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On Axiomatic Foundations for Qualitative Decision Theory - A Possibilistic Approach.

Adriana María Zapico

Foreword by Dr. Lluís Godo Lacasa

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Foreword by
Dr. Lluís Godo Lacasa
Institut d'Investigació en Intel·ligència Artificial
Consell Superior d'Investigacions Científiques

Volume Author
Adriana María Zapico
Institut d'Investigació en Intel·ligència Artificial
Consell Superior d'Investigacions Científiques
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A mis padres.

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Foreword

Qualitative decision theory under uncertainty is receiving an increasing interest within both Artificial Intelligence and Decision Analysis. Possibility theory offers a formal framework to respresent uncertainty in those domains where uncertainty is basically of ordinal, qualitative nature, and hence non-additive as opposed to probability theory.

In 1995 Dubois and Prade proposed a first axiomatic system à la Von Neumann and Morgenstern for a possibility-based decision theory in a simple framework where only finite linear scales of uncertainty and utility are assumed. In this approach, decisions are represented by possibility distributions on consequences, also named possibilistic lotteries, and they axiomatically characterize preference relations induced by a pessimistic and optimistic qualitative utilities. These qualitative utilities are particular kinds of aggregations of the utilities of single consequences, weighted by their uncertainty levels, and they are defined only from the ordering of the scales and an order reversing operation.

Following this seminal wortk, in this monograph the author systematically explores two different kinds of extensions of the model. First, the author considers natural generalizations of the possibilistic utilities by means of the use of t-norm like operations on the uncertainty scale. Second, the author considers more general settings than the original one, moving from linear to partially ordered uncertainty and utility scales and from totally consistent to partially inconsistent uncertainty representations. The author provides axiomatic characterizations, always à la Von Neumann and Morgenstern, for the preference relations induced in all of those situations. Moreover, she also provides an axiomatic basis for a possibilistic Case-based Decision theory. In a whole, I believe the book represents a remarkable step further in providing sound and well founded axiomatic basis for the relatively new paradigm of possibility-based decision theory.

This monograph is based on the author's Ph.D. dissertation, which I had the enjoyable opportunity to supervise.

Lluís Godo
IIIA - CSIC, Bellaterra, September 2003

Abstract

Representational issues of preferences in the framework of a possibilistic (ordinal) decision model under uncertainty are analysed. In this framework, uncertainty and preference are measured on different (finite) lattice structures, ranging from lineal scales to general distributive lattices. These structures are required to be commensurate. In this context, decisions can be ranked according to their expected utility in terms of generalised Sugeno integrals where t-norms and t-conorms play a role. For these generalised utility functions we provide axiomatic characterisations. Moreover, we propose how to extend the utility functions to cope with belief states that may be partially inconsistent and we show their usefulness to provide elements for a qualitative case-based decision methodology. Finally, we provide characterisations of the refinement orderings involving the utility functions proposed and we also propose a new framework with a weaker commensurability hypotheses.

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Sometimes, it is not easy to translate the feelings and the ideas in words, to say thanks without forgetting anybody, and of course, in English, this is a task nearly impossible for me. . . I will do it in my language:
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me ha dado la risa,
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los dos materiales
que forman mi canto
y el canto de todos
que es mi propio canto.
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que me ha dado tanto.
(V. Parra)

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Chapter 1

Decision under Uncertainty

We begin this Chapter giving a short introduction to situate our work. Next, in Sections 1.2 and 1.3, we give an outline of the goals and main contributions of the thesis, and we link them with already published papers that summarise our work. Finally, in Section 1.4 we describe the structure of this Ph.D. dissertation.

1.1 Introduction

Decision making is a daily activity which is involved in most of the acts we usually do. Several areas, such as Artificial Intelligence, Operation Research, Game Theory, Social Psychology and others are frequently interested in models for *Decision Making*.

Decision Theory (DT) may be understood in a broad sense and therefore related to different issues like individual decision making or Game Theory. Bacharach and Hurley (1991) observed that

> "It (*Decision Theory*) is about the ways in which decisions are related to the *Decisions Maker*'s aims and to her beliefs about how her options serve her aims."

There are two aspects that the different *DT* interpretations have in common:

- The subject of *Decision Theory* is the rational agent.
- The goal of *Decision Theory* is to have abstract theories of rational agency. That is, to obtain systematic constructions deduced from an axiomatic setting that are independent of the decision making domain.

Taking a decision amounts to choose, according to some criteria, the "best" of a set of available alternatives taking into account the available knowledge.

There are many approaches to rational decision making, however, many of them agree on the fact that the selection of decisions is determined by two factors: the *Decision Maker's preference on consequences* and the *information or belief about the current state of affairs the Decision Maker (DM for short) has*.

Usual assumptions in the different proposals for decision making theories are:

- *rationality hypothesis*: the Decision Maker is interested in maximising his utilities.

- *the feasibility of representing DM's preference relation $\preccurlyeq$ on consequences by a preference function on them*, i.e. the existence of a function $u : X \to (U, \leq_U)$, X being *the set of consequences* and $(U, \leq_U)$ the *preference valuation set,* such that

$$x \preccurlyeq y \qquad \text{iff} \qquad u(x) \leq_U u(y),$$

  is assumed. Usually, it is supposed $U = \mathbb{R}$.

We are interested in those models that assume the existence of a mapping u representing Decision Maker's preference on consequences. Hence, a problem of decision making may be represented by a 4-tuple $< S,\ X,\ D,\ u >$ with S being the *set of states or situations*, X the *set of consequences or outcomes*, and D is the *set of available decisions or alternatives*.

As it was mentioned, decision making depends on the available knowledge. For example, if a precise description of situations is available and each decision d on S is represented as a function $d : S \to X$ providing the consequence of the decision in each situation, we may apply this simple decision making model (see Figure 1.1):

> Given a situation s_0 and a set of available decisions D, *a best decision* will be a maximal element of D with respect to the order $\preccurlyeq_{s_0}$ induced by preferences on the consequences, $\preccurlyeq_{s_0}$ being defined as
>
> $$d \preccurlyeq_{s_0} d' \quad \text{iff} \quad u(d(s_0)) \leq_U u(d'(s_0)). \tag{1.1}$$

But in the real world, we may be faced with incomplete or ill-specified decision problems in which we cannot apply on (1.1) to define an order in D. For example, we may be in one of the following cases:

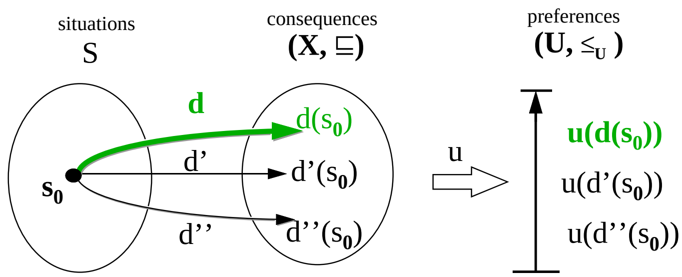
- the decision is precisely defined, but the real situation is imprecisely known (i.e. the actual state may be represented by a probability or a possibility distribution π_0 on the situations).

- s_0 is precisely known but d is imprecise (i.e. the actual consequence of d may be represented by a possibility distribution on the consequences).

- s_0 is precisely known but d is only partially known, i.e. d is partially defined.

In these cases, the simple model has to be extended to take decisions in an uncertain context.

