Base Belief Change for Finitary Monotonic Logics

Pere Pardo[1], Pilar Dellunde[1,2], and Lluís Godo[1]

[1] Institut d'Investigació en Intel·ligència Artificial (IIIA - CSIC)
08193 Bellaterra, Spain
[2] Universitat Autònoma de Barcelona (UAB) 08193 Bellaterra, Spain

Abstract. We slightly improve on characterization results already in the literature for base revision. We show that consistency-based partial meet revision operators can be axiomatized for any sentential logic $\mathcal{S}$ satisfying finitarity and monotonicity conditions (neither the deduction theorem nor supraclassicality are required to hold in $\mathcal{S}$). A characterization of limiting cases of revision operators, full meet and maxichoice, is also offered. In the second part of the paper, as a particular case, we focus on the class of graded fuzzy logics and distinguish two types of bases, naturally arising in that context, exhibiting different behavior.

Introduction

This paper is about (multiple) base belief change, in particular our results are mainly about base revision, which is characterized for a broad class of logics. The original framework of Alchourrón, Gärdenfors and Makinson (AGM) [1] deals with belief change operators on deductively closed theories. This framework was generalized by Hansson [9, 10] to deal with *bases*, i.e. arbitrary set of formulas, the original requirement of logical closure being dropped. Hansson characterized revision and contraction operators in, essentially, monotonic compact logics with the deduction theorem property. These results were improved in [11] by Hansson and Wassermann: while for contraction ([11, Theorem 3.8]) it is shown that finitarity and monotony of the underlying logic suffice, for revision (Theorem [11, Theorem 3.17]) their proof depends on a further condition, *Non-contravention*: for all sentences φ, if $\neg\varphi \in \mathrm{Cn}_{\mathcal{S}}(T \cup \{\varphi\})$, then $\neg\varphi \in \mathrm{Cn}_{\mathcal{S}}(T)$.

In this paper we provide a further improvement of Hansson and Wassermann's results by proving a characterization theorem for base revision in any finitary monotonic logic. Namely, in the context of partial meet base revision, we show that *Non-contravention* can be dropped in the characterization of revision if we replace the notion of unprovability (remainders) by consistency in the definition of partial meet, taking inspiration from [4]. This is the main contribution of the paper, together with its extension to the characterization of the revision operators corresponding to limiting cases of selection functions, i.e. full meet and maxichoice revision operators.

In the second part of the paper, as a particular class of finitary monotonic logics, we focus on graded fuzzy logics. We introduce there a distinction in basehood and observe some differences in the behavior of the corresponding base revision operators.

This paper is structured as follows. First we introduce in Section 1 the necessary background material on logic and partial meet base belief change. Then in Section 2 we set out the main characterization results for base revision, including full meet and maxichoice revision operators. Finally in Section 3 we briefly introduce fuzzy graded logics, present a natural distinction between bases in these logics (whether or not they are taken to be closed under truth-degrees) and compare both kinds of bases.

1 Preliminaries on theory and base belief change

We introduce in this section the concepts and results needed later. Following [6], we define a logic $\mathcal{S}$ as a finitary and structural consequence relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathbf{Fm}) \times \mathbf{Fm}$, for some algebra of formulas $\mathbf{Fm}$[3].

Belief change is the study of how some theory T (non-necessarily closed, as we use the term) in a given language L can adapt to new incoming information $\varphi \in L$ (inconsistent with T, in the interesting case). The main operations are: *revision*, where the new input must follow from the revised theory, which is to be consistent, and *contraction* where the input must not follow from the contracted theory. In the classical paper [1], by Alchourrón, Gärdenfors and Makinson, partial meet revision and contraction operations were characterized for closed theories in, essentially, monotonic compact logics with the deduction property[4]. Their work put in solid grounds this newly established area of research, opening the way for other formal studies involving new objects of change, operations (see [14] for a comprehensive list) or logics. We follow [1] and define change operators by using partial meet: *Partial meet* consists in (i) generating all logically maximal ways to adapt T to the new sentence (those subtheories of T making further information loss logically unnecessary), (ii) selecting some of these possibilities, (iii) forming their meet, and, optionally, (iv) performing additional steps (if required by the operation). Then a set of axioms is provided to capture these partial meet operators, by showing equivalence between satisfaction of these axioms and being a partial meet operator[5]. In addition, new axioms may be introduced to characterize the limiting cases of selection in step (ii), full meet

