Chapter 11
On possibilistic modal logics defined over MTL-chains

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Dedicated to Petr Hájek

Abstract In this paper we revisit a 1994 paper by Hájek et al. where a modal logic over a finitely-valued Łukasiewicz logic is defined to capture possibilistic reasoning. In this paper we go further in two aspects: first, we generalize the approach in the sense of considering modal logics over an arbitrary finite MTL-chain, and second, we consider a different possibilistic semantics for the necessity and possibility modal operators. The main result is a completeness proof that exploits similar techniques to the ones involved in Hájek et al.'s previous work.

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11.1 Introduction

In this paper, as our humble homage to Petr Hájek, our aim is to revisit Hájek et al.'s paper[1] [14] where a modal logic over a finitely-valued Łukasiewicz logic is

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[1] Actually, [14] was for F. Esteva and L. Godo the first joint paper with P. Hájek.

defined to capture possibilistic reasoning. In this paper we go further in two aspects: first, by generalizing Hájek's approach in the sense of considering modal logics over an arbitrary finite MTL-chain, and second, by considering a slightly different possibilistic semantics for the necessity and possibility modal operators.

Indeed, in [14] the authors defined a modal logic to reason about possibility and necessity degrees[2] of many-valued propositions. This logic was a generalization of the so-called *Possibilistic logic* (see e.g. [6, 5]), a well-known uncertainty logic to reasoning with graded beliefs on classical propositions by means of necessity and possiblity measures. Possibilistic logic deals with weighted formulas (φ, r), where φ is a classical proposition and $r \in [0,1]$ is a weight, interpreted as a lower bound for the necessity degree of φ. The semantics of these degrees is defined in terms of possibility distributions $\pi : \Omega \to [0,1]$ on the set Ω of classical interpretations of a given propositional language. A possibility distribution π on Ω ranks interpretations according to its plausibility level: $\pi(w) = 0$ means that w is rejected, $\pi(w) = 1$ means that w is fully plausible, while $\pi(w) < \pi(w')$ means that w' is more plausible than w. A possibility distribution $\pi : \Omega \to [0,1]$ induces a pair of dual possibility and necessity measures on propositions, defined respectively as:

$$\Pi(\varphi) = \sup\{\pi(w) \mid w \in \Omega, w(\varphi) = 1\}$$
$$N(\varphi) = \inf\{1 - \pi(w) \mid w \in \Omega, w(\varphi) = 0\} .$$

They are dual in the sense that $\Pi(\varphi) = 1 - N(\neg\varphi)$ for every proposition φ. From a logical point of view, possibilistic logic can be seen as a sort of graded extension of the non-nested fragment of the well-known modal logic of belief KD45.[3]

When we go beyond the classical framework of Boolean algebras of events to generalized algebras of many-valued events, one has to come up with appropriate extensions of the notion of necessity and possibility measures for many-valued events, as explored in [3]. A natural generalization, and indeed the one that we will focus on for the main result in this paper, is to consider Ω as the set of propositional interpretations of some many-valued calculi defined by a t-norm $\odot$ and its residuum $\Rightarrow$. Then, a possibility distribution $\pi : \Omega \to [0,1]$ induces the following generalized possibility and necessity measures over many-valued propositions:

$$\Pi(\varphi) = \sup\{\pi(w) \odot w(\varphi) \mid w \in \Omega\}$$
$$N(\varphi) = \inf\{\pi(w) \Rightarrow w(\varphi) \mid w \in \Omega\} .$$

Actually, these definitions agree with the ones commonly used in many-valued modal logics (see for example [2] and the references therein) in the particular case where the many-valued accessibility relations R in Kripke-style frames (W,R) (i.e.,. binary $[0,1]$-valued relations $R : W \times W \to [0,1]$) are indeed defined by possibility distributions $\pi : W \to [0,1]$ by putting $R(w,w') = \pi(w')$ for any $w, w' \in W$.

[2] In the sense of Possibility Theory [4].

[3] In fact, as it is explained in Section 2, two-valued possibility and necessity measures over classical propositions can be taken as an alternative semantics for the modal operators in the system KD45.

Structure of the paper.

After this introduction, Section 2 contains a rather extensive overview of related work. The main contribution of the paper is the fuzzy modal system given in Section 3 that is shown to properly capture the intended possibilistic semantics for the modal operators. The result has a limited scope since it only aplies to modal logics over finite MTL-chains (expanded with truth-constants and the Monteiro-Baaz's $\triangle$ operator) and to a language with finitely many variables. The axioms and completeness proof are natural generalizations of the ones for the system MVKD45 in [14], where the assumption about finitely-many variables is also adopted. In that paper the semantics of the possibility modal operator is defined in terms of a 'sup - min' combination of possibility values of worlds and truth-values of formulas. Here the semantics of the possibility modal operator is defined as a 'sup - $\odot$' combination, where $\odot$ is the monoidal operation of the MTL-chain. As in [14], we make an extensive use of the so-called maximal elementary conjunctions, which are definable in our setting. Admittedly, this makes the resulting axiomatization not very elegant. The paper ends with Section 4 stating some conclusions and an open problem.

11.2 Related work on modal approaches to possibilistic logics

When reviewing the literature on logical formalizations of different kinds of possibilistic reasoning, one can indentify two classes of systems according to the kind of language used and the possibilistic semantics of modal necessity and possibility operators, namely modal-like two-tiered logics and full modal logics. In this section we provide a brief overview of the most relevant ones for our purposes in each class.

11.2.1 Two-tiered logics

By two-tiered logics we refer to systems whose language is defined in a two-level manner: non-modal formulas are formulas from a given propositional logic L_1 (e.g. classical propositional logic) and then modal formulas are propositional combinations (according to a second logic L_2) of atomic modal formulas of the kind $\Box\varphi$ and $\Diamond\varphi$, where $\varphi \in L_1$. In this way, the language of these systems allow neither formulas with nested modalities (e.g. $\Box \Diamond \varphi$ is not allowed) nor formulas mixing both non-modal and modal subformulas (e.g. $\varphi \to \Box\psi$ with $\varphi, \psi \in L_1$ is not allowed). In all these systems, models can be considered under the form of a possibility distribution $\pi : \Omega \to [0,1]$ on the set Ω of propositional evaluations for the logic L_1 (either classical or many-valued).

Among logics falling in this class we can find the following ones:

(i) The logic MEL [1] corresponds to the case of L_1 and L_2 being both classical propositional logic (CPL) and where models are subsets E of the set Ω of classical interpretations for the language of L_1, i.e. $E \subseteq \Omega$. The two-valued possibility distribution corresponding to a model $E \subseteq \Omega$ is nothing but its characteristic function, i.e. the mapping $\pi_E : \Omega \to \{0,1\}$ where $\pi_E(w) = 1$ if $w \in E$, and $\pi_E(w) = 0$ otherwise. Atomic modal formulas are evaluated in a model E as follows:

$$E \models \Box\varphi \quad \text{if} \quad w(\varphi) = 1 \text{ for all } w \in E.$$

A complete axiomatization of MEL, which indeed can be seen as a fragment of the modal logic KD, is given by the following set of additional axioms and rule to those of CPL:

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
(D) $\Diamond\top$
necessitation: if φ is a CPL tautology, derive $\Box\varphi$

In this logic, one can only express two-valued possibilities and necessities, i.e. that a proposition is certainly true ($\Box\varphi$), certainly false ($\Box\neg\varphi$), possibly true ($\Diamond\varphi$) or possibly false ($\Diamond\neg\varphi$). The epistemic status "unknown" can be represented as $\Diamond\varphi \wedge \Diamond\neg\varphi$, or equivalenty $\neg\Box\varphi \wedge \neg\Box\neg\varphi$.

