Logical Cryptanalysis with WDSat

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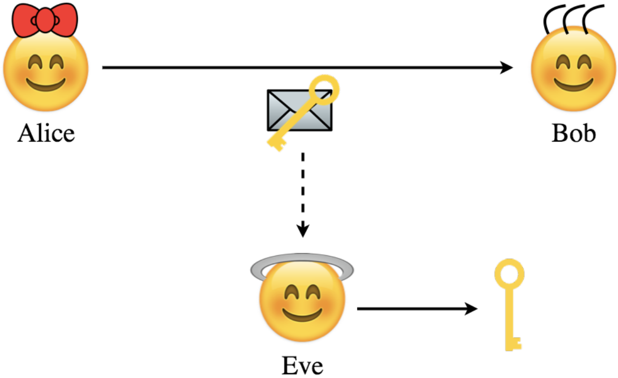
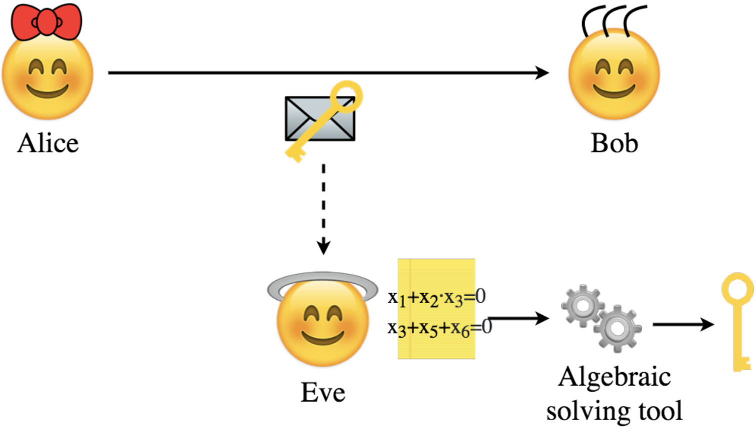


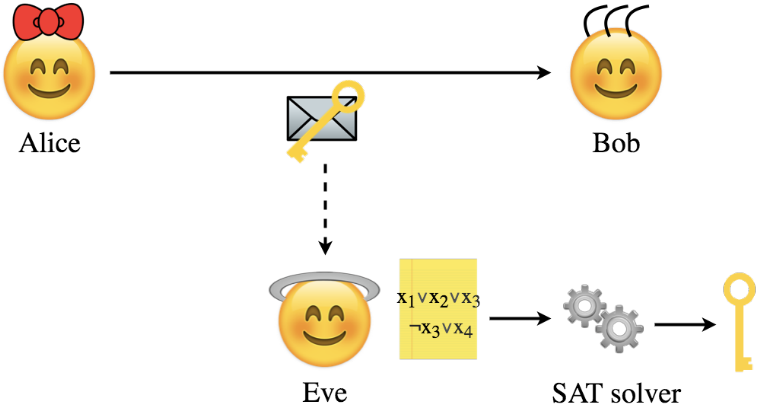
Goal

Determine minimum cryptographic key length requirements.

Algebraic cryptanalysis



Logical cryptanalysis



The multivariate polynomial problem

Example. A multivariate polynomial system of three equations in three variables

$$\mathbf{x}_1 + \mathbf{x}_2 \cdot \mathbf{x}_3 = 0$$
$$\mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_2 + \mathbf{x}_3 = 0$$
$$\mathbf{x}_1 + \mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{x}_3 + \mathbf{x}_2 \cdot \mathbf{x}_3 = 0.$$

At the core of algebraic cryptanalysis: finding a solution to the multivariate polynomial system results in recovering the secret key or the plaintext.

The degree-two case is the underlying problem in one of the five families of post-quantum cryptographic schemes.

From the algebraic model to the CNF-XOR model

Variables in $\mathbb{F}_2$:
$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6$.

