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THE THEORY OF MODULES OF SEPARABLY CLOSED FIELDS 1

PILAR DELLUNDE1, FRANÇOISE DELON, AND FRANÇOISE POINT2

Abstract. We consider separably closed fields of characteristic $p > 0$ and fixed imperfection degree as modules over a skew polynomial ring. We axiomatize the corresponding theory and we show that it is complete and that it admits quantifier elimination in the usual module language augmented with additive functions which are the analog of the $p$-component functions.

§1. Introduction. We will denote by $SCF_{p,e}$ the first-order theory of separably closed fields of characteristic $p > 0$ and imperfection degree $e \in \omega \cup \{\infty\}$ in the language of fields. Y. Ershov showed that $SCF_{p,e}$ is complete (see [5]). Moreover, whenever $e$ is finite, if one adds new constants for the elements of a chosen $p$-basis and the $p^e$ unary functions sending an element to its $p$-components over this basis, one gets quantifier elimination in this extended language (see for instance [4], Proposition 27).

Let $K$ be a model of $SCF_{p,e}$; we will consider additive reducts of this field. This kind of structures was first considered in the light of Zil'ber conjecture on the possible geometries on strongly minimal sets. In the case where $K$ is algebraically closed and $f(x)$ is any polynomial with coefficients in $K$, Gary Martin proved that either $f(x)$ is affine or multiplication is definable in an expansion of $(K, +, f)$ by finitely many scalar multiplications (see Theorem 3.7 in [7]). (A polynomial is additive if it is a linear combination of $X^q$'s, where $q$ is a power of $p$ and it is affine if it differs by a constant from an additive polynomial.) This specific result is of interest to us because its proof uses only the fact that $K$ is separably closed.

Here, we investigate the first-order theory of $K$ regarded as a module over the skew polynomial ring $R = \mathbb{F}_p(B)[t; \alpha]$ (see [2]), where $B$ is a $p$-basis of $K$ (hence a set of cardinality $e$), $\mathbb{F}_p(B)$ is the fraction field of the polynomial ring $\mathbb{F}_p[B]$ and $\alpha$ the Frobenius map (i.e., $p^{th}$ power). Note that, in the context of modules, the cardinality of $B$ is important, also when infinite: it defines the language in which we work. We extract a series of properties of these fields when viewed as $R$-modules and we show that the corresponding theory of modules over $R$ is model-complete (and decidable). In the resulting structures, the decomposition of an element over the $p$-basis can be expressed and we will extend the ordinary language of modules with the analog of the $p$-component functions. Let $T_e$ denotes the

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theory in this expanded language of the structures we are considering. The analysis
proceeds first in investigating the torsion part, which comes down to the question
whether we can describe in that weaker language the behaviour of roots of additive
polynomials. Second, we show that any positive primitive formula is equivalent to a
positive quantifier-free formula. Then we show that the index of one p.p. definable
subgroup in another is either 1 or $\infty$ in any torsion-free summand of a model of our
theory. This suffices to prove on one hand that $T_e$ admits quantifier elimination,
and on the other hand that it is the theory of separably closed fields of characteristic
$p$ and imperfection degree $e$ in that weaker language.

In a second article (see [3]), we will show that (as in the field case [11]) for $e > 0$
$T_e$ is non-superstable (though stable) and we will give a partial description of the
closed subset of the Ziegler spectrum corresponding to this theory. Then we will
characterize the types as certain submodules. We will identify the types of (finite)
$U$-rank and we will show that we don’t have elimination of imaginaries but p.p.
elimination of imaginaries.

§2. Preliminaries. Fix a prime number $p$ and a cardinal number $e$.

DEFINITION 1. Let $\mathbb{F}_p(B)$ be the fraction field of the polynomial ring $[\mathbb{F}_p[B], where
$B = \{b_i : i \in e\}$. Let $R = \mathbb{F}_p(B)[t; \alpha]$ be the skew polynomial ring in $t$ over $\mathbb{F}_p(B)$
associated with the Frobenius map $\alpha$ (see [2]). The domain of $R$ is the set of
polynomials in $t$ of the form $q(t) = t^n.a_n + \cdots + t.a_1 + a_0$, where the $a_i$’s belong
to $\mathbb{F}_p(B)$, with multiplication defined by the commutation rule $a.t = t.a^n$, for any
$a \in \mathbb{F}_p(B)$. If $a_n$ is different from zero, we call $n$ the degree $\text{deg}(q)$ of $q(t)$ and $q$
is called monic if $a_n$ equals 1. We also set $\text{deg}(0) = -\infty$. We have the usual rule
$\text{deg}(q.r) = \text{deg}(q) + \text{deg}(r)$, and so the right Euclidean division algorithm holds.

Substituting $x^{p^e}$ in $q(t)$ for $t^i$ for all $i$ (see Remark 1, item 3), the resulting
ordinary polynomials are exactly the additive polynomials. They have been first
considered by O. Ore who called them $p$-polynomials (see [8]). Equipped with
addition and composition, the set of additive polynomials forms a ring. As noted
by Ore, an ordinary polynomial having only simple roots is an additive polynomial
iff its roots form an $\mathbb{F}_p$-vector space (see Theorem 8, p. 565 in [8]). This entails
that any polynomial divides (in the ordinary sense) some additive polynomial (see
Chapter 3, p. 581 in [8]).

Let us recall a few properties of this ring, which can be found in Chapter 2
of [2]. By Proposition 2.11 in [2], $R$ is an integral domain where the right ideals
are principal. The proof is straightforward using the right Euclidean division
algorithm. This entails that this ring is right Ore (see Corollary 1.3.7 in [2]), but
whenever $e \neq 0$, it is not left Ore i.e., there exist two non zero elements $x_1$ and
$x_2$ such that $R.x_1 \cap R.x_2 = \{0\}$. (For instance take the elements $t$ and $t.b$, for
some $b \in B$.) So, it contains the free algebra over $\mathbb{F}_p$ on two generators $x_1$ and
$x_2$ (see Proposition 1.6.6 in [2]). We don’t know whether the theory $T_R$ of all right
$R$-modules is undecidable.

§3. Axiomatization. Finite imperfection degree. For $S$ a ring, the language of
right $S$-modules is $\mathcal{L}_S = \{+,-,0,-r : r \in S\}$, where for any $r \in S$ and $x$ in a right
$S$-module, $x \cdot r$ denotes the scalar multiplication of $x$ by the ring element $r$. Let $T_S$ be the theory of all right $S$-modules in this language.

We extend the usual $R$-module language by adding new unary functions which will be existentially $\mathcal{L}_R$-definable in the theory we will be considering (in case $e$ is finite, they will be definable by a positive primitive formula).

Let $e$ be finite and let $B = \{b_0, \ldots, b_{e-1}\}$.

**Definition 2.** Let $\mathcal{L} = \mathcal{L}_R \cup \{\lambda_i : i \in p^e\}$, where the $\lambda_i$'s are unary functions.

Given $i \in p^e$, let $m_i$ be the monomial $b_0^{i(0)} \cdots b_{e-1}^{i(e-1)}$ (we identify here $p^e$ with the set of all maps from $\{0, \ldots, e-1\}$ to $\{0, \ldots, p-1\}$). We will call the $m_i$'s $p$-monomials.

**Definition 3.** Let $T_e$ be the following $\mathcal{L}$-theory:
1. $T_R$ the theory of all right $R$-modules,
2. For each $\varphi \in R$ with $\varphi(0) \neq 0$, $\forall x \exists y, x = y \cdot \varphi$.
3. $\exists^{\varphi} x_i$ (the $x_i$'s are linearly independent over $\mathbb{F}_p$) $\wedge \bigwedge_i x_i \cdot \varphi = 0$. For each $\varphi \in R$ of degree $d$ with $\varphi(0) \neq 0$.
4. $\forall x x = \sum_{i \in p^e} \lambda_i(x) \cdot t \cdot m_i$.
5. $\forall x \forall(y_i)_{i \in p^e} (x = \sum_{i \in p^e} x_i \cdot t \cdot m_i \rightarrow \bigwedge_i x_i = \lambda_i(x))$.

In the sequel, we find it useful to express some of our results for subtheories of $T_e$, namely:

**Notation 3.1.** Let $T_{sep}$ be the $\mathcal{L}_R$-theory consisting of the axiom schemes 1, 2, 3 above. Let $T_{free}$ be the $\mathcal{L}$-theory consisting of the axiom schemes 1, 2, 4, 5. Note that the class of models of $T_{free}$ is closed under direct summands (in the class of $\mathcal{L}$-structures) and direct products (the axioms are Horn sentences). Let $T_\lambda$ be the $\mathcal{L}$-theory consisting of the axiom schemes 1, 4, 5.

**Remark 1.**
1. Note that we could have written axiom schemes 2 and 3 in a shorter way as: $\forall x \exists^{p^e} y (x = y \cdot \varphi)$, for each $\varphi \in R$ of degree $d$ with $\varphi(0) \neq 0$.
2. Observe that if $M$ is a model of $T_\lambda$, we have $M = \bigoplus_{i \in p^e} M \cdot t \cdot m_i$, as $\mathbb{F}_p$-vector-spaces. Note that an $\mathcal{L}_R$-substructure of a model of $T_\lambda$ is itself a model of axioms 4, 5 if and only if it is an $\mathcal{L}$-substructure.
3. Let $q$ be an element of $R$ of the form $q = t^a a_n + \cdots + t a_1 + a_0$. We denote by $q[x^p]$ the polynomial $a_n x^{tp^e} + \cdots + a_1 x^p + a_0 x$. We say that $q$ is separable if $q(0) \neq 0$. This terminology is justified by the fact that $q$ is separable as defined above if and only if $q[x^p]$ is separable in the classical sense, that is, its formal derivative is non-zero.