(ii) While keeping $L_1 = L_2 = CPL$, a natural generalization of MEL is to allow graded possibilities and necessities. This is done in [7], where the authors define what they call *Generalized Possibilistic logic*, GPL for short. To deal with graded possibility and necessity they fix a finite scale of uncertainty values $U = \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$ and for each value $\lambda \in U \setminus \{0\}$ introduce a pair of modal operators $\Box_\lambda$ and $\Diamond_\lambda$. In this case models (epistemic states) are possibility distributions $\pi : \Omega \to U$ on the set Ω of classical interpretations for the language L_1 with values in U, and the evaluation of the modal formulas is as follows:

$$\pi \models \Box_\lambda \varphi \quad \text{if} \quad N_\pi(\varphi) = \min\{1 - \pi(w) \mid w(\varphi) = 0\} \geqslant \lambda.$$

The dual possibility operators are defined as $\Diamond_\lambda \varphi = \neg\Box_{(1-\lambda)^+}\neg\varphi$, where the superscript $^+$ refers to the succesor. The semantics of $\Diamond_\lambda \varphi$ is the natural one, i.e. $\pi \models \Diamond_\lambda \varphi$ whenever the possibility degree of φ induced by π, $\Pi(\varphi) = \max\{\pi(w) \mid w(\varphi) = 1\}$, is at least λ. A complete axiomatization of GPL is given in [7], an equivalent and shorter axiomatization is given by the following additional set of axioms and rules to those of CPL:

$\Box_\lambda(\varphi \to \psi) \to (\Box_\lambda \varphi \to \Box_\lambda \psi)$
$\Diamond_1 \top$
$\Box_{\lambda_1} \varphi \to \Box_{\lambda_2} \varphi$, if $\lambda_1 \geqslant \lambda_2$
necessitation: if φ is a CPL tautology, derive $\Box_1 \varphi$.

(iii) Another kind of systems that have been proposed in the literature are the ones by Hájek et al. [13, 11]. Here the idea is a bit different since it is based on a formalization where L_1 is still CPL but L_2 is Łukasiewicz infinitely-valued logic. The idea here is interpreting the modality $\Box$ as a fuzzy modality in the sense that a formula $\Box\varphi$ (where φ is a classical, Boolean proposition of L_1) is a fuzzy formula whose degree of truth in a given model is taken as the necessity degree of φ. Then Łukasiewicz logic is used to build compound expressions (out of atomic modal ones) and to reason about the truth-degrees of those fuzzy propositions. The logic can be augmented by the introduction of rational truth constants to allow explicitly reasoning with necessity and possibility degrees. A complete axiomatization is given by axioms of Łukasiewicz logic plus the following ones on modalities, where $\Diamond\varphi$ is defined as $\neg_Ł\Box\neg\varphi$ (we add the subindex Ł to differentiate the connectives of Łukasiewicz logic from the ones of CPL) :

$$\Box(\varphi \to \psi) \to_Ł (\Box\varphi \to_Ł \Box\psi)$$
$$\Diamond\top$$

necessitation: if φ is a CPL tautology, then derive $\Box\varphi$.

(iv) Finally, we mention that the latter fuzzy logic-based approach has been generalized to reason about the necessity and possibility of *fuzzy events* [6], where a fuzzy event refers to a proposition (modulo logical equivalence) in a given fuzzy logic [3]. In these systems both logics L_1 and L_2 refer to two (possibly different) fuzzy logics and address different generalizations of the notion of necessity and possibility degrees of a fuzzy proposition. In general, if the fuzzy logic L_1 is the logic of a (continuous) t-norm $*$ and its residuum $\Rightarrow$, possibilistic models are given by possibility distributions $\pi : W \to [0,1]$, where now W is the set of L_1 interpretations, that evaluate the necessity and possibility degree of a proposition φ from L_1 as follows:

$$\|\Box\varphi\|_\pi = \inf\{\pi(w) \Rightarrow w(\varphi) \mid w \in W\}$$
$$\|\Diamond\varphi\|_\pi = \sup\{\pi(w) * w(\varphi) \mid w \in W\}.$$

Basically, two choices of L_1 and L_2 have been addressed in the literature, namely taking L_1 and L_2 to be some variants of Łukasiewicz logic [9] or some variants of Gödel logic [3]. For instance, in the former case, when L_1 and L_2 coincide with the $(k+1)$-valued Łukasiewicz logic $Ł_k$ expanded with truth-constants, the following is a complete set of additional axioms and inference rules to those of $Ł_k$ for both modal and non-modal formulas:

$$\Box(\varphi \wedge_Ł \psi) \leftrightarrow_Ł (\Box\varphi \wedge_Ł \Box\psi)$$
$$\Diamond\top$$
$$\Box(\bar{r} \oplus \varphi) \leftrightarrow_Ł (\bar{r} \oplus \Box\varphi), \text{ for } r \in \{0, 1/k, \ldots, (k-1)/k, 1\}$$

where $\oplus$ refers to the strong disjunction of Łukasiewicz logics and $\bar{r}$ denotes the truth constant of value r. The interested reader is referred to [10] for a general treatment of this kind of logics.

11.2.2 Full modal systems

As it regards to full modal systems capturing possibilistic semantics, either within a classical or fuzzy logic approach, one can find less proposals in the literature. Next we point out two approaches.

The most basic "possibilistic" system is indeed the classical modal logic KD45. As it is well-known, the logic KD45 is axiomatized by the modal axioms

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
(D) $\Diamond\top$
(4) $\Box\varphi \to \Box\Box\varphi$
(5) $\Diamond\varphi \to \Box\Diamond\varphi$

and the necessitation inference rule for $\Box$, and it is sound and complete for the class of Kripke models $M = (W, e, R)$, where W is a non-empty set of worlds, $e : W \times Var \to \{0,1\}$ provides a truth-evaluation of variables in each world, and the accessibility relation $R \subseteq W \times W$ is actually of the form $R = W \times E$ with $\emptyset \neq E \subseteq W$, see e.g. [15]. Hence, R can be equivalently described by a two-valued possibility distribution $\pi_E : W \to \{0,1\}$ with $\pi_E(w) = 1$ if $w \in E$, and $\pi_E(w) = 0$ otherwise. This yields the following truth-evaluation for modal formulas:

$$(M, w) \models \Box\varphi \qquad \text{if} \qquad (M, w') \models \varphi \text{ for each } w' \in E,$$

which clearly shows that it does not depend on the particular world where it is evaluated but only on the whole model.

The other directly related full modal system that we would like to refer to, and that is in fact the main motivation for this paper, is 1994 Hájek et al.'s paper [14], where a modal account of a certain notion of necessity and possibility of fuzzy events is provided. In particular the logic MVKD45, that we describe below, is developed over the finitely-valued Łukasiewicz logic $Ł_k$ (with truth-values in the set $S_k = \{0, 1/k, \ldots, (k-1)/k, 1\}$) expanded with some unary operators to deal with truth-constants.

Let us summarize here the main ingredients of the logic and the given axiomatization. The language of MVKD45 is that of $Ł_k$ built from a finite set of propositional variables $Var = \{p_1, \ldots, p_n\}$ and connectives $\to$ and $\neg$, expanded with two modal operators $\Box$ and $\Diamond$. Actually, in finitely-valued Łukasiewicz modal logics, one could consider only one of them since the other is definable by duality: e.g. $\Box\varphi$ is $\neg\Diamond\neg\varphi$. Interestingly enough, for all truth-values $r \in S_k$, the unary connectives (r), such that the value of $(r)\varphi$ is 1 if the value of φ is r and 0 otherwise, are definable in $Ł_k$. We also use expressions $(\leqslant r)\varphi$ and $(\geqslant r)\varphi$ to denote the disjunctions $\bigvee_{i \in S_k : i \leqslant r}(i)\varphi$ and $\bigvee_{i \in S_k : i \geqslant r}(i)\varphi$, respectively.