Propositional variables:
$x_1, x_2, x_3, x_4, x_5, x_6$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$$\mathbf{x}_1 + \mathbf{x}_2 \cdot \mathbf{x}_4 + \mathbf{x}_5 \cdot \mathbf{x}_6 + 1 = 0$$
$$\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_4 + \mathbf{x}_5 + 1 = 0$$
$$\mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_2 \cdot \mathbf{x}_4 = 0$$
$$\mathbf{x}_2 + \mathbf{x}_5 + \mathbf{x}_2 \cdot \mathbf{x}_4 + \mathbf{x}_5 \cdot \mathbf{x}_6 + 1 = 0$$
$$\mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_6 + 1 = 0$$

$$(x_1 \oplus (x_2 \wedge x_4) \oplus (x_5 \wedge x_6)) \wedge$$
$$(x_1 \oplus x_2 \oplus x_4 \oplus x_5) \wedge$$
$$(x_3 \oplus x_4 \oplus (x_2 \wedge x_4) \oplus \top) \wedge$$
$$(x_2 \oplus x_5 \oplus (x_2 \wedge x_4) \oplus (x_5 \wedge x_6)) \wedge$$
$$(x_3 \oplus x_4 \oplus x_6)$$

Multiplication in $\mathbb{F}_2$ $(\cdot)$ becomes the logical AND operation $(\wedge)$ and addition in $\mathbb{F}_2$ $(+)$ becomes the logical XOR $(\oplus)$.

Add new variable $x_{2,4}$ to substitute the conjunction $x_2 \wedge x_4$.

Transform the constraint

$$x_{2,4} \Leftrightarrow (x_2 \wedge x_4)$$

into CNF.

From the algebraic model to the CNF-XOR model

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_{2,4}, x_{5,6}$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$$(x_1 \oplus (x_2 \wedge x_4) \oplus (x_5 \wedge x_6)) \wedge$$
$$(x_1 \oplus x_2 \oplus x_4 \oplus x_5) \wedge$$
$$(x_3 \oplus x_4 \oplus (x_2 \wedge x_4) \oplus \top) \wedge$$
$$(x_2 \oplus x_5 \oplus (x_2 \wedge x_4) \oplus (x_5 \wedge x_6)) \wedge$$
$$(x_3 \oplus x_4 \oplus x_6)$$

$$(\neg x_{2,4} \vee x_2) \wedge$$
$$(\neg x_{2,4} \vee x_4) \wedge$$
$$(\neg x_2 \vee \neg x_4 \vee x_{2,4}) \wedge$$
$$(\neg x_{5,6} \vee x_5) \wedge$$
$$(\neg x_{5,6} \vee x_6) \wedge$$
$$(\neg x_5 \vee \neg x_6 \vee x_{5,6}) \wedge$$
$$(x_1 \oplus x_{2,4} \oplus x_{5,6}) \wedge$$
$$(x_1 \oplus x_2 \oplus x_4 \oplus x_5) \wedge$$
$$(x_3 \oplus x_4 \oplus x_{2,4} \oplus \top) \wedge$$
$$(x_2 \oplus x_5 \oplus x_{2,4} \oplus x_{5,6}) \wedge$$
$$(x_3 \oplus x_4 \oplus x_6)$$

The WDSat solver

WDSat algorithm

Based on the Davis-Putnam-Logemann-Loveland (DPLL) algorithm.

Three reasoning modules

- **CNF module** : Performs unit propagation on CNF-clauses.
- **XORSET module** : Performs unit propagation on the parity constraints. When all except one literal in a XOR clause is assigned, we infer the truth value of the last literal according to parity reasoning.
- **XORGAUSS module** : Performs Gaussian elimination on the XOR system.

OR-clauses are stored as bit-vectors comprised of three parts.

$$x_1 \vee x_3$$

0	0	1	0	0	0	1	0	0
value					*weight*			*sat*

Value

The arithmetic sum of the literals in the clause in their dimacs representation.

Weight

The number of unassigned literals left in the clause.

Sat slot

Set to 1 when the clause is already satisfied by one of its assigned literals, and to 0 otherwise.

Example.

$\neg x_1 \vee \neg x_2 \vee x_4$ $((-1-2+4 \ll 3)+3) \ll 1$	0	0	0	0	1	0	1	1	0
$x_1 \vee x_3$ $((1+3 \ll 3)+2) \ll 1$	0	0	1	0	0	0	1	0	0

Compressed CNF Reasoning

Example.

$\neg x_1 \vee \neg x_2 \vee x_4$ $((-1-2+4 \ll 3)+3) \ll 1$	0	0	0	0	1	0	1	1	0
$x_1 \vee x_3$ $((1+3 \ll 3)+2) \ll 1$	0	0	1	0	0	0	1	0	0

Set x_1 to FALSE.