As it has been mentioned, if there is no uncertainty, we may rank decisions applying (1.1). However, there are many problems in which the available information is poor. That is, we are in an uncertain decision making context. In these cases, a representation for uncertainty may be given or not. If no uncertainty representation is given, we may consider different criteria like those that evaluate a decision in terms of its worst possible consequence, its best one, or in terms of some weighted aggregation of them



Ranking decisions induced by consequences:

$d' \precsim_{s_0} d$ iff $d'(s_0) \precsim d(s_0)$ iff $u(d'(s)) \leq_U u(d(s))$

Figure 1.1: Decision without uncertainty: a simple model.

(for more details of some of these criteria you may see, for example, (Wald, 1950; Hurwicz, 1951; Luce and Raiffa, 1957)).

Other alternatives emerge from considering that fuzzy measures can be applied to model uncertainty (Grabisch, 97) (see Figure 1.2). In this case, another component is added to the 4-tuple modelling the problem. Now, we are considering $< S, X, D, u, \mu >$, where $\mu : S \to V$ is a fuzzy measure, V being an uncertainty scale.

Some particular kinds of fuzzy measures are Probability, Possibility and Necessity measures (Wang and Klir, 1992).

The basic references in classical *Decision Theory under Uncertainty* are Von Neumann and Morgenstern's *Expected Utility Theory* (1944), and the version of Savage (1972), characterising preference relations under uncertainty and the rationality hypothesis. Both approaches assume that *uncertainty is represented by probability distributions*.

Von Neumann and Morgenstern assume a probability distribution P encoding uncertainty on situations. Then, each decision induces a probability distribution on X defined as

$$P_d(x) = \sum_{s \in S | d(s) = x} P(s).$$

They consider *each decision as identified with its associated probability distribution*

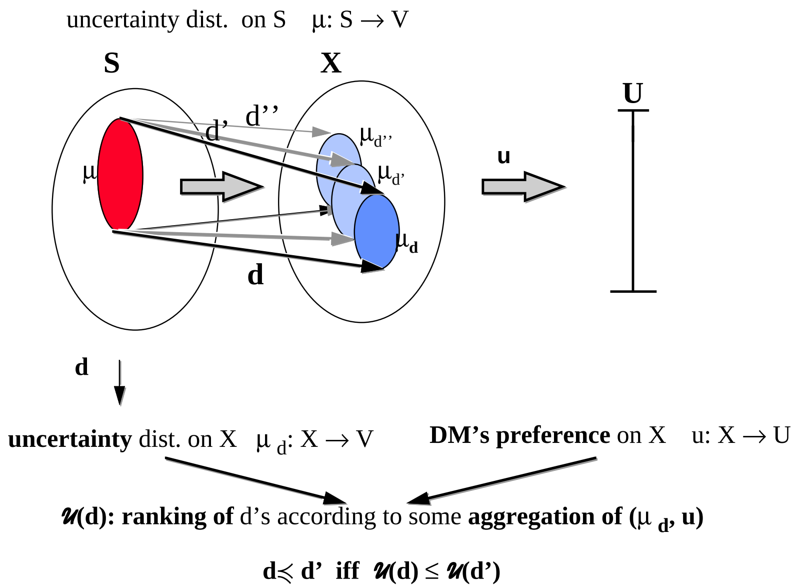


Figure 1.2: Decision Model with Uncertainty Representation

on X. So, to rank decisions they consider:

$$d \preccurlyeq d' \quad \text{iff} \quad P_d \preccurlyeq P_{d'}. \tag{1.2}$$

Hence, they focus on utility functions for probability distributions on consequences.

Distributions are ranked in terms of their expected value with respect to Decision Maker's preferences on consequences. That is, if numerical preferences $u : X \to \mathbb{R}$ are assigned to consequences, then, distributions are ranked as follows:

$$P_d \preccurlyeq P_{d'} \quad \text{iff} \quad E(P_d, u) \leq E(P_{d'}, u), \tag{1.3}$$

where

$$E(P_d, u) = \sum_{x \in X} P_d(x) u(x)$$

is the expected value of u with respect to the probability distribution P_d.

They propose to extend the initial model considering (1.3) instead of (1.1). Namely, Von Neumann and Morgenstern postulate that the "best" decisions, according to *Expected Utility Theory (EUT)*, are those whose corresponding probability distributions maximise the expected utility of u.

Savage (1972) proposes a somewhat different framework for *EUT*. He axiomatically characterises the preference relation *on acts* of *Decision Makers* that behave as *EUT* agents, i.e. that satisfy

$$d \preccurlyeq d' \qquad \text{iff} \qquad E(P, u \circ d) \leq E(P, u \circ d') \tag{1.4}$$

with $u : X \to \mathbb{R}$ (representing *DM*'s preferences on consequences) and $P : S \to [0, 1]$ being a probability distribution derived from the axiomatic setting. That is, Savage's version of (1.1) is (1.4), which is the same as considering (1.2) together with (1.3).

The classical axiomatic frameworks of Utility Theory have actually been questioned rather early, challenging some of the postulates leading to the expected utility criterion. Noticeably, Allais (1953) and later Ellsberg (1961) laid bare the existence of cases where a systematic violation of the expected utility criterion could be observed. Some of these violations were due to a cautious attitude of *Decision Makers*.

Another problem with *EUT* is that it needs numerical probabilities for each state and numerical utilities for all possible consequences. Sometimes, this assumption is too strong if there is only incomplete or poor available information. In these cases, a more qualitative approach is needed.

Another model is proposed by Gilboa and Schmeidler (1995). They claim that Decision Making under uncertainty is, at least, partly case-based. They suggest that people choose acts based on their performance in the past and they propose a *Case-Based Decision Theory (CBDT)*.

As Doyle and Thomason (1999) comment in a recent paper, there are many experiences showing that usually people explain and make their decisions with partial, generic and "uncertain" information. Hence, a qualitative approach may give tools for representing this decision making behaviour. Doyle and Thomason summarise main proposals on *Qualitative Decision Theory*. Among them, we find those models that use *Possibility Theory* as uncertainty formalism, in which two alternatives emerge: *à la Von Neumann and Morgenstern*, initiated by Dubois and Prade (1995), or *à la Savage*. Dubois et al. (1997h) propose a Savage's approach in a possibilistic framework and Sabbadin (1998a) develops this approach in his Ph.D. thesis.

In this Ph.D. we will follow the former approach, an axiomatic framework that is a qualitative counterpart to Von Neumann and Morgenstern's Expected Utility Theory. It makes use of qualitative/ordinal preference and uncertainty which are valued on finite sets,that are commensurate, and equipped with the maximum, minimum and an order-reversing operations. This *Qualitative Decision Theory* appears as the natural decision theory related to *Possibility Theory*.

1.2 Goals

We focus our work on representational issues of preferences in a framework of a possibilistic (ordinal) decision model under uncertainty, in the Von Neumann and Morgenstern's style.

Working Framework

We will assume the following working hypotheses

- We will deal with individuals' preferences.
- Rationality hypothesis, i.e. *DM* will try to maximise his benefit.
- The feasibility of representing *DM*'s preference relation on consequences by a preference function u on them is assumed. But, instead of choosing u as a real-function, we consider that it is defined over a *finite* set U of qualitative/ordinal values.
- Uncertainty is assumed of being of possibilistic nature, and it is measured on a *finite* set of qualitative/ordinal values V.
- One-shot decision problems.

We will be interested in different (finite) lattice structures where to measure preferences and uncertainty, ranging from lineal scales to general distributive lattices with involution.