The semantics of the modal operators is as follows. Models are possibilistic Kripke structures of the form $M = (W, e, \pi)$, where W is a non-empty set of possible worlds, $e : W \times Var \to S_k$ is an evaluation of propositional variables for each possible world and $\pi : W \to S_k$ is a normalized possibility distribution on W. Truth-evaluation of formulas are defined inductively in the usual way (we omit the reference to the model M):

- if $\varphi \in Var$, $\|\varphi\|_w = e(w, \varphi)$,
- if φ is a propositional combination, $\|\varphi\|_w$ is defined using the corresponding truth functions of the $Ł_k$ connectives,
- $\|\Box\varphi\|_w = \min\{\max(1 - \pi(w'), \|\varphi\|_{w'}) \mid w' \in W\}$,
- $\|\Diamond\varphi\|_w = \max\{\min(\pi(w'), \|\varphi\|_{w'}) \mid w' \in W\}$.

Note that this possibilistic semantics is a bit different from the general one we considered in item (iv) of the previous subsection. Actually this semantics was already proposed by Dubois, Prade et al. (see e.g. [6])Ê for generalizing necessity and possibility measures over fuzzy sets using Kleene-Dienes implication and minimum respectively.

The following axiomatization provided in [14] to capture this semantics makes heavy use of *maximally elementary conjunctions*. Given the finite set of propositional variables $Var = \{p_1, \ldots, p_n\}$, maximally elementary conjunctions (m.e.c.'s for short) are formulas of the kind $(r_1)p_1 \wedge \ldots \wedge (r_n)p_n$. The set of m.e.c.'s will be denoted **mec**. Axioms of MVKD45 are those of $Ł_k$ plus:

- Axioms of KD45:
  $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
  $\Box\varphi \leftrightarrow \Box\,\Box\varphi$
  $\Diamond\varphi \leftrightarrow \Box\,\Diamond\varphi$
  $\Diamond\top$
- $(r)\Box\varphi \leftrightarrow \Box(r)\Box\varphi$, $(r)\Diamond\varphi \leftrightarrow \Box(r)\Diamond\varphi$
- Possibilistic axioms:
  $((r)\Diamond\varphi \wedge E) \to (\leqslant r)(\varphi \wedge \Diamond E)$, with $E \in$ **mec**
  $(r)\Diamond\varphi \to \bigvee_{E\in\mathbf{mec}}(\geqslant r)\Diamond(E \wedge (r)(\varphi \wedge \Diamond E))$, with $r > 0$

Deductions rules are modus ponens and necessitation for $\Box$ (from φ infer $\Box\varphi$). In [14], the authors showed that this axiomatization is sound and complete with respect to the possibilistic semantics introduced above.

11.3 Extending the logic of a finite MTL-chain with possibilistic modal operators

In this section our aim is to generalize the above logic MVKD45 from [14]. On the one hand we consider more general many-valued propositional logics, namely we consider logics of finite linearly-ordered MTL algebras rather than only finitely-valued Łukasiewicz logics. But on the other hand, we consider a different semantics for the modal necessity and possibility operators than the one used in [14] and recalled in Section 2.2. Actually, the semantics adopted here is consistent with the one taken in [2], using the monoidal operation and its residuum to evaluate the possibility and necessity operators respectively rather than using the min operation and Kleene-Dienes implication as in [14]. Main consequences of these changes are that the necessity and possibility operators are no longer dual with respect to the negation of the logic, and that the necessity operator does not satisfy axiom (K).

In what follows, let $\mathbf{A} = (A, \wedge, \vee, \odot, \Rightarrow, 0, 1)$ denote a *finite* MTL-chain. Our modal logic will be defined on top of $\Lambda(\mathbf{A}^c_\triangle)$, the finitely-valued propositional logic of the finite MTL-chain $\mathbf{A}$ expanded with the Monteiro-Baaz's $\triangle$ operator and with truth constants $\overline{r}$ for each $r \in A$. Thus, the language of $\Lambda(\mathbf{A}^c_\triangle)$ is defined from a set of propositional variables using connectives $\wedge, \&, \rightarrow$ and $\triangle$, and truth constants $\overline{r}$.

For our purposes, we can consider the logic $\Lambda(\mathbf{A}^c_\triangle)$ as the consequence relation specified by: a formula φ logically follows from a set of formulas Γ, written $\Gamma \models_{\mathbf{A}^c_\triangle} \varphi$, whenever for each evaluation v of formulas into A such that $v(\psi) = 1$ for all $\psi \in \Gamma$, then $v(\varphi) = 1$ as well. Here, by evaluation we mean a mapping interpreting the connectives $\wedge, \&, \rightarrow$ into the algebra operations $\wedge, \odot, \Rightarrow$ respectively, the connective $\triangle$ into the function $\triangle : A \rightarrow A$ such that $\triangle(1) = 0$ and $\triangle(a) = 0$ for $a \neq 1$, and interpreting each truth-constant $\overline{r}$ into the value $r \in A$.

Then, we extend the language of $\Lambda(\mathbf{A}^c_\triangle)$ with two modal operators $\Box$ and $\Diamond$. Actually, we assume *our modal language to be generated from a finite set Var* $=$ $\{p_1, \ldots, p_n\}$ *of propositional variables* together with the connectives[4] $\wedge, \&, \rightarrow$, truth-constants $\overline{r}$ (for each $r \in A$) and unary operators $\triangle$, $\Box$ and $\Diamond$.

11.3.1 Semantics

Definition 11.1. An $\mathbf{A}$-*valued possibilistic Kripke frame* is a pair $F = \langle W, \pi \rangle$ where W is a non empty set (whose elements are called *worlds*) and π is a normalized A-valued unary relation (i.e., $\pi : W \longrightarrow A$ and there exists $w \in W$ such that $\pi(w) = 1$) called *possibility distribution*. ⊣

Actually, any $\mathbf{A}$-valued possibilistic Kripke frame $F = (W, \pi)$ can be considered as a usual Kripke frame $F = (W, R_\pi)$, where the A-valued binary accessibility relation $R_\pi : W \times W \rightarrow A$ is defined by $R_\pi(w, w') = \pi(w')$. Moreover, R_π is clearly serial, transitive and euclidean in the following generalized sense:

Serial: for every $w \in W$, there is $w' \in W$ such that $R_\pi(w, w') = 1$
Transitive: for every $w, w', w'' \in W$, $R_\pi(w, w') \odot R_\pi(w', w'') \leqslant R_\pi(w, w'')$
Euclidean: for every $w, w', w'' \in W$, $R_\pi(w, w') \odot R_\pi(w, w'') \leqslant R_\pi(w', w'')$.

Definition 11.2. An $\mathbf{A}$-*valued possibilistic Kripke model* (or simply a *possibilistic Kripke model*) is a 3-tuple $K = \langle W, e, \pi \rangle$ where $\langle W, \pi \rangle$ is an $\mathbf{A}$-valued possibilistic Kripke frame and e is a map, called *valuation*, assigning to each variable in *Var* and each world in W an element of A (i.e., $e : W \times Var \longrightarrow A$). We will say that K is *finite* when W is so. ⊣

[4] Other connectives are defined as usual in MTL, for instance $\neg\varphi$ is $\varphi \rightarrow \overline{0}$, $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$, and $\varphi \leftrightarrow \psi$ is $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

If $K = \langle W, e, \pi \rangle$ is a possibilistic Kripke model, the map e can be uniquely extended to a map $\| \cdot \|_{K,w} : Fm \longrightarrow A$ assigning to each formula and each world in $w \in W$ an element of A satisfying:

- $\|p\|_{K,w} = e(w,p)$ for each $p \in Var$,
- $\| \cdot \|_{K,w}$ is an algebraic homomorphism for the connectives $\wedge, \vee, \&, \rightarrow, \triangle$,
- $\|\overline{r}\|_{K,w} = r$, for every $r \in A$,
- and the following rules for evaluating modal formulas

$$\|\Diamond\varphi\|_{K,w} = \max_{u \in W}\{\pi(u) \odot \|\varphi\|_{K,u}\}, \tag{Sem-$\Diamond$}$$

$$\|\Box\varphi\|_{K,w} = \min_{u \in W}\{\pi(u) \Rightarrow \|\varphi\|_{K,u}\}. \tag{Sem-$\Box$}$$

Notice that the truth-evaluations for modal formulas starting with $\Diamond$ or $\Box$ do not depend on the particular world w but only on W and π. Also we define $\|\varphi\|_K = \min\{\|\varphi\|_{K,w} \mid w \in W\}$. When $\|\varphi\|_K = 1$ (resp. $\|\varphi\|_{K,w} = 1$) we will also write $K \models \varphi$ (resp. $(K,w) \models \varphi$). Finally, we define the notion of (local) logical consequence as follows: for any set of formulas $\Gamma \cup \{\varphi\}$, φ follows from Γ, denoted $\Gamma \models \varphi$, whenever for any model $K = (W, e, \pi)$ and world $w \in W$, if $(K,w) \models \psi$ for every $\psi \in \Gamma$ then $(K,w) \models \varphi$.