[3] That is, $\mathcal{S}$ satisfies (1) If $\varphi \in \Gamma$ then $\Gamma \vdash_{\mathcal{S}} \varphi$, (2) If $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_{\mathcal{S}} \varphi$, (3) If $\Gamma \vdash_{\mathcal{S}} \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash_{\mathcal{S}} \psi$ then $\Delta \vdash_{\mathcal{S}} \varphi$ (*consequence relation*); (4) If $\Gamma \vdash_{\mathcal{S}} \varphi$ then for some finite $\Gamma_0 \subseteq \Gamma$ we have $\Gamma_0 \vdash_{\mathcal{S}} \varphi$ (*finitarity*); (5) If $\Gamma \vdash_{\mathcal{S}} \varphi$ then $e[\Gamma] \vdash_{\mathcal{S}} e(\varphi)$ for all substitutions $e \in Hom(\mathbf{Fm}, \mathbf{Fm})$ (*structurality*). We will use throughout the paper relational $\vdash_{\mathcal{S}}$ and functional $\mathrm{Cn}_{\mathcal{S}}$ notation indistinctively, where $\mathrm{Cn}_{\mathcal{S}}$ is the consequence operator induced by $\mathcal{S}$. We will further assume the language of $\mathcal{S}$ contains symbols for conditional $\rightarrow$ and *falsum* $\overline{0}$.

[4] That is, logics satisfying the Deduction Theorem: $\varphi \vdash_{\mathcal{S}} \psi$ iff $\vdash_{\mathcal{S}} \varphi \rightarrow \psi$.

[5] Other known formal mechanisms defining change operators can be classified into two broad classes: *selection*-based mechanisms include selection functions on remainder

and maxichoice selection types. Finally, results showing the different operation types can be defined each other are usually provided too.

A *base* is an arbitrary set of formulas, the original requirement of logical closure being dropped. Base belief change, for the same logical framework than AGM, was characterized by Hansson (see [9], [10]). The results for contraction and revision were improved in [11] (by Hansson and Wassermann): for contraction ([11, Theorem 3.8]) it is shown that finitarity and monotony suffice, while for revision ([11, Theorem 3.17]) their proof depends on a further condition, *Non-contravention*: for all sentences φ, if $\neg\varphi \in \mathrm{Cn}_{\mathcal{S}}(T \cup \{\varphi\})$, then $\neg\varphi \in \mathrm{Cn}_{\mathcal{S}}(T)$. Observe this condition holds in logics having (i) the deduction property and (ii) the structural axiom of Contraction[6]. We show *Non-contravention* can be dropped in the characterization of revision if we replace unprovability (remainders) by consistency in the definition of partial meet.

The main difference between base and theory revision is syntax-sensitivity (see [12] and [3] for a discussion): two equivalent bases may output different solutions under a fixed revision operator and input (compare e.g. $T = \{p, q\}$ and $T' = \{p \wedge q\}$ under revision by $\neg p$, which give $\{\neg p, q\}$ and $\{\neg p\}$ respectively). Another difference lies in maxichoice operations: for theory revision it was proved in [2] that: non-trivial revision maxichoice operations $T \circledast \varphi$ output complete theories, even if T is far from being complete. This was seen as an argument against maxichoice. For base belief change, in contrast, the previous fact is not the case, so maxichoice operators may be simply seen as modeling optimal knowledge situations for a given belief change problem.

2 Multiple base revision for finitary monotonic logics.

Partial meet was originally defined in terms of unprovability of the contraction input sentences: *remainders* are maximal subsets of T failing to imply φ. This works fine for logics with the deduction theorem, where remainders and their consistency-based counterparts (defined below) coincide. But, for the general case, remainder-based revision does not grant consistency and it is necessary to adopt the consistency-based approach. Observe we also generalize revision operators to the *multiple* case, where the input of revision is allowed to be a base, rather than just a single sentence.

Definition 1. *([15], [4]) Given some monotonic logic $\vdash_{\mathcal{S}}$, let T_0, T_1 be theories. We say T_0 is* consistent *if $T_0 \nvdash_{\mathcal{S}} \overline{0}$, and define the set* $\mathrm{Con}(T_0, T_1)$ *of* subsets *of* T_0 maximally consistent with T_1 *as follows: $X \in Con(T_0, T_1)$ iff:*

sets and incision functions on kernels; *ranking*-based mechanisms include entrenchments and systems of spheres. For the logical framework assumed in the original developments (compact -and monotonic- closure operators satisfying the deduction property), all these methods are equivalent (see [14] for a comparison). These equivalences between methods need not be preserved in more general class of logics.

[6] If $T \cup \{\varphi\} \vdash_{\mathcal{S}} \varphi \to \overline{0}$, then by the deduction property $T \vdash_{\mathcal{S}} \varphi \to (\varphi \to \overline{0})$; i.e. $T \vdash_{\mathcal{S}} (\varphi \& \varphi) \to \overline{0}$. Finally, by transitivity and the axiom of contraction, $\vdash_{\mathcal{S}} \varphi \to \varphi \& \varphi$, we obtain $T \vdash_{\mathcal{S}} \varphi \to \overline{0}$.

(i) $X \subseteq T_0$,
(ii) $X \cup T_1$ *is consistent, and*
(iii) For any X' *such that* $X \subsetneq X' \subseteq T_0$, *we have* $X' \cup T_1$ *is inconsistent*

Now we prove some properties[7] of $\mathrm{Con}(\cdot,\cdot)$ which will be helpful for the characterization theorems of base belief change operators for arbitrary finitary monotonic logics.