**Proposition 3.1.** $T_e$ is consistent.

**Proof.** Let $K$ be a separably closed field of characteristic $p$ and imperfection degree $e$. We interpret $B$ as a $p$-basis of $K$. We define the action of $t$ on $K$ as the Frobenius map and so for instance we get for $y \in K$ and $\varphi \in R$, $y \cdot \varphi = q[y^p]$. We interpret the unary functions $\lambda_i$ as the $p$-components on $B$. Since $B$ is a $p$-basis, $K$
is a model of axioms 4 and 5 and since $K$ is separably closed, it satisfies $T_{sep}$ and so $K$ is a model of $T_v$.

Observe that with the same interpretation of the action of $t$ and of the $\lambda_i$’s, any field extension of $\mathbb{F}_p(B)$ admitting $B$ as a $p$-basis, is a model of $T_{\lambda}$.

**NOTATION 3.2.**

1. Given $q \in R$, we will define $\sqrt{q}$. First, for $a = \sum_{i \in p^e} a_i^p \cdot m_i \in \mathbb{F}_p(B)$, where the $a_i$’s belong to $\mathbb{F}_p(B)$, set $a^{1/p} := \sum_{i \in p^e} a_i^{1/p} \cdot m_i$. (Observe that $(a^p)^{1/p} = a$, but except if $a \in \mathbb{F}_p(B)^p$ and $a$ are distinct.) Then, for $q = \sum_{j=0}^n t^j a_j \in R$ with $a_j \in \mathbb{F}_p(B)$, set $\sqrt{q} := \sum_{j=0}^n t^{j/p} a_j^{1/p}$. Iteration $m$ times of $\sqrt{q}$ is denoted $\sqrt[m]{q}$.

2. Let $q = \sum_{i=0}^n t^i a_i \in R$ with $a_j = \sum_{i \in p^e} a_{ji}^p \cdot m_i$, $a_{ji} \in \mathbb{F}_p(B)$. For $i \in p^e$, set 
$$q_{(i)} := \sum_{j=0}^n t^{j/p} a_{ji}^{1/p}.$$ So, $q = \sum_{i \in p^e} q_{(i)} \cdot m_i$.

3. Any nonzero $q \in R$ can be written $q = t^m.(t^n.a_n + \cdots + t.a_1 + a_0) \in R$, with $a_0, \ldots, a_n \in \mathbb{F}_p(B)$ and $a_n.a_0 \neq 0$. Define $v(q) = m$ and write $q = t^v(q).q_s$. Not that $q_s$ is separable.

**REMARK 2.** Let $q \in R$. Then

1. $t.q_{(i)} = \sqrt[q_{(i)}]{t}$, for each $i \in p^e$.
2. If $q$ is separable, then for some $i \in p^e$, $\sqrt[q_{(i)}]{q}$ is also separable.
3. $t.q = \sum_{i \in p^e} \sqrt[q_{(i)}]{q} \cdot t^m \cdot m_i$.

We state now some preliminary lemmas on the $\mathcal{L}'$-terms. First, we introduce a notation for the composite of the functions $\lambda_i$.

**NOTATION 3.3.** For any $i_1, \ldots, i_m \in p^e$, let $\lambda_{i_1, \ldots, i_m} := \lambda_{i_1} \circ \cdots \circ \lambda_{i_m}$. Each $x$ in some model $M$ of $T_{\lambda}$ can be written uniquely as

$$x = \sum_{i_1, \ldots, i_m \in p^e} \lambda_{i_1}(\ldots(\lambda_{i_1}(x)) \ldots) \cdot t^{m \cdot m_{i_m}^{p-1}} \cdots \cdot t^{m \cdot m_{i_1}}.$$

We will index by $p^{em} = p^{me} = (p^m)^e$ the $p$-components iterated $m$ times and letting 

$$m_{(i_1, \ldots, i_m)} = m_{i_m}^{p_{i_m}^{m_{i_m}^{p_{i_m}^{p_{i_1}^{i_m}}}}} \cdots \cdot m_i$$

we get 

$$x = \sum_{k \in p^{me}} \lambda_k(x) \cdot t^{m \cdot m_k}.$$ We extend our notation to write an element $q \in R$ as: $q = \sum_{k \in p^{me}} q(k) \cdot m_k$. Iterating the above Remark, we get that $t^m.q = \sum_{k \in p^{me}} \sqrt[m]{q(k)} \cdot t^m \cdot m_k$.

**LEMMA 3.2.** Let $M$ be a model of $T_{\lambda}$. Then,

1. For any $u, v \in M$ and $i \in p^e$, $\lambda_i(u + v) = \lambda_i(u) + \lambda_i(v)$.
2. For any $u \in M$, $d = \sum_{r \in p^e} a_r^p \cdot m_r \in \mathbb{F}_p(B)$ and $i \in p^e$,

$$\lambda_i(u \cdot d) = \sum_{\delta \in 2^e} \left( \sum_{r,s \in p^e \cap (r+s = i + \delta)} \lambda_r(u) \cdot d_s \cdot \prod_{j \in e} b_{ij}^{\delta(j)} \right).$$
3. For any \( u \in M \), let \( d = \sum_{s \in p^i} d_s \cdot m_s \in F_p(B) \), \( k \) an integer \( \geq 1 \) and \( i \in p^\alpha \),

\[ \lambda_i(u \cdot t^k \cdot d) = u \cdot t^{k-1} \cdot d_i. \]

**Proof.** Immediate calculation from the definition of the functions \( \lambda_i \).

The above Lemma shows in fact that any model of \( T_\alpha \) is a module over the ring \( R[\lambda_i : i \in p^\alpha] \) where the multiplication rules are given by 2. and 3. above (see [1]).

**Corollary 3.3.** Let \( M \) be a model of \( T_\alpha \). Then any \( \mathcal{L} \)-term \( t(x_0, \ldots, x_k) \) is in \( M \) equal to a term of the form

\[ \sum_{j=0}^{k} \sum_{i \in p^m} \lambda_i(x_j) \cdot d_{ji} \]

for some integer \( m \geq 1 \) and \( d_{ji} \)'s in \( R \).

**Proof.** Using Lemma 3.2 we show, by induction on the length \( m \) of the tuple \( j \in p^m \), that \( \lambda_j(x \cdot r) = \sum_{i \in p^m} \lambda_i(x) \cdot r_i \), for some \( r_i \)'s in \( R \).

**Lemma 3.4.** Let \( m \in \mathbb{N}, m > 0 \), and \( q_j, q_j' \in R \), with \( q_j = t^m \cdot q_j' \). Then the equation \( \sum_j y_j \cdot q_j = u \) is equivalent to

\[ \bigwedge_{i \in p^m} \sum_j y_j \cdot \sqrt[m]{(q_j')_{(i)}} = \lambda_i(u) \]

in any model \( M \) of \( T_\alpha \).

**Proof.** By Remark 2 and Notation 3.3, \( q_j' = \sum_{i \in p^m} (q_j')_{(i)} \cdot m_i \) and

\[ \sum_j y_j \cdot t^m \cdot q_j = \sum_{i \in p^m} \left( \sum_j y_j \cdot \sqrt[m]{(q_j')_{(i)}} \right) \cdot t^m \cdot m_i. \]

Therefore, if \( \sum_j y_j \cdot q_j = u \), since \( M \) is model of \( T_\alpha \), we have

\[ \bigwedge_{i \in p^m} \sum_j y_j \cdot \sqrt[m]{(q_j')_{(i)}} = \lambda_i(u). \]

The converse is clear.

**Proposition 3.5.** Let \( M \) be a model of \( T_\alpha \), then the torsion part \( M_{tor} \) of \( M \) is:

\[ \{ x \in M : x \cdot q = 0, \text{ for some separable } q \in R \} \]

and \( M_{tor} \) is an \( \mathcal{L} \)-substructure of \( M \). Moreover, if \( M \) is a model of \( T_{free} \) (respectively \( T_e \)), then \( M_{tor} \) is a model of \( T_{free} \) (respectively \( T_e \)).

**Proof.**

1. Since \( R \) is a right Ore ring, the set of torsion elements forms a submodule. Let \( x_1 \) and \( x_2 \) belong to \( M_{tor} \) with \( x_1 \cdot q_1 = 0 \), \( x_2 \cdot q_2 = 0 \), \( q_1, q_2 \in R - \{0\} \). Then there exist \( a_1, a_2 \in R - \{0\} \) such that \( q_1 \cdot a_1 = q_2 \cdot a_2 = q \neq 0 \) and so \( (x_1 + x_2) \cdot q = 0 \). For \( r \in R \), let \( s = \text{lcm}(r, q_1) = r r_1 \) for some \( r_1 \in R - \{0\} \), then \( x_1 \cdot r \cdot r_1 = x_1 \cdot s = 0 \) and so \( x_1 \cdot r \) belongs to \( M_{tor} \).
2. Assume that \( x \in M_{tor} \) with \( x \cdot q = 0 \), for some \( q \in R - \{0\} \) with \( v(q) > 0 \). Let 
\[ q = t^v(q) \cdot q_v. \]
By Remark 2(3) and axioms 4, 5, \( \bigwedge_{i \in p^n} x \cdot t^{v(q) - 1} \cdot \sqrt{(q_v)_{(i)}} = 0. \)
Now, since \( q_v \) is separable, by Remark 2(2), for some \( i_0 \), \( \sqrt{(q_v)_{(i)}} \) is separable. 
By iterating this procedure we obtain \( x \cdot q' = 0 \), for some \( q' \) separable.
3. Let \( q = \sum_i t^i \cdot a_i \) be a separable element of \( R \) and suppose that \( x \cdot q = 0. \)
Then writing \( q \cdot a_0^{-1} = 1 + t \cdot q' \), we get 
\[ x = - \sum_{i \in p^n} x \cdot \sqrt{q'_{(i)}} \cdot t \cdot m_i, \]
so \( \lambda_i(x) = x \cdot \sqrt{q'_{(i)}} \) belongs to the \( R \)-submodule generated by \( x \).
4. So, \( M_{tor} \) is an \( \mathcal{L} \)-substructure of \( M \). To check that it is a model of \( T_{free} \) (respectively \( T_{p} \)) whenever \( M \) is, in both cases, it is enough to check axiom scheme 2. Let \( x \in M_{tor}, q \in R - \{0\} \) and \( y \) such that \( x = y \cdot q \). Since \( R \) is a domain, \( y \) belongs to \( M_{tor} \).