Call *reduced* a possibilistic Kripke model $K = (W, e, \pi)$ such that for any worlds $w, w' \in W$, if $e(w, \cdot) = e(w', \cdot)$ then $w = w'$, and hence $\pi(w) = \pi(w')$.[5] Since we are assuming that both the set of propositional variables Var and the MTL-chain $\mathbf{A}$ are finite, it holds that there is a finite number of reduced models and all of them have a finite number of worlds as well. Next lemma shows that we can actually restrict ourselves to consider the subclass of reduced possibilistic Kripke models.

Proposition 11.1. *For any possibilistic Kripke model K there is a reduced model K' such that $\|\varphi\|_K = \|\varphi\|_{K'}$ for any formula φ.*

Proof: Let $K = (W, e, \pi)$ be a possibilistic Kripke model and define an equivalence relation on W as follows: $w \cong w'$ whenever $e(w,p) = e(w',p)$ for all propositional variables $p \in Var$. We will denote by $[w]$ the equivalence class of w. Let us define the model $K' = (W', e', \pi')$ as follows:

1. $W' = W / \cong$
2. for each $w \in W$, $e'([w], p) = e(w,p)$ for all $p \in Var$
3. $\pi' : W' \rightarrow [0,1]$ is the mapping defined as $\pi'([w]) = \max\{\pi(w') \mid w' \in [w]\}$.

Clearly, K' is reduced. Let us check by induction that, for any formula φ and any $w \in W$, $\|\varphi\|_{K,w} = \|\varphi\|_{K',[w]}$. Indeed, this is obvious for φ being a propositional variable. The inductive steps for the propositional connectives are also clear, and the interesting steps are the cases of the modal operators:

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[5] We use the notation $e(w, \cdot)$ to denote the function $p \in Var \longmapsto e(w,p) \in A$.

- Let $\varphi = \Box\psi$. Then, using the induction hypothesis, we have the following chain of equalities:

  $\|\Box\varphi\|_{K,w} = \min_{w\in W}\{\pi(w) \Rightarrow \|\psi\|_{K,w}\} = \min_{w\in W}\{\pi(w) \Rightarrow \|\psi\|_{K',[w]}\} = \min_{w\in W}\{(\max_{w'\in[w]}\pi(w')) \Rightarrow \|\psi\|_{K',[w]}\} = \min_{w\in W}\{\pi'([w]) \Rightarrow \|\psi\|_{K',[w]}\} = \min_{[w]\in W'}\{\pi'([w]) \Rightarrow \|\psi\|_{K',[w]}\} = \|\Box\varphi\|_{K',[w]}$.

  We point out that the third equality is an easy consequence of the inclusion $\{\pi(w') \Rightarrow \|\psi\|_{K',[w']} : w' \in W\} \supseteq \{(\max_{w'\in[w]}\pi(w')) \Rightarrow \|\psi\|_{K',[w]} : w \in W\}$.
- Let $\varphi = \Diamond\psi$. Then, using the induction hypothesis, we have the following chain of equalities:

  $\|\Diamond\varphi\|_{K,w} = \max_{w\in W}\{\pi(w) \odot \|\psi\|_{K,w}\} = \max_{w\in W}\{\pi(w) \odot \|\psi\|_{K',[w]}\} = \max_{w\in W}\{(\max_{w'\in[w]}\pi(w')) \odot \|\psi\|_{K',[w]}\} = \max_{w\in W}\{\pi'([w]) \odot \|\psi\|_{K',[w]}\} = \max_{[w]\in W'}\{\pi'([w]) \odot \|\psi\|_{K',[w]}\} = \|\Diamond\varphi\|_{K',[w]}$.

This ends the proof. $\dashv$

This last result, together with the fact that there are only finitely many reduced models, has the following relevant consequences.

Corollary 11.1. *Modulo semantical equivalence, there are only a finite number of different formulas. Therefore, if Γ is a possibly infinite set of formulas, then there exists a finite subset $\Gamma_0 \subseteq \Gamma$ that is semantically equivalent to Γ in the following sense: for any formula φ, $\Gamma \models \varphi$ iff $\Gamma_0 \models \varphi$.*

Throughout the rest of the paper, we will make use of the following notation conventions:

- $(1)\varphi$ will denote the formula $\triangle\varphi$, and $(0)\varphi$ will denote the formula $\triangle\neg\varphi$,
  $(r)\varphi$ will denote the formula $\triangle(\overline{r} \leftrightarrow \varphi)$ (when $r \notin \{0,1\}$),
  $(\geqslant r)\varphi$ will denote the formula $\triangle(\overline{r} \to \varphi)$
  $(> r)\varphi$ will denote the formula $(\geqslant r)\varphi \wedge \neg(r)\varphi$
  $(\leqslant r)\varphi$ will denote the formula $\triangle(\varphi \to \overline{r})$
  $(< r)\varphi$ will denote the formula $(\leqslant r)\varphi \wedge \neg(r)\varphi$
- Propositional combinations of formulas of the kind $(r)\varphi$, where φ is an arbitrary formula, will be called *B-formulas* (for Boolean formulas)
- As in MVD45, *maximally elementary conjunctions* (m.e.c.'s) are B-formulas that are conjunctions of the form $\bigwedge_{i=1,\ldots,n}(r_i)p_i$ (remember that $p_1,\ldots,p_n$ are the finitely many fixed propositional variables). We will keep denoting by **mec** the set of all m.e.c.'s.

Note that for each B-formulas φ and ψ, the formulas $\varphi\wedge\psi$ and $\varphi\&\psi$ are equivalent, that is, $\|(\varphi\wedge\psi) \leftrightarrow (\varphi\&\psi)\|_K = 1$ for all possibilistic Kripke models K.

Next lemma shows some useful tautologies of the class of possibilistic Kripke models.

Lemma 11.1. *The following equivalences are tautologies for the class of all Possibilistic Kripke models:*

1. $\varphi \leftrightarrow \bigwedge_{r\in A}((r)\varphi \to \overline{r})$
2. $\Box\varphi \leftrightarrow \bigwedge_{r\in A}(\Diamond(r)\varphi \to \overline{r})$
3. $\Box(\varphi \to \overline{r}) \leftrightarrow (\Diamond\varphi \to \overline{r})$
4. $(0)\Diamond\varphi \leftrightarrow (1)\Box\neg\varphi$
5. $(r)\Diamond\varphi \leftrightarrow \Big((1)\Box(\varphi \to \overline{r}) \wedge (<1)\Box(\varphi \to \overline{r^-})\Big)$, *if $r > 0$ and r^- is the predecessor of r .*

Proof: 1. Obvious.

