Lower Bounds for QCDCL via Formula Gauge

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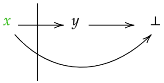
Initial situation

- QCDCL is the most used method for QBF solving.
  - Extension of CDCL.
- We want to determine if a given QBF in conjunctive normal form (short: QCNF) is true or false.
  - If the QCNF is false, we want to return a (long-distance) Q-resolution refutation.
- In the context of lower bounds, we will concentrate on false formulas only.

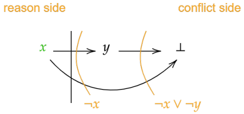
Initial situation

- A QCDCL run can be represented via implication graphs or trails.



$$\mathcal{T} = (\mathbf{x}, y, \bot)$$

- From each conflict we can learn a clause:



$$\frac{\neg x \vee \neg y \qquad \neg x \vee y}{\neg x}$$

Initial situation

- It was already shown that (nondeterministic) CDCL and Resolution are equivalent:

Pipatsrisawat, Darwiche 2010

For each Resolution refutation of a CNF ϕ there exists a CDCL refutation ι of ϕ with $|\iota| \in \mathcal{O}(n^3|\pi|)$, where n is the number of variables.

- However, this does not hold in the case of QBF:

Beyersdorff, B. 2021

QCDCL and Q-resolution are incomparable.

Initial situation

- A resulting question: What is hard for QCDCL? How can we achieve hard formulas for QCDCL, whose hardness does not depend on
  - propositional hardness or
  - hardness in long-distance Q-resolution?
- In a nutshell: Is there a lower bound technique *especially* for QCDCL?

Our inspiration

- There exists a formula for which hardness in QCDCL was already shown:

Definition (Janota 2015)

The QCNF $\mathtt{CR}_n$ (*Completion Principle*) consists of the prefix

$$\exists x_{(1,1)}, \ldots, x_{(n,n)} \forall u \exists a_1, \ldots, a_n, b_1 \ldots, b_n$$

and the matrix

$$\begin{array}{rl} x_{(i,j)} \vee u \vee a_i & \neg a_1 \vee \ldots \vee \neg a_n \\ \neg x_{(i,j)} \vee \neg u \vee b_j & \neg b_1 \vee \ldots \vee \neg b_n \end{array}$$

for $i, j = 1, \ldots, n$.

Our inspiration

$$\exists x_{(1,1)}, \dots, x_{(n,n)} \forall u \exists a_1, \dots, a_n, b_1 \dots, b_n$$

$$x_{(i,j)} \vee u \vee a_i \quad \neg a_1 \vee \dots \vee \neg a_n$$

$$\neg x_{(i,j)} \vee \neg u \vee b_j \quad \neg b_1 \vee \dots \vee \neg b_n$$

- A winning strategy for the universal player:
  - Case 1: For all i there exists a j such that $x_{(i,j)}$ is set to false. Then set u to false.
  - Case 2: There exists an i such that for all j the variable $x_{(i,j)}$ is set to true. Then set u to true.

Our inspiration

Theorem (Janota 2016)

$\mathtt{CR}_n$ is hard for QCDCL.

- Problem: This result depends on the learning scheme and the formula $\mathtt{CR}_n$ itself.

Question

Can we generalize the method of this result, such that it holds for a bigger class of formulas and for any learning scheme?

A generalized lower bound for QCDCL

Our result

For each QCNF Φ that fulfils a certain property, there exists a number $\mathrm{gauge}(\Phi)$ such that each QCDCL refutation of Φ has size $2^{\Omega(\mathrm{gauge}(\Phi))}$.

What is this certain property?

- From now on, let us restrict ourselves to Σ_3^b QCNFs with the prefix $\exists X \forall U \exists T$.

Definition

Let Φ be a QCNF of the form $\exists X \forall U \exists T \cdot \phi$. We call a clause C in the variables of Φ

- *X-clause*, if $\text{var}(C) \cap X \neq \emptyset$, $\text{var}(C) \cap U = \emptyset$ and $\text{var}(C) \cap T = \emptyset$,
- *T-clause*, if $\text{var}(C) \cap X = \emptyset$, $\text{var}(C) \cap U = \emptyset$ and $\text{var}(C) \cap T \neq \emptyset$,
- *XT-clause*, if $\text{var}(C) \cap X \neq \emptyset$, $\text{var}(C) \cap U = \emptyset$ and $\text{var}(C) \cap T \neq \emptyset$.

