Modal systems based on many-valued logics

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Abstract

We propose a general semantic notion of modal many-valued logic. Then, we explore the difficulties to characterize this notation in a syntactic way and analyze the existing literature with respect to this framework.

Keywords: modal many-valued logic, modal fuzzy logic.

1 Introduction

The purpose of this research is the search of a syntactic notion of *modal many-valued logic*[1] that generalizes the notion of (normal) modal logic [5, 1]. Our search is motivated by semantic issues. That is, we understand modal many-valued logics as logics defined by Kripke frames (possibly with many-valued accessibility relations) where every world follows the rules of a many-valued logic, this many-valued logic being the same for every world. The reader will find the details of this approach in Section 2.

Throughout the paper we will show the reader the difficulties of this search and we will try to specify which conditions should satisfy this syntactic notion. We will also review the works in the literature that fits inside our framework.

Unfortunately, we have not been able to found a syntactic characterization of the notion of modal many-valued logic and so the question remains open.

2 A semantic approach

In this section we start giving the definition of the modal many valued logic $Log_{\Box}(\mathbf{A}, \mathsf{F})$ associated with an algebra $\mathbf{A}$ and a class of $\mathbf{A}$-valued Kripke frames F.

The language of this new logic is, by definition, the propositional language generated by a set Var of propositional variables[2] together with the connectives given by the algebraic signature of $\mathbf{A}$ expanded with a new unary connective: the necessity[3] operator $\Box$. The set of formulas of the resulting language will be denoted by $Fm_{\Box}$.

We point out that the intended meaning of the universe A is a set of truth-values. The only requirements in our definition will be that the algebra $\mathbf{A}$ is a complete lattice, and that the algebraic language of $\mathbf{A}$ contains, besides meet $\wedge$ and join $\vee$, a constant 1 and an implication $\rightarrow$.

We stress that these conditions are quite weak and a lot of well-known algebras satisfy them, for instance, complete FL-algebras [20] and complete BL-algebras [13]. Hence, in particular we can consider that $\mathbf{A}$ is any of the three basic continuous t-norm algebras: *Lukasiewicz algebra* $[\mathbf{0},\mathbf{1}]_{\text{Ł}}$, *product algebra* $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ and *Gödel algebra* $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$. We also note that due to the fact that the free algebra with countable generators (i.e., the Lindenbaum-Tarski algebra) of any of the previous varieties of algebras is not complete it is not included in our framework.

An $\mathbf{A}$-*valued Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ where W is a set (of *worlds*) and R is a binary relation valued in A (i.e., $R : W \times W \longrightarrow A$) called *accessibility relation*. It is said that the Kripke frame is *classical* in case that the range of R is included in $\{0, 1\}$[4]. Whenever $\mathbf{A}$ is fixed, we will denote by Fr and CFr the classes of all $\mathbf{A}$-valued Kripke frames and all $\mathbf{A}$-valued classical Kripke frames. For the rest of the paper we will mainly focus on these two classes since they provide in some sense minimal logics.

Before introducing $Log_{\Box}(\mathbf{A}, \mathsf{F})$ we need to define what

[1] In particular, modal fuzzy logics will be inside this class.

[2] In most cases it is assumed that $Var = \{p_0, p_1, p_2, \ldots\}$.

[3] Later on we will give some ideas about how to develop these ideas with the possibility operator $\Diamond$.

[4] Here 0 means the minimum of $\mathbf{A}$ and 1 its maximum.

is an **A**-valued Kripke model. An **A**-*valued Kripke model* is a triple $\mathfrak{M} = \langle W, R, e\rangle$ where $\langle W, R\rangle$ is an **A**-valued Kripke frame and e is a map, called *valuation*, assigning to each variable in Var and each world in W an element of A. The map e can be uniquely extended to a map $\bar{e} : Fm_\Box \times W \longrightarrow A$ satisfying that:

- $\bar{e}$ is an algebraic homomorphism, in its first component, for the connectives in the algebraic signature of **A**, and
- $\bar{e}(\Box\varphi, w) = \bigwedge\{R(w, w') \to \bar{e}(\varphi, w') : w' \in W\}$.