2. $\|\Box\varphi\|_K = \bigwedge_{w\in W}\{\pi(w) \Rightarrow \|\varphi\|_{K,w}\} =$
$\bigwedge_{r\in A}(\bigwedge\{\pi(w) \Rightarrow r \mid w \in W, \|\varphi\|_{K,w} = r\}) =$
$\bigwedge_{r\in A}(\bigwedge_{w\in W}\{\pi(w) \Rightarrow (\|(r)\varphi\|_{K,w} \Rightarrow r)\}) =$[6]
$\bigwedge_{r\in A}(\bigwedge_{w\in W}\{(\pi(w) \odot \|(r)\varphi\|_{K,w}) \Rightarrow r\}) =$[7]
$\bigwedge_{r\in A}\{(\bigvee_{w\in W}\{\pi(w) \odot \|(r)\varphi\|_{K,w}\}) \Rightarrow r\} =$
$\bigwedge_{r\in A}\{\|\Diamond(r)\varphi\|_K \Rightarrow r\} =$
$\|\bigwedge_{r\in A}(\Diamond(r)\varphi \to \overline{r})\|_K$.
3. $\|\Box(\varphi \to \overline{r})\|_K = \bigwedge_{w\in W}\{\pi(w) \Rightarrow (\|\varphi\|_{K,w} \Rightarrow r)\} =$
$\bigwedge_{w\in W}\{(\pi(w) \odot \|\varphi\|_{K,w}) \Rightarrow r\} =$
$(\bigvee_{w\in W}\{(\pi(w) \odot \|\varphi\|_{K,w})\} \Rightarrow r =$
$\|\Diamond\varphi \to \overline{r}\|_K$.
4. Taking $r = 0$ in item 3 we get that $\Box(\neg\varphi) \leftrightarrow \neg\Diamond\varphi$ is a tautology, and hence in particular $(1)\Box\neg\varphi \leftrightarrow (0)\Diamond\varphi$ as well.
5. If $r > 0$ then the claim directly follows from the observation that for any formula ψ, $(r)\psi$ is equivalent to $(1)(\psi \to \overline{r}) \wedge (<1)(\psi \to \overline{r^-})$. Then, by item 3, we have that $(r)\Diamond\varphi$ is equivalent to $(1)(\Box\varphi \to \overline{r}) \wedge (<1)(\Box\varphi \to \overline{r^-})$.

⊣

Taking into account that item 1 of Lemma 11.1 gives that

$$\Diamond\varphi \leftrightarrow \bigwedge_{r\in A}((r)\Diamond\varphi \to \overline{r})$$

is a tautology, items 2, 4 and 5 of the same lemma tell us that, due to the presence of the truth-constants, the modal operators $\Box$ and $\Diamond$ are indeed inter-definable:

$\Box\varphi$ as $\bigwedge_{r\in A}(\Diamond(r)\varphi \to \overline{r})$, and
$\Diamond\varphi$ as $(>0)\Box\neg\varphi \wedge \Big(\bigwedge_{r\in A\setminus 0}\Big((1)\Box(\varphi \to \overline{r}) \wedge (<1)\Box(\varphi \to \overline{r^-})\Big) \to \overline{r}\Big)$.

Indeed, the latter is obtained by noticing that, by the above equivalence, $\Diamond\varphi$ is equivalent to $((0)\Diamond\varphi \to \overline{0}) \wedge (\bigwedge_{r\in A\setminus 0}((r)\Diamond\varphi \to \overline{r}))$, and then by applying item 4 to the first conjunct and item 5 to the second conjunct.

[6] Here we use the fact the equation $x \Rightarrow (y \Rightarrow z) = (x \odot y) \Rightarrow z$ holds in every MTL-chain.

[7] Here we use the fact the equation $(x_1 \Rightarrow y) \wedge (x_2 \Rightarrow y) = (x_1 \vee x_2) \Rightarrow y$ holds in every MTL-chain.

11.3.2 Syntax

Assuming a Hilbert style axiomatization (with modus ponens as unique inference rule) of $\Lambda(\mathbf{A})$ (i.e., the propositional logic of the MTL-chain $\mathbf{A}$), one can get an axiomatization of $\Lambda(\mathbf{A}^c_\triangle)$, its expansion with the Baaz-Monteiro $\triangle$ operator and canonical truth-constants, by adding (cf. [2, Prop. A.12]):

- the well-known axioms and necessitation rule for $\triangle$ (see e.g. [11]),
- the following book-keeping axioms:[8]

  $(\overline{r}\&\overline{s}) \leftrightarrow \overline{r \odot s}$
  $(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow s}$
  $(\overline{r} \wedge \overline{s}) \leftrightarrow \overline{\min(r,s)}$
  $\triangle\overline{r} \leftrightarrow \overline{\triangle r}$,

- and the witnessing axiom:

  $\bigvee_{r\in A}(\varphi \leftrightarrow \overline{r})$.

The last book-keeping axiom guarantees that truth-constants in the logic behave as *canonical* ones, in the sense that each truth-constant $\overline{r}$ is actually interpreted as the value r in $\mathbf{A}$.

Taking this into account, next we define an axiomatic system for the modal expansion of $\Lambda(\mathbf{A}^c_\triangle)$ that will be shown to be sound and complete with respect to the class of possibilistc Kripke models defined above.

Definition 11.3. The logic $Pos(\mathbf{A}^c_\triangle)$ has the following axioms:

- Axioms of $\Lambda(\mathbf{A}^c_\triangle)$
- Axioms from KD45:

(4) $\Box\varphi \leftrightarrow \Box\Box\varphi$
(5) $\Diamond\varphi \leftrightarrow \Box\Diamond\varphi$
(D) $\Diamond\top$

(4') $(r)\Box\varphi \leftrightarrow \Box(r)\Box\varphi$, for each $r \in A$
(5') $(r)\Diamond\varphi \leftrightarrow \Box(r)\Diamond\varphi$, for each $r \in A$

- Possibilistic axioms (for each $r \in A$):

(NΠ) $\Box(\varphi \to \overline{r}) \leftrightarrow (\Diamond\varphi \to \overline{r})$
(Π1) $((r)\Diamond\varphi \wedge E) \to (\leqslant r)(\varphi\&\Diamond E)$, with $E \in \mathbf{mec}$
(Π2) $(r)\Diamond\varphi \to \bigvee_{E\in\mathbf{mec}}(\geqslant r)\Diamond(E \wedge (r)(\varphi\&\Diamond E))$, with $r > 0$

---

[8] Notice that these axioms could also be expressed as the following B-formulas:

$$(r\odot s)(\overline{r}\&\overline{s}),\quad (r\Rightarrow s)(\overline{r}\to\overline{s}),\quad (\min(r,s))(\overline{r}\wedge\overline{s}),\quad (\triangle r)\triangle\overline{r}\quad \text{and}\quad \bigvee_{r\in A}(r)\varphi.$$

However, the adopted formulation makes less use of the $\triangle$ connective.

Deductions rules of $Pos(\mathbf{A}^c_\triangle)$ are modus ponens, necessitation for $\triangle$ (from φ derive $\triangle\varphi$) and monotonicity for $\Box$: if $\varphi \to \psi$ is a theorem, infer $\Box\varphi \to \Box\psi$.

The notion of proof in $Pos(\mathbf{A}^c_\triangle)$, denoted $\vdash$, is defined from the above axioms and rules (notice that the application of the monotonicity rule for $\Box$ is restricted to theorems, in contrast to the other two rules). $\dashv$

Axioms $(\Pi 1)$ and $(\Pi 2)$ actually capture the semantics of the $\Diamond$ operator defined in (Sem-$\Diamond$) as a maximum of values. If $\Diamond\varphi$ takes value r, $(\Pi 1)$ tells us that each element in the maximum must be less of equal than r, while $(\Pi 2)$ expresses that the maximum is actually attained. Notice also that each m.e.c. E correspond to a possible world w, and hence the value of $\Diamond E$ corresponds to the possibility distribution on w.