Lemma 1. *Let* $\mathcal{S}$ *be some finitary logic and* T_0 *a theory. For any* $X \subseteq T_0$, *if* $X \cup T_1$ *is consistent, then* X *can be extended to some* Y *with* $Y \in \mathrm{Con}(T_0, T_1)$.

Proof. Let $X \subseteq T_0$ with $X \cup T_1 \nvdash_{\mathcal{S}} \overline{0}$. Consider the poset $(T^*, \subseteq)$, where $T^* = \{Y \subseteq T_0 : X \subseteq Y \text{ and } Y \cup T_1 \nvdash_{\mathcal{S}} \overline{0}\}$. Let $\{Y_i\}_{i\in I}$ be a chain in $(T^*, \subseteq)$; that is, each Y_i is a subset of T_0 and consistent with T_1. Hence, $\bigcup_{i\in I} Y_i \subseteq T_0$; since $\mathcal{S}$ is finitary, $\bigcup_{i\in I} Y_i$ is also consistent with T_1 and hence is an upper bound for the chain. Applying Zorn's Lemma, we obtain an element Z in the poset with the next properties: $X \subseteq Z \subseteq T$ and Z maximal w.r.t. $Z \cup \{\varphi\} \nvdash_{\mathcal{S}} \overline{0}$. Thus $X \subseteq Z \in \mathrm{Con}(T, \varphi)$.

Remark 1. Considering $X = \emptyset$ in the preceding lemma, we infer: if T_1 is consistent, then $\mathrm{Con}(T_0, T_1) \neq \emptyset$.

For simplicity, we assume that the input base T_1 (to revise T_0 by) is consistent. Now, the original definition of selection functions is modified according to the consistency-based approach.

Definition 2. *Let* T_0 *be a theory. A* selection function *for* T_0 *is a function*

$$\gamma : \mathcal{P}(\mathcal{P}(\mathbf{Fm})) \setminus \{\emptyset\} \longrightarrow \mathcal{P}(\mathcal{P}(\mathbf{Fm})) \setminus \{\emptyset\}$$

such that for all $T_1 \subseteq \mathbf{Fm}$, $\gamma(\mathrm{Con}(T_0, T_1)) \subseteq \mathrm{Con}(T_0, T_1)$ *and* $\gamma(\mathrm{Con}(T_0, T_1))$ *is non-empty.*

Thus, selection functions and revision operators are defined relative to some fixed base T_0. Although, instead of writing $\circledast^{T_0} T_1$, we use the traditional infix notation $T_0 \circledast T_1$ for the operation of revising base T_0 by T_1.

2.1 Base belief revision.

The axioms we propose (inspired by [4]) to characterize (multiple) base revision operators for finitary monotonic logics $\mathcal{S}$ are the following, for arbitrary sets T_0, T_1:

[7] Note that $\mathrm{Con}(T_0, T_1)$ cannot be empty, since if input T_1 is consistent, then in the worst case, we will have $\emptyset \subseteq T_0$ to be consistent with T_1.

(F1) $T_1 \subseteq T_0 \circledast T_1$ (Success)
(F2) If T_1 is consistent, then $T_0 \circledast T_1$ is also consistent. (Consistency)
(F3) $T_0 \circledast T_1 \subseteq T_0 \cup T_1$ (Inclusion)
(F4) For all $\psi \in \mathbf{Fm}$, if $\psi \in T_0 - T_0 \circledast T_1$ then,
there exists T' with $T_0 \circledast T_1 \subseteq T' \subseteq T_0 \cup T_1$
and such that $T' \nvdash_{\mathcal{S}} \overline{0}$ but $T' \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$) (Relevance)
(F5) If for all $T' \subseteq T_0$ ($T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0} \Leftrightarrow T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0}$)
then $T_0 \cap (T_0 \circledast T_1) = T_0 \cap (T_0 \circledast T_2)$ (Uniformity)

Given some theory $T_0 \subseteq \mathbf{Fm}$ and selection function γ for T_0, we define the partial meet revision operator $\circledast_\gamma$ for T_0 by $T_1 \subseteq \mathbf{Fm}$ as follows:

$$T_0 \circledast_\gamma T_1 \;=\; \bigcap \gamma(\mathrm{Con}(T_0, T_1)) \cup T_1$$

Definition 3. *Let $\mathcal{S}$ be some finitary logic, and T_0 a theory. Then $\circledast : \mathcal{P}(\mathbf{Fm}) \to \mathcal{P}(\mathbf{Fm})$ is a* revision operator *for T_0 iff for any $T_1 \subseteq \mathbf{Fm}$, $T_0 \circledast T_1 = T_0 \circledast_\gamma T_1$ for some selection function γ for T_0.*

Lemma 2. *The condition* $\mathrm{Con}(T_0, T_1) = \mathrm{Con}(T_0, T_2)$ *is equivalent to the antecedent of Axiom* (F5)

$$\forall T' \subseteq T_0 \; (T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0} \Leftrightarrow T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0})$$

Proof. (If-then) Assume $\mathrm{Con}(T_0, T_1) = \mathrm{Con}(T_0, T_2)$ and let $T' \subseteq T_0$ with $T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0}$. By Lemma 1, T' can be extended to $X \in \mathrm{Con}(T_0, T_1)$. Hence, by assumption we get $T' \subseteq X \in \mathrm{Con}(T_0, T_2)$ so that $T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0}$ follows. The other direction is similar. (Only if) This direction follows from the definition of $\mathrm{Con}(T_0, \cdot)$.