First, following Dubois and Prade's proposal, we shall assume (finite) linear uncertainty and preference scales. We shall consider two qualitative criteria that generalise the well-known maximin and maximax criteria, making them more realistic. They are suited to one-shot decisions and they are not based on the notion of mean value, but take the form of medians.

The *first goal* will be to improve the axiomatic characterisations of these pessimistic and optimistic orderings. These functions are utility functions in the sense that they not only preserve the preference ordering but the max-min mixture on $\Pi(X)$, the set of *normalised possibility distributions on* X, as well.

Besides max-min mixtures of possibility distributions, we consider other mixtures involving t-conorms and t-norms. For each t-norm $\top$ and conorm $\perp$ on V, we will be interested in $\perp$-$\top$ mixtures that combine two possibility distributions π_1 and π_2 into a new one, denoted $M_{\top,\perp}(\pi_1, \pi_2; \lambda, \mu)$, with $\lambda, \mu \in V$ and $\lambda \perp \mu$ = 1, defined as

$$M_{\top,\perp}(\pi_1, \pi_2; \lambda, \mu)(x) = (\lambda \top \pi_1(x)) \perp (\mu \top \pi_2(x)).$$

We shall require these mixtures to satisfy a form of reduction of lotteries, this will lead to restrict ourselves to $\max - \top$ mixtures (Dubois et al., 1996b). So, for each t-norm $\top$ on V, we may consider Possibilistic Mixture.

Thus, a *second goal* will be to characterise the behaviour of functions that preserve these possibilistic mixtures. Moreover, we will look for preference relations on $(\Pi(X), M_\top)$ that are representable by these generalised utility functions.

The direct application of these models for case-based decision problems may have unsatisfactory results because of the possibly non-normalised distributions involved. So, a *third goal* will be to extend the models to deal with these type of problems.

There are actual problems where the available information may be only partially ordered, for example, preference on consequences may be given in terms of a vectorial function over a product of linear scales if preference is expressed in terms of the marginal preferences. To be able to deal with these types of problems, a further extension of the model will be analysed. We will propose utility functions, representing pessimistic and optimistic criteria, defined in terms of partially ordered preferences on consequences where uncertainty may also be measured on lattices. Therefore, a *last goal* will be to characterise these orderings and the preference relations representable by them as well.

1.3 Contributions

Our approach, as already mentioned first outlined by Dubois and Prade (1995), is focused on an axiomatic framework to *Possibilistic Decision Theory* that may be regarded a qualitative counterpart to Von Neumann and Morgenstern's Expected Utility Theory.

First, we consider (finite) qualitative/ordinal preference and uncertainty linear scales, equipped with the *maximum, minimum* and an *order-reversing* operations, that are commensurate. This *commensurateness hypothesis* means that we are assuming the existence of an onto order-preserving mapping $h : V \to U$.

Under these hypotheses Dubois and Prade proposed a first axiomatic setting to characterise the preference relation induced by a pessimistic qualitative utility which is expressed in terms of the preference on consequences and the "possibilistic" lotteries on S, S being the finite set of situations.

We provide an improvement of Dubois and Prade's axiomatic setting together with the representation theorem of preference relations induced by a pessimistic utility function defined as

$$QU^{-}(\pi|u) = \min_{x \in X} \max(n(\pi(x)), u(x)),$$

with $n = n_U \circ h,\ n_U$ being the reversing involution in U.

Sometimes, this criterion may be too conservative and we may be interested in an optimistic criterion, like requiring π *to make at least one of the good consequences highly plausible*, at least to some extent. This behaviour is reflected assessing a degree of intersection between the fuzzy set of possible consequences and the preferred ones. That is, we shall also consider the utility function

$$QU^{+}(\pi|u) = \max_{x \in X} \min(h(\pi(x)), u(x)).$$

We adequate the axiomatic setting given for pessimistic utilities, to represent this optimistic behaviour, providing the respective representation theorem.

We show that both qualitative functions are utility functions, in the sense that they not only represent the given preference relation, but they preserve the internal operation as well.

To sum up, two qualitative criteria are axiomatised in this setting: a pessimistic one and an optimistic one, respectively obeying an uncertainty aversion axiom and an uncertainty-attraction axiom. As it is said, these criteria generalise the well-known maximin and maximax criteria, making them more realistic.

As also mentioned, we have been also concerned with $\max -\top$ mixtures on $\Pi(X)$. Thus, we have been also interested in the behaviour of functions that preserve these possibilistic mixtures.

We propose the following generalised qualitative utility functions, which are extensions of the qualitative utility QU^- and QU^+:

$$\begin{aligned} GQU^-(\pi) &= \min_{x_i \in X} n(\pi(x_i) \top \lambda_i), \\ GQU^+(\pi) &= \max_{x_i \in X} h(\pi(x_i) \top \mu_i), \end{aligned}$$

where $n(\lambda_i) = u(x_i) = h(\mu_i)$, with $n = n_U \circ h$, $h : V \to U$ being an onto order-preserving mapping, verifying a further *coherence condition w.r.t.* $\top$ to guarantee the correctness of the above definition, namely:

$$h(\lambda) = h(\mu) \Rightarrow h(\alpha \top \lambda) = h(\alpha \top \mu), \quad \forall \alpha, \lambda, \mu \in V.$$

These generalised utility functions may result in different orderings from the ones associated with QU.

We characterise the preference relations on $\Pi(X)$ that are representable by the above generalised qualitative utilities GQU^- and GQU^+.

One of the possible applications of these decision models is for case-based decision problems, where a memory of cases M, summarising the behaviour of decisions in previous situations, is assumed to be available as well as a similarity function on situations $Sim : S \times S \to V$.

We propose to estimate to what extent a consequence x can be considered plausible, in a current situation s_0 after taking a decision d, in terms of the extent to which the current situation s_0 is similar to situations in which x was experienced after taking the decision d.

This amounts to assume, for each case (s, d, x) in a memory M, a principle stating that

> "The more similar s_0 is to s, the more plausible x is a consequence of d at s_0".

This kind of guiding meta-rule has been recently considered in (Dubois et al., 1997a) for case-based reasoning. According to this principle, given a memory of cases M, if a similarity relation is available in the set of situations, the following possibility distribution $\pi_{d,s_0} : X \to V$ on the set of consequences can be derived

$$\pi_{d,s_0}(x) = \max\{Sim(s_0, s) | \, (s, d, x) \in M\},$$

where, by convention, we take $\max \emptyset = 0$.

Then, given a preference function on the set of consequences $u : X \to U$, *the utility $U^-_{s_0}(d)$ of decision d can be estimated, in terms of its associated distribution*.

However, these distributions may result non-normalised, and the direct application of the utility functions mentioned up to now may result in unsatisfactory results.

In order to cope with these problems, following the proposal of (Dubois et al., 1997a), *we obtain new criteria modifying the utility functions previously mentioned with a level of uncertainty, which correspond to the degree of inconsistency of the distributions. Hence, we extend the model to include non-normalised distributions providing the axiomatic characterisations of these utilities*.

In some case-based decision problems, as it is noticed by Gilboa and Schmeidler (1996), the evaluation of the utility of a decision may involve not only the behaviour of this act in previous situations but other decisions as well. In order to deal with this type of problems, we propose to apply the principle:

> "The more similar are (s_0, d) and (s, d'), the more plausible x is a consequence of d at s_0".

