Constraint Satisfaction and Constraint Programming

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Overview

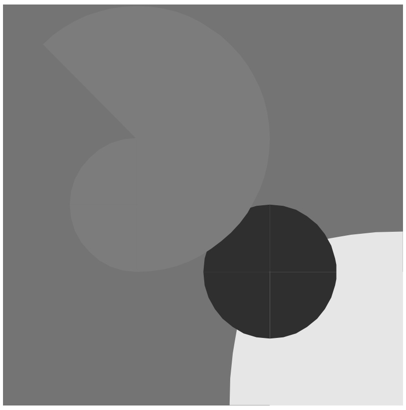
Introduction

Constraint Satisfaction
- Search
- Inference
- Hybrids

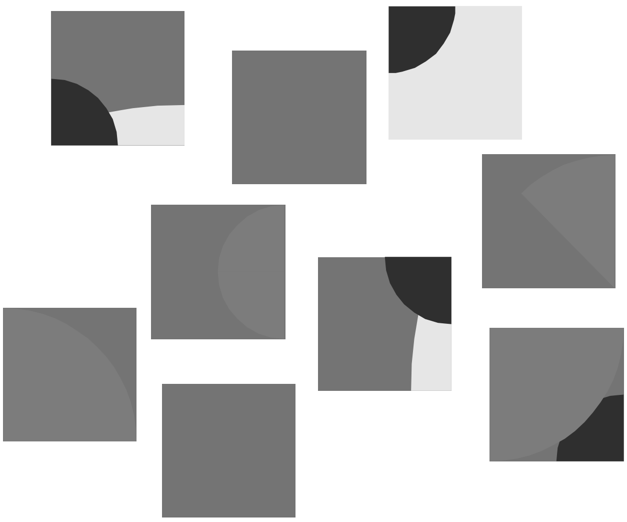
Constraint Programming
- Modelling with Constraints
- Optimization
- Existing Solvers

Summary

Modern Art

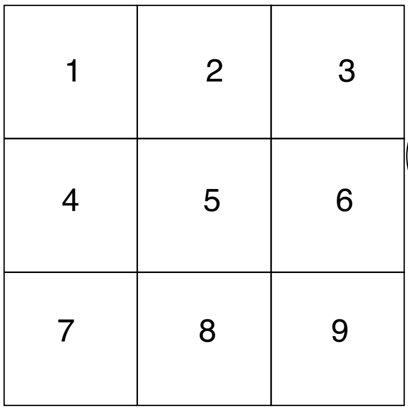


Modern Art: Accident



How can we reconstruct the painting?

Modern Art: Reconstruction

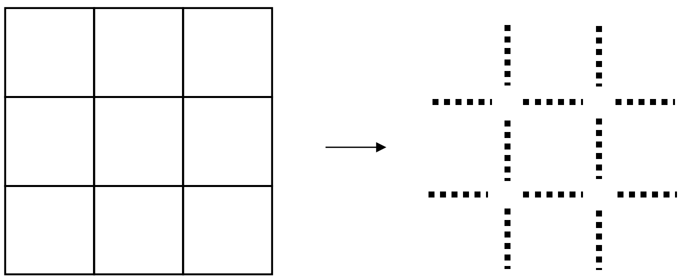


1. Locate pieces in grid slots
2. Two adjacent slots must have the same color pattern on the contact edge
3. Find a globally consistent arrangement

constraint between each pair of adjacent slots

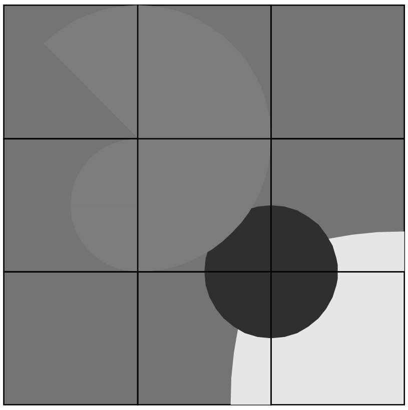
Modern Art: All Constraints

12 constraints



Solution: assignment satisfying **every** constraint

Modern Art: Solution



Conclusions from Modern Art

Constraint problems: most of the knowledge can be expressed in terms of constraints among problem elements

One constraint:

- Involves a subset of problem elements
- Declares permitted (or forbidden) value combinations
- Provides a local view of the whole problem

Solution:

- Satisfies every constraint
- Global view of the whole problem
- Process: from local to global consistency

