

4.2.3. Experimental computation of the base scale

In Section 4.2.2, we have seen how, thanks to k parameter, we can use different domain scales for the fuzzy numbers we are handling. But there is another key point when using transformations and the following question arises. Should the increase/decrease be the same and not depend on the meaning of the fuzzy number? or, in other words, should the increase be the same irrespective of the fact that we are dealing with ages or with distance? Up to here, we have considered that the user could change such increase/decrease through the scale factor k. In this section we are going to see, in an experimental way, how we can adjust the transformation model to each problem domain.

Let us think, for example, that when dealing with ages, the user is prepared to admit that *approximately* 40 *years* is any age in the interval $[38,42]$ (spread is 2) but when talking about distance in kilometers, *approximately* 7 *kilometers* is any distance in the interval $[6,8]$ (spread 1).

Taking these comments into account, it seems to be reasonable that k parameter is not only dependent on the domain scale but also on the concept the fuzzy number is representing. In this sense, k should have the form

$$k = k_0 \cdot \lambda,$$

where λ is the relation between the scale we are using and the base scale (as in the previous section) and k_0 is the correction factor that depends on the meaning and that allows us to determine the reference scale. Let us see now how k_0 could be experimentally calculated considering that we are working in a base scale, i.e. $\lambda = 1$.

Let us suppose we have the real value $\{(x, x, 0, 0), \alpha\}$ with $\alpha < 1$ (see Fig. 9(a)).

We could ask the user to what point he is prepared to relax x as to accept it as completely true. Let us suppose the user says that he will accept it as completely true if we *enlarge* x at both sides with a spread c, as shown in Fig. 9(b), becoming the initial real value the fuzzy number $\{(x, x, c, c), 1\}$. If it is so,

$$c = 0 + \frac{\Delta}{k_0} \text{ and } \Delta = \frac{1 - \alpha}{1 \cdot \alpha}$$

and, subsequently, value k_0 is

$$k_0 = \frac{1 - \alpha}{c \cdot \alpha} \text{ with } c \neq 0.$$

For example, if we have the real value $\{(5, 5, 0, 0), 0.9\}$ and the user admits it as completely true if we transform it into $\{(5, 5, 0.5, 0.5), 1\}$, as shown in Fig. 10, then the computed value for k_0 is

$$k_0 = \frac{1 - 0.9}{0.5 \times 0.9} = \frac{0.1}{0.45} = \frac{1}{4.5} = 0.22.$$

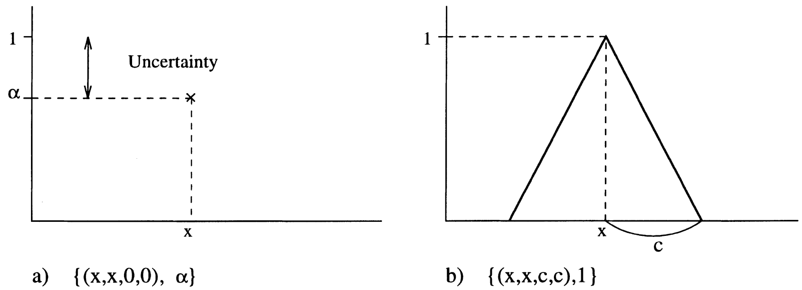


Fig. 9. Relaxation of x according to user's credibility.

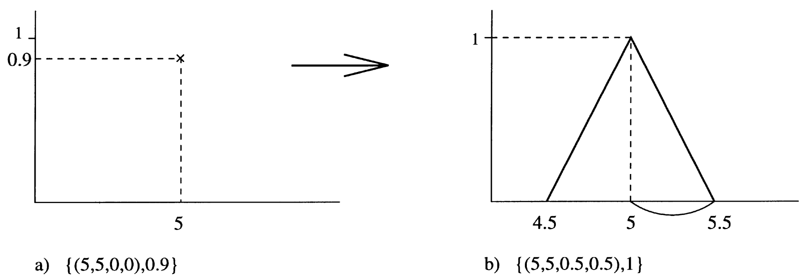


Fig. 10. Relaxation of value 5 for being true.