Although the functions e and $\bar{e}$ are different there will be no confusion between them, and so sometimes we will use the same notation e for both.

Following the same definitions than in the Boolean modal case [5, 1] it is clear how to define *validity* of a $Fm_\Box$-formula in an **A**-valued Kripke model and in an **A**-valued Kripke frame.

Now we are ready to introduce the *modal many-valued logic* $Log_\Box(\mathbf{A}, \mathsf{F})$. It is defined as the set of formulas $\varphi \in Fm_\Box$ satisfying that for every **A**-valued Kripke model $\langle W, R, e\rangle$ over a frame $\langle W, R\rangle$ in F and for every world w in W, it holds that $e(\varphi, w) = 1$.

Remark 1 *For the sake of simplicity in this paper we restrict ourselves to adding the necessity operator $\Box$, but analogously we could have considered a possibility operator ruled by the condition*[5]

$$e(\Diamond\varphi, w) = \bigvee\{R(w, w') \odot e(\varphi, w') : w' \in W\}.$$

Note 2 *We stress that for the case that* **A** *is the Boolean algebra of two elements all previous definitions correspond to the standard terminology in the field of modal logic (cf. [5, 1]). As far as the authors know the first one to talk about this way of extending the valuation e into the modal many-valued realm was* M. FITTING *in [9].*

Up to now we have considered a logic as a set of formulas. Besides this way to consider logics, it is also common to consider them as consequence relations, e.g., [2]. Following this approach next we define two different consequence relations.

The *modal many-valued local consequence* $\models_{l(\mathbf{A},\mathsf{F})}$ associated with an algebra **A** and a class of **A**-valued Kripke frames F is defined by the following equivalence:

$$\Gamma \models_{l(\mathbf{A},\mathsf{F})} \varphi \quad \text{iff}$$

for every **A**-valued Kripke model $\langle W, R, e\rangle$ over a frame $\langle W, R\rangle$ in F and for every world w in W, it holds that if $e(\gamma, w) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi, w) = 1$.

The *modal many-valued global consequence* $\models_{g(\mathbf{A},\mathsf{F})}$ associated with an algebra **A** and a class of **A**-valued Kripke frames F is given by the following definition:

$$\Gamma \models_{g(\mathbf{A},\mathsf{F})} \varphi \quad \text{iff}$$

for every **A**-valued Kripke model $\langle W, R, e\rangle$ over a frame $\langle W, R\rangle$ in F, it holds that if $e(\gamma, w) = 1$ for every $\gamma \in \Gamma$ and every world w in W, then $e(\varphi, w) = 1$ for every world w in W.

We point out that the set of theorems of both consequence relations is precisely the set $Log_\Box(\mathbf{A}, \mathsf{F})$.

3 Differences with the modal Boolean case

General Considerations. Let us assume that we have fixed an algebra **A** satisfying the previous requirements. In order to simplify our discussion we will also assume that it is a FL_{ew}-algebra [20][6], i.e., a residuated lattice. This hypothesis will allow us to use the residuation law.

In order to find a successful syntactic definition of the notions introduced in the previous section[7] first of all we would need to settle a completeness theorem for the logics introduced in the previous section. In particular, we should know how to axiomatize the minimal one $Log_\Box(\mathbf{A}, \mathsf{Fr})$. What formulas must we add to an axiomatization of the many-valued logic defined by **A** in order to obtain a complete axiomatization of $Log_\Box(\mathbf{A}, \mathsf{Fr})$?