To prove soundness of $Pos(\mathbf{A}^c_\triangle)$ with respect to the possibilistic Kripke semantics, we need first to prove some auxiliary results in the next lemma.

Lemma 11.2. *Let $K = (W, e, \pi)$ be a possibilistic Kripke model. Then the following conditions hold:*

1. *For each $w \in W$ and formula φ, there is a unique m.e.c. E and truth-value $r \in A$ such that $(K, w) \models E \wedge (r)\varphi$.*
2. *For each formula φ and every m.e.c. E, if $(K, w) \models E$ for some $w \in W$, then there exists a unique value r such that $K \models E \to (r)\varphi$.*
3. *For any m.e.c. E, formula φ and value r, it holds $K \models (>0)\Diamond((r)\varphi \wedge E) \to (E \to (r)\varphi)$.*

Proof: Items 1 and 2 are easy. As for item 3, let $w \in W$, and assume $(K, w) \models (>0)\Diamond((r)\varphi \wedge E)$. Then necessarily $\|\Diamond((r)\varphi \wedge E)\|_w > 0$, i.e. there exists $w' \in W$ such that $\pi(w') \odot \|(r)\varphi \wedge E\|_{w'} > 0$. Since $(r)\varphi \wedge E$ is a B-formula, the latter holds iff $\pi(w') > 0$ and $\|(r)\varphi \wedge E\|_{w'} = 1$. Therefore $(K, w') \models E$, and by item 2, r is actually the unique value such that $K \models E \to (r)\varphi$, and hence in particular, $(K, w) \models E \to (r)\varphi$. So we have proved that for any $w \in W$, $(K, w) \models (>0)\Diamond((r)\varphi \wedge E) \to (E \to (r)\varphi)$. $\dashv$

Theorem 11.1 (Soundness). *The logic $Pos(\mathbf{A}^c_\triangle)$ is sound with respect to the class of possibilistic Kripke models.*

Proof: Notice that, as observed before, possibilistic Kripke models can be considered as many-valued Kripke models with a transitive, euclidean and serial accessibility relation. Therefore, from general results in [16], axioms (4), (5) and (D) are automatically sound. Moreover, the related axioms with truth constants (4') and (5') are also sound as an easy computation shows, and the soundness of axiom (NΠ) is just item 3 of Lemma 11.1. Next we prove soundness of axioms $(\Pi 1)$ and $(\Pi 2)$.

$(\Pi 1)$: Assume there is $w \in W$ such that $(K, w) \models (r)\Diamond\varphi \wedge E$, otherwise the result is trivial. Then $\|\Diamond E\|_w = \max\{\pi(w') \mid w' \models E\} = \pi(w)$ since w can only be the unique world in W such that $w \models E$. Then $\|\varphi \& \Diamond E\|_w = \|\varphi\|_w \odot \|\Diamond E\|_w = \|\varphi\|_w \odot \pi(w) \leqslant \max\{\|\varphi\|_{w'} \odot \pi(w') \mid w' \in W\} = \|\Diamond\varphi\| = r$.

($\Pi 2$): Assume $r > 0$, otherwise it is obvious. If $r = \|\Diamond\varphi\|$, then there is $w_0 \in W$ and a m.e.c. E such that $r = \|\varphi\|_{w_0} \odot \pi(w_0)$, $w_0 \models E$ and $\pi(w_0) = \|\Diamond E\|$. Therefore $r = \|\varphi\|_{w_0} \odot \|\Diamond E\|$. Thus $\|E \wedge (r)(\varphi \& \Diamond E)\|_{w_0} = 1$ and hence $\|E \wedge (r)(\varphi \& \Diamond E)\|_{w_0} \odot \pi(w_0) \geqslant \|\varphi\|_{w_0} \odot \pi(w_0) = r$. Consequently, we have $\|\Diamond(E \wedge (r)(\varphi \& \Diamond E))\| \geqslant r$.

Modus ponens and necessitation for $\triangle$ are trivially sound, and finally, it is also easy to show that monotonicity inference rule for $\Box$ is also sound when applied to valid implications. $\dashv$

It is worth mentioning that the well-known axiom (K) is not sound in general (although it is indeed sound for B-formulas), except for the case when the finite MTL-chain **A** is a Gödel chain (see [2, Corollay 3.13] for details). Actually this is the main difference with the modal system studied in [14], since there the semantics of the necessity operator $\Box$ (defined in Section 2.2), using min instead of $\odot$, makes axiom (K) sound for every MTL-chain **A**.

11.3.3 Completeness

To finish this section our aim is to show that the logic $Pos(\mathbf{A}^c_\triangle)$ also enjoys strong completeness with respect to the semantics defined in the previous Section 11.3.1. Let us remind that our axiomatization is already complete with respect to the non-modal semantics given by the chain $\mathbf{A}^c_\triangle$, see [2, Prop. A.12] for more details. Moreover, for instance, the $\triangle$-deduction theorem

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ iff } \Gamma \vdash \triangle\varphi \to \psi$$

can be straightforwardly proved to hold by induction on the rules of our axiomatization (take into account the monotonicity rule only applies to theorems).

Definition 11.4. A theory is a set of B-formulas. A theory T is consistent if $T \not\vdash \overline{0}$. A theory T is complete if for each B-formula ψ, either $T \vdash \psi$ or $T \vdash \neg\psi$. Moreover, we say that two complete theories T and T' are equivalent, written $T \approx T'$, if for each r and φ, $T \vdash (r)\Diamond\varphi$ iff $T' \vdash (r)\Diamond\varphi$. $\dashv$

Note that any inconsistent theory is complete, and all inconsistent theories are equivalent. On the other hand, using classical techniques, one can show that any consistent theory T can be always extended to a complete and consistent super-theory $T^\star \supseteq T$.

Next we will prove some lemmas necessary for the completeness proof.

Lemma 11.3. *(a) If φ is a B-formula, $\vdash \varphi \leftrightarrow (1)\varphi$, $\vdash \neg\varphi \leftrightarrow (0)\varphi$ and $\vdash \varphi \vee \neg\varphi$.*
(b) If φ is a B-formula and $0 < r < 1$, then $(r)\varphi \vdash \overline{0}$.
(c) T is complete and consistent iff for every formula φ there exists a unique r such that $T \vdash (r)\varphi$.

*(d) For each complete and consistent theory T there is a unique $E \in$ **mec** such that $T \vdash E$. We will denote such a unique m.e.c. E_T.*

Proof: (a) Taking into account that $(1)\varphi$ is $\triangle\varphi$ and $(0)\varphi$ is $\triangle\neg\varphi$, and that B-formulas are propositional combinations of formulas starting with $\triangle$, it turns out that the considered formulas are already tautologies in $\mathrm{MTL}_\triangle$ (considering all the $\Box$-formulas and $\Diamond$-formulas as propositional variables), and hence, by completeness of $\mathrm{MTL}_\triangle$, they are also provable in $\mathrm{MTL}_\triangle$, and thus in $\Lambda(\mathbf{A}^c_\triangle)$ as well.

(b) This is an immediate consequence that our axiomatization is complete with respect to the non-modal semantics over the chain $\mathbf{A}^c_\triangle$.

(c) Suppose T is complete and $T \not\vdash (r)\varphi$ for each $r \in A$. Then $T \vdash \neg(r)\varphi$ for each $r \in A$, hence $T \vdash \bigwedge_{r\in A} \neg(r)\varphi$, in other words, $T \vdash \neg \bigvee_{r\in A}(r)\varphi$. But this is in contradiction with the witnessing axiom $\bigvee_{r\in A}(\varphi \leftrightarrow \bar{r})$. Conversely, let ψ be a B-formula and let r the unique value such that $T \vdash (r)\psi$ given by the assumption. By (b), it follows that $r \in \{0,1\}$, hence either $T \vdash (1)\psi$ or $T \vdash (0)\psi$, i.e. either $T \vdash \psi$ or $T \vdash \neg\psi$.