$\neg x_1 \vee \neg x_2 \vee x_4$ $\vert 1$	0	0	0	0	1	0	1	1	1
x_3 $-(((1 \ll 3)+1) \ll 1)$	0	0	0	1	1	0	0	1	0

Example.

$\neg x_1 \vee \neg x_2 \vee x_4$ $((-1-2+4 \ll 3)+3) \ll 1$	0	0	0	0	1	0	1	1	0
$x_1 \vee x_3$ $((1+3 \ll 3)+2) \ll 1$	0	0	1	0	0	0	1	0	0

Set x_1 to FALSE.

$\neg x_1 \vee \neg x_2 \vee x_4$ $\|1$	0	0	0	0	1	0	1	1	1
x_3 $-(((1 \ll 3)+1) \ll 1)$	0	0	0	1	1	0	0	1	0

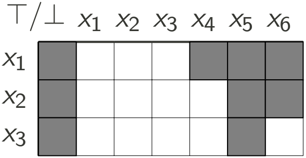
Propagation x_3 is set to TRUE.

WDSat - XORGAUSS module

- All variables in an XOR-clause belong to the same equivalence class.
- We choose one literal from the equivalence class to be the representative.
- Property: a representative of an equivalence class will never be present in another equivalence class.

XOR-clauses	Equivalence classes
$x_1 \oplus x_4 \oplus x_5 \oplus x_6$	$x_1 \Leftrightarrow x_4 \oplus x_5 \oplus x_6 \oplus \top$
$x_1 \oplus x_2 \oplus x_4 \oplus \top$	$x_2 \Leftrightarrow x_5 \oplus x_6 \oplus \top$
$x_2 \oplus x_3 \oplus x_6 \oplus \top$	$x_3 \Leftrightarrow x_5 \oplus \top$

- Implementation: A compact *EC* structure.



WDSat - XORGAUSS module

- All variables in an XOR-clause belong to the same equivalence class.
- We choose one literal from the equivalence class to be the representative.
- Property: a representative of an equivalence class will never be present in another equivalence class.

	XOR-clauses	Equivalence classes
	$x_1 \oplus x_4 \oplus x_5 \oplus x_6$	$x_1 \Leftrightarrow x_4 \oplus x_5 \oplus x_6 \oplus \top$
$x_2 \oplus x_5 \oplus x_6$	~~$x_1 \oplus x_2 \oplus x_4 \oplus \top$~~	$x_2 \Leftrightarrow x_5 \oplus x_6 \oplus \top$
	$x_2 \oplus x_3 \oplus x_6 \oplus \top$	$x_3 \Leftrightarrow x_5 \oplus \top$

- Implementation: A compact *EC* structure.

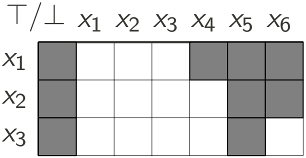
	$\top/\bot$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	■				■	■	■
x_2	■					■	■
x_3	■					■	

WDSat - XORGAUSS module

- All variables in an XOR-clause belong to the same equivalence class.
- We choose one literal from the equivalence class to be the representative.
- Property: a representative of an equivalence class will never be present in another equivalence class.

	XOR-clauses	Equivalence classes
	$x_1 \oplus x_4 \oplus x_5 \oplus x_6$	$x_1 \Leftrightarrow x_4 \oplus x_5 \oplus x_6 \oplus \top$
$x_2 \oplus x_5 \oplus x_6$	~~$x_1 \oplus x_2 \oplus x_4 \oplus \top$~~	$x_2 \Leftrightarrow x_5 \oplus x_6 \oplus \top$
$x_3 \oplus x_5$	~~$x_2 \oplus x_3 \oplus x_6 \oplus \top$~~	$x_3 \Leftrightarrow x_5 \oplus \top$

- Implementation: A compact *EC* structure.