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Some Definitions

Constraint Network (CN): (X, D, C)

- $X = \{x_1, x_2, \ldots, x_n\}$ variables
- $D = \{d_1, d_2, \ldots, d_n\}$ domains (finite)
- $C = \{c_1, c_2, \ldots, c_r\}$ constraints

$c \in C$ $var(c) = \{x_i, x_j, \ldots, x_k\}$ scope

$rel(c) \subseteq d_i \times d_j \times .. \times d_k$ permitted tuples

Constraint Satisfaction Problem (CSP):

- CN solving: assignment satisfying every constraint
- NP-complete task

Relevance

CSP: formal model to express problems

Many problems can be represented as CSP:

- Academic problems:
  - SAT, Graph coloring, N-queens, . . .
- Real problems:
  - Scheduling, Resource allocation, Routing, ….

Many AI tasks can be modeled as CSP:

- Automated reasoning
- Planning
- Spatial and temporal inference

Running Example: n-queens

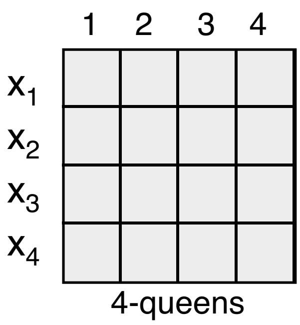
GOAL: Locate n queens in an n x n chessboard, such that they do not attack each other

Formulation:

- Variables: one queen per row
- Domains: available columns
- Constraints:

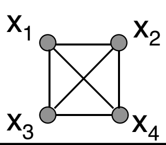
different columns and different diagonals

$x_i \neq x_j$ $\quad\quad$ $| x_i - x_j | \neq | i - j |$



4-queens

Constraint Graph:



Backtrack Search

Strategy:

- Build a partial solution:
  - A partial consistent assignment
- Extend consistently the partial solution
  - One new assigned variable each time
- If no consistent extension:
  - Backtrack: change a previous assignment

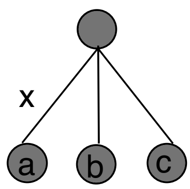
Variables:

- Past ∈ partial solution (assigned)
- Future ∉ partial solution (unassigned)

Tree Search

State space: explored as a tree

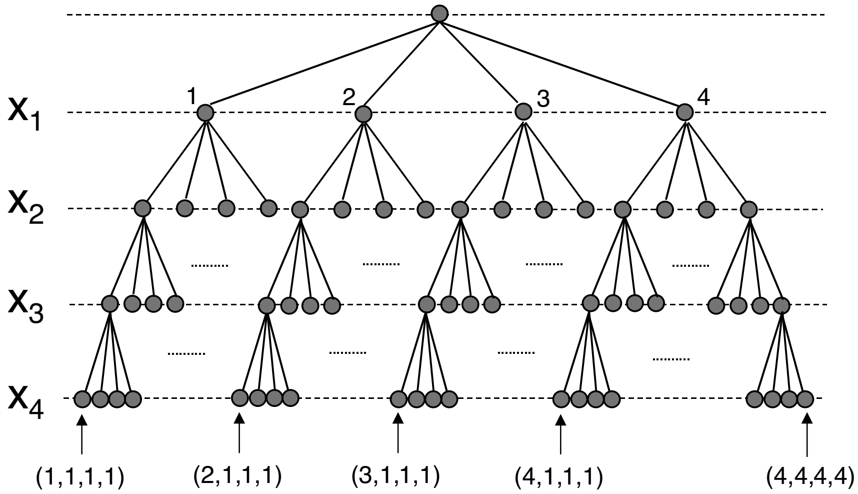
- root: empty
- one variable per level
- sucessors of a node:
  - one sucessor per value of the variable
  - meaning: variable ← value



Tree:

- each branch defines an assignment
- depth n (number of variables)
- branching factor d (domain size)

Search tree for 4-queens



Backtracking Algorithm

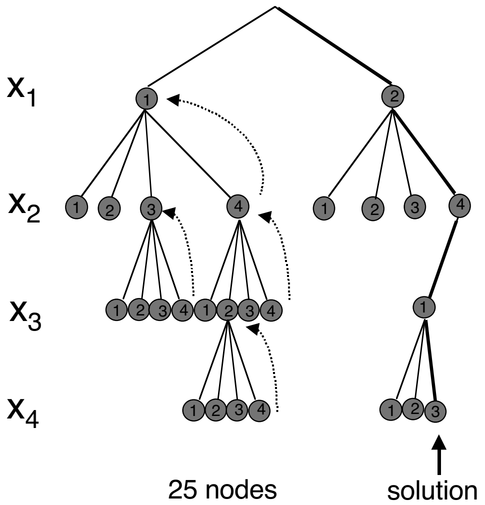
Depth-first tree traversal (DFS)