Finally, we are in conditions to prove the main characterization result for partial meet revision.

Theorem 1. *Let $\mathcal{S}$ be a finitary monotonic logic. For any $T_0 \subseteq \mathbf{Fm}$ and function $\circledast : \mathcal{P}(\mathbf{Fm}) \to \mathcal{P}(\mathbf{Fm})$:*

$$\circledast \textit{ satisfies } (\mathrm{F1}) - (\mathrm{F5}) \text{ iff } \circledast \textit{ is a revision operator for } T_0$$

Proof. (Soundness) Given some partial meet revision operator $\circledast_\gamma$ for T_0, we prove $\circledast_\gamma$ satisfies (F1) – (F5).

(F1) – (F3) hold by definition of $\circledast_\gamma$. (F4) Let $\psi \in T_0 - T_0 \circledast_\gamma T_1$. Hence, $\psi \notin T_1$ and for some $X \in \gamma(\mathrm{Con}(T_0, T_1))$, $\psi \notin X$. Simply put $T' = X \cup T_1$: by definitions of $\circledast_\gamma$ and Con we have (i) $T_0 \circledast_\gamma T_1 \subseteq T' \subseteq T_0 \cup T_1$ and (ii) T' is consistent (since T_1 is). We also have (iii) $T' \cup \{\psi\}$ is inconsistent (otherwise $\psi \in X$ would follow from maximality of X and $\psi \in T_0$, hence contradicting our previous step $\psi \notin X$). (F5) We have to show, assuming the antecedent of(F5), that $T_0 \cap (T_0 \circledast_\gamma T_1) = T_0 \cap (T_0 \circledast_\gamma T_2)$. We prove the $\subseteq$ direction only since the other is similar. Assume, then, for all $T' \subseteq T_0$,

$$T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0} \;\Leftrightarrow\; T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0}$$

and let $\psi \in T_0 \cap (T_0 \circledast_\gamma T_1)$. This set is just $T_0 \cap (\bigcap \gamma(Con(T_0, T_1)) \cup T_1)$ which can be transformed into $(T_0 \cap \bigcap \gamma(Con(T_0, T_1)) \cup (T_0 \cup T_1)$, i.e. $\bigcap \gamma(Con(T_0, T_1)) \cup (T_0 \cup T_1)$ (since $\bigcap \gamma(Con(T_0, T_1)) \subseteq T_0$). Case $\psi \in \bigcap \gamma(Con(T_0, T_1))$. Then we use Lemma 2 upon the assumption to obtain $\bigcap \gamma(Con(T_0, T_1)) = \bigcap \gamma(Con(T_0, T_2))$, since γ is a function. Case $\psi \in T_0 \cap T_1$. Then $\psi \in X$ for all $X \in \gamma(Con(T_0, T_1))$, by maximality of X. Hence, $\psi \in \bigcap \gamma(Con(T_0, T_1))$. Using the same argument than in the former case, $\psi \in \bigcap \gamma(Con(T_0, T_2))$. Since we also assumed $\psi \in T_0$, we obtain $\psi \in T_0 \cap (T_0 \circledast_\gamma T_2)$.
(Completeness) Let $\circledast$ satisfy (F1) – (F5). We have to show that for some selection function γ and any T_1, $T_0 \circledast T_1 = T \circledast_\gamma T_1$. We define first

$$\gamma(\mathrm{Con}(T_0, T_1)) = \{X \in \mathrm{Con}(T_0, T_1) : X \supseteq T_0 \cap (T_0 \circledast T_1)\}$$

We prove that (1) γ is well-defined, (2) γ is a selection function and (3) $T_0 \circledast T_1 = T \circledast_\gamma T_1$.

(1) Assume (i) $\mathrm{Con}(T_0, T_1) = \mathrm{Con}(T_0, T_2)$; we prove that $\gamma(\mathrm{Con}(T_0, T_1)) = \gamma(\mathrm{Con}(T_0, T_2))$. Applying Lemma 2 to (i) we obtain the antecedent of (F5). Since $\circledast$ satisfies this axiom, we have (ii) $T_0 \cap (T_0 \circledast T_1) = T_0 \cap (T_0 \circledast T_2)$. By the above definition of γ, $\gamma(\mathrm{Con}(T_0, T_1)) = \gamma(\mathrm{Con}(T_0, T_2))$ follows from (i) and (ii).