We say that Φ fulfils the XT-property if ϕ contains no XT-clauses as well as no unit T-clauses and there do not exist two T-clauses that are resolvable.

What is this certain property?

Definition

We say that Φ fulfils the XT-property if ϕ contains no XT-clauses as well as no unit T-clauses and there do not exist two T-clauses that are resolvable.

- Intuitively, this says that there is no direct connection between the X- and T-variables, i.e., Φ does not contain clauses with X- and T-variables, but no U-variables.
- Important: This property is "hereditary", that means every learned clause will fulfil this property, as well.
  $\rightarrow$ This property will hold during the whole QCDCL run.

gauge(Φ)

Our result

For each Σ_3^b QCNF Φ that fulfils the XT-property, there exists a number gauge(Φ) such that each QCDCL refutation of Φ has size $2^{\Omega(\text{gauge}(\Phi))}$.

- What is gauge(Φ)?

Definition

For a Σ_3^b QCNF Φ with prefix $\exists X \forall U \exists T$ let W_Φ be the set of all Q-resolution derivations π from Φ of some X-clause such that π only contains T-resolution and reduction steps. We define the gauge of Φ as

$$\text{gauge}(\Phi) := \min\{|C| : C \text{ is the root of some } \pi \in W_\Phi\}.$$

gauge(Φ)

Definition

For a Σ_3^b QCNF Φ with prefix $\exists X \forall U \exists T$ let W_Φ be the set of all Q-resolution derivations π from Φ of some X-clause such that π only contains T-resolution and reduction steps. We define the gauge of Φ as

$$\text{gauge}(\Phi) := \min\{|C| : C \text{ is the root of some } \pi \in W_\Phi\}.$$

- Intuitively, gauge(Φ) is the minimal number of X-literals that are necessarily piled up in a Q-resolution derivation in which we want to get rid of all T-literals.

gauge(Φ)

Definition (Janota 2015)

The QCNF $\mathtt{CR}_n$ (*Completion Principle*) consists of the prefix

$$\exists x_{(1,1)}, \ldots, x_{(n,n)} \forall u \exists a_1, \ldots, a_n, b_1 \ldots, b_n$$

and the matrix

$$\begin{array}{rl} x_{(i,j)} \vee u \vee a_i & \neg a_1 \vee \ldots \vee \neg a_n \\ \neg x_{(i,j)} \vee \neg u \vee b_j & \neg b_1 \vee \ldots \vee \neg b_n \end{array}$$

for $i, j = 1, \ldots, n$.

- $\mathtt{CR}_n$ fulfils the XT-property and it holds $\text{gauge}(\mathtt{CR}_n) = n$.
  $\rightarrow$ $\mathtt{CR}_n$ is hard for QCDCL.

Another example

Definition

The formula $\texttt{Equality}_n$ is defined as the QCNF with the prefix

$$\exists x_1 \dots x_n \forall u_1 \dots u_n \exists t_1 \dots t_n$$

and the matrix

$$x_i \lor u_i \lor t_i \qquad \neg t_1 \lor \dots \lor \neg t_n$$
$$\neg x_i \lor \neg u_i \lor t_i$$

for $i = 1, \dots, n$.

- $\texttt{Equality}_n$ fulfils the XT-property and it holds $\texttt{gauge}(\texttt{Equality}_n) = n$.
  $\rightarrow$ $\texttt{Equality}_n$ is hard for QCDCL.

Our result

Theorem

For each Σ_3^b QCNF Φ that fulfils the XT-property, every QCDCL refutation of Φ has size $2^{\Omega(\mathrm{gauge}(\Phi))}$.

- With this technique, one can show that formulas like
  - $\mathtt{CR}_n$
  - $\mathtt{Equality}_n$
  - $\mathtt{ENarrow}_n$

  are hard for QCDCL under arbitrary learning schemes.

Fin

Thanks for listening.