**Corollary 3.6.** Let \( K \) be a separably closed field with \( p \)-basis \( B \). Let us denote by \( \mathbb{F}_p(B)^{sep} \) the separable closure of \( \mathbb{F}_p(B) \). Then \( K_{tor} = \mathbb{F}_p(B)^{sep} \).

**Proof.** Since any polynomial divides an additive polynomial (see [8] Chapter 3), we have that \( \mathbb{F}_p(B)^{sep} \subseteq K_{tor} \). Now \( K_{tor} \subseteq \mathbb{F}_p(B)^{sep} \) since Proposition 3.5 tells us that \( x \in K_{tor} \) implies that \( x \cdot q = 0 \) for some separable \( q \).

**Remark 3.** The proof of Proposition 3.5 may be generalized as follows. Let \( A, C \) be two \( \mathcal{L} \)-structures models of \( T_p \) and suppose that \( A \subseteq C \). Let \( u \) belong to \( C \) and assume that there exists \( q \in R - \{0\} \) such that \( u \cdot q \) belongs to \( A \). Then there exists a separable element \( r \in R \) such that \( u \cdot r \) belongs to \( A \).

### §4. Axiomatization. Infinite imperfection degree.

In this section, we will extend our previous results to the infinite imperfection case. Let \( B \) be infinite. Let \( \mathcal{L}_\infty = \mathcal{L}_R \cup \{ \lambda_i^b : i \in p^n, b \in B^n, n \in \omega - \{0\} \} \), where the \( \lambda_i^b \)'s are unary functions. We fix an enumeration \( m_{n,i}(\bar{x}) \) of the monomials of the form \( x_0^{i(0)} \ldots x_n^{i(n-1)} \), with \( n \in \mathbb{N} \) and \( i \in p^n \). As in the finite imperfection degree case, we will call the elements of the form \( m_{n,i}(\bar{b}) \) where \( \bar{b} \in B^n, p \)-monomials.

**Definition 4.** Let \( T_\infty \) be the axiom schemes 1, 2, 3 as in the finite imperfection degree case together with the following schemes:

4. \( \forall x \left( \bigvee_{i \in p^n} \lambda_i^b(x) \neq 0 \rightarrow x = \sum_{i \in p^n} \lambda_i^b(x) \cdot t \cdot m_{n,i}(\bar{b}) \right), \)

   for each \( \bar{b} = (b_0, \ldots, b_{n-1}) \in B^n, n \in \mathbb{N} - \{0\} \).

5. \( \forall x \forall (x_i)_{i \in p^n} \left( x = \sum_{i \in p^n} x_i \cdot t \cdot m_{n,i}(\bar{b}) \rightarrow \bigwedge_{i \in p^n} x_i = \lambda_i^b(x) \right), \)

   for each \( \bar{b} = (b_0, \ldots, b_{n-1}) \in B^n, n \in \mathbb{N} - \{0\} \).

A model of \( T_\infty \) is \( K_0 = \mathbb{F}_p(B)^{sep} \), where the functions \( \lambda_i^b \) are interpreted as follows. If \( x \notin K_0(\bar{b}) \), then \( \lambda_i^b(x) = 0 \), for any \( i \in p^n \). For \( x = \sum_i x_i^p \cdot m_{n,i}(\bar{b}) \) for some \( x_i \)'s in \( K_0 \), then \( \lambda_i^b(x) = x_i \). So, \( T_\infty \) is consistent.

Similarly to the finite imperfection case, we will denote by \( T_{\lambda,\infty} \) the \( \mathcal{L}_\infty \)-theory consisting of axiom scheme 1 together with axiom schemes 4.\( \infty \) and 5.\( \infty \) and by \( T_{free,\infty} \) the \( \mathcal{L}_\infty \)-theory consisting of \( T_{\lambda,\infty} \) together with axiom scheme 2. Observe
that if $M$ is a model of $T_{\lambda, \infty}$, then for any integer $n$ and $\vec{b} \in B^n$ the sum of the $\mathbb{F}_p$-vector spaces $M \cdot t \cdot m_{i,j}(\vec{b})$ in $M$ is direct. And, for $x \in M$, $\lambda^\infty_i(x) \neq 0$ for some $\vec{b} \in B^n$ and some $i \in p^n$ if and only if $x$ is a nonzero element of the direct sum

$$\bigoplus_{i \in p^n} M \cdot t \cdot m_{i,j}(\vec{b})$$

for some $\vec{b} \in B^n$.

In order to extend Notation 3.2 to the case when $e$ is infinite, note that each $q \in R$ belongs to $\mathbb{F}_p(\vec{b})[t; \alpha]$ for some $n \in \mathbb{N}$ and $\vec{b} \in B^n$. Then we obtain analogous statements to Remark 2 and Proposition 3.5.

Now we need to introduce a "tree property" which indicates for a specific element the width of the tree of its (non-trivial) $p$-components.

**NOTATION 4.1.** For any sequences $\vec{b} \in B^n$ and $\vec{i} = (i_1, \ldots, i_m) \in p^n \times \cdots \times p^n$, set $\lambda^\infty_i := \lambda^n_{i_m} \circ \cdots \circ \lambda^n_{i_1}$. As in Notation 3.3, we will identify the set $(p^n)^m$ with the set of maps from $n$ to $p^m$, and for $i$ such map, define $m_{n,i}(\vec{b}) := \prod_{j \in n} b^{i(j)}_j$.

From now on, we drop the “bar” above the index “$i$”.

**DEFINITION 5.** Let $M$ be an $\mathcal{L}_\infty$-structure, $u \in M$, $\vec{b} \in B^m$, $m$ an integer $\geq 1$. We say that $u$ has the $(\vec{b}, m)$-tree property (t.p.): if $u = \sum_{i \in p^m} \lambda^\infty_i(u) \cdot t^m \cdot m_{n,i}(\vec{b})$.

Note that this property is expressible by a positive quantifier-free formula.

**REMARK 4.** Let $M$ be a model of $T_{\lambda, \infty}$, $u \in M$, $\vec{b} = (b_0, \ldots, b_{n-1}) \in B^n$, $m$ an integer $\geq 1$. The following are equivalent:

1. $u$ has the $(\vec{b}, m+1)$-t.p.
2. $u$ has the $(\vec{b}, 1)$-t.p. and for any $k \leq m$, any $i = (i_1, \ldots, i_k)$, $\lambda^\infty_i(u)$ has the $(\vec{b}, 1)$-t.p.
3. $u$ has the $(\vec{b}, 1)$-t.p. and for any $i \in p^n$, $\lambda^\infty_i(u)$ has the $(\vec{b}, m)$-t.p.

The proofs of the next three Lemmas are immediate using the definition of the functions $\lambda^\infty_i$.

**LEMMA 4.1.** Let $M$ be a model of $T_{\lambda, \infty}$, $u, v \in M$ and $d \in \mathbb{F}_p(B)$, $d = \sum_{i \in p^n} d^{i_j}_j \cdot m_{n,i}(\vec{b})$, for some $\vec{b} = (b_0, \ldots, b_{n-1}) \in B^n$ and $d_i$’s in $\mathbb{F}_p(B)$. Then,

1. If $u$ and $v$ have the $(\vec{b}, 1)$-t.p. then for any $i \in p^n$,

$$\lambda^\infty_i(u + v) = \lambda^\infty_i(u) + \lambda^\infty_i(v).$$

2. For any $i \in p^n$, $k$ integer $\geq 1$,

$$\lambda^\infty_i(u \cdot t^k \cdot d) = u \cdot t^{k-1} \cdot d_i.$$

3. For any $i \in p^n$,

$$\lambda^\infty_i(u \cdot d) = \sum_{\delta \in p^n} \sum_{r+s=i-p^k} \lambda^\infty_r(u) \cdot d_s \cdot \prod_{j \in n} b^{\delta(j)}_j.$$
**LEMMA 4.3.** Let \( q_j, q_j' \in R, q_j = t^m q_j' \) where \( m > 0 \). Assume that the \( q_j \)'s belong to \( \mathbb{F}_p(\bar{b})[t; \alpha] \) where \( \bar{b} \in B^n \). Then the equation \( \sum_j y_j \cdot q_j = u \) is equivalent to
\[
\bigwedge_{i \in \mathbb{P}^m} \sum_j y_j \cdot \overline{c^n(q_j')_i} = \lambda^b_i(u) \quad \text{and} \quad (u \text{ has the } (\bar{b}, m)\text{-t.p. property})
\]
in any model \( M \) of \( T_{\lambda, \infty} \).

§5. **The torsion part and divisible closure over a substructure.** The main part of this section will be devoted to the proof that the torsion submodules of any two models of \( T_c \) are isomorphic. Then we will see that we can adapt our proofs to show that any \( \mathscr{L} \)-substructure of a model of \( T_c \) can be extended in a canonical way to a submodel.

**DEFINITION 6.** Let \( q \) be an element of \( R \). \( q \) is said to be prime if it is non invertible, separable and if it cannot be written as a product of two non invertible elements of \( R \).

Fix an integer \( n \geq 1 \). Let \( N \) be an \( \mathscr{L}_R \)-structure, \( N_0 \) a substructure of \( N \) and \( u \in N - N_0 \) be such that \( u \cdot q = 0 \), for some separable \( q \in R \) of degree \( n \).

**LEMMA 5.1.** There is a monic separable polynomial \( q_u \in R \) with degree \( \leq n \) such that:

1. \( q_u \) is of degree minimal such that \( q_u \neq 0 \) and \( u \cdot q_u \in N_0 \);
2. For any \( r \in R \) such that \( u \cdot r \in N_0 \), \( q_u \) divides \( r \) on the right.