The fact that the famous modal axiom (K) (sometimes called *normality* axiom)

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \tag{K}$$

does not in general belong to $Log_\Box(\mathbf{A}, \mathsf{Fr})$ is what makes difficult even to suggest an axiomatization of $Log_\Box(\mathbf{A}, \mathsf{Fr})$. As a simple counterexample we can consider the logic $Log_\Box([\mathbf{0}, \mathbf{1}]_{\text{Ł}}, \mathsf{Fr})$ and the Kripke model given by $W = \{a, b\}$, $R(a, a) = 1$, $R(a, b) = 1/2$, $e(p_0, a) = 1$, $e(p_0, b) = 1/2$, $e(p_1, a) = 1$ and $e(p_1, b) = 0$. Then, $e(\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1), a) = 1/2$.

[5] The connective $\odot$ is what sometimes is called in the literature fusion, multiplicative conjunction, etc. (see [13, 20]).

[6] We remind the reader that in particular all BL-algebras [13] are FL_{ew}-algebras.

[7] In the modal Boolean case it is well-known the existence of modal logics that are Kripke frame incomplete. Hence, the searched definition of modal many-valued logic will have to include more logics that the ones introduced in Section 2.

It is easy to check that the *necessity rule*, from φ follows $\Box\varphi$, holds for $Log_\Box(\mathbf{A}, \mathsf{Fr})$. Another property that holds for $Log_\Box(\mathbf{A}, \mathsf{Fr})$ is the monotonicity of the necessity operator, i.e., if $\varphi \to \psi$ is in the logic then also $\Box\varphi \to \Box\psi$ is in the logic. Moreover, it is possible to see that

$$(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi)$$

is valid under our semantics. However, in general

$$(\Box\varphi \odot \Box\psi) \to \Box(\varphi \odot \psi)$$

and

$$(\Box(\varphi \to \psi) \odot \Box\varphi) \to \Box\psi$$

fail. The reader can notice that the last formula is equivalent to (K) thanks to the residuation law.

Although in general (K) does not belong to $Log_\Box(\mathbf{A}, \mathsf{Fr})$, let us remark two particular cases where (K) holds. For these two cases the difficulties disappear quite a lot as we will see in examples of Section 4. The first one is when $\odot$ and $\wedge$ coincide in the algebra $\mathbf{A}$. In particular this means that (K) belongs to $Log_\Box([\mathbf{0}, \mathbf{1}]_\mathbf{G}, \mathsf{Fr})$. And the second one is when F is the class of classical Kripke frames CFr, i.e., for any algebra $\mathbf{A}$ all $\mathbf{A}$-valued classical Kripke frames satisfy the normality condition. In particular this means that (K) belongs to $Log_\Box([\mathbf{0}, \mathbf{1}]_\text{Ł}, \mathsf{CFr})$ and $Log_\Box([\mathbf{0}, \mathbf{1}]_\mathbf{\Pi}, \mathsf{CFr})$.

Transfer Properties. We are going to show with three counterexamples that in general metalogical properties are lost when we move from the modal Boolean case to the modal many-valued one. This implies that in order to attack future problems for modal many-valued logics we will need to introduce new machinery, what makes this new field a really exciting and appealing one.

First of all we point out that the fact that two algebras $\mathbf{A}$ and $\mathbf{B}$ generate the same variety does not imply that $Log_\Box(\mathbf{A}, \mathsf{CFr}) = Log_\Box(\mathbf{B}, \mathsf{CFr})$. As a counterexample we can consider $\mathbf{A}$ as the standard Gödel algebra $[\mathbf{0}, \mathbf{1}]_\mathbf{G}$ and $\mathbf{B}$ as its subalgebra of universe $\{0\} \cup [1/2, 1]$. It is not hard to see that $\Box\neg\neg p \to \neg\neg\Box p$ belongs to $Log_\Box(\mathbf{B}, \mathsf{CFr})$ while fails in $Log_\Box(\mathbf{A}, \mathsf{CFr})$.