At each node:

- check every completely assigned constraint
- if consistent, continue DFS
- otherwise, prune current branch
- continue DFS

Complexity: $O(d^n)$

Backtracking on 4-queens

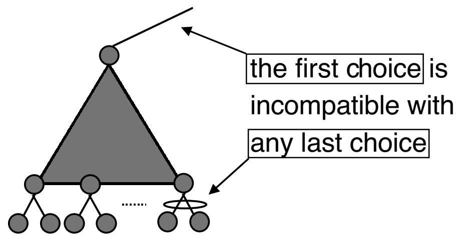


	1	2	3	4
x_1		Q		
x_2				Q
x_3	Q			
x_4			Q	

Problems of Backtracking

Thrashing:

- the same failure can be rediscovered an exponential number of times



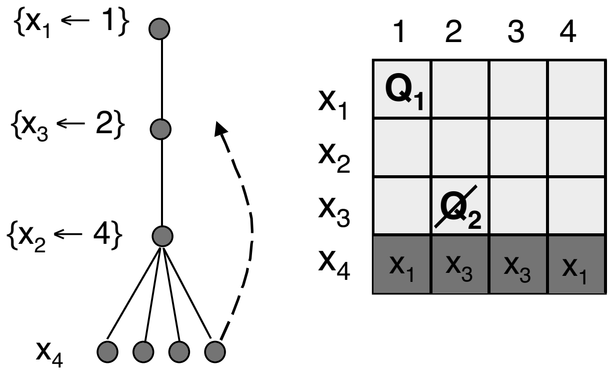
Solutions:

- check not completely assigned constraints: lookahead
- non-chronological backtracking: backjumping

Backjumping

Non-chronological backtracking:

- jumps to the last decision responsible for the dead-end
- intermediate decisions are removed



Inference

Inference: P ⟶ P'

legal operations on variables, domains, constraints

- P' is equivalent to P: Sol(P) = Sol(P')
- P' is presumably easier to solve than P
  - smaller search space
  - constraints are more explicit

Inference can be:

- complete: produces the solution
  adaptive consistency
- incomplete: requires further search
  arc consistency

Adaptive Consistency

Problem P, var x, C_x ={constraints on x}

Idea:

- Substitute C_x by a new constraint $\underline{c}$
- $\underline{c}$ summarizes the effect of C_x on P
- $\underline{c}$ does not mention x

now x is isolated: it can be eliminated

} variable elimination

Process:

problems:	P →	P' →	P'' →	. . . →	$P^{(n-1}$
#vars:	n	n-1	n-2	. . .	1

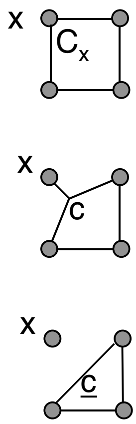
trivially solved

← solution without search

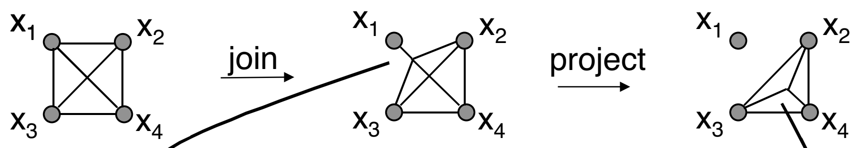
Variable Elimination

To eliminate var x:

- Join all constraints in $C_x \rightarrow c$
- Substitute C_x by c
- Project out variable x from $c \rightarrow \underline{c}$
- Substitute c by $\underline{c}$

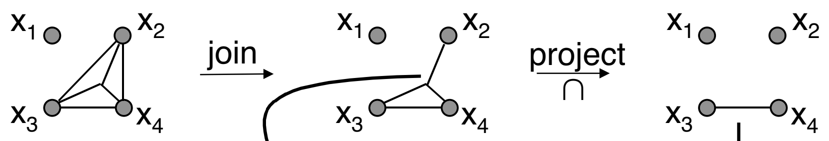


Example: 4-queens (x_1)