There are certain kind of decision problems where we are not able to measure uncertainty and/or preferences in such linearly ordered sets, but only in partially ordered ones. For example , we may have partially ordered uncertainty in case-based decision problems when the degrees of similarity on problems are only partially ordered. In this case, if we are not provided with an aggregation criterion for similarity vectors that summarises the criteria on an ordinal linear scale, we are not able to apply the previously mentioned models.

Hence, we are also interested in a qualitative decision model that let us make decisions in cases where the *DM*'s preferences on consequences are only partially ordered or when the uncertainty on the consequences is measured on a lattice.

In order to cope with some of these situations, we propose an extension of the model in two steps:

1. *preferences and/or uncertainty are measured on finite products of (finite) linear scales,*

2. *both preferences and uncertainty are graded on distributive lattices.*

Most of the contributions contained in this thesis have been reported in the following publications[1]:

- Lluís Godo and Adriana Zapico. On the Possibilistic-Based Decision Model: Characterisation of Preferences Relations under Partial Inconsistency.[2] In *Applied Intelligence*, 14(3), pages 319-333,2001.

- Didier Dubois, Lluís Godo, Henri Prade and Adriana Zapico.On the Possibilistic-Based Decision Model: From Decision under Uncertainty to Case-Based Decision.[3] In *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 7(6), pages 631-670,1999.

- Lluís Godo and Adriana Zapico. Generalised Qualitative Utility Functions for Representing Partial Preferences Relations. *Joint Conf. EUSFLAT-ESTYLF99*, pages 343–346, Mallorca, 1999.

- Adriana Zapico. Axiomatic Foundations for Qualitative/Ordinal Decisions with Partial Preferences. In *16th. International Joint Conf. on Artificial Intelligence (IJCAI'99)*, pages 132–137, Stockholm, 1999.

[1]Latter, two other other works were published in Dubois et al. (2000a) and Zapico (2001).
[2]This is a revised and extended version of the paper (Godo and Zapico, 2001).
[3]This is a revised and extended version of the papers (Dubois et al., 1998c) and (Dubois et al., 1998d).

- Didier Dubois, Lluís Godo, Henri Prade and Adriana Zapico. Making Decision in a Qualitative Setting: From Decision under Uncertainty to Case-Based Decision. In *6th International Conference on Principles of Knowledge Representation and Reasoning(KR'98)*, pages 594 – 605, Trento, 1998.

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- Didier Dubois, Lluís Godo, Henri Prade and Adriana Zapico. Possibilistic Representation of Qualitative Utility: An Improved Characterisation. In *7th Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'98)*, Paris, pages 180–187, 1998.

- Adriana Zapico and Lluís Godo. Axiomatic Foundations for Qualitative/Ordinal Decisions with Partially Ordered Preferences. Tech. Rep. IIIA 98/33.

- Adriana Zapico and Lluís Godo. On the Possibilistic-Based Decision Model: Preferences under Partially Inconsistent Belief States. In *ECAI'98 Workshop on Decision theory meets artificial intelligence: qualitative and quantitative approaches*, Brighton, pages 99–109, 1998.

- Adriana Zapico and Lluís Godo. On the Representation of Preferences in Possibilistic Qualitative Decision Theory. In *Jornades d' Intel.ligència Artificial: Noves Tendencies. Organised by the Catalan Society of Artificial Intelligence*, Lleida, pages 118–125, 1997.

There are some on going works that, although they are in the first steps, we understand that may result in further contributions:

- As it has been said, we are mainly interested in the representational issues of possibilistic decision model under uncertainty, however, the possible application of our model of course is of our interest. Two projects in which the *Institut d'Investigació en Intel.ligència Artificial (IIIA- CSIC)* is now involved give us the context for beginning the analysis of the support that the models could provide. Up to now we are in the first steps of the analysis.

- We propose to weaken the commensurability hypothesis, non-requiring h to be onto. *We provide the characterisations of these orderings for finite linear scales*.

- In some problems it may be not enough to rank distribution taking into account one criterion, for example the pessimistic criterion, and we may be interested in refining it by another one (e.g. the optimistic criterion). We analyse the characterisation of some refinements involving the generalised qualitative criteria we have proposed.

1.4 Structure of the Thesis

The Thesis is structured as it is detailed below.

Chapter 1 contains a small introduction, the organisation of the memory and our goals and contributions.

In **Chapter 2,** we summarise some approaches to decision making under uncertainty, mainly the classical approach of Von Neumann and Morgenstern together with some alternative approaches, among which we are especially interested in *Possibilistic* and *Case-based Decision Theory*.

Expected Utility Theory has two approaches. In **Chapter 3,** we summarise the possibilistic view of these versions: Savage's possibilistic approach, developed by Sabbadin and Dubois et al. and Von Neumann and Morgenstern's approach, initially proposed by Dubois and Prade and which we extend in this work.

In **Chapter 4,** following the Von Neumann and Morgenstern's possibilistic approach, we propose an improvement of Dubois and Prade's axiomatic setting for qualitative decision criteria under uncertainty where only ordinal commensurate scales are required for assessing uncertainty and preference. These criteria generalise the well-known maximin and maximax criteria, making them more realistic.

Chapter 5: These criteria measure a degree of intersection/inclusion of π, the set of possible consequences, and u, the set of preferred consequences. In this Chapter we consider extended and alternative definitions of these operations, so that other utility functions are obtained. In particular, two ordinal utility functions that generalise previous ones are studied. We provide the characterisations of the preference relations induced by these functions.

Chapter 6: Up to this Chapter, we have been applying finite linear order scales to measure uncertainty and preferences. Now, we deal with decision problems that do not satisfy this linearity hypothesis. This point is developed through the memory in three steps. In this Chapter, we suppose that uncertainty and/or preferences are measured in a finite product of (finite) linear scales.

Secondly, in **Chapter 7,** uncertainty and/or preferences are measured on finite distributive lattices and utility functions are defined assuming that the only available operations are *minimum*, *maximum* and an *involution*.
Finally in a third step, we consider that other (t-norm-like) operations, different from *minimum* and *maximum*, are available. In particular, we consider finite, distributive, residuated lattices with involution as uncertainty and preference valuation sets. Consequently, the axiomatic decision model is extended to adequately cover these general algebraic structures as evaluation domains for the utility functions.

Chapter 8: In order to apply the models when the belief states are partially inconsistent, what may happen in case-based decision problems or when different sources of inconsistent information are available, the possibilistic decision framework is extended to cope with non-normalised distributions. Moreover, elements for a qualitative case-based decision methodology are proposed, with pessimistic and optimistic evaluations formally similar to the expressions which cope with uncertainty, up to modifying factors which handle the lack of normalisation of similarity evaluations. Also, we analyse the application of similarity functions involving acts for *Possibilistic Case-Based Decision Theory* following the proposal of Gilboa and Schmeidler.

Chapter 9: We describe some results obtained in the on going research, one related with the commensurability hypothesis between the uncertainty and preference values sets and the other with refinements of orderings are summarised here.

In **Chapter 10,** we show that our model may be applied for some decision making problems involved in two projects that are being developed in the *Institut d'Investigació en Intel.ligència Artificial (IIIA- CSIC)*.

Chapter 11: In this last Chapter of the memory we summarise the main contributions, we list the most interesting open problems left in this Ph.D., and describe research topics to be addressed in the near future.