(2) Since T_1 is consistent, by Remark 1 we obtain $\mathrm{Con}(T_0, T_1)$ is not empty; we have to show that $\gamma(\mathrm{Con}(T_0, T_1))$ is not empty either (since the other condition $\gamma(\mathrm{Con}(T_0, T_1)) \subseteq \mathrm{Con}(T_0, T_1)$ is met by the above definition of γ). We have $T_0 \cap T_0 \circledast T_1 \subseteq T_0 \circledast T_1$; the latter is consistent and contains T_1, by (F2) and (F1), respectively; thus, $(T_0 \cap T_0 \circledast T_1) \cup T_1$ is consistent; from this and $T_0 \cap T_0 \circledast T_1 \subseteq T_0$, we deduce by Lemma 1 that $T_0 \cap T_0 \circledast T_1$ is extensible to some $X \in \mathrm{Con}(T_0, T_1)$. Thus, exists some $X \in \mathrm{Con}(T_0, T_1)$ such that $X \supseteq T_0 \cap T_0 \circledast T_1$. In consequence, $X \in \gamma(\mathrm{Con}(T_0, T_1)) \neq \emptyset$.

For (3), we prove first $T_0 \circledast T_1 \subseteq T_0 \circledast_\gamma T_1$. Let $\psi \in T_0 \circledast T_1$. By (F3), $\psi \in T_0 \cup T_1$. Case $\psi \in T_1$: then trivially $\psi \in T_0 \circledast_\gamma T_1$ Case $\psi \in T_0$. Then $\psi \in T_0 \cap T_0 \circledast T_1$. In consequence, for any $X \in \mathrm{Con}(T_0, T_1)$, if $X \supseteq T_0 \cap T_0 \circledast T_1$ then $\psi \in X$. This implies, by definition of γ above, that for all $X \in \gamma(\mathrm{Con}(T_0, T_1))$ we have $\psi \in X$, so that $\psi \in \bigcap \gamma(\mathrm{Con}(T_0, T_1)) \subseteq T_0 \circledast_\gamma T_1$. In both cases, we obtain $\psi \in T_0 \circledast_\gamma T_1$.

Now, for the other direction: $T_0 \circledast_\gamma T_1 \subseteq T_0 \circledast T_1$. Let $\psi \in \bigcap \gamma(\mathrm{Con}(T_0, T_1)) \cup T_1$. By (F1), we have $T_1 \in T_0 \circledast T_1$; then, in case $\psi \in T_1$ we are done. So we may assume $\psi \in \bigcap \gamma(\mathrm{Con}(T_0, T_1))$. Now, in order to apply (F4), let X be arbitrary with $T \circledast T_1 \subseteq X \subseteq T_0 \cup T_1$ and X consistent. Consider $X \cap T_0$: since $T_1 \subseteq T_0 \circledast T_1 \subseteq X$ implies $X = X \cup T_1$ is consistent, so is $(X \cap T_0) \cup T_1$. Together with $X \cap T_0 \subseteq T_0$, by Lemma 1 there is $Y \in \mathrm{Con}(T_0, T_1)$ with $X \cap T_0 \subseteq Y$. In addition, since $T_0 \circledast T_1 \subseteq X$ implies $T_0 \circledast T_1 \cap T_0 \subseteq X \cap T_0 \subseteq Y$, we obtain $Y \in \gamma(\mathrm{Con}(T_0, T_1))$, by the definition of γ above. Condition $X \cap T_0 \subseteq Y$ also implies $(X \cap T_0) \cup T_1 \subseteq Y \cup T_1$. Observe that from $X \subseteq X \cup T_1$ and $X \subseteq T_0 \cup T_1$ we infer that $X \subseteq (X \cup T_1) \cap (T_0 \cup T_1)$. From the latter being identical to $(X \cap T_0) \cup T_1$ and the fact that $(X \cap T_0) \cup T_1 \subseteq Y \cup T_1$, we obtain that $X \subseteq Y \cup T_1$. Since $\psi \in Y \in \mathrm{Con}(T_0, T_1)$, we have $Y \cup T_1$ is consistent with ψ, so its subset X is also consistent with ψ. Finally, we may apply *modus tollens* on Axiom (F4) to obtain that $\psi \notin T_0 - T_0 \circledast T_1$, i.e. $\psi \notin T_0$ or $\psi \in T_0 \circledast T_1$. But since the former is false, the latter must be the case.

Full meet and maxichoice base revision operators. The previous result can be extended to limiting cases of selection functions formally defined next.