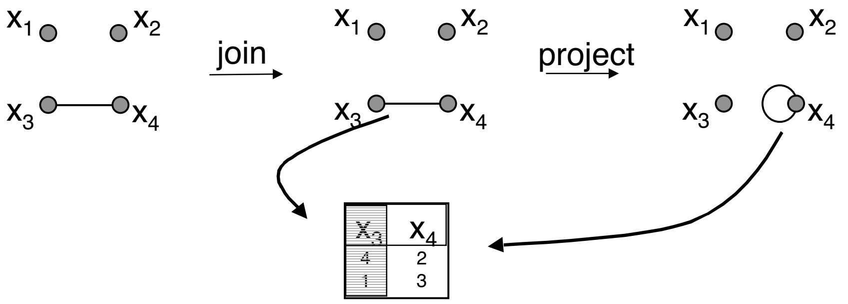
x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
1	3	2	2	3	1	4	4
1	3	2	3	3	1	4	2
1	3	4	2	3	1	4	1
1	3	4	3	3	1	2	4
1	4	2	2	3	1	2	2
1	4	2	3	3	1	2	1
1	4	4	2	4	2	3	3
1	4	4	3	4	2	3	2
2	4	1	1	4	2	1	3
2	4	1	3	4	2	1	2
2	4	1	4	4	1	3	3
2	4	3	1	4	1	3	2
2	4	3	3	4	1	1	3
2	4	3	4	4	1	1	2

Example: 4-queens (x_2)



x_2	x_3	x_4
~~1~~	~~3~~	~~2~~
1	4	2
~~1~~	~~4~~	~~4~~
~~4~~	~~2~~	~~3~~
4	1	3
~~4~~	~~1~~	~~1~~

Example: 4-queens (x_3)



x_3	x_4
4	2
1	3

Example: All Solutions 4-queens

	x_4	x_3	x_2	x_1
SOLUTION 1:	2	4	1	3

x_4
2
3

x_3	x_4
4	2
1	3

x_2	x_3	x_4
1	3	2
1	4	2
1	4	4
4	2	3
4	1	3
4	1	1

x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
1	3	2	3	3	1	4	4
1	3	2	3	3	1	4	2
1	3	4	2	3	1	4	1
1	3	4	3	3	1	2	4
1	4	2	2	3	1	2	2
1	4	2	3	3	1	2	1
1	4	4	2	4	2	3	3
1	4	4	3	4	2	3	2
2	4	1	1	4	2	1	3
2	4	1	3	4	2	1	2
2	4	1	4	4	1	3	3
2	4	3	1	4	1	3	2
2	4	3	3	4	1	1	3
2	4	3	4	4	1	1	2

Example: All Solutions 4-queens

	x_4	x_3	x_2	x_1
SOLUTION 2:	3	1	4	2

x_4
2
3

x_3	x_4
4	2
1	3

x_2	x_3	x_4
1	3	2
1	4	2
1	4	4
4	2	3
4	1	3
4	1	1

x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
1	3	2	3	3	1	4	4
1	3	2	3	3	1	4	2
1	3	4	2	3	1	4	1
1	3	4	3	3	1	2	4
1	4	2	2	3	1	2	2
1	4	2	3	3	1	2	1
1	4	4	2	4	2	3	3
1	4	4	3	4	2	3	2
2	4	1	1	4	2	1	3
2	4	1	3	4	2	1	2
2	4	1	4	4	1	3	3
2	4	3	1	4	1	3	2
2	4	3	3	4	1	1	3
2	4	3	4	4	1	1	2

Arc Consistency

- c is arc-consistent iff: every possible value of every variable in var (c) appears in rel(c)

- If c is not arc-consistent because $a \in d_x$:
  - a will not be in any solution
  - a can be removed: $d_x \leftarrow d_x - \{a\}$ (domain filtering)
  - if d_x becomes empty, P has no solution

  } inference

- P is arc-consistent iff: every constraint is arc-consistent

- If P is arc-consistent $\not\rightarrow$ P has solution

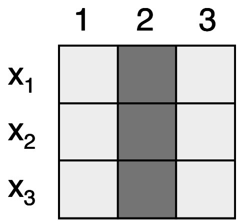
incomplete inference !!

