

# A Product Modal Logic

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Fuzzy modal logics are a family of logics that are still under research for their understanding. Several papers have been published on this issue, treating different problems about the fuzzy modal logics (see for instance [CR10], [CR11], [BEGR11], [HT06,HT06] or [CMRR13]). However, the study of the product modal logics, which we understand as logics that arise from Kripke structures whose relation and universes are evaluated over the product standard algebra, has remained undone. We present here some results to partially fill that gap for Kripke semantics with crisp accessibility relations, together with a characterization of a strongly complete infinitary product logic with truth-constants. We consider that the characterization and understanding of the product modal logics could open the possibility of studying the more general case of BL modal logics.

## 1 Propositional strong completeness

Propositional Product logic is finite strong complete but not strong complete with respect to the standard product chain over the real unit interval as proved in [Háj98]. In [Mon06], Montagna defined a logical system, an axiomatic extension of the BL logic with storage operator  $*$  and an infinitary rule

$$(R_M) \frac{\chi \vee (\varphi \rightarrow \psi^k), \text{ for all } k}{\chi \vee (\varphi \rightarrow \psi^*)},$$

that is proved to be strong complete (for infinite theories) with respect to the standard BL chains over the real unit interval. In particular, the expansion of Product Logic with the infinitary rule and Monteiro-Baaz Delta operator is complete with respect to the standard Product algebra over the real unit interval with Delta.

On the other hand, in [SCE<sup>+</sup>06], the addition of rational truth constants to product logic was studied, and it was proven that the extension of product logic with the  $\Delta$  axioms from before and natural axiomatization for the constants is finitely strong standard complete with respect to the canonical standard product algebra (where the rational constants are interpreted by its name, in  $[0, 1]_{\mathbb{Q}}$ ).

We let  $\Pi^*$  be the logic defined by the following axioms and rules:

- Axioms of  $\Pi$  (product propositional logic) (see for instance [Háj98])

- Axioms referring to rational constants over product logic [SCE<sup>+</sup>06]
- Axioms of the  $\Delta$  operator ([Háj98]) plus
 
$$\Delta \bar{c} \leftrightarrow \bar{\delta}(c), \text{ for each } c \in [0, 1]_{\mathbb{Q}}$$
 with  $\bar{\delta}(1) = 1$  and  $\bar{\delta}(x) = 0$  for  $x < 1$ .
- Rules of modus ponens and necessitation for  $\Delta$ : from  $\varphi$  derive  $\Delta\varphi$
- The infinitary rules

$$(\mathbf{R}_1) \frac{\bar{c} \rightarrow \varphi, \text{ for all } c \in (0, 1)_{\mathbb{Q}}}{\varphi} \quad (\mathbf{R}_2) \frac{\varphi \rightarrow \bar{c}, \text{ for all } c \in (0, 1)_{\mathbb{Q}}}{\neg\varphi}$$

Considering that these two last rules imply the archimedeanicity of the algebras associated to a logic closed by them, if we follow the usual precourse of extending a theory to another one complete and we let it be closed under  $R_1$  and  $R_2$ , we obtain that its Lindembaum sentence algebra is an archimedean product chain (with canonical constants). Following some ideas from [Mon06] and the results about product algebras available at [CT00], we can equally prove the strong completeness of  $\Pi^*$  with respect to the canonical standard product algebra,

$$\Gamma \vdash_{\Pi^*} \varphi \text{ iff } \Gamma \models_{[0,1]_{\Pi}} \varphi.$$

It is interesting that this logic has a natural behaviour, in the sense that the Deduction Theorem (with  $\Delta$ ) is still valid, i.e.  $\Gamma \cup \{\alpha\} \vdash_{\Pi^*} \varphi$  iff  $\Gamma \vdash_{\Pi^*} \Delta\alpha \rightarrow \varphi$ .

## 2 A modal product logic

Our aim is to define a modal logic over the standard product algebra with canonical rational truth-constants and the  $\Delta$  operator, by introducing the two usual modalities  $\Box$  and  $\Diamond$ , and with Kripke semantics defined by structures with crisp accessibility relations.

**Definition 1.** A *Crisp Product Kripke model (PK-model)* is a structure  $\mathcal{M} = \langle W, R, e \rangle$  where:

- $W$  a non-empty set of objects (worlds);
- $R \subseteq W \times W$  (a crisp accessibility relation);
- $e: W \times \mathcal{V} \rightarrow [0, 1]$  a truth-evaluation of propositional variables  $\mathcal{V}$  in each world  $[0, 1]_{\Pi}$ .