Setting x_6 to TRUE

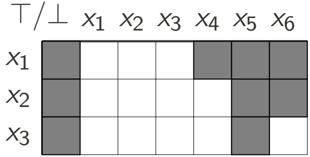
Algorithm 1 Function INFER_NON_REPRESENTATIVE(ul, tv, F)

Input: Propositional variable ul, truth value tv, the propositional formula F

Output: The EC structure is modified.

```
add ul to R.
if tv = TRUE then
    FLIP_CONSTANT(EC[ul]).
end if
set ul to 1 in EC[ul].
for each r in R do
    if ul is set to 1 in EC[r] then
        EC[r] ← EC[r] ⊕ EC[ul].
        if all variable bits in EC[r] are set to 0 then
            if the constant bit in EC[r] is set to 1 then
                add r to XG_propagation_stack.
            else
                add ¬r to XG_propagation_stack.
            end if
        end if
    end if
end for
set ul to 0 in EC[ul].
```

Before execution:



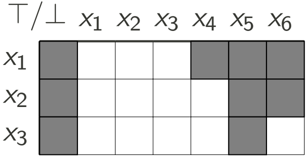
Setting x_6 to TRUE

Algorithm 2 Function INFER_NON_REPRESENTATIVE(ul, tv, F)

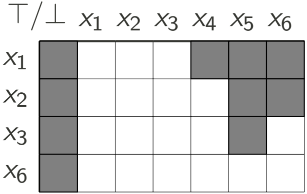
Input: Propositional variable ul, truth value tv, the propositional formula F
Output: The EC structure is modified.

1: add ul to R.
2: **if** tv =TRUE **then**
3: FLIP_CONSTANT($EC[ul]$).
4: **end if**
5: set ul to 1 in $EC[ul]$.
6: **for each** r in R **do**
7: **if** ul is set to 1 in $EC[r]$ **then**
8: $EC[r] \leftarrow EC[r] \oplus EC[ul]$.
9: **if** all variable bits in $EC[r]$ are set to 0 **then**
10: **if** the constant bit in $EC[r]$ is set to 1 **then**
11: add r to $XG\_propagation\_stack$.
12: **else**
13: add $\neg r$ to $XG\_propagation\_stack$.
14: **end if**
15: **end if**
16: **end if**
17: **end for**
18: set ul to 0 in $EC[ul]$.

Before execution:



After line 3:



Setting x_6 to TRUE

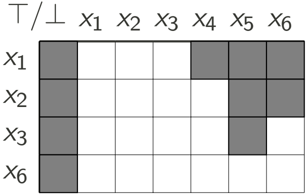
Algorithm 3 Function INFER_NON_REPRESENTATIVE(ul, tv, F)

Input: Propositional variable ul, truth value tv, the propositional formula F

Output: The EC structure is modified.

1: add ul to R.
2: **if** tv =TRUE **then**
3: FLIP_CONSTANT($EC[ul]$).
4: **end if**
5: set ul to 1 in $EC[ul]$.
6: **for each** r in R **do**
7: **if** ul is set to 1 in $EC[r]$ **then**
8: $EC[r] \leftarrow EC[r] \oplus EC[ul]$.
9: **if** all variable bits in $EC[r]$ are set to 0 **then**
10: **if** the constant bit in $EC[r]$ is set to 1 **then**
11: add r to $XG\_propagation\_stack$.
12: **else**
13: add $\neg r$ to $XG\_propagation\_stack$.
14: **end if**
15: **end if**
16: **end if**
17: **end for**
18: set ul to 0 in $EC[ul]$.

After line 3:



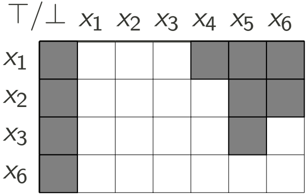
Setting x_6 to TRUE

Algorithm 4 Function INFER_NON_REPRESENTATIVE(ul, tv, F)

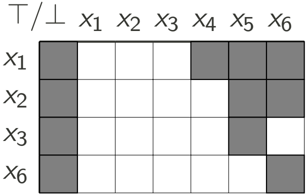
Input: Propositional variable ul, truth value tv, the propositional formula F
Output: The EC structure is modified.

```
1:  add ul to R.
2:  if tv = TRUE then
3:      FLIP_CONSTANT(EC[ul]).
4:  end if
5:  set ul to 1 in EC[ul].
6:  for each r in R do
7:      if ul is set to 1 in EC[r] then
8:          EC[r] ← EC[r] ⊕ EC[ul].
9:          if all variable bits in EC[r] are set to 0 then
10:             if the constant bit in EC[r] is set to 1 then
11:                 add r to XG_propagation_stack.
12:             else
13:                 add ¬r to XG_propagation_stack.
14:             end if
15:         end if
16:     end if
17: end for
18: set ul to 0 in EC[ul].
```

After line 3:



After line 5:



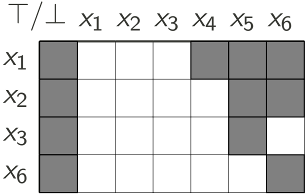
Setting x_6 to TRUE

Algorithm 5 Function INFER_NON_REPRESENTATIVE(*ul*, *tv*, *F*)

Input: Propositional variable *ul*, truth value *tv*, the propositional formula *F*
Output: The *EC* structure is modified.

