A Fast Algorithm for SAT in Terms of Formula Length

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Our contribution

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    - Branch and Search: New branching rules.
    - Measure and Conquer: Some assumptions on weights.

Outline

Problem Definition

The Satisifiability Problem

Given a CNF formula $\mathcal{F} = C_1 \wedge \cdots \wedge C_m$ on n boolean variables $x_1, \cdots, x_n$, decide if there is an assignment to $x_1, \cdots, x_n$ that makes $\mathcal{F} = 1$.

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Example

- $\mathcal{F}_1 = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_4}) \wedge (x_3 \vee x_4)$.
  Solution: $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1 \Rightarrow \mathcal{F}_1$ is satisifiable.

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  Solution: $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1 \Rightarrow \mathcal{F}_1$ is satisifiable.
- $\mathcal{F}_2 = (x_1 \vee x_2) \wedge (\overline{x_1} \vee x_2) \wedge (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_2})$.
  Solution: Not exist $\Rightarrow \mathcal{F}_2$ is not satisifiable.

Background

The SAT problem has been extensively and intensively studied in many fields:

- heuristic algorithms
- randomized algorithms
- approximation algorithms
- exact and parameterized algorithms
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There are three popular parameters to measure the running time of algorithms for SAT:

Parameter/Measure	Best Running Time Bound
n: the number of variables	$O^*(2^n)$ [1]
m: the number of clauses	$O^*(1.2226^m)$ [2]
L: the number of literals (length)	$O^*(1.0646^L)$ [3]

Strong Exponential Time Hypothesis: The SAT Problem can not be solved in time $O^*(2^n)$.

Our contribution is improving the running time bound in terms of the number of literals (formula length).

[1] Strong Exponential Time Hypothesis (SETH)
[2] AAAI'2021 Chu, Xiao and Zhang
[3] This paper

Background

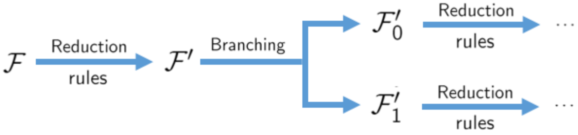
Table: Previous and our upper bound for SAT

Running time bounds	References
$O^*(1.0927^L)$	Van Gelder 1988
$O^*(1.0801^L)$	Kullmann and Luckhardt 1997
$O^*(1.0758^L)$	Hirsch 1998
$O^*(1.074^L)$	Hirsch 2000
$O^*(1.0663^L)$	Wahlström 2005
$O^*(1.0652^L)$	Chen and Liu 2009
$O^*(1.0646^L)$	This paper 2021

Our Algorithm - Overview

Our algorithm is a standard branch-and-search algorithm (Davis–Putnam–Logemann–Loveland (DPLL) algorithm):

- We first apply some reduction rules to reduce the instance.
  - Reduce the size (measure) of the formula and bring us some properties
  - Take polynomial time
- When no reduction rules can be applied, we will search for a solution by branching.
  - Assign value(s) to variable(s) or literal(s)
  - Exponentially increase the running time



Preliminaries

- (i,j)-literal: a literal z is called an (i,j)-literal in a formula $\mathcal{F}$ if z appears i times and $\overline{z}$ appears j times in the formula $\mathcal{F}$.
  (i^+,j)-literal, (i,j^+)-literal, (i^+,j^+)-literal, ...
- Degree: For a variable x in a formula $\mathcal{F}$, the degree of it, denoted by $deg(x)$, is the number of times it appears in the formula. We say a variable is an i-variable if the degree of it is i.
  i^+-variable, i^--variable, ...
- Length: The length of a clause C, denoted by $|C|$, is the number of literals in it. We call a clause k-clause if the length of it is k.
  k^+-clause, ...

$$\mathcal{F} = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_4}) \wedge (x_3 \vee x_4) \wedge (\overline{x_1} \vee x_3 \vee \overline{x_4})$$

x_3 is a $(2,1)$-literal, $deg(x_3) = 3$.