Definition 4. *A revision operator for* T_0 *is* full meet *if it is generated by the identity selection function* $\gamma_{\mathrm{fm}} = \mathsf{Id}$*:* $\gamma_{\mathrm{fm}}(\mathrm{Con}(T_0, T_1)) = \mathrm{Con}(T_0, T_1)$*; that is,*

$$T_0 \circledast_{\mathrm{fm}} T_1 = (\bigcap \mathrm{Con}(T_0, T_1)) \cup T_1$$

A revision operator for T_0 *is* maxichoice *if it is generated by a selection function of type* $\gamma_{\mathrm{mc}}(\mathrm{Con}(T_0, T_1)) = \{X\}$*, for some* $X \in \mathrm{Con}(T_0, T_1)$*, and in that case* $T_0 \circledast_{\gamma_{\mathrm{mc}}} T_1 = X \cup T_1$*.*

To characterize *full meet* and *maxichoice* revision operators for some theory T_0 in any finitary logic, we define the next additional axioms:

(FM) For any $X \subseteq \mathbf{Fm}$ with $T_1 \subseteq X \subseteq T_0 \cup T_1$
$X \nvdash_{\mathcal{S}} \overline{0}$ implies $X \cup (T_0 \circledast T_1) \nvdash_{\mathcal{S}} \overline{0}$

(MC) For all $\psi \in \mathbf{Fm}$ with $\psi \in T_0 - T_0 \circledast T_1$ we have
$T_0 \circledast T_1 \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$

Theorem 2. *Let* $T_0 \subseteq \mathbf{Fm}$ *and* $\circledast$ *be a function* $\circledast : \mathcal{P}(\mathbf{Fm})^2 \to \mathcal{P}(\mathbf{Fm})$*. Then the following hold:*

(fm) $\circledast$ *satisfies* (F1) – (F5) *and* (FM) iff $\circledast = \circledast_{\gamma_{\mathrm{fm}}}$
(mc) $\circledast$ *satisfies* (F1) – (F5) *and* (MC) iff $\circledast = \circledast_{\gamma_{\mathrm{mc}}}$

Proof. We prove (fm) first. (Soundness): We know $\circledast_{\gamma_{\mathrm{fm}}}$ satisfies (F1) – (F5) so it remains to be proved that (FM) holds. Let X be such that $T_1 \subseteq X \subseteq T_0 \cup T_1$ and $X \nvdash_{\mathcal{S}} \overline{0}$. From the latter and $X - T_1 \subseteq (T_0 \cup T_1) - T_1 \subseteq T_0$ we infer by Lemma 1 that $X - T_1 \subseteq Y \in \mathrm{Con}(T_0, T_1)$, for some Y. Notice $X = X' \cup T_1$ and that for any $X'' \in \mathrm{Con}(T_0, T_1)$ $X'' \cup T_1$ is consistent and

$$T_0 \circledast_{\gamma_{\mathrm{fm}}} T_1 = (\bigcap \mathrm{Con}(T_0, T_1)) \cup T_1 \subseteq X' \subseteq X''$$

Hence $X \subseteq X''$, so that $T_0 \circledast_{\gamma_{\mathrm{fm}}} T_1 \cup X \subseteq X''$. Since the latter is consistent, $T_0 \circledast_{\mathrm{fm}} T_1 \cup X \nvdash_{\mathcal{S}} \overline{0}$. (Completeness) Let $\circledast$ satisfy (F1) – (F5) and (FM). It suffices to prove that $X \in \gamma(\mathrm{Con}(T_0, T_1)) \Leftrightarrow X \in \mathrm{Con}(T_0, T_1)$; but we already know that $\circledast = \circledast_{\gamma}$, for selection function γ (for T_0) defined by: $X \in \gamma(\mathrm{Con}(T_0, T_1)) \Leftrightarrow T_0 \cap T_0 \circledast T_1 \subseteq X$. It is enough to prove, then, that $X \in \mathrm{Con}(T_0, T_1)$ implies $X \supseteq T_0 \cap T_0 \circledast T_1$. Let $X \in \mathrm{Con}(T_0, T_1)$ and let $\psi \in T_0 \cap T_0 \circledast T_1$. Since $\psi \in T_0$ and $X \in \mathrm{Con}(T_0, T_1)$, we have by maximality of X that either $X \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$ or $\psi \in X$. We prove the former case to be impossible: assuming it we would have $T_1 \subseteq X \cup T_1 \subseteq T_0 \cup T_1$. By (FM), $X \cup T_1 \cup (T_0 \circledast T_1) \nvdash_{\mathcal{S}} \overline{0}$. Since $\psi \in T_0 \circledast T_1$, we would obtain $X \cup \{\psi\} \nvdash_{\mathcal{S}} \overline{0}$, hence contradicting the case assumption; since the former case is not possible, we have $\psi \in X$. Since X was arbitrary, $X \in \mathrm{Con}(T_0, T_1)$ implies $X \subseteq T_0 \cap T_0 \circledast T_1$ and we are done.
For (mc): (Soundness) We prove (MC), since (F1) – (F5) follow from $\circledast_{\gamma_{\mathrm{mc}}}$ being a partial meet revision operator. Let $X \in \mathrm{Con}(T_0, T_1)$ be such that $T_0 \circledast_{\gamma_{\mathrm{mc}}} \varphi =$