Therefore, there is an unique monic separable polynomial \( q_u \in R \) of minimal degree such that \( u \cdot q_u \in N_0 \).

Suppose furthermore that \( N_0 \) contains all elements of \( N_{tor} \) annihilated by some separable element of \( R \) of degree \( < n \), then \( q_u \) is prime.

**PROOF.** Let \( I \) be the set of polynomials \( r \) such that \( u \cdot r \in N_0 \). By hypothesis, \( I \) contains a separable polynomial \( q \). Since \( N_0 \) is a right \( R \)-module, \( I \) is a right ideal and since \( R \) is right principal, there is \( q_u \) a monic polynomial generating \( I \), moreover such polynomial is unique. Since \( q \) is separable, \( q_u \) is separable.

The last assertion under the additional hypothesis on \( N_0 \) is clear.

For \( c \in N \), let \( \mathbb{F}_p(B)[c] \) the \( \mathbb{F}_p(B) \)-vector subspace of \( N \) generated by \( c \) and we denote by \( N_0(c)_R \) (respectively \( (c)_R \)) the \( R \)-submodule of \( N \) generated by \( N_0 \cup \{ c \} \) or \( N_0 \cup c \) (respectively \( c \)).

**COROLLARY 5.2.** Let \( u \in N - N_0 \), as above, and let \( n_1 \) be the degree of \( q_u \), then \( N_0(u)_R = N_0 \oplus \mathbb{F}_p(B)[u] \oplus \mathbb{F}_p(B)[u \cdot t] \oplus \cdots \oplus \mathbb{F}_p(B)[u \cdot t^{n_1 - 1}] \) as \( \mathbb{F}_p(B) \)-vector spaces, i.e., \( N_0(u)_R \) is isomorphic to the direct sum of \( N_0 \) and \( n_1 \) copies of \( \mathbb{F}_p(B) \).

**PROOF.** Since \( u \cdot q_u \in N_0 \) and \( q_u \) is of degree minimal such, one has \( N_0(u)_R = N_0 + \mathbb{F}_p(B)[u, u \cdot t, \ldots, u \cdot t^{n_1 - 1}] \) and \( \mathbb{F}_p(B)[u \cdot t^d] \cap (N_0 + \mathbb{F}_p(B)[u, u \cdot t, \ldots, u \cdot t^{d - 1}]) = \{0\} \) for any \( d < n_1 \).

Let \( N, M \) be two \( R \)-modules, models of \( T_{sep} \), and suppose that \( N_0 \) and \( M_0 \) are two isomorphic submodules of \( N \) and \( M \) respectively which we assume to contain all the elements (in respectively \( N, M \)) annihilated by some separable element of \( R \) of degree \( < n \). We denote by \( j \) this isomorphism and we want to extend it. Let \( u \in N_{tor} - N_0 \) be such that \( u \cdot q = 0 \), for some separable \( q \in R \) of degree \( n \). Let
Let \( u \in \mathbb{F}_p(B)^{sep} \) be any element. Using analogous arguments we can show that it is an isomorphism. Suppose that \( u \cdot r = q \cdot s \) for some \( r, s \in R \). Then, we have \( r = q \cdot s \) and, since \( j \) is an isomorphism and by choice of \( \bar{u} \),

\[
0 = j(u_0 - u_0' + u \cdot q_u \cdot s) = j(u_0 - j(u_0' + j(u \cdot q_u) \cdot s) = j(u_0) - j(u_0') - \bar{u} \cdot (r - r').
\]

Consequently \( j(u_0) + \bar{u} \cdot r = j(u_0') + \bar{u} \cdot r' \).

**Notation 5.1.** Let \( N \) be an \( R \)-module. Let \( N_{tor, sep} \) be the set of elements of \( N \) annihilated by some separable element of \( R \) of degree \( \leq n \) for some \( q \in R, q(0) \neq 0 \).

**Proposition 5.4.** Let \( N \) be any model of \( T_{sep} \). Then \( N_{tor, sep} \) is isomorphic as an \( R \)-module to \( \mathbb{F}_p(B)^{sep} \). Given two models \( N, M \) of \( T_{sep} \) with \( N \subseteq M \), we have \( N_{tor, sep} = M_{tor, sep} \).

**Proof.** By induction we build two chains of submodules of \( N_{tor, sep} \) and \( \mathbb{F}_p(B)^{sep} \) respectively, \( (N_i : l \in \omega) \) and \( (M_i : l \in \omega) \), such that for any \( l \in \omega, N_i \) and \( M_i \) contain all the elements of respectively \( N \) and \( \mathbb{F}_p(B)^{sep} \) annihilated by some separable element of \( R \) of degree \( \leq l \). We define simultaneously a chain of isomorphisms \( (h_l : l \in \omega) \) such that for any \( l \in \omega, h_l : N_i \to M_i \). Take \( N_0 = M_0 = \{0\} \) and \( h_0 \) the identity. Suppose inductively that we have defined \( N_l, M_l, h_l \). Let \( (u_i : i \in \omega) \) be an enumeration of the elements of \( N_{tor} \) annihilated by the separable elements of \( R \) of degree \( l \). Using \( 5.3 \) we build by induction two chains of submodules of \( N_{tor, sep} \) and \( \mathbb{F}_p(B)^{sep} \) respectively, \( (N^0_{l+1} : i \in \omega) \) and \( (M^0_{l+1} : i \in \omega) \), and a chain of isomorphisms \( (h^0_{l+1} : i \in \omega) \) such that \( N^0_{l+1} = N_l, M^0_{l+1} = M_l, h^0_{l+1} = h_l \) and for any \( i \in \omega, h^0_{l+1} : N^0_{l+1} \to M^0_{l+1} \).

The number of elements in a model annihilated by a given separable element in \( R \) is finite and described by \( T_{sep} \); this also gives the second assertion of the statement.

**Corollary 5.5.** Let \( N \) be a model of \( T_e \). Then, \( N_{tor} \approx \mathbb{F}_p(B)^{sep} \).

**Proof.** By Proposition 3.5, we have that \( N_{tor} = N_{tor, sep} \). So, we may apply the preceding proposition.

Let \( P \) be the set of elements of \( R \) corresponding to primitive additive polynomials (see [8]) w.r. to \( \mathbb{F}_p(B)^{sep} \); i.e., \( q \in P \) iff \( q(0) \neq 0 \) and there is an element \( u \in \mathbb{F}_p(B)^{sep} \)
such that \( u \cdot q = 0 \) but, for all polynomial \( r[x^p] \) properly dividing \( q[x^p] \) in the usual sense, one has \( u \cdot r \neq 0 \). Such an element \( u \) is called a primitive root of \( q \). Note that O. Ore (see p. 582 in [2]) asked for necessary and sufficient conditions for an additive polynomial to have primitive roots. In particular, he gave an example of an additive polynomial without any.

**Corollary 5.6.** Let \( q \) be a separable element of \( R \) and let the \( r_i[x^p] \)'s to be all monic additive polynomials properly dividing \( q[x^p] \). Then \( q \) belongs to \( \mathcal{P} \) iff \( T_{\text{sep}} \vdash \exists x (x \cdot q = 0 \land \bigwedge_i x \cdot r_i \neq 0) \).

Now we want to generalize Proposition 5.5 to the following situation. Let \( A \) be an \( \mathcal{L}_R \)-substructure (respectively \( \mathcal{F} \)-substructure) of a model \( M \) of \( T_{\text{sep}} \) (respectively \( T_t \)).

We will show in both cases that there is a canonical extension of \( A \) to a model of \( T_{\text{sep}} \) (respectively \( T_t \)). Even though in the first case it will be an \( \mathcal{L}_R \)-structure and the second case an \( \mathcal{F} \)-structure, the domain of both extensions will be the same and so we will denote it by the same letter.

By abuse of language, one might say that Proposition 5.4 was the case \( A = \{0\} \). We need the following Lemma.

**Lemma 5.7.** The set \( R_0 \) of separable elements of the skew polynomial ring \( R \) is a right denominator set: i.e., \( \forall r \in R, \forall r_0 \in R_0, r.R_0 \cap r_0.R \neq \emptyset \).

**Proof.** Given two elements \( r \) and \( q \) of \( R \), O. Ore constructs explicitly their least common multiple (as in the commutative case) (see [9]). One then checks that if \( r \) is separable, then \( \text{lcm}(r, q) = r.r_1 = q.q_1 \) with \( r_1, q_1 \in R \) and \( q_1 \) separable.

**Proposition 5.8.** Let \( M, N \) be models of \( T_{\text{sep}} \) containing, respectively, isomorphic \( \mathcal{L}_R \)-substructures \( A, B \). Then we may extend this partial isomorphism to a minimal submodel \( A_{\text{sep}} \) of \( M \) containing \( A \).

**Proof.** By Proposition 5.5, we know that the separable torsion submodules of \( M \) and \( N \) are isomorphic. Since \( A_{\text{tor,sep}} = M_{\text{tor,sep}} \cap A \), we may extend this isomorphism to an isomorphism from \( A + M_{\text{tor,sep}} \) to \( B + N_{\text{tor,sep}} \).

To get a model of \( T_{\text{sep}} \), we need to close these substructures by adding solutions of equations of the form (*) \( u \cdot s = a \), where \( a \in A \) and \( s \) is a separable element of \( R \). Setting \( A_0 = A + M_{\text{tor}} \), then \( A_1 \) is constructed solving equations of the form (*) with \( a \in A_0 \).

Let \( \phi \) be the isomorphism between \( A_0 \) and \( B_0 \), and suppose that there exists an element \( a \) in \( A_0 \) which is not divisible in \( A_0 \) by a separable polynomial, say \( q \). Since \( M \) is a model of \( T_{\text{sep}} \), there exists \( u \) belonging to \( M \) such that \( u \cdot q = a \). Let now \( s' \) be a separable polynomial of degree minimal such that \( u \cdot s' \) belongs to \( A_0 \). Let \( a' = u \cdot s' \). The set of solutions of the equation \( a' = x \cdot s' \) is equal to \( u + \{ x : x \cdot s' = 0 \} \), which means that \( A_0 \) contains none of them. Let \( v \in N \) be such that \( v \cdot s' = \phi(a') \). Since \( A_0 \) is isomorphic to \( B_0 \), \( v \) does not belong to \( B_0 \).