Preliminaries

If we assign value 1 to literal $x(x = 1, \overline{x} = 0)$, then

- All clauses containing literal x will be removed from the formula.
- All literals $\overline{x}$ will be removed from the clauses.

We use $\mathcal{F}_{x=1}$ to indicate the formula after assigning $x = 1$.

$$\mathcal{F} = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_4}) \wedge (x_3 \vee x_4) \wedge (\overline{x_1} \vee x_3 \vee \overline{x_4})$$

$$\mathcal{F}_{x_3=1} = (x_1 \vee x_2) \wedge (\overline{x_2} \vee \overline{x_4})$$

Reduction Rules

R-Rule 1 (Elimination of duplicated literals). *If a clause C contains duplicated literals z, remove all but one z in C.* $\mathcal{F}' \wedge (zzD) \to \mathcal{F}' \wedge (zD)$.

R-Rule 2 (Elimination of subsumptions). *If there are two clauses C and D such that $C \subseteq D$, remove clause D.* $\mathcal{F}' \wedge C \wedge D \to \mathcal{F}' \wedge C$.

R-Rule 3 (Elimination of tautology). *If a clause C contains two opposite literals z and $\bar{z}$, remove clause C.* $\mathcal{F}' \wedge (z\bar{z}C) \wedge D \to \mathcal{F}' \wedge D$.

R-Rule 4 (Elimination of 1-clauses and pure literals). *If there is a 1-clause $\{x\}$ or a $(1^+, 0)$-literal x, assign $x = 1$.* $\mathcal{F}' \wedge (x) \to \mathcal{F}'_{x=1}$.

And some other reduction rules... (**R-Rule 6** $\sim$ **R-Rule 10**)

A CNF-formula is called reduced, if none of reduction rules can be applied on it.

Reduction Rules

Lemma

In a reduced CNF-formula $\mathcal{F}$, all variables are 3^+-variables.

Lemma

In a reduced CNF-formula $\mathcal{F}$, if there is a 2-clause xy, then no other clause in $\mathcal{F}$ contains xy, $\overline{x}y$, or $x\overline{y}$.

And some other properties...

Our Algorithm

For a literal x, we have two kinds of branching:

- simple branching: $x = 1$ and $x = 0$
- strong branching: $x = 1$ & $C = 0$ and $x = 0$, where x is a $(1, i)$-literal and xC is the only clause containing literal x.

Our Algorithm

Algorithm 1: SAT($\mathcal{F}$)

Input: a CNF-formula $\mathcal{F}$
Output: 1 or 0 to indicate the satisfiability of $\mathcal{F}$
Step 1. If $\mathcal{F} = \emptyset$, return 1. If $\mathcal{F}$ contains an empty clause, return 0.
Step 2. If $\mathcal{F}$ is not a reduced CNF-formula, iteratively apply the reduction rules to reduce it.
Step 3. If there is a d-variable x with $d \geq 6$, return SAT($\mathcal{F}_{x=1}$)$\vee$SAT($\mathcal{F}_{x=0}$).
Step 4. If there is a $(1,4)$-literal x (assume xC is the only clause containing x), return SAT($\mathcal{F}_{x=1\ \&\ C=0}$)$\vee$SAT($\mathcal{F}_{x=0}$).
Step 5. If there is a 5-variable x contained in a 2-clause, return SAT($\mathcal{F}_{x=1}$)$\vee$SAT($\mathcal{F}_{x=0}$).
Step 6. If there is a 5-variable x contained in a 4^+-clause, return SAT($\mathcal{F}_{x=1}$)$\vee$SAT($\mathcal{F}_{x=0}$).
Step 7. If there is a clause containing both a 5-variable x and a 4^--variable, return SAT($\mathcal{F}_{x=1}$)$\vee$SAT($\mathcal{F}_{x=0}$).
Step 8. If there are still some 5-variables, then $\mathcal{F} = \mathcal{F}^* \wedge \mathcal{F}'$, where $\mathcal{F}^*$ is a 3-CNF with $var(\mathcal{F}^*)$ be the set of 5-variables in $\mathcal{F}$ and $var(\mathcal{F}^*) \cap var(\mathcal{F}') = \emptyset$. We return SAT($\mathcal{F}^*$) $\wedge$ SAT($\mathcal{F}'$) and solve $\mathcal{F}^*$ by using the 3-SAT algorithm by Liu [14].
Step 9. If there is a $(1,3)$-literal x (assume xC is the only clause containing x), return SAT($\mathcal{F}_{x=1\ \&\ C=0}$)$\vee$SAT($\mathcal{F}_{x=0}$).
Step 10. If there is a $(2,2)$-literal x, return SAT($\mathcal{F}_{x=1}$)$\vee$SAT($\mathcal{F}_{x=0}$).
Step 11. Apply the algorithm by Wahlström [18] to solve the instance.