So then, whereas λ depends only on the scale we are doing the transformation on, k_0 permits the user to specify the changes magnitude depending directly on the domain of the problem we are tackling.

As a conclusion of the experimental computation of k, we can say that:

1. The translation of certainty into imprecision is valid for the concrete problem we are tackling, as it has been elicitated experimentally by the user.
2. We can adjust the transformation to particular problems and domains.
3. It is obvious that the experiment for obtaining k_0 should be repeated many times using different and separate domain values and different certainty levels. The idea is to obtain an average k_0, that is, if we do n experiments and k_0^i is the value obtained in experiment i, then

$$k_0 = \sum_{i=1}^{n} k_0^i \cdot \frac{1}{n}.$$

5. Management of uncertain fuzzy data

As pointed out in the introduction, there are two main approaches to deal with uncertain fuzzy numbers. From a semantical point of view and for our purposes, it seems to be more reasonable to truncate at level α than to *extend* the support set to the whole domain. Nevertheless, truncating has the inconvenience that non-normalized fuzzy numbers must be handled. To be able to work always with normalized fuzzy numbers, we use the transformation function introduced in this paper setting $\alpha = 1$. This function will convert an uncertain imprecise value (A, α) to an A' which is normalized and provide the same information as the original A value truncated to an α level. All these processes can be summarized as follows:

1. We start from a fuzzy number A and a certainty level α attached to it.
2. We truncate A at level α assuming that A height is its certainty value. We build A^α.
3. To take advantage of normalized fuzzy number properties, we transform A^α into $A^{\mathscr{T}}$ (normalized version of A^α using our transformation function).

We are going to illustrate this process with an example.

Let A be a trapezoidal number expressed as $A = \{(m_1, m_2, a, b), 1\}$ with an associated certainty level α. A truncated to level α is

$$A^\alpha = \{(m_1 - a \cdot (1 - \alpha), m_2 + b \cdot (1 - \alpha), a \cdot \alpha, b \cdot \alpha), \alpha\}$$

and the transformation of A^α is

$$A^{\mathscr{T}} = \left\{ \left(m_1 - a \cdot (1 - \alpha), m_2 + b \cdot (1 - \alpha), a \cdot \alpha + \frac{1 - \alpha}{k \cdot \alpha}, b \cdot \alpha + \frac{1 - \alpha}{k \cdot \alpha} \right), 1 \right\}.$$

It can be proved that the direct expression for computing $A^{\mathscr{T}}$ from A and α is

$$A^{\mathscr{T}} = \left(\frac{1-\alpha}{\alpha} \otimes S\right) \oplus \left(\alpha \otimes \left(A \oplus \left(\frac{1-\alpha}{\alpha} \otimes \mathrm{Supp}(A)\right)\right)\right),$$

where $\oplus$ and $\otimes$ are the fuzzy extensions of sum and product operators, S the fuzzy number expressed by $S = \{(0, 0, 1/k, 1/k), 1\}$ and $\mathrm{Supp}(A)$ the support set of A expressed by the fuzzy number $\mathrm{Supp}(A) = \{(m_1 - a, m_2 + b, 0, 0), 1\}$.

The advantage of this process is obvious. Uncertainty and imprecision are included in the fuzzy number itself and there is no need to develop or use different mechanisms from those already introduced for normalized fuzzy numbers.

5.1. An approximate reasoning interpretation

As we mentioned in the introduction, the problem of the co-existence of both uncertainty and imprecision can be formulated by means of the compositional rule of inference. In fact, some approaches that solve this problem make use of well-known implication functions on the certainty degree and the rule consequent

If the certainty degree is 1, *then the value is A*

Certainty degree is α

$A^{\mathscr{T}}$

The starting point of our approach is quite different since we have transformed A in such a way that the information provided by A truncated at the certainty level is preserved. Anyway, we are going to prove that this approach is very close to the approximate reasoning one. In fact, we are proving that for the case of trapezoidal fuzzy numbers, the whole process of transformation applied is an implication function in the sense of Trillas and Valverde [15]. To do that, we are going to find the expression which summarizes the whole process of transformation of A with certainty α into $A^{\mathscr{T}}$. This expression is calculated in the following property.