(d) Since T is complete and consistent, by (c), for each propositional variable p_i there is a unique r_i such that $T \vdash (r_i)p_i$, hence T proves the m.e.c. $\bigwedge_{i=1,\ldots,n}(r_i)p_i$. $\dashv$

Lemma 11.4. *The following conditions hold:*

(a) For any B-formula φ and any consistent theory T, if $T \vdash (>0)\Diamond\varphi$ then φ is consistent (i.e., $\varphi \not\vdash \bar{0}$).
(b) For any B-formula φ, $\vdash \Diamond\neg\varphi \to \neg\Box\varphi$.
(c) For any B-formulas φ and ψ, $\vdash \Box\varphi \to (\Diamond\psi \to \Diamond(\varphi \wedge \psi))$.

Proof: (a) Assume φ is inconsistent. Then by the $\triangle$-deduction theorem, $\vdash \triangle\varphi \to \bar{0}$, i.e. $\vdash \neg\triangle\varphi$, but since φ is a B-formula, $\vdash \varphi \leftrightarrow \triangle\varphi$, we have $\vdash \neg\varphi$. By the rule of necessitation for $\Box$, $\vdash \Box\neg\varphi$, hence (by axiom $(N\Pi)$), we have $\vdash \neg\Diamond\varphi$, i.e. $\vdash (0)\Diamond\varphi$, and hence $T \vdash (0)\Diamond\varphi$ as well. But this is in contradiction with the hypothesis that $T \vdash (>0)\Diamond\varphi$.

b) Let φ be a B-formula. Taking $r = 0$, axiom $(N\Pi)$ gives $\neg\Diamond\neg\varphi \leftrightarrow \Box\neg\neg\varphi$, but if φ is a B-formula $\neg\neg\varphi$ is equivalent to φ. Hence $\vdash \neg\neg\Diamond\neg\varphi \leftrightarrow \neg\Box\varphi$, and since $\psi \to \neg\neg\psi$ is a theorem of MTL, we thus have $\vdash \Diamond\neg\varphi \to \neg\Box\varphi$.

(c) For B-formulas, by (a) of Lemma 11.3, φ and ψ we have that ψ is equivalent to $(\psi \wedge \neg\varphi) \vee (\psi \wedge \varphi)$. Hence $\vdash \Diamond\psi \leftrightarrow (\Diamond(\psi \wedge \neg\varphi) \vee \Diamond(\psi \wedge \varphi))$, hence $\vdash \Diamond\psi \to (\Diamond(\neg\varphi) \vee \Diamond(\psi \wedge \varphi))$, and by (b), $\vdash \Diamond\psi \to (\neg\Box\varphi \vee \Diamond(\psi \wedge \varphi))$. Now, using that $(\neg\varphi \vee \psi) \to (\varphi \to \psi)$ is a theorem of MTL, we also have $\vdash \Diamond\psi \to (\Box\varphi \to \Diamond(\varphi \wedge \psi))$, and hence $\vdash \Box\varphi \to (\Diamond\psi \to \Diamond(\varphi \wedge \psi))$ as well.

$\dashv$

Lemma 11.5. *Let T_0 be a complete and consistent theory and let $T_0 \vdash (r)\Diamond\varphi$. Then, the following conditions hold:*

*(a) For any theory $T \approx T_0$ and for any $E \in$ **mec**, $T \vdash E \to (\leqslant r)(\varphi \& \Diamond E)$.*
*(b) There is a theory $T \approx T_0$ and $E \in$ **mec** such that $T \vdash E \wedge (r)(\varphi \& \Diamond E)$.*

Proof: (a) Using (b) of Lemma 11.3, there is a unique value r such that $T \vdash (r)\Diamond\varphi$. If E is a m.e.c., then both $(r)\Diamond\varphi$ and E are B-formulas and then $(r)\Diamond\varphi \wedge E$ is equivalent to $(r)\Diamond\varphi \,\&\, E$. Then axiom $(\Pi 1)$ can be equivalently expressed as $((r)\Diamond\varphi \,\&\, E) \to (\leqslant r)(\varphi\&\Diamond E)$, and this equivalent in turn to $(r)\Diamond\varphi \to (E \to (\leqslant r)(\varphi\&\Diamond E))$. Now by applying modus ponens to the latter and $(r)\Diamond\varphi$, we have $T \vdash E \to (\leqslant r)(\varphi\&\Diamond E)$.

(b) Assume $T_0 \vdash (r)\Diamond\varphi$ with $r > 0$. Then, by modus ponens with axiom $(\Pi 2)$, we get $T_0 \vdash \bigvee_{E\in\mathbf{mec}}(\geqslant r)\Diamond(E \wedge (r)(\varphi\&\Diamond E))$. Since T_0 is complete[9], for some E, $T_0 \vdash (\geqslant r)\Diamond(E \wedge (r)(\varphi\&\Diamond E))$, and since $r > 0$ we have that $T_0 \vdash (>0)\Diamond(E \wedge (r)(\varphi\&\Diamond E))$.

Let D denote $E \wedge (r)(\varphi\&\Diamond E)$, and let $H = \{(s)\Diamond\psi \mid T_0 \vdash (s)\Diamond\psi\}$ be the set of B-formulas of the kind $(s)\Diamond\psi$ provable from T_0. We are going to prove that D is consistent with H. Let H_f be the conjunction of an arbitrary finite subset of H. Obviously, $T_0 \vdash \Box H_f$. Since both D and H_f are Boolean, by (c) of Lemma 11.4, we have $\vdash \Box H_f \to (\Diamond D \to \Diamond(D\wedge H_f))$, and by modus ponens, $T_0 \vdash \Diamond D \to \Diamond(D\wedge H_f)$, and thus $T_0 \vdash (>0)\Diamond D \to (>0)\Diamond(D\wedge H_f)$ as well. But $T_0 \vdash (>0)\Diamond D$, so again by modus ponens, $T_0 \vdash (>0)\Diamond(D\wedge H_f)$. Hence, by (a) of previous Lemma 11.4, $D\wedge H_f$ is consistent. We have thus proved that D is consistent with any arbitrary finite conjunction H_f of H, therefore $\{D\}\cup H$ is consistent. Finally consider T to be a completion of $\{D\}\cup H$. This theory clearly proves D (i.e. $T \vdash E \wedge (r)(\varphi\&\Diamond E)$) and T proves the same formulas of the kind $(s)\Diamond\psi$ than T_0, that is, $T \approx T_0$.

Finally, assume $T_0 \vdash (0)\Diamond\varphi$. Let T be any theory such that $T \approx T_0$. Then by (c) of Lemma 11.3 we have $T \vdash E_T$, and taking $r = 0$ in (a) above, we have $T \vdash E_T \to (0)(\varphi\&\Diamond E_T)$. Therefore, $T \vdash (0)(\varphi\&\Diamond E_T)$ and the statement is proved. ⊣

Corollary 11.2. *Let T_0 be a complete and consistent theory. Then, for any formula φ, $T_0 \vdash (r)\Diamond\varphi$ iff the following two conditions hold:*

(a) For any theory $T \approx T_0$, $T \vdash (\leqslant r)(\varphi\&\Diamond E_T)$,
(b) There is a theory $T_\varphi \approx T_0$ such that $T_\varphi \vdash (r)(\varphi\&\Diamond E_{T_\varphi})$.