Secondly we notice that it can happen that two classes F_1 and F_2 of classical Kripke frames have different modal many-valued logics for an algebra $\mathbf{A}$ while for the case of the Boolean algebra of two elements they share the same logic. Why? It is well-known that the modal logic $S4$ is generated both by the class F_1 of finite quasi-orders (perhaps fails the antisymmetric property) and the class F_2 of infinite partial orders. However, $\Box\neg\neg p \to \neg\neg\Box p$ belongs to $Log_\Box([\mathbf{0}, \mathbf{1}]_\mathbf{G}, \mathsf{F}_1)$ while fails in $Log_\Box([\mathbf{0}, \mathbf{1}]_\mathbf{G}, \mathsf{F}_2)$.

Table 1: Fitting systems ($A = \{t_1, \ldots, t_n\}$)

$$\varphi \Rightarrow \varphi$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\varphi_1 \supset \varphi_2, \varphi_2 \supset \varphi_3 \Rightarrow \varphi_1 \supset \varphi_3$$

$$\frac{\Gamma, t_i \supset \varphi \Rightarrow \Delta, t_i \supset \psi \quad \text{for every } i \in \{1, \ldots, n\}}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$\frac{\Gamma, \psi \supset t_i \Rightarrow \Delta, \varphi \supset t_i \quad \text{for every } i \in \{1, \ldots, n\}}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$\Rightarrow \varphi \wedge \psi \supset \varphi$$
$$\Rightarrow \varphi \wedge \psi \supset \psi$$
$$\varphi_3 \supset \varphi_1, \varphi_3 \supset \varphi_2 \Rightarrow \varphi_3 \supset (\varphi_1 \wedge \varphi_2)$$

$$\Rightarrow \varphi \supset \varphi \vee \psi$$
$$\Rightarrow \psi \supset \varphi \vee \psi$$
$$\varphi_1 \supset \varphi_3, \varphi_2 \supset \varphi_3 \Rightarrow (\varphi_1 \vee \varphi_2) \supset \varphi_3$$

$$\Rightarrow t_i \supset t_j \quad \text{, if } t_i \leq t_j$$
$$t_i \supset t_j \Rightarrow \quad \text{, if } t_i \not\leq t_j$$

$$(\varphi_1 \wedge \varphi_2) \supset \varphi_3 \Rightarrow \varphi_1 \supset (\varphi_2 \supset \varphi_3)$$
$$\varphi_1 \supset (\varphi_2 \supset \varphi_3) \Rightarrow (\varphi_1 \wedge \varphi_2) \supset \varphi_3$$

$$\varphi \Rightarrow (\mathit{true} \supset \varphi)$$
$$(\mathit{true} \supset \varphi) \Rightarrow \varphi$$

$$\frac{\Rightarrow \varphi}{\Rightarrow \Box\varphi}$$

$$\Rightarrow \Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$$
$$\Rightarrow (t_i \supset \Box\varphi) \equiv \Box(t_i \supset \varphi), \text{ if } i \in \{1, \ldots, n\}$$

Lastly we remark that it is possible to have that $Log_\Box(\mathbf{2}, \mathsf{F})$ enjoys the finite Kripke frame property while $Log_\Box(\mathbf{A}, \mathsf{F})$ does not. A counterexample is given by the standard Gödel algebra $[\mathbf{0}, \mathbf{1}]_\mathbf{G}$ and the class F of classical quasi-orders. The failure of the finite Kripke frame property of $Log_\Box([\mathbf{0}, \mathbf{1}]_\mathbf{G}, \mathsf{F})$ is witnessed, for instance, by the formula $\Box\neg\neg p \to \neg\neg\Box p$.

4 Examples in the literature

In the last years there has been a growing number of papers about combining modal and many-valued logics. Some approaches differ from ours, like [6, 19], but others stay as particular cases of our framework. Among the ones that fit in our framework we can cite [9, 10, 14, 11, 12, 15, 4].