$X \cup T_1$ and let $\psi \in T_0 - T_0 \circledast_{\gamma_{\mathrm{mc}}} T_1$. We have $\psi \notin X \cup T_1 = T_0 \circledast T_1$. Since $\psi \in T_0$ and $X \in \mathrm{Con}(T_0, T_1)$, $X \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$. Finally $T_0 \circledast T_1 \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$. (Completeness) Let $\circledast$ satisfy (F1) – (F5) and (MC). We must prove $\circledast = \circledast_{\gamma_{\mathrm{mc}}}$, for some maxichoice selection function γ_{mc}. Let $X, Y \in \mathrm{Con}(T_0, T_1)$; we have to prove $X = Y$. In search of a contradiction, assume the contrary, i.e. $\psi \in X - Y$. We have $\psi \notin \bigcap \gamma(\mathrm{Con}(T_0, T_1))$ and $\psi \in X \subseteq T_0$. By MC, $T_0 \circledast T_1 \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$. Since $T_0 \circledast T_1 \subseteq X$, we obtain $X \cup \{\psi\}$ is also inconsistent, contradicting previous $\psi \in X \nvdash_{\mathcal{S}} \overline{0}$. Thus $X = Y$ which makes $\circledast = \circledast_{\gamma_{\mathrm{mc}}}$, for some maxichoice selection function γ_{mc}.

3 The case of graded fuzzy logics.

The characterization results for base revision operators from the previous section required weak assumptions (monotony and finitarity) upon the consequence relation $\vdash_{\mathcal{S}}$. In particular these results hold for a wide family of systems of mathematical fuzzy logic. The distinctive feature of these logics is that they cope with graded truth in a compositional manner (see [8]). Graded truth may be dealt implicitly, by means of comparative statements, or explicitly, by introducing truth-degrees in the language. Here we will focus on a particular kind of fuzzy logical languages allowing for explicit representation of truth-degrees, that will be referred as *graded fuzzy logics*, and which are expansions of t-norm logics with countable sets of truth-constants, see e.g. [5]. These logics allow for occurrences of truth-degrees, represented as new propositional atoms $\overline{r}$ (one for each $r \in \mathcal{C}$) in any part of a formula. These truth-constants and propositional variables can be combined arbitrarily using connectives to obtain new formulas. The graded language obtained in this way will be denoted as $\mathbf{Fm}(\mathcal{C})$. A prominent example of a logic over a graded language is Hájek's Rational Pavelka Logic **RPL** [8], an extension of Łukasiewicz logic with rational truth-constants in $[0, 1]$; for other graded extensions of t-norm based fuzzy logics see e.g. [5]. In t-norm based fuzzy logics, due to the fact that the implication is residuated, a formula $\overline{r} \to \varphi$ gets value 1 under a given interpretation e iff $r \leq e(\varphi)$. In what follows, we will also use the signed language notation (φ, r) to denote the formula $\overline{r} \to \varphi$.

If $\mathcal{S}$ denotes a given t-norm logic, let us denote by $\mathcal{S}(\mathcal{C})$ the corresponding expansion with truth-constants from a suitable countable set $\mathcal{C}$ such that $\{0, 1\} \subset \mathcal{C} \subseteq [0, 1]$. For instance if $\mathcal{S}$ is Łukasiewicz logic and $\mathcal{C} = \mathbb{Q} \cap [0, 1]$, then $\mathcal{S}(\mathcal{C})$ would refer to **RPL**. For these graded fuzzy logics, besides the original definition of a base as simply a set of formulas, it makes sense to consider another natural notion of basehood, where bases are closed by lower bounds of truth-degrees. We call them $\mathcal{C}$-closed bases.

Definition 5. *(Adapted from [9]) Given some (monotonic) t-norm fuzzy logic $\mathcal{S}$ with language $\mathbf{Fm}$ and a countable set $\mathcal{C} \subset [0, 1]$ of truth-constants, let $T \subseteq \mathbf{Fm}(\mathcal{C})$ be a base in $\mathcal{S}(\mathcal{C})$. We define $\mathrm{Cn}_{\mathcal{C}}(T) = \{(\varphi, r') : (\varphi, r) \in T,$ for $r, r' \in \mathcal{C}$ with $r \geq r'\}$. A base $T \subseteq \mathbf{Fm}(\mathcal{C})$ is called $\mathcal{C}$-closed when $T = \mathrm{Cn}_{\mathcal{C}}(T)$.*

Notice that, using Gerla's framework of abstract fuzzy logic [7], Booth and Ricther [4] define revision operators for bases which are closed with respect to truth-values in some complete lattice W.

The following results prove $\circledast_\gamma$ operators preserve $\mathcal{C}$-closure, thus making $\mathcal{C}$-closed revision a particular case of base revision under Theorem 1.