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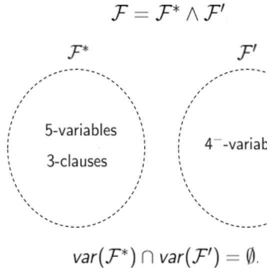

[4] Liu, S.: Chain, generalization of covering code, and deterministic algorithm for k-SAT.(ICALP 2018)

Algorithm SAT($\mathcal{F}$)

Property: Now all variables have a degree ≤ 4.

Step 9. If there is a $(1, 3)$-literal x (assume xC is the only clause containing x), return SAT($\mathcal{F}_{x=1\ \&\ C=0}$)$\vee$SAT($\mathcal{F}_{x=0}$).

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Property: Now all variables have a degree exactly 3.
Step 11. Apply the algorithm with time $O^*(1.1279^{(d-2)n}) = O^*(1.1279^n)$ by Wahlström[5] to solve the instance.

[5] Wahlström, M.: Faster exact solving of SAT formulae with a low number of occur-rences per variable. (SAT2005)

Our Algorithm

Algorithm SATSolver($\mathcal{F}$)
INPUT: a CNF formula $\mathcal{F}$
OUTPUT: a report whether $\mathcal{F}$ is satisfiable

1. $\mathcal{F} = \text{Reduction}(\mathcal{F})$;
2. pick a $d(\mathcal{F})$-variable x;
3. **if** $d(\mathcal{F}) > 5$ **then**
   return SATSolver($\mathcal{F}[x]$) $\vee$ SATSolver($\mathcal{F}[\overline{x}]$);
4. **else if** $d(\mathcal{F}) > 3$ **then**

   4.1 **if** x is a $(2,2)$-variable with clauses $x\overline{y}_1z_1$, xz_2z_3, $\overline{x}y_1$, and $\overline{x}y_2$ such that y_1 is a 4-variable and y_2 is a 3-variable **then**
   let $\overline{y}_2C_0$ be a clause containing $\overline{y}_2$;
   return SATSolver($\mathcal{F}[C_0 = \text{true}]$) $\vee$ SATSolver($\mathcal{F}[C_0 = \text{false}]$);

   4.2 **if** both x and $\overline{x}$ are 2^+-literals **then**
   return SATSolver($\mathcal{F}[x]$) $\vee$ SATSolver($\mathcal{F}[\overline{x}]$);

   4.3 **else** (* assume the only clause containing $\overline{x}$ is $\overline{x}z_1 \cdots z_h$ *)
   return SATSolver($\mathcal{F}[x]$) $\vee$ SATSolver($\mathcal{F}[\overline{x}, \overline{z}_1, \ldots, \overline{z}_h]$);
5. **else if** $d(\mathcal{F}) = 3$ **then**
   Apply the algorithm by Wahlström [12];
6. **else return** true;

Chen and Liu's algorithm[6] in 2009

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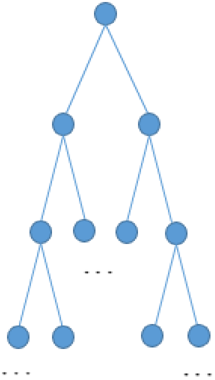
[6] Chen, J., Liu, Y.: An improved SAT algorithm in terms of formula length. (WADS2009)

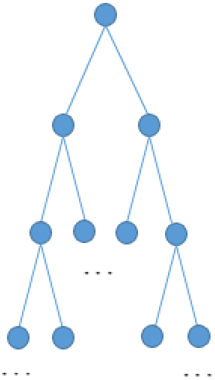
Running Time Bound Analysis

To determine the worst-case running time of a branching algorithm, we can analyze the size of the search tree generated in the algorithm.