**Claim.** \( s' \) is of degree minimal such that \( v \cdot s' \) belongs to \( B_0 \).

**Proof.** Let \( r \) be an element of \( R \) of degree minimal such that \( v \cdot r \) belongs to \( B_0 \). Then \( r \) divides \( s' \) i.e., there exists \( r_1 \in R \) such that \( s' = r \cdot r_1 \). Set \( b_0 = v \cdot r \). Since \( \phi \) is an isomorphism between \( A_0 \) and \( B_0 \), there exists \( c_0 \) in \( A_0 \) with \( \phi(c_0) = b_0 \). Now \( a' = u \cdot s' = u \cdot r \cdot r_1 = c_0 \cdot r_1 \). So, \( u \cdot r - c_0 \) belongs to \( \ker(r_1) \) which is included in \( A_0 \). Therefore, \( u \cdot r \) belongs to \( A_0 \) and by choice of \( s' \), \( \deg(r) = \deg(s') \).
Extend $\phi$ on $(u)_R + A_0$ by setting $\bar{\phi}(u \cdot r + a_0) = v \cdot r + \phi(a_0)$, where $r$ belongs to $R$ and $a_0$ to $A_0$. This is well defined: suppose that $u \cdot r + a_0 = u \cdot r' + a_1$ i.e., $u \cdot (r - r')$ belongs to $A_0$. Then $s'$ divides $r - r'$ i.e., $(r - r') = s'.s_1$. Therefore, $u \cdot (r - r') = a' \cdot s_1 = a_1 - a_0$, so $\phi(a' \cdot s_1) = \phi(a_1 - a_0)$ and $v \cdot (r - r') = v \cdot s' \cdot s_1 = \phi(a') \cdot s_1 = \phi(a' \cdot s_1)$. Thus, $v \cdot (r - r') = \phi(a_1) - \phi(a_0)$. This map is injective by the claim.

We repeat this process in order to get an $R$-module $A_1$ in which all elements of $A_0$ are divisible by each separable element of $R$.

**Claim.** $A_1$ is a model of $T_{sep}$.

**Proof.** Let $v \in A_1$ and $u \in M$ be such that $u \cdot q = v$, where $q \in R$ and $q(0) \neq 0$. We are going to show that $u$ belongs to $A_1$. By construction of $A_1$, there exists $u_i$'s in $M, r_i, q_i$'s in $R$ with $q_i(0) \neq 0$ such that $v = \sum_{i \geq 1} u_i \cdot r_i$ and $u_i \cdot q_i \in A_0$. Since $R$ is right Ore, there exist $s_1 \in R$ with $s_1(0) \neq 0$ (see Lemma 5.7). $t_1 \in R - \{0\}$ such that $r_1 \cdot s_1 = q_1 \cdot t_1$. So we have $v \cdot s_1 = \sum_{i \geq 1} u_i \cdot r_i \cdot s_1 = u_1 \cdot q_1 \cdot t_1 + \sum_{i \geq 2} u_i \cdot r_i \cdot s_1$. Iterating, we get first that there exist $s_2, t_2 \in R - \{0\}$ with $s_2(0) \neq 0$ such that $r_2 \cdot s_1 \cdot s_2 = q_2 \cdot t_2$. So we get $v \cdot s_1 \cdot s_2 = a + \sum_{i \geq 3} u_i \cdot r_i \cdot s_1 \cdot s_2$, where $a \in A_0$. Finally we get that there exists $s \in R$ with $s(0) \neq 0$ such that $v \cdot s \in A_0$. So $u \cdot q \cdot s \in A_0$, which implies that $u \in A_1$.

**Corollary 5.9.** Let $M, N$ be two models of $T_e$ containing two isomorphic $S$-substructures $A, B$, respectively. Then we may extend this partial isomorphism to $A'_{sep}$ which is a minimal submodel of $M$ containing $A$.

**Proof.** We keep the same notations as in the proof above. We assume now that $A$ is an $S$-structure and $M$ is a model of $T_e$. We first check that $A_1$ is an $S$-structure: if $u \cdot s = a$ for some separable element $s$ of $R$, then $\lambda_i(u)$ belongs to the $R$-submodule generated by $u$ and $\lambda_j(a)$. see proof (item 3) of Proposition 3.5, which also implies that the isomorphism $\bar{\phi}$ above commutes with the functions $\lambda_i$'s. Thus $A_1 = A'_{sep}$ is the required minimal submodel.

**Remark 5.** Note that in the above Corollary, we could have replaced $T_e$ by any extension of $T_{free}$ which specifies for each separable polynomial of $R$ the number of elements annihilated by it.

§6. **Quantifier elimination in the case of finite perfection degree.** First, we note that the proof of the Proposition in [6] p. 176, adapts to right Euclidean rings. Let us recall the definition.

**Definition 7.** Let $S$ be a domain, it is a right Euclidean ring if there exists a function $\delta$ from $S$ to $\mathbb{N}$ such that $\forall p_1 \forall p_2 \exists q \exists r(p_1 = p_2 \cdot q + r \text{ with } \delta(r) < \delta(p_2))$ (see [6] p. 143).

**Proposition 6.1.** Let $S$ be a right Euclidean ring and $A$ be a matrix with coefficients in $S$. Then there exist invertible matrices $P, Q$ ($P$ with coefficients in $\{0, 1\}$), such that $P AQ$ is lower triangular.

**Proof.** For convenience of the reader, let us indicate the main steps of the proof, which goes by induction on the number of lines of the $n \times m$ matrix $A$. As usual, a lower triangular matrix $C$ is of the form $C = (c_{ij})$, where $c_{ij} = 0$ if $i < j$. 


1. Let $P_{ij} = 1 - e_{ii} - e_{jj} + e_{ij} + e_{ji}$, where $1$ is the identity matrix and $e_{ij}$ the matrix with $1$ in position $(i, j)$ and $0$ in the other positions. Note that $P_{ij}$ is invertible, $P_{ij}^2 = 1$.

2. Let $T_{ij}(c) = 1 + e_{ij}c$, where $c \in S$ and $i \neq j$. The inverse of $T_{ij}(c)$ is $T_{ij}(-c)$.

3. $P_{ij}A$: exchange the lines $i, j$ in $A$.

4. $A.P_{ij}$: exchange the columns $i, j$ in $A$.

5. $A.T_{ij}(c)$: the $j^{th}$ column of $A$ is replaced by the sum of the $i^{th}$ column $\times c$ of $A$ and the $j^{th}$ column of $A$.

If $A$ is non zero, let $a_{ij}$ be a non zero element of minimal $\delta$ in matrix $A$. In the product $P_{i1}.A.P_{1j}$, the element $a_{ij}$ is in the $(1, 1)$-position. Thus, we may assume that the initial matrix $A$ has the property that the element in the $(1, 1)$-position is of minimal $\delta$. Then we will show that by multiplying $P_{i1}.A.P_{1j}$ by matrices of the form $P_{sr}$ and $T_{hk}(c)$ (on the right), we will obtain a matrix which first line only consists of zero's except at the $(1, 1)$-position. So let us consider an element $a_{1k}$, with $k \neq 0$. Performing the Euclidean division, there exist $q_k, r_k \in S$ such that $a_{1k} = a_{11}.q_k + r_k$ with $\delta(r_k) < \delta(a_{11})$. In the product $P_{i1}.A.P_{1j}.T_{1k}(-q_k)$ the element at the $(1, k)$-position is $r_k$. Either it is equal to 0, so we go on applying the process above to another element on the first line, or one multiplies the matrix obtained so far by $P_{ik}$. Now the element at the $(1, 1)$-position is the (nonzero) element $r_k$ with $\delta(r_k) < \delta(a_{11})$. Therefore, the process terminates after finitely many steps and finally one obtains a matrix $A'$ of the form

$$\begin{pmatrix}
a & 0 \\
. & B
\end{pmatrix}$$

with $a$ a nonzero element and $B$ a $(n - 1) \times (m - 1)$ matrix to which one applies the induction hypothesis. Note that $A'$ is equal to the product of $A$ by invertible matrices of the form $P_{ij}$ or $T_{lk}(c)$, $c \in S$, and note that on the right we only multiply by matrices of the form $P_{ij}$. Note that the reasoning does not depend on whether $a_{ij}$ is a non zero element of minimal $\delta$ within the $i^{th}$ line, or in the whole matrix $A$.

**Definition 8.** A lower triangular $n \times m$-matrix of co-rank $\ell$ is of the form $(A_1, 0)$ where $A_1$ is a lower triangular $n \times k$-matrix $(k \leq n, m)$ with only non zero elements on its diagonal and $0$ is a zero $n \times (m - k)$-matrix, with $\ell = n - k$.

Proposition 6.1 may be reformulated as follows.

**Corollary 6.2.** Any $n \times m$-matrix $A \neq 0$ is equivalent to a lower triangular matrix of co-rank $\ell$, with $n - \ell > 0$.

Now we will apply the above proposition in our setting, namely to the skew polynomial ring $R$. We note the following.

**Corollary 6.3.** Suppose that some line of $A$ contains a separable element of $R$, then the element $a_{11}$ on the diagonal of the lower triangular matrix $P.A.Q$, given by Proposition 6.1, is separable.

**Proof.** It suffices to note that while performing the Euclidean algorithm: $a_{1k} = a_{11}.q_k + r_k$, if $a_{11}(0) = 0$ and $a_{1k}(0) \neq 0$, then $r_k(0) \neq 0$ and if $a_{11}(0) \neq 0$, either $r_k = 0$ and the element at the $(1, k)$-position is zero, so we keep $a_{11}$ which is
separable, or we switch the positions of \( r_k \) and \( a_{11} \) and we are back to the first case whenever \( r_k(0) = 0 \).