Chapter 2

Decision Theory: Some Approaches

A problem of decision making may be represented by a 4-tuple $< S, X, D, u >$ being S the *set of states or situations*, X the *set of consequences or outcomes*. As it was said, we are interested in those models that assume the existence of a mapping u representing *Decision Maker*'s preference on consequences. Finally, D is the set of available decisions or alternatives, where decisions are functions d:$S \to X$.

As it was mentioned, if there is no uncertainty, we may rank decisions applying (1.1) (see Figure 1.1), that is,

$$d \preccurlyeq_{s_0} d' \quad \text{iff} \quad u(d(s_0)) \leq_U u(d'(s_0)).$$

However, there are many problems in which the available information is poor. That is, we are in an uncertain decision making context. In these cases, a representation for uncertainty may be given or not. If any uncertainty representation is given, we may consider different criteria like those that evaluate a decision in terms of its worst possible consequence, its best one, or as weighted aggregation of them. Some of these models are introduced in the first Section.

Other alternatives emerge from considering that fuzzy measures can be applied to model uncertainty (Grabisch, 97) (see Figure 1.2). In this case, another component is added to the 4-tuple modelling the problem.
Now, we are considering $< S, X, D, u, \mu >$ where μ:$S \to V$ is a *fuzzy measure*, V being an uncertainty scale. Let us recall the definition of *fuzzy measures*.

Definition 1
A fuzzy measure *(Grabisch, 97) on a finite set X is a set function μ:$\mathcal{P}(X) \to [0,1]$ satisfying*

- *$\mu(\emptyset)$ = 0 and $\mu(X)$ = 1,*
- *$A \subset B \subseteq X$ implies $\mu(A) \leq \mu(B)$.*

Some particular fuzzy measures are *Probability*, *Possibility* and *Necessity* ones. *Possibility measures*, Π, are fuzzy measures which also satisfy that

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B)),$$

while *Necessity measures N* satisfy

$$N(A \cap B) = \min(N(A), N(B)),$$

and *Probability measures P* satisfy

$$P(A \cup B) = P(A) + P(B) \quad \text{ if } A \cap B = \emptyset.$$

The classical model for decision making under uncertainty is Von Neumann and Morgenstern's Expected Utility Theory (*EUT*) (1944), and Savage's version (1972), which uses probability measures to model uncertainty about the state of the world.

This probabilistic model has some drawbacks, in Section 2.4 we summarise some alternatives that lead to some of these problems.

Another model is proposed by Gilboa and Schmeidler, from a case-based view, which also is summarised in Section 2.3.

Possibility theory provides other alternatives (Dubois and Prade, 1995; Dubois et al., 1997e). As we are mainly interested in them, since our work is developed in a possibilistic framework, we introduce these models in the next Chapter with more detail.

Next, we introduce some decision models where uncertainty representation is not available, while in Section 2.2 *Expected Utility Theory* is summarised. In Section 2.3, a Case-Based approach suggested by Gilboa and Schmeidler is introduced, while other approaches are briefly commented in the last Section.

2.1 Decision Models without Uncertainty Representation

Luce and Raiffa (1957) gather some criteria to choose decisions when the states are uncertain and no uncertainty representation is given. These criteria[1], as well as the *maximax* criterion are detailed below.

Wald's Criterion: *Maximin*

Wald (1950) suggests a conservative criterion that evaluates each act in terms of its worst consequences. Next, he chooses the act with greatest payoff, i.e. the "best decision" is

$$d' = argmax_{d \in D} \ (\min_{s \in S}(u(d(s)))).$$

[1]Notice that in some of them S and D are assumed as being finite.

Maximax Criterion

The dual optimistic criterion evaluates each act in terms of its best consequences choosing the act with great payoff, i.e. the "best decision" is

$$d' = argmax_{d\in D}\ (\max_{s\in S}(u(d(s)))).$$

Hurwicz's Criterion

Hurwicz (1951) proposes an intermediate criterion that combines the best and worst consequences. Indeed, for each $\alpha \in [0, 1]$ (the so called pessimist-optimist index), each act d is associated with an α-index, i.e.

$$\alpha \cdot (\min_{s\in S}(u(d(s)))) + (1-\alpha) \cdot (\max_{s\in S}(u(d(s)))).$$

The best decision would be the one with the higher α-index. Note, that if $\alpha = 1$, then we recover *maximin* criterion, while for $\alpha = 0$, it results in *maximax* criterion.

"Principle of Insufficient Reason" Criterion

This principle, formulated by Bernoulli (1738), asserts that in the case that one is "completely ignorant" about the real state, one may consider that all states are equally probable. Following this principle, each act is evaluated in terms of its expected utility, that is, for each d,

$$\frac{\sum_{s\in S} u(d(s))}{|S|},$$

choosing the act with greatest payoff, where $|S|$ denotes the cardinality of the set S.

2.2 Classical Approaches: *Expected Utility Theory*

The basic references in classical Decision Theory are Von Neumann and Morgenstern's Expected Utility Theory (1944), and the version of Savage (1972), characterising preference relations under uncertainty and the rationality hypothesis. Both approaches to decision making under uncertainty assume that *uncertainty is represented by probability distributions*. In this Section we recall them, especially Von Neumann and Morgenstern's version.

2.2.1 Von Neumann and Morgenstern's Expected Utility Theory

Von Neumann and Morgenstern suppose that uncertainty on real situation is represented by a single probability distribution P on S, $P{:}S \to [0, 1]$, S being *the set of situations*. A *decision* or *act* d on S is represented by a function d: $S \to X$ which provides the consequence of the decision in each situation.

Then, each decision induces a probability distribution on X defined as

$$P_d(x) = \sum_{s \in S | d(s) = x} P(s).$$

Von Neumann and Morgenstern consider each decision d as identified with its associated probability P_d, so for ranking decisions they consider:

$$d \preccurlyeq d' \quad \text{iff} \quad P_d \preccurlyeq P_{d'}. \tag{2.1}$$

Hence, they focus on utility functions on distributions on consequences.

Distributions are ranked in terms of their expected value with respect to *Decision Maker*'s preferences on consequences. That is, if numerical preferences, $u{:}X \to \mathbb{R}$, are assigned to consequences, they define

$$P_d \preccurlyeq P_{d'} \quad \text{iff} \quad E(P_d, u) \leq E(P_{d'}, u). \tag{2.2}$$

With

$$E(P_d, u) = \sum_{x \in X} P_d(x) u(x) \tag{2.3}$$

the expected value of u with respect to the probability distribution P_d.

They propose to extend the initial model considering (2.2) instead of (1.1).

Let $\wp$ denote *the set of probability distributions on* X. Let us introduce the notion of *binary probabilistic lottery*. Let A, B be two events and $\alpha \in [0, 1]$, the *binary lottery* which is the combination of these two events with α, denoted by

$$\alpha \odot A \oplus (1 - \alpha) \odot B,$$

is the prospect of considering that the first occurs with a probability α, and B occurs with the remaining probability $1 - \alpha$. In general, if l and l' are lotteries, then

$$\alpha \odot l \oplus (1 - \alpha) \odot l'$$

is a compound lottery. Thus, any (compound) probabilistic lottery decomposes as a finite sequence of compositions of binary lotteries, in a tree-like form. The set of probabilistic lotteries on X will be denoted by $\mathcal{L}(X)$.