Property 1. *Let A be a trapezoidal fuzzy number expressed as* $A = (m_1, m_2, a, b)$ *and let us suppose that the uncertainty level of A is* α. *Let A truncated to level be* A^{α} *and the normalized version of* A^{α} *be* $A^{\mathscr{T}}$. *In these conditions, it is verified that*

$\forall v \in \mathbb{R} \mid \mu_A(v) > 0,$

$$\mu_{A^{\mathcal{T}}}(v) = \begin{cases} 1 & \alpha \leqslant \mu_A(v), \\ \min\left\{ \dfrac{a}{a\cdot\alpha + \frac{\Delta}{k}} \cdot \mu_A(v) + \dfrac{\Delta}{a\cdot\alpha + \Delta + \frac{\Delta}{k}}, \dfrac{b}{b\cdot\alpha + \frac{\Delta}{k}} \cdot \mu_A(v) + \dfrac{\Delta}{b\cdot\alpha + \Delta + \frac{\Delta}{k}} \right\}, \end{cases}$$

where $\Delta = \Delta(\alpha, 1) = (1-\alpha)/\alpha$ *and k is the scale factor.*

Proof. Proving this property is quite direct taking into account that the α-cut of A has the following membership function

$$\mu_{A^\alpha}(v) = \begin{cases} \alpha & \text{if } \mu_A(v) \geqslant \alpha, \\ \mu_A(v) & \text{otherwise.} \end{cases}$$

So then, $A^{\mathcal{T}} = \{(m_{1\alpha}, m_{2\alpha}, a\cdot\alpha + (\Delta/k), b\cdot\alpha + (\Delta/k)), 1\}$, where $m_{1\alpha} = m_1 - a.(1-\alpha) = m_1 - a + a\cdot\alpha$ and $m_{2\alpha} = m_2 + b\cdot(1-\alpha) = m_2 + b - b\cdot\alpha$.

The graphical representation of these trapezoidal fuzzy numbers is shown in Fig. 11.

From Fig. 11 we can directly obtain that $\mu_{A^{\mathcal{T}}}(v) = 1$ if $\alpha \leqslant \mu_A(v)$. Let us see now the case where this condition is not held. In this situation, $\alpha > \mu_A(v) > 0$ and $v \in (m_1 - a, m_{1\alpha}]$ or $v \in (m_{2\alpha}, m_{2\alpha} + b]$. Besides,

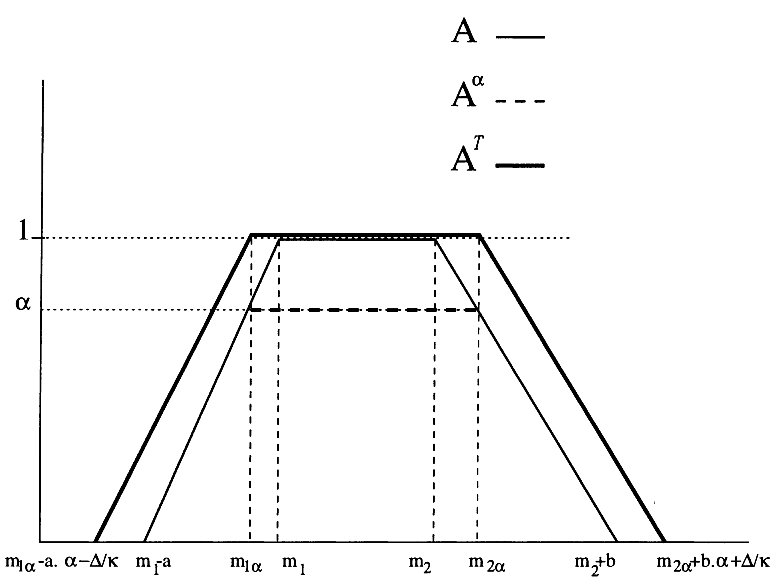


Fig. 11. Graphical representation of A, A^α and $A^{\mathcal{T}}$.