\textbf{Definition 9.} An \( n \times m \)-matrix is \textit{lower triangular separable} (l.t.s.) if it is of the form \((A_1, 0)\) where \( A_1 \) is a lower triangular \( n \times k \)-matrix \((k \leq n, m)\) with only separable elements on its diagonal and 0 is a zero \( n \times (m - k)\)-matrix.

\textbf{Lemma 6.4.} Let \( A \) be an \( n \times k \)-matrix with coefficients in \( R \). Then the system of equations \( \check{y} \cdot A = \check{u} \) is equivalent in any model \( M \) of \( T \), to a system

\[ \check{y} \cdot P \cdot B = \check{t}(\check{u}), \]

where \( P \) is a permutation matrix, \( B \) is a lower triangular separable \( n \times k' \)-matrix and \( \check{t} \) is a tuple of \( \mathcal{L} \)-terms.

\textbf{Proof.} We will prove the Lemma by induction on the number \( n \) of lines of matrix \( A \). Let \( a_{\ell t} \) be a coefficient of minimal valuation in matrix \( A \), say \( w \), and assume it is strictly positive. Multiplying \( A \) on the left by the matrix \( P_{\ell k} \), we get this element on the first line of the resulting product.

Set \( \check{z} = \check{y} \cdot P_{\ell k}^{-1} \cdot u_{\ell} \). Now, \( \sum z_j \cdot a_{j \ell} = u_{\ell} \) is equivalent by Lemma 3.4 to

\[ \bigwedge_{i \in p^w} \lambda_i(u_{\ell}) = \sum_j z_j \cdot \sqrt[\rho^w]{(a_{j \ell})_{(i)}}, \]

with \( a_{j \ell} \in R \) defined by \( a_{j \ell} = t^{w}.a_{j \ell}' \). We have

\[ a_{j \ell} = \sum_i (a_{j \ell})_{(i)} \cdot m_i = t^{w}.\sum_i (a_{j \ell})_{(i)} \cdot m_i = \sum_i \sqrt[\rho^w]{(a_{j \ell})_{(i)}} \cdot t^{w}.m_i \]

and \( \sqrt[\rho^w]{(a_{j \ell})_{(i)}} \) is separable for at least one \( i \). This amounts to replacing each column of \((P_{\ell k} \cdot A)\) by \( p^w \)-columns and \( u_{\ell} \) by the \( p^w \)-tuple \( \lambda_*(u_{\ell}) = (\lambda_i(u_{\ell}))_{i \in p^w} \). The system \( \check{z} \cdot P_{\ell k} \cdot A = \check{u} \) is equivalent to \( \check{z} \cdot \check{A} = \lambda_*(\check{u}) \), where the matrix \( \check{A} \) has a separable element on its first line.

Now by the above Corollary, one obtains a matrix \( \check{C} \) of the form

\[ \begin{pmatrix} a & \check{0} \\ \check{a} & C \end{pmatrix} \]

with \( a \) a separable element, by multiplying the matrix \( \check{A} \) on the right by matrices of the form \( P_{ij} \) and \( T_{ij}(c) \), for some \( c \in R \) (since the non zero element of \( \check{A} \) is already on the first line we do not need to multiply \( \check{A} \) on the left by some \( P_{ij} \)'s). Let \( Q \) be the product of these invertible matrices and set \( \check{Q} = (Q_1, Q_2) \), where \( Q_1 \) is the first column of \( Q \). So, the system \( \check{y} \cdot A = \check{u} \) is equivalent to \( \check{z} \cdot \check{C} = \lambda_*(\check{u}) \cdot Q \), which in turn is equivalent to

\[ \sum_i z_i \cdot a_i = \lambda_*(\check{u}) \cdot Q_1 \land (z_2, \ldots, z_n) \cdot C = \lambda_*(\check{u}) \cdot Q_2, \]

where \( a_1 = a, \ (a_{i \geq 2 \leq n}) = \check{a} \) and the matrix \( C \) has \( n - 1 \) lines. We apply the induction hypothesis to \( C \).

\textbf{Corollary 6.5.} Every p.p. \( \mathcal{L}_R \)-formula is equivalent to a conjunction of atomic \( \mathcal{L} \)-formulas modulo \( T \).
Proof. Let \( \phi(x_1, \ldots, x_m) \) be a p.p. formula of the form \( \exists y_1, \ldots, y_k (\overline{x} \cdot B = \overline{y} \cdot A) \), where \( A, B \) are non zero matrices with coefficients in \( R \). By Lemma 6.4, there exist a permutation matrix \( P \), an l.t.s. \( k \times n \)-matrix \( \tilde{A} \) and \( n \) \( \mathcal{L} \)-terms \( t_i, 1 \leq i \leq n \), such that:

\[
\phi(\overline{x}) \iff \exists y_1, \ldots, y_k (t_1(\overline{x}), \ldots, t_n(\overline{x})) = \overline{y} \cdot P \cdot \tilde{A}.
\]

Set \( \overline{y} \cdot P = \overline{y}' \). Assume that matrix \( \tilde{A} \) is of co-rank \( \ell \) and write it as \( \tilde{A} = (A_1, 0) \), where \( A_1 \) is an l.t.s. \( k \times (k - \ell) \)-matrix.

We have

\[
\phi(\overline{x}) \iff \exists y_1', \ldots, y_k' (t_1(\overline{x}), \ldots, t_{k-\ell}(\overline{x})) = \overline{y}' \cdot A_1 \land (t_{k-\ell+1}(\overline{x}), \ldots, t_n(\overline{x})) = \overline{0}.
\]

**Corollary 6.6.** Let \( M \) be a model of \( T_{free} \). Then \( M_{tor} \) is a pure submodule.

**Proof.** We have shown (see Proposition 3.5) that \( M_{tor} \) is an \( \mathcal{L} \)-substructure. So, we may apply the above Corollary.

In the next Lemma, we will use the following terminology. An \( \mathcal{L}_R \)-inequation in variables \( x_1, \ldots, x_n \) and parameters \( \overline{v} \) is a basic formula of the form \( \sum_{j} x_j \cdot r_j \neq t(\overline{v}) \), where \( r_i \in R, 1 \leq i \leq n \), and \( t \) is an \( \mathcal{L}_R \)-term. We will say that this inequation is nontrivial if for some \( i, r_i \neq 0 \).

**Lemma 6.7.** Let \( T \) be the theory of all torsion-free right \( R \)-modules satisfying axiom scheme 2. Let \( \Sigma(\overline{x}, \overline{u}) \) be a system of equations in \( \mathcal{L}_R \) of the form \( \overline{x} \cdot A = \overline{u} \), where \( A \) is a non zero l.t.s. \( m \times k \)-matrix of co-rank \( \ell \), and \( \overline{x} = (x_1, \ldots, x_m), \overline{u} = (u_1, \ldots, u_{m-\ell}, 0) \) are variables. Then, for any system \( \Gamma(\overline{x}, \overline{v}) \) of \( \mathcal{L}_R \)-inequations in \( \overline{x} \) and parameters \( \overline{v} \), there exists a system of \( \mathcal{L}_R \)-inequations \( \Gamma'(x_{m-\ell+1}, \ldots, x_m, \overline{u}, \overline{v}) \) such that \( \Sigma(\overline{x}, \overline{u}) \land \Gamma(\overline{x}, \overline{v}) \) is equivalent in \( T \) to \( \Sigma(\overline{x}, \overline{u}) \land \Gamma'(x_{m-\ell+1}, \ldots, x_m, \overline{u}, \overline{v}) \), with the convention that if \( \ell = 0 \), the system of inequations is of the form \( \Gamma'(\overline{u}, \overline{v}) \).

Moreover, we can split the set \( \Gamma' \) into two subsets, \( \Gamma'_0(x_{m-\ell+1}, \ldots, x_m, \overline{u}, \overline{v}) \) consisting of non-trivial inequations and \( \Gamma'_1(\overline{u}, \overline{v}) \) of trivial ones such that for any model \( M \) of \( T \), for any \( \overline{a}, \overline{c} \in M \) satisfying \( \Gamma'_1(\overline{a}, \overline{c}), \Sigma(\overline{x}, \overline{a}) \land \Gamma'(\overline{x}, \overline{c}) \) has realizations in any non zero \( R \)-submodule \( M_0 \) of \( M \) containing \( (\overline{a})_{sep} \).

**Proof.** We proceed by induction on \( m \). Write the matrix \( A \) as \( (\overline{a}, A_1) \), where \( \overline{a} \) is the first column of \( A \). We associate pairwise the first equation \( \sum_j x_j \cdot a_i = u_1 \) of \( \Sigma \) and the \( j \)th inequation of \( \Gamma \) : \( \sum_j x_j \cdot b_{ij} \neq t_j(\overline{v}) \), where \( t_j(\overline{v}) \) is an \( \mathcal{L}_R \)-term. Since \( R \) is right Ore, for each \( j \) such that \( b_{ij} \neq 0 \), there exist \( c_j, b_j \in R - \{0\} \) such that \( a_1 \cdot c_j = b_{ij} \cdot b_j \). The system:

\[
\begin{align*}
\sum_i x_i \cdot a_i \cdot c_j &= u_1 \cdot c_j \\
\sum_i x_i \cdot b_{ij} \cdot b_j &\neq t_j(\overline{v}) \cdot b_j
\end{align*}
\]

is equivalent in any torsion-free module to

\[
\begin{align*}
\sum_i x_i \cdot a_i &= u_1 \\
\sum_{i>1} x_i \cdot (b_{ij} \cdot b_j - a_i \cdot c_j) &\neq t_j(\overline{v}) \cdot b_j - u_1 \cdot c_j.
\end{align*}
\]

Let us denote by \( \Gamma''(x_2, \ldots, x_m, \overline{u}, \overline{v}) \) the set of inequations so obtained together with all the inequations of \( \Gamma \) where the coefficient of \( x_1 \) is equal to 0. Either \( k = 1 \) and so \( \Gamma' := \Gamma'' \) is the required set of inequations, or \( k \geq 2 \), so we apply the induction
hypothesis. The system \((x_2, \ldots, x_m) \cdot A_1 = (u_2, \ldots, u_{m-\ell}, 0) \land \Gamma''(x_2, \ldots, x_m, \bar{u}, \bar{v})\) is equivalent to \((x_2, \ldots, x_m) \cdot A_1 = (u_2, \ldots, u_{m-\ell}, 0) \land \Gamma'(x_{m-\ell+1}, \ldots, x_m, \bar{u}, \bar{v})\), for a system \(\Gamma'\) of inequations. Finally we obtain that the system \(\Sigma \land \Gamma\) is equivalent to \(\Sigma \land \Gamma'\).