1: add *ul* to *R*.
2: **if** *tv* = TRUE **then**
3: FLIP_CONSTANT($EC[ul]$).
4: **end if**
5: set *ul* to 1 in $EC[ul]$.
6: **for each** *r* in *R* **do**
7: **if** *ul* is set to 1 in $EC[r]$ **then**
8: $EC[r] \leftarrow EC[r] \oplus EC[ul]$.
9: **if** all variable bits in $EC[r]$ are set to 0 **then**
10: **if** the constant bit in $EC[r]$ is set to 1 **then**
11: add *r* to *XG_propagation_stack*.
12: **else**
13: add $\neg r$ to *XG_propagation_stack*.
14: **end if**
15: **end if**
16: **end if**
17: **end for**
18: set *ul* to 0 in $EC[ul]$.

After line 5:



XORGAUSS infer algorithm

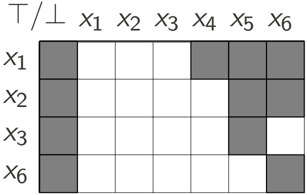
Setting x_6 to TRUE

Algorithm 6 Function INFER_NON_REPRESENTATIVE(ul, tv, F)

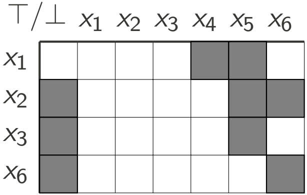
Input: Propositional variable ul, truth value tv, the propositional formula F
Output: The EC structure is modified.

```
 1: add ul to R.
 2: if tv = TRUE then
 3:     FLIP_CONSTANT(EC[ul]).
 4: end if
 5: set ul to 1 in EC[ul].
 6: for each r in R do
 7:     if ul is set to 1 in EC[r] then
 8:         EC[r] ← EC[r] ⊕ EC[ul].
 9:         if all variable bits in EC[r] are set to 0 then
10:             if the constant bit in EC[r] is set to 1 then
11:                 add r to XG_propagation_stack.
12:             else
13:                 add ¬r to XG_propagation_stack.
14:             end if
15:         end if
16:     end if
17: end for
18: set ul to 0 in EC[ul].
```

After line 5:



After line 8:



XORGAUSS infer algorithm

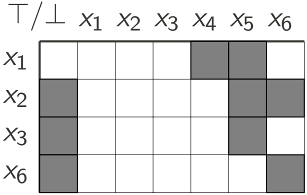
Setting x_6 to TRUE

Algorithm 7 Function INFER_NON_REPRESENTATIVE(ul, tv, F)

Input: Propositional variable ul, truth value tv, the propositional formula F
Output: The EC structure is modified.

1: add ul to R.
2: **if** tv = TRUE **then**
3:     FLIP_CONSTANT($EC[ul]$).
4: **end if**
5: set ul to 1 in $EC[ul]$.
6: **for each** r in R **do**
7:     **if** ul is set to 1 in $EC[r]$ **then**
8:         $EC[r] \leftarrow EC[r] \oplus EC[ul]$.
9:         **if** all variable bits in $EC[r]$ are set to 0 **then**
10:             **if** the constant bit in $EC[r]$ is set to 1 **then**
11:                 add r to *XG_propagation_stack*.
12:             **else**
13:                 add $\neg r$ to *XG_propagation_stack*.
14:             **end if**
15:         **end if**
16:     **end if**
17: **end for**
18: set ul to 0 in $EC[ul]$.