If we have a probability distribution P on a set $\{x_1, x_2, x_3\}$, observe that we may see it as a compound lottery. Indeed, if $p_j = P(x_j)$, we have that

$$P \quad \longleftrightarrow \quad p_1 \odot x_1 \oplus (p_2 + p_3) \odot \left(\frac{p_2}{p_2 + p_3} \odot x_2 \oplus \frac{p_3}{p_2 + p_3} \odot x_3 \right).$$

Thus, in general, any probability distribution on a *finite* set, may be seen as a compound lottery, that is, as a sequence of binary lotteries.

On the other hand, the so-called *probabilistic mixture* operation is defined on $\wp$ as the convex linear combination of probability distributions on X. Namely, if P and Q are probability distributions on X and $\alpha \in [0, 1]$, the probabilistic mixture of P and Q with respect to α is the probability distribution (P, Q, α) defined as

$$(P, Q, \alpha)(x) = \alpha \cdot P(x) + (1 - \alpha) \cdot Q(x).$$

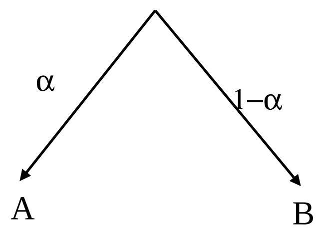

Figure 2.1: The binary probabilistic lottery of A and B with α and β

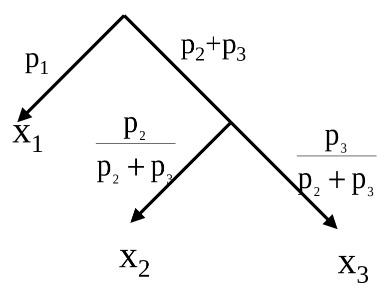

Figure 2.2: The lottery $p_1 \odot x_1 \oplus (p_2+p_3) \odot \left(\frac{p_2}{p_2+p_3} \odot x_2 \oplus \frac{p_3}{p_2+p_3} \odot x_3\right)$.

Since each probabilistic distribution on X can be identified with a probabilistic lottery, the probabilistic mixture operation can be seen as an operation between lotteries as well. Indeed, if we formally define a combination operation on lotteries

$$\mathcal{C} : \mathcal{L}(X) \times \mathcal{L}(X) \times [0,1] \to \mathcal{L}(X)$$

as

$$\mathcal{C}(l, l', \alpha) = \alpha \odot l \oplus (1-\alpha) \odot l',$$

it turns out that if P and Q are probability distributions identifiable with lotteries l_P and l_Q respectively, then the lottery corresponding to the probability mixture (P, Q, α), i.e. $l_{(P,Q,\alpha)}$, is nothing but $\mathcal{C}(l_P, l_Q, \alpha)$. Therefore, from now on, we shall identify the set $\wp$ of probability distributions on X equipped with the probabilistic mixture operation with the set $\mathcal{L}(X)$ of lotteries on X equipped with the operation $\mathcal{C}$ for combining lotteries (for more details about mixtures, including hybrid ones, you may see (Dubois et al., 2000b)).

Definition 2

- *Given* $\sqsubseteq$ *a preference relation on* $\wp$, *let* f *be a function from* $\wp$ *to* $\mathbb{R}$. *We say that*

$$(f \text{ represents } \sqsubseteq) \quad \textit{iff} \quad (\forall P, Q \in \wp)(\, P \sqsubseteq Q \Leftrightarrow f(P) \leq f(Q)).$$

- *Given a set* $\mathcal{A}$, *with an internal operation and a preference relation on it, a* utility function *over* $\mathbb{R}$, ut:$\mathcal{A} \to \mathbb{R}$, *is a function that represents the preference relation and also preserves the internal operation.*

Considering the probabilistic mixture as the internal operation on $\wp$, vonNeumann and Morgenstern (1944) characterise the preference relations on *probability distributions on consequences* of *Decision Makers* that behave as *EUT* agents. Indeed, they propose the following axiomatic setting on ($\wp$, $\preccurlyeq$):

- AxA: $\preccurlyeq$ is a total pre-order (i.e. $\preccurlyeq$ is reflexive, transitive and complete).
- $AxB.1$: $P \prec Q \ \Rightarrow\ P \prec (P, Q, \alpha)$, with $0 < \alpha < 1$.
- $AxB.2$: $P \succ Q \ \Rightarrow\ P \succ (P, Q, \alpha)$, with $0 < \alpha < 1$.
- $AxB.3$: $P \prec T \prec Q \ \Rightarrow\ \exists \alpha \in (0, 1)$ s.t. $(P, Q, \alpha) \prec T$.
- $AxB.4$: $P \succ T \succ Q \ \Rightarrow\ \exists \alpha \in (0, 1)$ s.t. $(P, Q, \alpha) \succ T$.
- $AxC.1$ *(commutativity)*: $(P, Q, \alpha) = (Q, P, \alpha)$.
- $AxC.2$*("lottery" reduction)*(see Figure 2.3):

$$((P, Q, \beta), Q, \alpha) = (P, Q, \alpha.\beta).$$

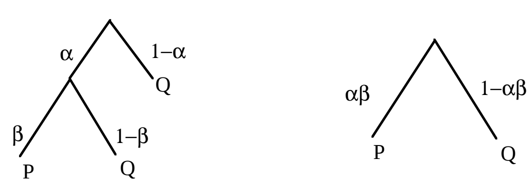

Figure 2.3: Probabilistic mixture reduction

AxA establishes that the *Decision Maker* is able to order all lotteries from worst to best. $AxB.1$ and $AxB.2$ is likeness convexity, that is, they establish that if Q is at least as preferred as P, then even a chance of Q is least as preferred as P, and Q is least as preferred as each combination of P and Q. An assumption of continuity is expressed by $AxB.3$ and $AxB.4$, while $AxC.1$ says that it is irrelevant the order in which the constituents involved are named. Finally, the reduction axiom expresses how second order lottery may coincide with a first order one. They proved the following theorem, which provides foundations for the *Expected Utility Theory*:

Theorem 2.1 (Von Neumann - Morgenstern)
A relation on $(\wp, \preccurlyeq)$ *satisfies the previous axiomatic setting if and only if there exists a function* $ut : \wp \to \mathbb{R}$ *such that*

$$P \preccurlyeq Q \Leftrightarrow ut(P) \leq ut(Q)$$

and

$$ut(P, Q, \alpha) = \alpha \cdot ut(P) + (1 - \alpha) \cdot ut(Q).$$

Moreover, ut *is unique up to a linear transformation.*

2.2.2 Savage's Version

Savage (1972) proposes a somewhat different framework for *EUT*, he axiomatically characterises the preference relation *on acts* of *Decision Makers* that behave as *EUT* agents, i.e. that satisfy

$$d \preccurlyeq d' \qquad \text{iff} \qquad E(P, u \circ d) \leq E(P, u \circ d') \tag{2.4}$$

with $u{:}X \to \mathbb{R}$ (representing *DM*'s preferences on consequences) and $P{:}S \to [0, 1]$ being a probability distribution. That is, his version of (1.1) is (2.4).

For a detailed explanation you may see (Savage, 1972), however, let us briefly summarise his proposal. Generally speaking, the axiomatic setting establishes that the preference is a complete pre-order ($Sav1$).

His characteristic axiom, the "*sure principle thing*" ($Sav2$), establishes that the choice between two alternatives must be unaffected by the value of outcomes corresponding to states for which both alternatives have the same payoff.