Let \(\Gamma'_1(\bar{u}, \bar{v})\) consist of all the trivial inequations in \(\Gamma'\) and \(\Gamma'_0(x_m-\ell+1, \ldots, x_m, \bar{u}, \bar{v})\) of the other ones.

Let \(M\) be a model of \(T\), let \(\bar{a}, \bar{c}\) be in \(M\) and assume that \(\Gamma'_1(\bar{a}, \bar{c})\) holds. We will construct a solution \(\bar{x}\) in \(M\) of \(\Sigma \land \Gamma\). Let us index the elements of \(\Gamma'_1\) by \(J\) and let \(i_j\) be the first \(i\) such that \(d_{ij} \neq 0\). Let \(m_1\) be the maximal such \(i_j, j \in J\), and let \(J_1\) be the subset \(\{j \in J: i_j = m_1\}\) of \(J\). If the set \(\{i: m_1 < i \leq m\}\) is not empty, choose arbitrarily in \(M_0\) the components \(x_{m_1+1}, \ldots, x_m\) of \(\bar{x}\). Then, choose \(x_{m_1} \in M_0\) satisfying the following inequations: \(\bigwedge_{j \in J_1} x_{m_1} \cdot d_{m_1 j} = f_j(\bar{u}, \bar{v}) - \sum_{i > m_1} x_i \cdot d_{ij}\) (there exists such an \(x_{m_1}\) in \(M_0\), since there are only finitely many forbidden choices and \(M_0\) is infinite). Then, let \(m_2\) be the maximal \(i_j, j \in J - J_1\), and let \(J_2\) be the subset \(\{j \in J - J_1: i_j = m_2\}\) of \(J - J_1\). If the set \(\{i: m_2 < i \leq m\}\) is not empty, choose arbitrarily in \(M_0\) the components \(x_{m_2+1}, \ldots, x_{m_1-1}\) of \(\bar{x}\). Then, choose \(x_{m_2} \in M_0\) satisfying the following inequations: \(\bigwedge_{j \in J_2} x_{m_2} \cdot d_{m_2 j} = f_j(\bar{u}, \bar{v}) - \sum_{i > m_2} x_i \cdot d_{ij}\) (there exist such an \(x_{m_2}\) in \(M_0\), since there are only finitely many forbidden choices and \(M_0\) is infinite). Then, let \(m_3\) be the maximal \(i_j, j \in J - J_1 - J_2\), and let \(J_3\) be the subset \(\{j \in J - J_1 - J_2: i_j = m_3\}\) of \(J - J_1 - J_2\). If the set \(\{i: m_3 < i \leq n\}\) is not empty, choose arbitrarily in \(M_0\) the components \(x_{m_3+1}, \ldots, x_{m_2-1}\) of \(\bar{x}\). Then, choose \(x_{m_3} \in M_0\) satisfying the following inequations: \(\bigwedge_{j \in J_3} x_{m_3} \cdot d_{m_3 j} = f_j(\bar{u}, \bar{v}) - \sum_{i > m_3} x_i \cdot d_{ij}\). Proceed similarly up to the \((m + 1 - \ell)\)th-component of \(\bar{x}\), then use the system of equations \(A \cdot \bar{x} = \bar{a}\) to find successively, using axiom scheme 2, the first \(m - \ell\) components of \(\bar{x}\) in \(M_0\). It is at this point that we use the fact that \(M_0\) contains \(\langle \bar{a} \rangle^{sep}\).

**Lemma 6.8.** Let \(\phi, \psi\) be two p.p. \(L_{R;}\)-formulas defining two subgroups with \(\phi\) implying \(\psi\) in \(T_R\). Suppose that there exists a torsion-free \(L\)-structure \(M\) modelling \(T_{free}\) with \([\phi(M) : \psi(M)] > 1\). Then, for any non-trivial torsion-free \(L\)-structure \(N\) modelling \(T_{free}, [\phi(N) : \psi(N)] = \infty\).

**Proof.** Let \(a \in M\) be such that \((\phi(a) \land \neg \psi(a))\). By Corollary 6.5, there exist finitely many \(L\)-terms \(t_k(x)\) (respectively \(s_l(x)\)) such that \(\phi(x)\) is equivalent modulo \(T_{free}\) to \(\bigwedge_{k} t_k(x) = 0\) (respectively \(\psi(x)\) is equivalent (modulo \(T_{free}\)) to \(\bigwedge_{l} s_l(x) = 0\)). \(\exists x \in M(\phi(x) \land \neg \psi(x))\) is equivalent to \(\bigvee_{k \in L} \exists x \in M(\bigwedge_{k} t_k(x) = 0 \land s_l(x) \neq 0)\). There is some \(m \geq 1\) such that the terms \(t_k(x)\) and \(s_l(x)\) are modulo \(T_{free}\) of the form \(\lambda_j(x) \cdot d_j\) with \(d_j \in R\) (see Corollary 3.3). Let us use new variables \(y_j\) for the \(\lambda_j(x)\)'s. Identify \(p^{me}\) with \(\{0, \ldots, p^{me} - 1\}\) and set \(\tilde{y} = (y_0, \ldots, y_{p^{me} - 1})\), and let \(t'_k(\tilde{y}) = \sum_j y_j \cdot t_{kj}\) (respectively \(s'_l(\tilde{y}) = \sum_j y_j \cdot s_{lj}\)) be the terms obtained by substituting \(y_j\) for \(\lambda_j(x)\), from \(t_k(x)\) (respectively \(s_l(x)\)).

The system \(\bigwedge_{k} t'_k(\tilde{y}) = 0\) can be written as \(\tilde{y} \cdot A = 0\) and by Lemma 6.4, it is equivalent (modulo \(T_{free}\)) to the system \(\tilde{y} \cdot P \cdot \tilde{A} = 0\), where the matrix \(P\) is a permutation matrix and \(\tilde{A}\) is an l.t.s. matrix of non zero co-rank \(\ell\) since \(\phi(\psi(M)) > 1\). For ease of notation, set \(n = p^{me}\), \(\tilde{z} = \tilde{y} \cdot P\), and \(\sum_{0 \leq j < n} y_j \cdot s_{lj} = \sum_{0 \leq j < n} z_j \cdot \tilde{s}_{lj}\) for some \(\tilde{s}_{lj}\)'s in \(R\). Now, the system \(S_l(\tilde{z})\) below, which is assumed to be consistent for each \(l \in L\):

\[
\begin{align*}
\tilde{z} \cdot \tilde{A} &= 0 \\
\sum_{0 \leq j < n} z_j \cdot \tilde{s}_{lj} &\neq 0,
\end{align*}
\]
is equivalent by Lemma 6.7 to a system of the form:
\[
\begin{cases}
\sum_{n-\ell<j<n} z_j \cdot s_{ij} = 0 \\
\sum_{n-\ell<j<n} z_j \cdot s_{ij} \neq 0
\end{cases}
\]
where the inequation is non-trivial by consistency of \(\phi \land \neg \psi\).

Let now \(N\) be a torsion-free \(\mathcal{L}\)-structure, modelling \(T_{\text{free}}\). Suppose we have constructed \(d\) elements \(x(1), \ldots, x(d)\) satisfying \(\phi\) and such that all \(x(i), x(i) - x(j), i \neq j\). do not satisfy \(\psi\). Denote by \(\psi_{ij}(i), 1 \leq i \leq d\), the \(\lambda_{ij}\)-components, \(0 \leq j \leq n - 1\) of \(x(i)\). We are looking for an element \(x(d + 1)\) satisfying \(\phi\) and such that \(x(d + 1)\) and 
\[
x(d + 1) = \sum_{0 \leq j < n} y_j \cdot t^m \cdot m_j
\]
such that \(\bar{z} := (y_j)_{0 \leq j < n} \cdot P\) satisfies one of the following systems \(S_{l,f}(\bar{z}, \bar{v}(1), \ldots, \bar{v}(d)) := \{\bar{z} \cdot \bar{A} = 0\} \cup \Gamma_{l,f}(\bar{z}, \bar{v}(1), \ldots, \bar{v}(d))\), where \(l \in L, f = (f(1), \ldots, f(d)) \in L^d\) and \(\Gamma_{l,f}(\bar{z}, \bar{v}(1), \ldots, \bar{v}(d))\) consists of:
\[
\begin{cases}
\Lambda_{1 \leq i \leq d} \sum_{n-\ell<j<n} z_j \cdot s_{ij} \neq 0 \\
\Lambda_{1 \leq i \leq d} \sum_{n-\ell<j<n} z_j \cdot s_{ij} \neq 0
\end{cases}
\]
It suffices to apply Lemma 6.7 with \(\bar{a} = \bar{0}\) and \(\bar{c} = \bar{v}(1), \ldots, \bar{v}(d))\); note that in this case \(\Gamma_f\) is empty.