Proof: From left to right, it is a direct consequence of the previous lemma by considering for each complete and consistent theory T the corresponding m.e.c. E_T as defined in (c) of Lemma 11.3. For the other direction we reason as follows. Assume conditions (a) and (b) hold and assume further that $T_0 \vdash (s)\Diamond\varphi$ with $s \neq r$. If $s < r$, then applying the 'left to right part' to $T_0 \vdash (s)\Diamond\varphi$ we would get (a): for every $T \approx T_0$, $T \vdash (\leqslant s)(\varphi\&\Diamond E_T)$, and this would contradict (b): $T_\varphi \vdash (r)(\varphi\&\Diamond E_T)$. In a similar way, if $s > r$, then the application of the 'left to right part' to $T_0 \vdash (s)\Diamond\varphi$ would give (b): $T' \vdash (s)(\varphi\&\Diamond E_T)$ for some $T' \approx T_0$, which would be in contradiction with (a): $T \vdash (\leqslant r)(\varphi\&\Diamond E_T)$ for all $T \approx T_0$. ⊣

From these lemmas we are ready to prove strong completeness but first we define a sort of canonical model that will be used in the completeness proof.

Definition 11.5. Let T_0 be a complete and consistent theory. For each $r \in A$ and formula φ such that $T_0 \vdash (r)\Diamond\varphi$, let T_φ be the complete theory such that $T_\varphi \approx T_0$

[9] Remind that a complete theory is prime in the classical sense for B-formulas.

and $T_\varphi \vdash (r)(\varphi \& \Diamond E_{T_\varphi})$ (as guaranteed by (b) of Corollary 11.2). Then we define the following possibilistic Kripke model

$$K_0 = (W_0, e_0, \pi_0)$$

where

- $W_0 = \{T_0\} \cup \{T_\varphi \mid \varphi \text{ formula}\}$ is the set of worlds,
- $e_0 : W_0 \times Var \to A$ is defined by $e_0(T, p) = s$ whenever $T \vdash (s)p$,[10] and
- $\pi_0 : W_0 \to A$ is defined by $\pi_0(T) = s$ if $T \vdash (s)\Diamond E_T$. ⊣

Note that, so defined, there is at least some $T \in W_0$ such that $\pi_0(T) = 1$. Indeed, since $T_0 \vdash (1)\Diamond\top$, the theory $T_\top$ is such that $T_\top \vdash (1)(\top \& \Diamond E_\top)$, i.e. $T_\top \vdash (1)\Diamond E_\top$. Then, by definition, $\pi_0(T_\top) = 1$. Therefore, $K_0 = (W_0, e_0, \pi)$ is indeed a possibilistic Kripke model according to Definition 11.2. Moreover, it is a finite model, i.e. W_0 is finite. Indeed, W_0 contains at most as many theories T as m.e.c.s E in **mec**, and it is clear that **mec** is a finite set.

The truth-evaluation of a formula φ in a world $T \in W_0$, $\|\varphi\|_{T,K_0}$, is defined as usual (see the paragraph after Definition 11.2). In particular, for any formula ψ we have

$$\|\Diamond\psi\|_{K_0} = \max_{T \in W_0}\{\pi(T) \odot \|\psi\|_{T,K_0}\}.$$

Lemma 11.6. *(Truth Lemma) For each formula ψ, value r and $T \in W_0$,*

$$T \vdash (r)\psi \text{ iff } \|\psi\|_{T,K_0} = r.$$

Proof: The proof is by induction, the interesting induction step being for $\psi = \Diamond\varphi$.

Assume first $T \vdash (r)\Diamond\varphi$, and reason as follows. Since $T \in W_0$, i.e. $T \approx T_0$, by definition $T_0 \vdash (r)\Diamond\varphi$ as well. Then by Corollary 11.2, this happens if and only if: (a) $T' \vdash (\leqslant r)(\varphi \,\&\, \Diamond E_{T'})$ for any $T' \approx T_0$, and (b) there exists $T_\varphi \in W_0$ such that $T_\varphi \vdash (r)(\varphi \,\&\, \Diamond E_{T_\varphi})$. Then this is in turn equivalent to the following equalities:

$$\begin{aligned} r &= \max\{s \mid T' \vdash (s)(\varphi \,\&\, \Diamond E_{T'}), T' \in W_0\} \\ &= \max\{s_1 \odot s_2 \mid T' \vdash (s_1)\varphi,\ T' \vdash (s_2)\Diamond E_{T'}, T' \in W_0\} \\ &= \max\{s_1 \odot s_2 \mid s_1 = \|\varphi\|_{T',K_0}, s_2 = \pi_0(T'), T' \in W_0\} \\ &= \max\{\|\varphi\|_{T',K_0} \odot \pi_0(T') \mid T' \in W_0\} \\ &= \|\Diamond\varphi\|_{K_0}. \end{aligned}$$

Note that in the third equality we apply the induction hypothesis to $(s_1)\varphi$.

For the right-to-left implication, assume $\|\Diamond\varphi\|_{T,K_0} = r$. Since we have already proved the converse implication, we know that $T \nvdash (r')\Diamond\varphi$ for every $r' \neq r$. Since $T \in W_0$, it is complete and consistent, and by (c) of Lemma 11.3, we get that $T \vdash (r)\Diamond\varphi$. ⊣

[10] This definition is sound due to (c) of Lemma 11.3.

Theorem 11.2 (Strong Completeness). *$Pos(\mathbf{A}^c_\triangle)$ is strongly complete with respect to the class of A-valued possibilistic Kripke frames, that is, the following conditions are equivalent for any set of formulas $\Gamma \cup \{\varphi\}$:*

(1) $\Gamma \vdash \varphi$
(2) $\Gamma \models \varphi$
(3) For any reduced *(and thus finite) possibilistic Kripke model $K = (W, e, \pi)$ and $w \in W$, if $\|\psi\|_{w,K} = 1$ for all $\psi \in \Gamma$, then $\|\varphi\|_{w,K} = 1$.*

Proof: (1) $\Rightarrow$ (2) is soundness (Theorem 11.1) and (2) $\Rightarrow$ (3) is trivial. As for (3) $\Rightarrow$ (1), assume $\Gamma \not\vdash \varphi$. Then $\{(1)\psi \mid \psi \in \Gamma\} \cup \{(<1)\varphi\}$ is consistent, hence it can be extended to a complete theory T_0. It is clear that $T_0 \vdash (1)\psi$ for every $\psi \in \Gamma$. Moreover, since T_0 is complete, $T_0 \vdash (r)\varphi$, for some $r < 1$. We then build a possibilistic Kripke model $K_0 = (W_0, e, \pi)$ like in Definition 11.5, hence with W_0 being finite. Then, by Lemma 11.6, $\|\psi\|_{T_0,K_0} = 1$ for all $\psi \in \Gamma$ and $\|\varphi\|_{T_0,K_0} = r$, and hence $\|\varphi\|_{T_0,K_0} = r < 1$. $\dashv$

Completeness with respect to reduced models trivially implies that for every finite number of propositional variables *n*, the corresponding finitary consequence $\vdash$ relation is *decidable*. To conclude, we would also like to notice that the above strong completeness result could also be obtained from a weak completeness result (i.e. completeness for theorems) taking into account Corollary 11.1 and the $\triangle$-deduction theorem.

11.4 Conclusions and further work

In this short paper we have shown how the approach of [14] can be easily adapted to define a many-valued modal system that capture reasoning with natural generalizations of possibility and necessity measures over many-valued formulas in a general finite setting.

As recalled in Section 2, in the classical framework, when the possibility distributions and the accessibility relations are crisp ($\{0,1\}$-valued), possibilistic systems correspond to the classical modal system KD45, which is sound and complete with respect to the class of Kripke frames with serial, transitive and euclidean accessibility relations. In other words, in the classical setting the tautologies of KD45-models are the same than the tautologies of possibilistic models.

This result extends without difficulty to the many-valued framework when the accessibility relations and the possibility distributions remain $\{0,1\}$-valued. However it is currently unknown whether it also extends in the general many-valued case, when the accessibility relations and possibility distributions are both many-valued (not necessarily finitely-valued, like in this paper). So the following problem remains *open*: is every tautology of the class of possibilistic models (as defined here in this paper) a tautology of the class Kripke models whose accessibility relations are serial, transitive and euclidean?

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