After line 8:



XORGAUSS infer algorithm

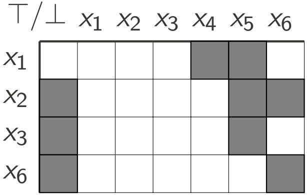
Setting x_6 to TRUE

Algorithm 8 Function INFER_NON_REPRESENTATIVE(ul, tv, F)

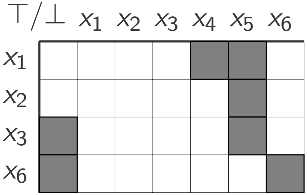
Input: Propositional variable ul, truth value tv, the propositional formula F
Output: The EC structure is modified.

```
1: add ul to R.
2: if tv = TRUE then
3:     FLIP_CONSTANT(EC[ul]).
4: end if
5: set ul to 1 in EC[ul].
6: for each r in R do
7:     if ul is set to 1 in EC[r] then
8:         EC[r] ← EC[r] ⊕ EC[ul].
9:         if all variable bits in EC[r] are set to 0 then
10:            if the constant bit in EC[r] is set to 1 then
11:                add r to XG_propagation_stack.
12:            else
13:                add ¬r to XG_propagation_stack.
14:            end if
15:        end if
16:    end if
17: end for
18: set ul to 0 in EC[ul].
```

After line 8:



After line 8:



XORGAUSS infer algorithm

Setting x_6 to TRUE

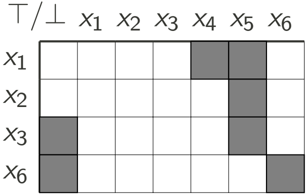
Algorithm 9 Function INFER_NON_REPRESENTATIVE(ul, tv, F)

Input: Propositional variable ul, truth value tv, the propositional formula F

Output: The EC structure is modified.

```
1: add ul to R.
2: if tv = TRUE then
3:     FLIP_CONSTANT(EC[ul]).
4: end if
5: set ul to 1 in EC[ul].
6: for each r in R do
7:     if ul is set to 1 in EC[r] then
8:         EC[r] ← EC[r] ⊕ EC[ul].
9:         if all variable bits in EC[r] are set to 0 then
10:            if the constant bit in EC[r] is set to 1 then
11:                add r to XG_propagation_stack.
12:            else
13:                add ¬r to XG_propagation_stack.
14:            end if
15:        end if
16:    end if
17: end for
18: set ul to 0 in EC[ul].
```

After line 8:



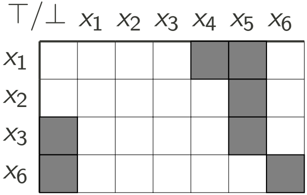
Setting x_6 to TRUE

Algorithm 10 Function INFER_NON_REPRESENTATIVE(ul, tv, F)

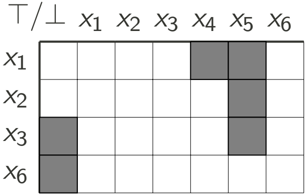
Input: Propositional variable ul, truth value tv, the propositional formula F
Output: The EC structure is modified.

1: add ul to R.
2: **if** tv =TRUE **then**
3: FLIP_CONSTANT($EC[ul]$).
4: **end if**
5: set ul to 1 in $EC[ul]$.
6: **for each** r in R **do**
7: **if** ul is set to 1 in $EC[r]$ **then**
8: $EC[r] \leftarrow EC[r] \oplus EC[ul]$.
9: **if** all variable bits in $EC[r]$ are set to 0 **then**
10: **if** the constant bit in $EC[r]$ is set to 1 **then**
11: add r to $XG\_propagation\_stack$.
12: **else**
13: add $\neg r$ to $XG\_propagation\_stack$.
14: **end if**
15: **end if**
16: **end if**
17: **end for**
18: set ul to 0 in $EC[ul]$.

After line 8:



After line 18.



Comparing different SAT approaches for solving Boolean polynomial systems with 50 quadratic equations over 25 variables.

- Results show an average of 100 runs.
- Running times are in seconds.

Input form	#Vars	#Clauses	Solver	Runtime	#Conflicts
CNF	8301	33006	MiniSat	11525.24	40718489
			Glucose	2384.99	10982657
			Kissat	2118.52	6622284
			Relaxed	3014.22	10353009
CNF-XOR	325	920	CryptoMiniSat	2870.81	9197978
			CryptoMiniSat + GE	594.48	2407635
			WDSat	57.85	14177200
			WDSat + GE	23.77	1046328
ANF	25	50	WDSat + XG-EXT	**0.82**	**21140**

Conclusion

- WDSat outperforms state-of-the-art SAT solvers for instances derived from dense Boolean polynomial systems.
- The compressed CNF reasoning module allows WDSat to handle polynomial systems of higher degree without compromising its performance.

WDSat on github

https://github.com/mtrimoska/WDSat