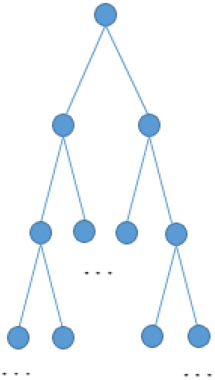
- First a measure μ is defined.
- We use $T(\mu)$ to indicate the maximum size or the number of leaves of the search tree for the input with the measure being at most μ.
- If the algorithm branches into l sub-branches with the measure decreasing at least a_i in the i-th sub-branch, we get a recurrence relation: $T(\mu) \leq T(\mu - a_1) + T(\mu - a_2) + \cdots + T(\mu - a_l)$.

Running Time Bound Analysis



- If the algorithm branches into l sub-branches with the measure decreasing at least a_i in the i-th sub-branch, we get a recurrence relation: $T(\mu) \leq T(\mu - a_1) + T(\mu - a_2) + \cdots + T(\mu - a_l)$.
- $[a_1, a_2, ..., a_l]$ is called a branching vector.
- The largest root of the function $f(x) = 1 - \sum_{i=1}^{l} x^{-a_i}$ is called the branching factor of the recurrence.
- $T(\mu) = O(\gamma^{\mu})$, where γ is the maximum branching factor of all branching factors.

Running Time Bound Analysis



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- Running time bound: $O^*(\gamma^{\mu})$.

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We introduce a weight to each variable in the formula according to the degree of the variable:

$$w : \mathbb{Z}^+ \to \mathbb{R}^+$$

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$$\mu(\mathcal{F}) = \sum_{x \in \mathcal{F}} w_{deg(x)}$$

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Let n_i denote the number of i-variables in $\mathcal{F}$. We also have

$$\mu(\mathcal{F}) = \sum_{i} n_i \cdot w_i$$

Measure and Conquer

If we ensure that $w_i \leq i$, then we have

$$\mu(\mathcal{F}) = \sum_i n_i \cdot w_i \leq \sum_i n_i \cdot i \leq L(\mathcal{F})$$

This tells us if we get a running time bound of $O^*(c^{\mu(\mathcal{F})})$ for a real number c, we also get a running time bound of $O^*(c^{L(\mathcal{F})})$.

Measure and Conquer

We also define $\delta_i = w_i - w_{i-1}$, this is roughly the weight of a literal with its corresponding variable have a degree of i.

For each branching rule, we will analyze how much the new measure $\mu(\mathcal{F})$ decreases in each sub-branch to get the running time bound.

An example:

$$\mathcal{F} = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_4}) \wedge (x_3 \vee x_4) \wedge (\overline{x_1} \vee x_3 \vee \overline{x_4})$$
$$\mathcal{F}_{x_3=1} = (x_1 \vee x_2) \wedge (\overline{x_2} \vee \overline{x_4})$$
$$\mathcal{F}_{x_3=0} = (\overline{x_2} \vee \overline{x_4}) \wedge (x_4) \wedge (\overline{x_1} \vee \overline{x_4})$$

The branching vector is:

$$[(w_3) + (\delta_2) + (\delta_3 + \delta_2), (w_3) + (\delta_2) + (\delta_2)]$$

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Adopt L as the measure

$w_4 = 4, w_3 = 3 \Rightarrow \delta_4 = 1$:

Branching vector	**Branching factor**
$[w_4 + 2w_3, w_4 + 6\delta_4]$	1.0718
$[w_4 + 2\delta_4, w_4 + 6\delta_4]$	1.0926