$$\mu_A(v) = \begin{cases} \dfrac{v-(m_1-a)}{a} & \text{if } v \in (m_1-a, m_{1\alpha}], \\ \dfrac{(m_2+b)-v}{b} & \text{if } v \in (m_{2\alpha}, m_2+b]. \end{cases}$$

Finally,

$$\mu_{A^{\mathscr{T}}}(v) = \begin{cases} \dfrac{v-(m_1-(a-\Delta/k))}{a\cdot\alpha+\Delta/k} & \text{if } v \in (m_1-(a-\Delta/k), m_{1\alpha}], \\ \dfrac{(m_2+b+\Delta/k)-v}{b.\alpha+\Delta/k} & \text{if } v \in (m_{2\alpha}, m_2+b+\Delta/k]. \end{cases}$$

Making operations, we have

$$\mu_{A^{\mathscr{T}}}(v) = \begin{cases} \dfrac{v-(m_1-a)\cdot a}{a\cdot(a\cdot\alpha+\Delta/k)} + \dfrac{\Delta/k}{a\cdot\alpha+\Delta/k} & \\ = \dfrac{a}{a\cdot\alpha+\Delta/k}\cdot\mu_A(v) + \dfrac{\Delta/k}{a\cdot\alpha+\Delta/k} & \text{if } v \in (m_1-(a-\Delta/k), m_{1\alpha}], \\ \dfrac{(m_2+b-v)\cdot b}{b\cdot(b\cdot\alpha+\Delta/k)} + \dfrac{\Delta/k}{b\cdot\alpha+\Delta/k} & \\ = \dfrac{b}{b\cdot\alpha+\Delta/k}\cdot\mu_A(v) + \dfrac{\Delta/k}{b\cdot\alpha+\Delta/k} & \text{if } v \in (m_{2\alpha}, m_2+b+\Delta/k]. \end{cases}$$

When $v \in (m_1-(a-\Delta/k), m_{1\alpha}]$ then,

$$\frac{a}{a\cdot\alpha+\Delta/k}\cdot\mu_A(v) + \frac{\Delta/k}{a\cdot\alpha+\Delta/k} \leqslant \frac{b}{b\cdot\alpha+\Delta/k}\cdot\mu_A(v) + \frac{\Delta/k}{b\cdot\alpha+\Delta/k}$$

and when $v \in (m_{2\alpha}, m_2+b+\Delta/k]$ then,

$$\frac{b}{b\cdot\alpha+\Delta/k}\cdot\mu_A(v) + \frac{\Delta/k}{b\cdot\alpha+\Delta/k} \leqslant \frac{a}{a\cdot\alpha+\Delta/k}\cdot\mu_A(v) + \frac{\Delta/k}{a\cdot\alpha+\Delta/k}.$$

Making the suitable operations we obtain the expression of the given transformation function

$$\mu_{A^{\mathscr{T}}}(v) = \begin{cases} 1 & \text{if } \alpha \leqslant \mu_A(v), \\ \min\left\{ \dfrac{a\cdot\alpha\cdot k\cdot\mu_A(v)-\alpha+1}{a\cdot\alpha^2\cdot k-\alpha+1}, \dfrac{b\cdot\alpha\cdot k\cdot\mu_A(v)-\alpha+1}{b\cdot\alpha^2\cdot k-\alpha+1} \right\}. \end{cases}$$

This result led us to define as transformation function for the fuzzy number A the function

$$\forall x,y \in (0,1], \quad I(x,y) = \begin{cases} 1 & \text{if } x \leqslant y, \\ \min\left\{ \dfrac{a\cdot x\cdot y-x+1}{a\cdot x^2-x+1}, \dfrac{b\cdot x\cdot y-x+1}{b\cdot x^2-x+1} \right\}. \end{cases} \qquad \square$$

The relationship of this result with the approximate reasoning is shown in the following property.

Property 2. *The function $I(x, y)$ previously defined is an implication function in the sense of Trillas and Valverde* [15].