Example: 3-queens

c_{12} is not arc-consistent
because value 2 of d_1

c_{12} is not arc-consistent
because value 2 of d_2

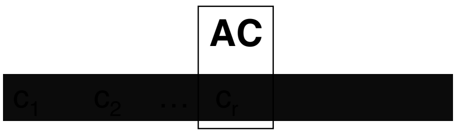
c_{23} is not arc-consistent
because value 2 of d_3



Constraint Propagation

- **AC**(c): procedure to make c arc consistent

- To make P arc-consistent, process each constraint ?



- But **AC**(c) may render other constraints arc-inconsistent

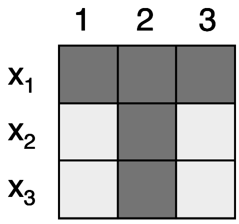
- To make P arc-consistent, iterate:
  - Apply **AC** on $\{c_1, c_2, \ldots, c_r\}$
  - Until no changes in domains: fix point

Example: 3-queens

value 2 of d_3 was removed
(to make c_{23} arc-consistent)

↓

this makes c_{13} arc-inconsistent



c_{13} is not arc-consistent
because value 1 of d_1

c_{13} is not arc-consistent
because value 3 of d_1 → domain d_1 empty

↓

no solution !!

Hybrids: Search + Inference

Idea:

- Search: backtracking (could be non-chronological)
- Inference: at each node, **AC** on some constraints
  - Future domains are pruned
  - Values no **AC** are eliminated

Effect:

- Future domains are reduced: less nodes to explore
- **AC** at each node: more work per node
- Very beneficial: reduces thrashing

Forward Checking

FC is a combination of:

- Search: backtracking
- Inference: at each node, **AC** on constraints with assigned and unassigned variables

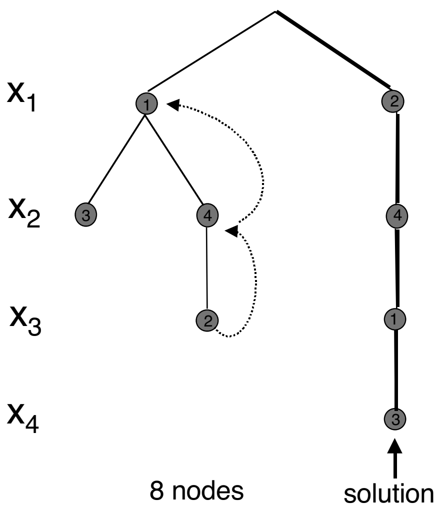
When a domain becomes empty :

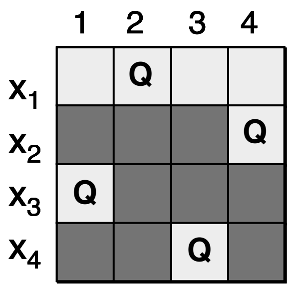
- No solutions following current branch
- Prune current branch and backtrack

Caution:

- Values removed by **AC** at level i, have to be restored when bactracking at level i or above

Example: FC on 4-queens





Maintaining Arc Consistency

MAC is a combination of:

- Search: backtracking
- Inference: at each node, **AC** on all constraints
- Preprocess: subproblems are **AC**

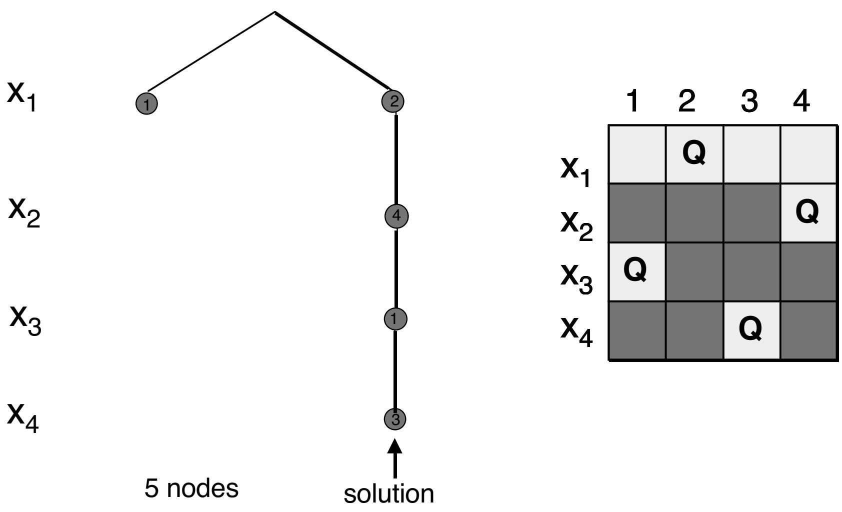
When a domain becomes empty :