Proposition 1. *If T_0, T_1 are $\mathcal{C}$-closed graded bases, for any partial meet revision operator $\circledast_\gamma$, $T_0 \circledast_\gamma T_1$ is also a $\mathcal{C}$-closed graded base.*

Proof. Since T_0 is $\mathcal{C}$-closed, by maximality of $X \in \gamma(\mathrm{Con}(T_0, T_1))$ we have X is also $\mathcal{C}$-closed, for any such X. Let $(\psi, s) \in \bigcap \gamma(\mathrm{Con}(T_0, T_1))$ and $s' <_{\mathcal{C}} s$ for some $s' \in \mathcal{C}$. Then $(\psi, s) \in X$ for any $X \in \gamma(\mathrm{Con}(T_0, T_1))$ implies $(\psi, s') \in X$ for any such X. Hence $\bigcap \gamma(\mathrm{Con}(T_0, T_1))$ is $\mathcal{C}$-closed. Finally, since T_1 is $\mathcal{C}$-closed, we deduce $\bigcap \gamma(\mathrm{Con}(T_0, T_1)) \cup T_1$ is also $\mathcal{C}$-closed.

Let $\mathcal{P}_{\mathcal{C}}(\mathbf{Fm})$ be the set of $\mathcal{C}$-closed sets of $\mathbf{Fm}$ sentences. We introduce an additional axiom (F0) for revision of $\mathcal{C}$-closed bases by $\mathcal{C}$-closed inputs:

(F0) $T_0 \circledast T_1$ is $\mathcal{C}$-closed, if T_0, T_1 are

Corollary 1. *Assume $\mathcal{S}$ and $\mathcal{C}$ are as before and let $\circledast : \mathcal{P}_{\mathcal{C}}(\mathbf{Fm}) \to \mathcal{P}(\mathbf{Fm})$. Then, $\circledast$ satisfies* (F0) – (F5) *iff for some selection function $\gamma, T_0 \circledast T_1 = T_0 \circledast_\gamma T_1$ for every $T_1 \in \mathcal{P}_{\mathcal{C}}(\mathbf{Fm})$.*

As shown in the next example, $\mathcal{C}$-closed revision makes a big difference[8] in **RPL**. (Recall that **RPL** negation function, defined in $[0, 1]$, is $n(x) = 1 - x$.)

Example 1. (In **RPL**) Let $\mathcal{C} = \mathbb{Q} \cap [0, 1]$, base $T = \{(p, 0.9), (p \to q, 0.9)\}$ and input $T' = \{(\neg q, 0.4)\}$.

1. (No $\mathcal{C}$-closure.) In this case, we have two maxichoice revision outputs: $\{(p, 0.9), (\neg q, 0.4)\}$, and $\{(p \to q, 0.9), (\neg q, 0.4)\}$; the remaining revision is full meet: $T \circledast_{fm} T' = T'$.
2. (Rational $\mathcal{C}$-closure) Consider base $T_0 = \mathrm{Cn}_{\mathcal{C}}(T)$ and input $T_1 = \mathrm{Cn}_{\mathcal{C}}(T')$. Maxichoice revisions $T_0 \circledast_{mc} T_1$ are of form: $T_0 \circledast_{mc} T_1 = \mathrm{Cn}_{\mathcal{C}}(\{(p, r), (p \to q, s), (\neg q, 0.4)\})$ for any r, s such that $r + s - 1 = 0.6$ and $r, s \leq 0.9$.
3. (Finite $\mathcal{C}$-closure) Under the finite set of truth-constants $\mathcal{C} = \{\frac{k}{10} : k \leq 10\}$ (i.e. with constants for $0, 0.1, \ldots, 0.9, 1$), $\mathcal{C}$-closure gives three maxichoice revisions: $r = 0.9, s = 0.7$; $r = s = 0.8$; and $r = 0.7, s = 0.9$ (for sets $T_0 \circledast_{mc} T_1$ defined above); the remaining operators are obtained by combining two maxichoice selections, giving $r = 0.8, s = 0.7$; $r = 0.7, s = 0.8$; and (full meet) $r = s = 0.7$.

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[8] Examples on syntax-sensitivity show that in base revision it is natural to prefer bases without conjunctive formulas, i.e. to prefer $\{\ldots, \varphi, \psi, \ldots\}$ rather than $\{\ldots, \varphi \wedge \psi, \ldots\}$. This is also the case for **RPL** conjunction &: we should rephrase $\varphi \equiv \overline{r}$ as the two formulas $\overline{r} \to \varphi, \overline{1 - r} \to \neg\varphi$, instead of the original definition in [8] of $\equiv$, which would give $(\overline{r} \to \varphi \,\&\, \varphi \to \overline{r})$. This way, we obtain $\mathrm{Cn}_{\mathcal{C}}(\{\varphi \equiv \overline{0.5}\}) \circledast \mathrm{Cn}_{\mathcal{C}}(\{\overline{0.7} \to \varphi\}) \vdash_{\mathbf{RPL}} \overline{0.7} \equiv \varphi$.

4 Conclusions.

We improved Hansson and Wassermann characterization of revision operators in a class of logics without the deduction property. Apart from the general theorem, standard results for full meet and maxichoice revision operators are also provided. Then we moved to the field of graded fuzzy logics, in contradistinction to the approach by Booth and Richter in [4]; their work inspired us to prove similar results for a more general logical framework, including t-norm based fuzzy logics from Hájek. Finally, we observed the differences between revision for bases if they are assumed to be closed under truth-degrees.

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