**Proposition 6.9.** The theory \(T_c\) admits quantifier elimination and it is complete.

**Proof.** First, any \(\mathcal{L}\)-formula is equivalent to an \(\mathcal{L}_R\)-formula in \(T_c\). Second, any theory of modules admits p.p. elimination i.e., every formula is equivalent to a boolean combination of positive primitive formulas modulo invariant sentences. An invariant sentence is one which specifies the index for one p.p. definable subgroup included in another (see [10], Chapter 2). By Corollary 6.5, it suffices to prove that the index (in \(\omega \cup \{\infty\}\)) of two p.p. definable subgroups (with one included in the other) is the same in any model of \(T_c\). Note that this last property implies that \(T_c\) is complete. We may always work in an \(\aleph_1\)-saturated model \(M\) of \(T_c\).

By Proposition 5.5, the torsion submodel \(M_{\text{tor}}\) is isomorphic to \(\mathbb{F}_p(B)^{\text{sep}}\) and in a nonprincipal ultrapower of \(M\), say \(M^*\), the corresponding ultrapower \(M^*_\text{tor}\) of \(M_{\text{tor}}\) is a direct summand as a pure-injective pure submodule (by Corollary 6.6) and \(M^*/M^*_\text{tor}\) is a torsion-free non zero model of \(T_{\text{free}}\). But by the Lemma above, the index of two p.p. definable subgroups (with one included in the other) is the same in any non-trivial torsion-free \(R\)-module purely embedded in a model of \(T_c\). This suffices by [10] Lemma 2.23.

**Remark 6.**

1. In fact, we proved that not only our theory \(T_c\) admits quantifier elimination but that any positive primitive \(\mathcal{L}\)-formula is equivalent to a conjunction of atomic \(\mathcal{L}\)-formulas. This last property appears in the theory of modules and is denoted by “\(\text{elim} - Q^+\)” (see [10] p. 319). In particular, it means that the p.p. definable functions are in fact definable by a conjunction of atomic formulas.

2. The same proof also gives that the theory of the class of torsion-free models of \(T_{\text{free}}\) admits q.e. in \(\mathcal{L}\) as well as any extension of \(T_{\text{free}}\) which specifies for each separable polynomial the finite number of elements it annihilates.

**Corollary 6.10.** The \(\mathcal{L}_R\)-reduct of \(T_c\) is model-complete. It is axiomatized by \(T_{\text{sep}}\) and \(\forall x \exists ! (x_i)_{i \in P^c} \cdot x = \sum x_i \cdot t \cdot m_i\).
THEORY OF MODULES OF SEPARABLY CLOSED FIELDS

The functions \( \lambda_i \)'s are \( \mathcal{L}_R \)-existentially definable in any model of \( T_e \) and \( T_e \) admits quantifier elimination in the language \( \mathcal{L} \) by the above proposition.

\[ \text{Corollary 6.11. } T_e \text{ has a prime model.} \]

\[ \text{Proof. } \text{Apply Propositions 5.4 and 6.9.} \]

\[ \text{Corollary 6.12. } T_e \text{ is decidable.} \]

\[ \text{Proof. } T_e \text{ is complete and recursively axiomatizable (a “natural” enumeration of } R \text{ provides a recursively enumerable axiomatisation of } T_e). \text{ The result may also be deduced from the fact that } T_e \text{ is the } \mathcal{L}-\text{reduct of the theory of the separably closed field } \mathbb{F}_p(B)^{sep} \text{ which is decidable.} \]

\section{Quantifier elimination in the case of infinite imperfection degree.}

As in the finite imperfection degree case, the torsion submodule of any model of \( T_\infty \) is isomorphic to \( \mathbb{F}_p(B)^{sep} \).

Any \( \mathcal{L}_\infty \)-formula is equivalent to an \( \mathcal{L}_R \)-formula in \( T_\infty \) since the \( \lambda_i \) functions are positively existentially \( \mathcal{L}_R \)-definable.

\[ \text{Lemma 7.1. } \text{Let } A \text{ be a } h \times k \text{-matrix with coefficients in } \mathbb{F}_p(b)(t; \alpha), \text{ for some tuple } b \in B^n. \text{ Then, there are } \mathcal{L}_\infty \text{-terms } (\tau_{i,j}(.))_{i=1}^h \text{ of the form } \tau_{i,j}(.) = \lambda_i^b \circ \cdots \circ \lambda_j^b (.), \text{ with } i,j \in p^{nm}, \text{ for all } j = 1, \ldots, r_1, \text{ a } k' \text{-tuple of } \mathcal{L}_R \text{-terms } i, \text{ a permutation matrix } P, \text{ an } l \times s, \text{ } h \times k' \text{-matrix } B, \text{ and integers } g_l, 1 \leq g_l \leq k, \text{ such that the system of equations } \tilde{y} \cdot A = \tilde{u} \text{ is equivalent, modulo } \mathcal{T}_{free, \infty}, \text{ to} \]

\[ \tilde{y} \cdot P^{-1} \cdot B = \tilde{t} \left( \left( \lambda_i^b \circ \cdots \circ \lambda_j^b (u_{g_l}) \right)_{i=1}^h \right) \wedge \bigwedge_{1 \leq i \leq s} \lambda_i^{b \cdot P^{-1} \cdot B} \circ \cdots \circ \lambda_j^{b \cdot P^{-1} \cdot B} (u_{g_l}) \text{ has the } (\tilde{b}, m_{i,j}) \text{-tp property.} \]

\[ \text{Proof. } \text{The proof is analogous to the proof of Lemma 6.4. Note that at each step, we look at a coefficient of minimal valuation and when } \tilde{v} \cdot Q = \tilde{w}, \text{ where } Q \text{ is an invertible matrix with coefficients in } \mathbb{F}_p(b)(t; \alpha), \text{ and if each component } w_l \text{ of } \tilde{w} \text{ has the } (\tilde{b}, m) \text{-tp property for some } m, \text{ then this also holds for each component } v_l \text{ of } \tilde{v}. \]

\[ \text{Proposition 7.2. Every primitive positive } \mathcal{L}_R \text{-formula is equivalent, modulo } \mathcal{T}_{free, \infty}, \text{ to a positive quantifier-free } \mathcal{L}_\infty \text{-formula.} \]

\[ \text{Proof. } \text{The proof is analogous to the proof of Corollary 6.5 with the additional information that the obtained terms occurring in the conjunction of atomic formulas equivalent to a given p.p. formula have the } (\tilde{b}, m) \text{-tp property. for some } m \text{ and tuple } b \in B^n, \text{ for some } n. \text{ Note that this last property is expressible by a positive quantifier-free } \mathcal{L}_\infty \text{-formula (see Definition 5).} \]

\[ \text{Lemma 7.3. } \text{Let } \phi, \psi \text{ be two p.p. } \mathcal{L}_R \text{-formulas respectively defining two subgroups with } \psi \text{ implying } \phi \text{ in } T_R. \text{ Suppose that there exists a torsion-free } \mathcal{L}_\infty \text{-structure } M \text{ modelling } \mathcal{T}_{free, \infty} \text{ with } [\phi(M) : \psi(M)] > 1. \text{ Then } [\phi(N) : \psi(N)] = \infty, \text{ for any non-trivial torsion-free } \mathcal{L}_\infty \text{-structure } N \text{ modelling } \mathcal{T}_{free, \infty}. \]

\[ \text{Proof. } \text{Again the proof of this lemma is analogous to the proof of Lemma 6.8. Let } t_k(x), s_k(x) \text{ be } \mathcal{L}_\infty \text{-terms such that } [\phi(x) \wedge \neg \psi(x)] \leftrightarrow \bigwedge_{k} t_k(x) = 0 \wedge s_k(x) \neq 0. \text{ Each } \mathcal{L}_\infty \text{-term } t_k \text{ (respectively } s_k \text{) is equivalent to a term of the form} \]

\[
\sum_i \lambda_i^b(x) \cdot r_i \quad \text{(respectively } \sum_j \lambda_j^b(x) \cdot s_j) , \text{ where } r_i, s_j \in R - \{0\} \quad \text{(see Lemma 4.1).}
\]
Notice that we may assume that the \( \lambda \) functions have the same superscript \( b \in B^n \). Also, for the clarity of the exposition, we reverted to the tuple notation for their subscripts which might not have the same length. We prove by induction on the maximal length \( r \) of \( i \) for functions \( \lambda_i \) occurring in \( \phi \) and \( \psi \) that, as in 6.8, a system \( \Sigma \) of the form \( \bigwedge_k t_k(x) = 0 \wedge s_k(x) \neq 0 \) has infinitely many “unequivalent” solutions if it has one. In other words there is a quantifier-free \( \mathcal{L}_R \)-formula \( \Theta \) such that \( \Sigma \) is equivalent to \( \Theta(\lambda_i^b(x) : i \text{ of length } \leq r) \). We know the result for \( r = 0 \) as Lemma 6.7 also holds for infinite \( e \). Now, for \( e > 0 \), \( \Sigma \) is equivalent to
\[
\left[ \Theta \left( \lambda_i^b(x) : \text{ \( i \) of length } \leq r \right) \wedge x = \sum_i \lambda_i^b(x) \cdot t \cdot m_n(i) \right] \forall \Theta \left( \left( \lambda_i^b(x) : \text{ \( i \) of length } < r \right) \cdot \tilde{0} \right).
\]
Using induction hypothesis, we have only to deal with the first term of the disjunction, and, by 6.7 again, if such a system has in some non zero torsion-free model \( N \) of \( T_{\text{free,} \infty} \) a solution \( x \) having the \((\tilde{b}, r)\)-t.p., then it has one in any such structure. Therefore, we may proceed as in the finite imperfection degree case to produce unequal solutions.

**Proposition 7.4.** \( T_{\infty} \) admits quantifier elimination and is complete; for any countable \( B_0 \subset B \), the corresponding reduct is decidable.

**Proof.** As in Corollary 6.12, we take a recursive presentation of the ring \( \mathbb{F}_p(B_0)[r : \alpha] \).

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    Carol Wood
    Stable URL: http://links.jstor.org/sici?sici=0022-4812%28197909%2944%3A3%5C412%5CNOTSOS%53E2.0.CO%5B2-7

NOTE: The reference numbering from the original has been maintained in this citation list.