Measure and Conquer

Adopt μ as the measure

If we set $w_5 = 5, w_4 = 3.84682, w_3 = 1.92341 \Rightarrow \delta_4 = 1.92341$:

Branching vector	**Branching factor**
$[w_4 + 2w_3, w_4 + 6\delta_4]$	1.0646
$[w_4 + 2\delta_4, w_4 + 6\delta_4]$	1.0646

Running Time Bound Analysis

Assumptions:

- $w_1 = w_2 = 0$
- $w_i \leq i (3 \leq i \leq 4), w_4 = 2w_3$
- $w_i = i (i \geq 5)$
- $\delta_i > 0 (i \geq 3)$
- ...

In **Step 8**, the literals of all 5-variables form a 3-SAT instance $\mathcal{F}^*$. We apply the $O^*(1.3279^n)$-time algorithm for 3-SAT to solve our problem, where n is the number of variables in the instance. Since $w_5 = 5$, we have that $n = \mu(\mathcal{F}^*)/w_5 = \mu(\mathcal{F}^*)/5$. So the running time for this part will be

$$O^*(1.3279^{\mu(\mathcal{F}^*)/w_5}) = O^*(1.0584^{\mu(\mathcal{F}^*)}).$$

Running Time Bound Analysis

In **Step 11**, all variables are 3-variables. We apply the $O^*(1.1279^n)$-time algorithm by Wahlström to solve this special case, where n is the number of variables. For this case, we have that $n = \mu(\mathcal{F})/w_3$. So the running time of this part is

$$O^*((1.1279^{1/w_3})^{\mu(\mathcal{F})}).$$

Table 2. The weight setting

$w_1 = w_2 = 0$	
$w_3 = 1.9234132344759123$	$\delta_3 = 1.9234132344759123$
$w_4 = 3.8468264689518246$	$\delta_4 = 1.9234132344759123$
$w_5 = 5$	$\delta_5 = 1.1531735310481754$
$w_i = i (i \geq 6)$	$\delta_i = 1 (i \geq 6)$

Table 3. The branching vector and factor for each step

Steps	Branching vectors	Branching factors
Step 3	$[w_6 + \delta_6, w_6 + 11\delta_6]$	1.0636
Step 4	$[w_5 + 2w_3, w_5 + 8\delta_5]$	1.0632
Step 5	$[w_5 + 3\delta_5, w_5 + 2w_3 + 5\delta_5]$ $[w_5 + 2\delta_5, w_5 + 4w_3 + 4\delta_5]$	1.0618 1.0636
Step 6	$[w_5 + 4\delta_5, w_5 + 7\delta_5]$	1.0636
Step 7	$[w_5 + 4\delta_5, w_5 + 5\delta_5 + w_3]$	1.0646
Step 8	$O^*((1.3279^{1/w_5})^{\mu})$	1.0584
Step 9	$[w_4 + 2w_3, w_4 + 6\delta_4]$	1.0646
Step 10	$[w_4 + 2\delta_4, w_4 + 6\delta_4]$	1.0646
Step 11	$O^*((1.1279^{1/w_3})^{\mu})$	1.0646

The best choice of w_i can be found by solving a quasi-convex program problem.

The Final Result

Table 2. The weight setting

$w_1 = w_2 = 0$	
$w_3 = 1.9234132344759123$	$\delta_3 = 1.9234132344759123$
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$$\Downarrow$$

$$c^\mu \leq c^{\mu-(w_6+\delta_6)} + c^{\mu-(w_6+11\delta_6)}$$
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$$w_1 = w_2 = 0, w_4 = 2w_3, \ldots$$

Some other assumptions as constraints

The Final Result

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Some other assumptions as constraints

Minimize c

The Final Result

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$$w_1 = w_2 = 0, w_4 = 2w_3, \ldots$$

Some other assumptions as constraints

Minimize c

Bottleneck: Step 7, 9, 10, 11.

The Final Result

Theorem

Our algorithm SAT($\mathcal{F}$) solves the SAT problem in $O^*(1.0646^L)$ time.

Thanks for listening!