Given the preference relation on acts $\preccurlyeq$ and an event B, he defines a *conditioned preference on acts* $\preccurlyeq_B$:

> "$d \preccurlyeq_B d'$ iff $f \preccurlyeq g$ for all f and g that agree with d and d', respectively, on B and with each other in the complement of B and $g \preccurlyeq f$ for all such pairs or for none".

He defines an event B as *null* iff $d \preccurlyeq_B d' \ \forall d, d'$.

From the preference on acts, Savage induces a preference relation $\leq$ on consequences, i.e.

$$\forall x, y \in X, \text{ if } d(s) = x, \forall s \in S, d'(s) = y, \forall s \in S, \text{ then } x \leq y \iff d \preccurlyeq d'.$$

$Sav3$: If $d(s) = x_1$ and $d'(s) = x_2$ $\forall s \in B$, B being not null, then $d' \preccurlyeq_B d$ iff $x_2 \leq x_1$.

He requires the preference relation induced on events[2] $\trianglelefteq$ to be complete ($Sav4$). While the preference induced on consequences is required to be non-trivial, i.e. there exists at least one pair x, x' such that x is less preferred than x' ($Sav5$).

These axioms let Savage prove that the preference relation on S is a *"qualitative probability"*, that is

- $QP1$: $\trianglelefteq$ is a total preorder on $\mathcal{P}(S)$. [3]
- $QP2$: $\forall B \subseteq S,\ \emptyset \trianglelefteq B,\ \emptyset \triangleleft S$.
- $QP3$: $\forall B, C, D$ s.t. $D \cap (B \cup C) = \emptyset,\ B\ \trianglelefteq C \iff (B \cup D) \trianglelefteq (C \cup D)$.

He also considers the following technical axioms:

- $Sav6$: if $d \prec d'$ and x is a consequence, then there exists a partition of S such that, if d or d' is so modified on any one element of the partition as to take the value x at every s there, other values being undisturbed, then the modified d remains less preferred than d', or d remains less preferred than the modified d', as the case may require.
- $Sav7$: if $d \preccurlyeq_B d'(s)\ \forall s \in B$, then $d \preccurlyeq_B d'$.

This axiomatic setting lets him characterise the preference relations *on acts* that are representable in terms of the expected value of a preference function on consequences with respect to the probability distribution on S. That is, Savage's theorem says: If $(D, \preccurlyeq)$ satisfies Savage's axioms, there exists one and only one probability measure on $S,\ P{:}\mathcal{P}(S) \to [0,1]$, where $\mathcal{P}(S)$ denotes the power set of S, and a preference function on consequences $u{:}X \to \mathbb{R}$ such that

$$d \preccurlyeq d' \iff E(P, u \circ d) \leq E(P, u \circ d').$$

Of course Savage's axioms are sound, i.e. given a probability distribution on S and a preference function on consequences u, the order induced in D by the expected utility (that is, the order defined in (2.4)) satisfies Savage's axioms.

2.3 Case-Based Decision Theory

Gilboa and Schmeidler (1995) claim that Decision Making under uncertainty is, at least, partly case-based. They suggest that people choose acts based on their performance in the past and they propose a case-based Decision Theory (CBDT).

People frequently reason establishing analogies between past cases and the one at hand. Applying Hume's principle (1748):

[2] $A \trianglelefteq B$ iff when $x' < x,\ xAx' \preccurlyeq xBx'$, with the *compound act of x and x' w.r.t.* $A \subseteq S$ defined as

$$xAx'(s) = \begin{cases} x, & \text{if} \quad s \in A \\ x'(s), & \text{if} \quad s \notin A. \end{cases}$$

[3] $\mathcal{P}(S)$ is the power set of S.

"From causes which appear similar we expect similar effects",

Gilboa and Schmeidler (1995) proposed a *Case-Based Decision Theory (CBDT)* .

This theory assumes available partial information about the possible consequences of decisions by having stored the performance of decisions taken in different past situations as a set (memory) *M* of decision problem instances of triples (cases) *(situation, decision, consequence)*, and a given *similarity* Sim on situations as primitive. The *Decision Maker*, in face of a new situation s_0, is proposed to choose a decision d which maximises a counterpart of classical expected utility, instead of (2.3) they consider,

$$U_{s_0,M}(d) = \sum_{(s,d,x)\in M} Sim(s_0, s) \cdot u(x). \tag{2.5}$$

Sim is a non-negative function which estimates the similarity of situations and u provides a numerical preference for each consequence x. Gilboa and Schmeidler axiomatically characterise the relations induced by this U-maximisation.

Observe that a difference with *EUT* is that, while in *EUT* the decision is evaluated on *all* possible states, in *CBDT* each decision is evaluated on *a different set of states*. Another one is that, for the utility function $U_{s_0,M}$ the similarity may not add to one, i.e. it may be that for any s_0

$$\sum_{(s,d,x)\in M} Sim(s_0, s) \neq 1.$$

Gilboa and Schmeidler (1996) have also proposed another utility function $V_{s_0,M}$, which is a modification of the previous one, replacing Sim with the similarity function Sim' defined as

$$Sim'(s, s_0) = \begin{cases} \frac{Sim(s,s_0)}{\sum_{(s',d,x)\in M} Sim(s',s_0)}, & \text{if} \quad \sum_{(s',d,x)\in M} Sim(s', s_0) \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

so,

$$V_{s_0,M}(d) = \sum_{(s,d,x)\in M} Sim'(s_0, s).u(x).$$

Observe that now, for each d either

$$\sum_{(s,d,x)\in M} Sim'(s_0, s) = 1 \quad \text{or} \quad \sum_{(s,d,x)\in M} Sim'(s_0, s) = 0.$$

Obviously, this model is still requiring numerical values for preferences and similarity degrees. Another property that sometimes may be a drawback is that their utility functions, as in *EUT*, compensate between good and bad results.

2.4 Other Approaches

The number of works on Decision under uncertainty is too big to try to summarise them here, and it is not the goal of this work. Nevertheless, we briefly mentioned some of them, those that are more related with different aspects of our work.

One of the problems of *EUT* is that it needs numerical probabilities for each state and numerical utilities for all possible consequences. Sometimes this assumption is too strong if there is only incomplete or poor available information. In these cases, a more qualitative approach is needed. Moreover, *EUT* is specially tailored for repeated decisions whose results accumulate additively. This is the underlying meaning of the averaging nature of expected utility. However, in the case of one-shot decisions or decisions whose individual results do not compensate each other, *EUT* does not yield a convincing criterion for rank-ordering decisions. This situation of non-additivity naturally occurs with qualitative information about the worth of consequences.

The classical axiomatic frameworks of utility theory have actually been questioned rather early, challenging some of the postulates leading to the expected utility criterion. Noticeably, Allais (1953) and later Ellsberg (1961) laid bare the existence of cases where a systematic violation of the expected utility criterion could be observed. Some of these violations were due to a cautious attitude of Decision-Makers.

More recently Gilboa (1987) and Schmeidler (1989) have advocated and axiomatised lower and upper expectations expressed by Choquet's integrals attached to non-additive numerical set-functions (corresponding to a family of probability measures) as a formal approach to utility that accounts for Ellsberg's paradox (see also (Sarin and Wakker, 1992)). One of these generalised expected utility criteria (the lower expectation) is also a numerical generalisation of the cautious Wald's criterion for decision under ignorance. Choquet integrals, especially the lower expectations, are mild versions of Wald criterion. The pessimistic (resp. optimistic) criterion, that we will characterise, can again be viewed as a refinement of Wald's criterion (resp. the maximax criterion), but the utility functions are qualitative, hence they reject the notion of averaging put forward by the classical theory, and also sanctioned by Choquet's integrals.