Next we will discuss the known axiomatizations in the literature of logics of the form $Log_\Box(\mathbf{A}, \mathsf{F})$ where $\mathbf{A}$ is non Boolean and F is the class of all Kripke frames or the class of all classical Kripke frames[8]. Indeed, the only known ones are for logics satisfying axiom (K), i.e., the authors are unaware of any axiomatization for a case where axioms (K) fails. This remains as a challenge.

A is a finite Heyting algebra. This case was considered by M. FITTING in [10, Section6]. The language includes constants for every element of the fixed algebra $\mathbf{A}$ (i.e., for every truth value), what simplifies the proofs and allows to give a unified presentation of the calculus to axiomatize $Log_\Box(\mathbf{A}, \mathsf{Fr})$. The last statement refers to the fact that all these calculus share the same schemes without constants. The calculus is given using sequents and can be found in Table 1. Completeness of this calculus means that $Log_\Box(\mathbf{A}, \mathsf{Fr})$ coincides with the set of formulas $\varphi \in Fm_\Box$ such that the sequent $\Rightarrow \varphi$ is derivable using the calculus in Table 1. We notice that using the constants it is very easy to see that $Log_\Box(\mathbf{A}, \mathsf{Fr}) \neq Log_\Box(\mathbf{A}, \mathsf{CFr})$. Other papers that study these cases are [17, 18, 16].

$\mathbf{Log}_\Box([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{Fr})$. This case has been studied by X. CAICEDO and R. RODRÍGUEZ in [4]. They have proved that this logic is axiomatized by the calculus given in Table 2. The proof is based on the construction of a canonical model[9], which indeed is classical. From here it follows that for every class of Kripke frames F, it holds that $Log_\Box([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{Fr}) = Log_\Box([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{CFr})$[10]. Therefore, for the case of $[\mathbf{0},\mathbf{1}]_\mathbf{G}$ we already know how to introduce the notion of *modal many-valued logic*: it is any set of $Fm_\Box$-formulas that contains the formulas in Table 2 and is closed under the rules in Table 2.

$\mathbf{Log}_\Box([\mathbf{0},\mathbf{1}]_\text{Ł}, \mathsf{CFr})$. The recent paper [15] by G. HANSOUL and B. TEHEUX axiomatizes the normal modal logic $Log_\Box([\mathbf{0},\mathbf{1}]_\text{Ł}, \mathsf{CFr})$ with the infinite calculus given in Table 3. The proof is based on the construction of a classical canonical model. Surprisingly this proof does not need the presence in the language of constants for every truth value. The trick to avoid the introduction of constants is based on a result of [21] (see [15, Definition 5.3]).

A slightly different approach. As we have claimed before it is unknown how to manage the resulting non-normal logics. One possibility to avoid this difficulty is to introduce graded modalities $\Box_t$ (where $t \in \mathbf{A}$) corresponding to the cuts of the many-valued accessibility relation, i.e., using

$$e(\Box_t\varphi, w) = \bigwedge\{e(\varphi, w') : R(w, w') \geq t\}$$

to extend the valuation. Then, it is easy to see that all modalities $\Box_t$ are normal. We notice that in some particular cases, axiomatizations for these graded modalities has been found in the literature (see for instance [8, 22, 3]). The case considered in [3] corresponds to consider the n-valued Łukasiewicz chain algebra $\mathbf{L_{n-1}}$ (see [7]) and having constants in the language for every element in the n-valued Łukasiewicz algebra. The axiomatization given in [3] is shown in Table 4. An interesting fact about this case is that $\Box$ is definable in the new language because

$$(\Box\varphi) \leftrightarrow \bigwedge\{t \to \Box_t\varphi : t \in \text{Ł}_{n-1}\}$$

is valid under our semantics.