The evaluation  $e$  is extended to all modal formula in  $Fm_{\Box}$  (with constants) by defining inductively at each world  $w$ , the evaluation of propositional connectives by their corresponding operation in the algebra (over the evaluated terms), and the evaluation of modal formulas as follows:

$$e(w, \Box\varphi) := \inf\{e(v, \varphi) : R w v = 1\}; \quad e(w, \Diamond\varphi) := \sup\{e(v, \varphi) : R w v = 1\}$$

We can consider different notions of truth, depending on the locality. We say  $\varphi$  is true in  $M$  at  $w$ , and write  $M \models_w \varphi$  iff  $e(w, \varphi) = 1$ .  $\varphi$  is valid in  $M$  ( $M \models \varphi$ ) iff  $M \models_w \varphi$  for any  $w \in W$ , and finally,  $\varphi$  is PK-valid iff  $M \models \varphi$  for any PK-model  $M$ .

Then, at a semantic level we will study the *local (product crisp) modal logic  $PK_l$* , defined by letting for all sets  $\Gamma \cup \{\varphi\}$  of modal formulas without canonical constants  $\Gamma \models_{PK_l} \varphi$  iff for every CPK-model  $M$  and for any world  $w \in W$ , if  $M \models_w \Gamma$  then  $M \models_w \varphi$ . To provide an axiomatization for  $PK_l$ , we will consider the following logic  $K_{\Pi}$ :

- Axioms and rules from  $\Pi^*$ .
- $(K) : \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ ;
- $(\mathbf{A}_{\Box}1) : (\bar{c} \rightarrow \Box\varphi) \leftrightarrow \Box(\bar{c} \rightarrow \varphi)$ ;
- $(\mathbf{A}_{\Box}2) : \Delta\Box\varphi \leftrightarrow \Box\Delta\varphi$ ;
- $(\mathbf{A}_{\Diamond}1) : \Box(\varphi \rightarrow \bar{c}) \leftrightarrow (\Diamond\varphi \rightarrow \bar{c})$ ;
- $(\mathbf{A}_{\Diamond}2) : \Diamond\Delta\varphi \rightarrow \Delta\Diamond\varphi$ ;
- $(\mathbf{NR}_{\Box}) : \frac{\varphi}{\Box\varphi}$ , applied only over theorems

The notion of proof in  $K_{\Pi}$ , denoted  $\vdash_{K_{\Pi}}$  is defined as follows: for any set of formulas  $\Gamma \cup \{\varphi\}$ ,  $\Gamma \vdash_{K_{\Pi}} \varphi$  iff there is a (possibly infinite) proof from  $\Gamma$  to  $\varphi$  using axioms and rules from  $K_{\Pi}$ .

In the logic  $K_{\Pi}$ , the Deduction Theorem keeps holding because the new modal rule only affects theorems, but having an infinitary logic, to proceed towards a completeness theorem it is necessary to prove that if  $\Gamma \vdash_{K_{\Pi}} \varphi$  holds, then  $\Box\Gamma \vdash_{K_{\Pi}} \Box\varphi$  holds as well, where  $\Box\Gamma$  is a shorthand for  $\{\Box\psi \mid \psi \in \Gamma\}$ .

Since we only added as modal rule one just affects theorems, we can move from modal derivations to propositional ones, just adding a new set of premises (modal theorems):  $\Gamma \vdash_{K_{\Pi}} \varphi$  iff  $\Gamma \cup Th_{K_{\Pi}} \vdash_{\Pi^*} \varphi$  where  $Th_{K_{\Pi}} := \{\theta : \emptyset \vdash_{K_{\Pi}} \theta\}$ . This result is crucial for being able to use a canonical model to prove completeness.

The canonical model can be defined as usual, fixing as universes all the  $\Pi^*$ -evaluations into the canonical standard product algebra (of variables and modal formulae) that satisfy the modal theorems, and defining the relation by  $R_c v w = 1$  if and only if

- $v(\Box\theta) = 1 \Rightarrow w(\theta) = 1$  for all  $\theta \in Fm$ ; and
- $v(\Diamond\theta) < 1 \Rightarrow w(\theta) < 1$  for all  $\theta \in Fm$ ;

It can be proven that the extension of the evaluation to modal formulae keeps satisfying the condition  $e(v, \varphi) = v(\varphi)$  for every  $\varphi$ , i.e., that in particular both  $v(\Box\varphi) = \inf\{w(\varphi) : R_c v w = 1\}$  and  $v(\Diamond\varphi) = \sup\{w(\varphi) : R_c v w = 1\}$ , and so, the model we define works properly for the completeness proof.

With this, we obtain that  $K_{\Pi}$  is a modal logic with truth-constants, complete with respect to the class of crisp Kripke models whose worlds evaluate formulas over the canonical standard product algebra.

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