Some remarks about standard first order tautologies

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Abstract

In this short note we gather some known results about tautologies of first order fuzzy logics with the standard semantics, mainly for logics of a continuous t-norms, and specially for Lukasiewicz, Product and Gödel logics, and we end up with an open problem. We summarize some completeness and satisfiability results but we will not deal with decidability and complexity issues. For general notions and results on first-order fuzzy logics we refer to [1, 6, 5].

The following facts are well-known (see e.g. Hájek’s book [4]):

– Gödel first order logic \( G\forall \) is the only one that is strong standard complete and thus standard semantics coincides with general semantics. This also implies that this logic is recursively axiomatizable, and the classical deduction-detachment theorem is valid in \( G\forall \).
– Lukasiewicz and Product first order logics, \( L\forall \) and \( \Pi\forall \), are not standard complete and while the set of tautologies of these logics with the general semantics is recursively axiomatizable, those with the standard semantics are not even recursively enumerable.

On the other hand, Hájek also proved that a formula is a standard tautology of \( L\forall \) iff, for each \( n \geq 2 \), it is a tautology of \( L_n\forall \), where \( L_n\forall \) denotes the \((n+1)\)-valued Lukasiewicz first order logic. But a similar result cannot be generalized to other first order fuzzy logics (see e.g. [3]), namely:

– If \([0, 1]_*\) is the standard chain defined by a continuous t-norm \( * \) different from the Lukasiewicz t-norm, then the formula (with witness axiom \((G\forall)\) for the universal quantifier),

\[
(\exists x)(P(x) \to (\forall y)P(y))
\]

is a tautology over any finite \( L_*\)-chain but it is not a tautology over \([0, 1]_*\). In particular, it does not hold in general that a formula is a standard tautology of \( G\forall \) (i.e. a tautology over \([0, 1]_c\)) iff, for each \( n \geq 2 \), is a \( G_{n\forall}\)-tautology (i.e. a tautology over \((n+1)\) element Gödel chain).
– Let \( * \) be a left-continuous (non-continuous) t-norm. Then the formula,

\[
(\forall x)(\chi \& \psi) \to (\chi \& (\forall x)\psi) \text{ where } x \text{ is not free in } \chi,
\]
is a tautology over any finite $L_\ast$-chain, but it is not a tautology over the
standard chain $[0, 1]_\ast$.

Nevertheless, for $\Pi \forall$ the following result holds [2]:

- A formula is a standard tautology (i.e. over $[0, 1]_{\Pi}$) if and only if it is a tautology over a
  one-element generated subchain of $[0, 1]_\Pi$.
- A formula is a tautology over a (quasi-discrete) product algebra over a set
  $\{1, 0\} \cup \{a^n \mid n \in \mathbb{N}\}$, for some $0 < a < 1$.

Taking into account the above behaviour of $L \forall$ and $\Pi \forall$ with respect to one-
element generated subchains of the standard one, one could ask whether this
behaviour generalizes to logics $L_\ast \forall$, where $\ast$ is an ordinal sum of Lukasiewicz and
product components, in the following sense: does the set of standard tautologies $L_\ast \forall$
coincide with the common tautologies of the family of logics $L_\ast \forall'$ where $\ast'$
is obtained from $\ast$ by replacing
- each Lukasiewicz component by a $L_n$ component, and
- each product component by a one-element generated product chain ?

However, an easy argument shows that the answer to this question is negative.
The reason is that each $L_\ast \forall'$ obviously satisfies the witness axiom for the existen-
tial quantifier $(C \exists x)$, while $L_\ast \forall$ does not. Indeed, let $a \in [0, 1]_\ast$ be an idempotent
variable different from 0 and 1 and take the $[0, 1]_\ast$-model $M = (\mathbb{N}, P_M)$, where
$P_M(n) = a_n$ with $(a_n)_{n \in \mathbb{N}}$ forming an increasing sequence with limit $a$ and such
that $a_n < a$ for all $n \in \mathbb{N}$. Then the value of the formula

$$(\exists x)((\exists y)P(y) \rightarrow P(x))$$

in the model $M$ is

$$\sup_n \left( \sup_{m \leq n} (a_m \rightarrow a_n) \right) = \sup_n (a \rightarrow a_n) = a \neq 1.$$

We finish with some short remarks about the relationships between the
1-satisfiability (SAT) and the positive-satisfiability (SAT$_{pos}$) problems for the
three main logics $G \forall$, $L \forall$ and $\Pi \forall$ with the standard semantics. For the first two
tactics the following holds:

- For $G \forall$, SAT is equivalent to SAT$_{pos}$.
  Indeed, taking into account the completeness result and the deduction-detachment theorem for $G \forall$, we have: $\varphi$ is not 1-satisfiable if and only if $\varphi \models_G \varphi \rightarrow 0$, i.e. $\varphi$ is not positively satisfiable.
- For $L \forall$, the previous equivalence is false since it is already false for proposi-
tional logic (for example, the formula $p \land \neg p$ is positively satisfiable but not
1-satisfiable).

Open problem. For the case of product logic, although SAT is equivalent
to SAT$_{pos}$ in the propositional case, whether this equivalence holds for $\Pi \forall$ is
currently an open problem. In fact, to solve this problem is equivalent to solve
the question of whether the validity of the following instance of the deduction-detachment theorem
\[ \varphi \models_{H \forall} \bar{0} \iff \models_{H \forall} \varphi \rightarrow \bar{0}. \]

holds within the standard semantics.

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**References**