Table 2: Inference Rules of $Log_\Box([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{Fr})$

$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
$\varphi \to (\psi \to \varphi)$
$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
$(\varphi \wedge (\psi \wedge \chi)) \to ((\psi \wedge \varphi) \wedge \chi)$
$(\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \wedge \psi) \to \chi)$
$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
$0 \to \varphi$
$\varphi \to (\varphi \wedge \varphi)$
$\neg\varphi \leftrightarrow (\varphi \to 0)$
$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\neg\neg\Box\varphi \to \Box\neg\neg\varphi$
From φ infer $\Box\varphi$
From φ and $\varphi \to \psi$, infer ψ

Table 3: Inference Rules of $Log_\Box([\mathbf{0},\mathbf{1}]_\text{Ł}, \mathsf{CFr})$

$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
$\varphi \to (\psi \to \varphi)$
$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\Box(\varphi \odot \varphi) \leftrightarrow (\Box\varphi \odot \Box\varphi)$
$\Box(\varphi \oplus \varphi) \leftrightarrow (\Box\varphi \oplus \Box\varphi)$
$\Box(\varphi \oplus \varphi^m) \leftrightarrow (\Box\varphi \oplus (\Box\varphi)^m)$ for every $m \geq 0$
From φ infer $\Box\varphi$
From φ and $\varphi \to \psi$, infer ψ

5 Main Open Problems

In opinion of the authors the main open problems in this field are the search of axiomatizations for $Log_\Box([\mathbf{0},\mathbf{1}]_\text{Ł}, \mathsf{Fr})$ and $Log_\Box([\mathbf{0},\mathbf{1}]_\mathbf{\Pi}, \mathsf{Fr})$ in case they are recursively axiomatizable. The main difficulties here are the lack of normality of these logics.

[8]Hence, we do not consider cases where the class of frames satisfy some extra conditions, e.g., reflexive and $\odot$-transitive frames. The reason why we do not talk about them is because we want to consider the minimal logics.

[9]This technique also gives strong completeness in the sense that the caculus in Table 2 axiomatizes $\models_{l([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{Fr})}$.

[10]We stress that this does not contradict facts like that $\Box p \to p$ and $\Box\neg p \to \neg p$ define, over $[\mathbf{0},\mathbf{1}]_\mathbf{G}$, the same class of classical Kripke frames while they define different classes of Kripke frames.

Table 4: Inference Rules for $\Box_0, \ldots, \Box_1$ $(t \in L_{n-1})$

$$
\begin{array}{c}
(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
\varphi \to (\psi \to \varphi) \\
((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi) \\
(\neg\varphi \to \neg\psi) \to (\psi \to \varphi) \\
(\neg\varphi) \leftrightarrow (\varphi \to 0) \\
(\varphi \odot \psi) \leftrightarrow \neg(\varphi \to \neg\psi) \\
n.\varphi \to (n-1).\varphi \\
(m.\varphi^{m-1})^n \leftrightarrow (n.\varphi^m), \quad 2 \le m \le n-2 \text{ and } m \nmid (n-1) \\
(t_i \to t_j) \leftrightarrow t_k, \quad \text{if } t_k = t_i \to t_j \\
\Box_t(\varphi \to \psi) \to (\Box_t\varphi \to \Box_t\psi) \\
(\Box_t\varphi \odot \Box_t\psi) \to \Box_t(\varphi \odot \psi) \\
\Box_{t_i}\varphi \to \Box_{t_j}\varphi, \quad \text{if } t_i \le t_j \\
\Box_0\varphi \to \varphi \\
\neg\Box_0\neg\varphi \to \Box_0\neg\Box_0\neg\varphi \\
\Box_{t_i}(t_j \to \varphi) \leftrightarrow (t_j \to \Box_{t_i}\varphi) \\
\text{From } \varphi \text{ infer } \Box_t\varphi \\
\text{From } \varphi \text{ and } \varphi \to \psi, \text{ infer } \psi
\end{array}
$$

Once there is an axiomatization for them (if any) it seems easy to find the right definition of modal many-valued logic. And once we know the definition of the class of modal many-valued logics the next step will be their study with all possible techniques: algebras[11], Kripke frames, Kripke models, sequent calculus, etc

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