- No solutions following current branch
- Prune current branch and backtrack

Caution:

- Values removed by **AC** at level i, have to be restored when bactracking at level i or above

Example: MAC on 4-queens



Search Heuristics

Dynamic variable selection:

- Variable may have different orders in branches
- Freedom to choose next variable

Heuristic:

1. Select the variable with minimum domain
   `domain`
2. Select the variable involved in most constraints
   `degree`

Combination: min (`domain / degree`)

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Modelling

Problem P as CSP:

- Several formulations are possible

- Select variables and domains
  - Search space size: $|d_1| \times |d_2| \times .. \times |d_n|$
  - Select formulation with smallest size

- Select constraints:
  - Number of constraints
  - Arity
  - **AC** cost
  - Pruning power

Constraints

Number:

- High: causes a high overhead
- Low: is preferred (compact representation)
- Keeping low number, some redundancy is advised

Arity: number of variables involved in a constraint

Arity and **AC**: trade-off

- arity(c) = k, AC(c) is $O(d^k)$
- high arity causes higher AC cost
- but AC on high arity constraints prunes more !!

Global Constraints

c is global iff:

- arity(c) > 2
- c is logically equivalent to $\{c_1, c_2, \ldots, c_k\}$ binary
- **AC**(c) prunes more than **AC**($c_1, c_2, \ldots, c_k$)

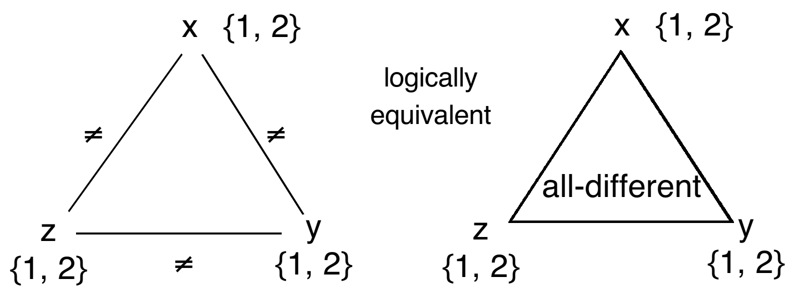
Propagation:

- specialized algorithms　　decrease AC complexity
- exploits the constraint semantics

Catalog:

- set of common structures to reuse
- best known algorithms for propagation

Example: all-different



3 binary constraints,
they are AC,
no pruning

1 ternary constraint,
it is not AC,
AC pruning → empty domain
no solution!!

Optimization

Constraint Optimization Problem: (X, D, C, F)
F(X) is a cost function

GOAL: min F(X), satisfying C

Solving method:

- Hybrid: search + consistency assigned constraints
  DFS

- When F(X) = Z*, add constraint F(X) < Z*

↑ Branch and Bound

Branch and Bound

Search: depth-first

At each node:

- Consistency on assigned constraints
- AC on (some) constraints (optional)
- Computes a lower bound of F(X): F(X)

Prunes current branch: when

- Inconsistent assigned constraint
- Empty domain (because AC)
- F(X) > Z* : no solution will improve Z*

Constraint Programmming

Declarative Programming: you declare

- Variables
- Domains
- Constraints

and ask the SOLVER to find a solution!!

SOLVER offers:

- Implementation for variables / domains / constraints
- Hybrid algorithm: backtracking + incomplete inference
- Global constraints + optimized AC propagation
- Empty domain detection
- Embedded heuristics

Constraint Logic Programming

Logic Programming:

- Depth-first search
- Unification: substitute equals by equals clauses/database

special case of constraint solving

substituted by

More general constraint solver → Constraint Logic Programming

Existing solvers:

- Chip, Eclipse, Mozart, Sictus Prolog (and many others)

Imperative Constraint Programming

Library to be included in your program

Provides:

- Special objects:
  - Variables / Domains / Constraints (global)
- Special functions to find:
  - One solution / the next solution

Existing Solvers:

- Ilog Solver, Choco

Summary

Constraint Satisfaction

- Search: backtracking
- Inference: complete / incomplete (AC)
- Hybrids: backtracking + AC

Constraint Programming

- Modelling: formulation / global constraints
- Optimization: branch and bound
- Existing Solvers: logical vs imperative CP

To know more . . .

Next week, slides and a list of references available at

http://www.iiia.csic.es/~pedro/