Proof.
1. I is a decreasing function in x. That is $\forall x, x' \mid x \leqslant x', I(x', y) \leqslant I(x, y)$. In fact,
   – if $x, x' \leqslant y$ the result is immediate.
   – if $x \leqslant y$ and $x' > y$ it is also immediate since $I(x', y) \leqslant 1$.
   – if $x, x' > y$ the expression is $(a \cdot y \cdot x - x + 1)/(a \cdot x^2 - x + 1)$ given that $x > y$. In this case, the numerator increases with x more slowly than the denominator and, therefore, the quotient is a decreasing function.
2. I is an increasing function in y. That is, $\forall y, y' \mid y \leqslant y', I(x, y) \leqslant I(x, y')$. This result is obvious since both functions (numerator and denominator) are linear with respect to y with positive coefficients. Therefore, the function will be increasing in y.
3. $I(0, y) = 1\ \forall y,\ 0 \leqslant y$ and $I(0, y) = 1$.
4. $I(1, y) = y$. Considering the following expression for $x = 1$, the result is directly obtained.
$$I(x,y) = \max\left\{0, \min\left\{\frac{a \cdot y - 1 + 1}{a - 1 + 1}, \frac{b \cdot y - 1 + 1}{b - 1 + 1}\right\}\right\} = y.$$
5. $I(x, I(y, z)) = I(y, I(x, z))$ (property of interchangeability).There are two cases:
   - $y \leqslant z$. In this case the right side of the property becomes $I(x, 1) = 1$ by applying properties 1 and 4, since $I(x, z) \geqslant I(1, z) = z$. Consequently, $y \leqslant z \leqslant I(x, z)$. On the other hand, the left side is also equal to 1, since we have that: $y \leqslant z$, $I(y, z) = 1$ and $I(x, 1) = 1$, which proves the equality.
   - $z < y$. There are two possibilities:
     – $x \geqslant y > z$. We are going to prove that: $x \leqslant I(y, z)$ and $y \leqslant I(x, z)$, and subsequently that $I(x, I(y, z)) = I(y, I(x, z)) = 1$. In fact, the following expressions are held: $x = I(1, x) \leqslant I(y, x) \leqslant I(y, z)$ and $y = I(1, y) \leqslant I(x, y) \leqslant I(x, z)$.
     – $y > z \geqslant x$. According to I properties, $I(x, z) = 1 \Rightarrow I(y, I(x, z)) = 1$. We are going to prove that $I(x, I(y, z)) = 1$. To do that, we must prove that $x \leqslant I(y, z)$. Let us suppose the contrary: $x > I(y, z)$, then $I(1, x) > I(y, z) \geqslant I(y, x)$ and this lead us to the expression $I(1, x) > I(y, x), y \leqslant 1$ which is in contradiction with the decreasing character of I.

So then, I is an implication function. □

6. Conclusions

The problem of imprecision and uncertainty management through fuzzy numbers has been addressed. Fuzzy numbers are a useful tool for representing imprecise information but, in many cases, this imprecise information can be given with an uncertainty degree In these cases, the fuzzy number includes additional information about the confidence of this information. In this paper we have proposed a method that allows us to transform the whole information (imprecision + uncertainty) into a new fuzzy number in such a way that:

- It maintains a principle of distribution of the imprecision based on a metric that takes into account the distance to the original concept, that is, the closer an element is to the concept the more is the increase of its membership to the mentioned concept (based on Zadeh's principle).
- It permits to adjust the results to the users' point of view by using the scale factor, i.e. we can adjust the transformation to particular problems and domains.
- It is interpreted from the approximate reasoning point of view and, subsequently, it guarantees sound results.
- It is easy to implement and, therefore, to be included as a data preprocessing module.
- The method, not only normalizes but equalizes. This is an important characteristic for problems where an accepted level of uncertainty exists (not necessarily 1).
- If β is set to 1, all the software developed for normalized fuzzy sets is re-usable and no new versions are necessary to treat uncertainty.

As future avenues for research we can mention:

1. To give a general expression for fuzzy numbers. In this paper we have only considered the case of trapezoidal fuzzy numbers but the results could be generalized for any kind of fuzzy number.
2. To use linguistic uncertainty instead of uncertainty levels.
3. To study how transformations affect the results obtained from arithmetic operations, matching or ranking of fuzzy values.

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