Hendon et al. (1994) assume that uncertainty on consequences is measured by belief functions. They assume as primitive a set of beliefs functions on consequences and a preference relation on it. In order to take decisions, they assume a probability distribution on the set of states S. Their hypothesis is that each decision assigns to each state not a consequence but a set of consequences. Hence, each decision is identified with a belief function on consequences. Then, they develop a model à la Von Neumann and Morgenstern.

Other alternatives have been proposed in the literature and steps to qualitative decision theory have been investigated in various directions by AI researchers in the last years. Some approaches are based on an all-or-nothing notions of utility and/or plausibility, e.g., Bonet and Geffner (1996), Brafman and Tennenholtz (1997). The latter clearly advocates Wald cautious criterion. Others, like Pearl (1993,1994), use integer-valued functions.

Bonet and Geffner (1996) propose a qualitative model based on rules, providing a semantics based on high probabilities and lexicographic preferences. They argue that the decision chosen is easy to justify on the basis of reasons for and against the decision. Input situations are modelled by a set of propositions and observations, while output situations are modelled as a set of goals, each one with its priority. A set of actions and action rules are assumed to be given, as well as a plausibility measure on situations whose values are: unlikely, plausible and likely. They classify goals in positive or negative taking into account if they are desired or not. A relative importance is defined on goals using its priorities and polarities (+ or -).

Boutilier (1994) proposes a modal conditional logic, whose semantics enables him to represent and reason with qualitative probabilities and preferences. He can represent conditional preferences, these being defeasible. He suggests to focus on the states with maximum plausibility only, a policy which Dubois et al. (1998a) argue that it leads to debatable decisions.

Brafman and Tennenholtz (1996,1997) propose four decision criteria: maximin, minimax, minimax regret and competitive ratio. These criteria use two parameters: a qualitative utility function defined on states and decisions, and local states. The Decision Maker's behaviours modelled by these criteria are characterised by an approach similar to Savage's.

For more details on Qualitative Decision Theory, a recent paper by Doyle and Thomason (1999) summarises main works on it. Among them we find those models that use Possibility Theory as uncertainty formalism, and two alternatives emerge: à la Von Neumann and Morgenstern, initiated by Dubois and Prade (1995), or à la Savage (Dubois et al. (1997h)) . Sabbadin (Sabbadin, 1998a) develops Savage's approach in a possibilistic framework in his Ph.D. thesis. As we are specially interested in the possibilistic framework, we devote next Chapter to a detailed review of these possibilistic approaches.

Another aspect of Decision under Uncertainty is Dynamic Decision Problems. In a qualitative setting, for example, there is an approach by Sabbadin et al. (1998b) proposing a generalisation of the possibilistic model of Dubois and Prade.

We may be interested not only in individuals preference as in the mentioned approaches but in working with the preference of a group. Models involving this second option are usually called Multiperson Decision Making models. There are many researchers working with qualitative information in the different topics that this type of problems involves. For example, Herrera et al. (1998) assume linguistic preference relations for expressing the opinions of individuals and linguistic values for expressing their respective power or importance degrees. In order to deal with non-weighted linguistic information, they propose the linguistic ordered weighted averaging (LOWA) operator, while to deal with weighted linguistic information, three operators of linguistic weighted information aggregation are used: the linguistic weighted disjunction (LWD) operator, the linguistic weighted conjunction (LWC) operator and the linguistic weighted averaging (LWA) operator. Godo and Torra (1998a) propose a method for aggregating qualitative information weighted with natural numbers, that is, they propose qualitative weighted means involving T-norms on the set of values. As it is mentioned, several issues are involved in Multiperson Decision Making models, for example, summaries of some models involving fuzzy aggregation of numerical preferences is provided by (Grabisch et al., 1998), for fuzzy preference in multiple criteria by Fodor et al. (1998), and applying fuzzy quantifiers by Kacprzyk and Nurmi (1998).

There are also some works applying fuzzy sets and possibility theory gathered in (Kacprzyk and Fedrizzi, 1990).

Chapter 3

Possibilistic Approaches: Antecedents

The following approaches are based on the hypothesis that uncertainty on states of the world is possibilistic in nature. They are possibilistic views of the *Expected Utility Theory*. The first one assumes a possibility distribution on situations is known and deals with preference relations on *possibilistic lotteries*, while in the second one, preference relations are defined on *decisions*. In both cases, the preference relations satisfying their axiomatic settings are representable by criteria with are expressible in terms of Sugeno integrals (Sugeno, 1977).

3.1 Possibilistic Qualitative Decision Theory à la Von Neumann and Morgenstern: Antecedents

Dubois and Prade (1995) have suggested a qualitative counterpart to Von Neumann and Morgenstern's *Expected Utility Theory*. As it was mentioned, they assume that uncertainty is of possibilistic nature, and they make use of *finite* qualitative preference and uncertainty scales equipped with the maximum, minimum and an order-reversing operations.

It is also assumed that the scales of uncertainty and preferences are commensurate. Dubois and Prade propose a characterisation of the preference relations that are representable by qualitative utility functions which are a generalisation of the maximin Wald's criterion (see Section 2.1 or (Wald, 1950)).

In order to introduce their proposal, let us first present some useful notation and definitions. S will denote a ***finite*** *set of situations* and X will denote a ***finite*** *set of consequences of acts*. A *decision or act* d on S is represented by a function $d{:}S \to X$, which provides the consequence of the decision in each possible situation.

V will denote a ***finite*** *linear scale of uncertainty*, with $\inf(V) = 0_V$, $\sup(V) = 1_V$. The belief state about which is the actual situation is supposed to be represented by a possibility distribution π:$S \to V$, with the following conventions:

$$\begin{array}{lll} \pi(s) & = 0_V & \text{means that state } s \text{ is rejected as impossible;} \\ \pi(s) & = 1_V & \text{means that } s \text{ is totally possible (=plausible).} \end{array}$$

Distinct states may simultaneously have a degree of possibility equal to 1_V. Flexibility in this description is modelled by letting $\pi(s)$ between 0_V and 1_V for some states s. Thus, the value $\pi(s)$ represents *the degree of possibility of the state* s, some states being more possible than others. Clearly, if S is the complete range of states, at least one of the elements of S should be fully possible, so that $\exists\ s,\ \pi(s) = 1_V$ (*normalisation*). In this Chapter, we only consider normalised possibility distributions.

A possibility distribution π is said to be *at least as specific as* π' if and only if for each state of affairs s: $\pi(s) \leq \pi'(s)$ (Yager, 1983). Then, π is at least as restrictive and informative as π'.

In the possibilistic framework extreme forms of partial knowledge can be captured, namely:

- *complete knowledge*: for some $s_0,\ \pi(s_0) = 1_V$ and $\pi(s) = 0_V\ \forall s \neq s_0$ (only state s_0 is possible).

- *complete ignorance*: $\pi(s) = 1_V, \forall s \in S$ (all states in S are possible).

$\Pi(S, V)$ will denote the *set of **normalised** possibility distributions on S over V*, i.e.

$$\Pi(S, V) = \{\pi : S \to V |\ \exists s \in S\ \pi(s) = 1_V\}.$$

Notation 3.1
For the sake of simplicity, we shall generally omit the reference to the uncertainty scale, that is, we shall use the notation $\Pi(S)$. Also for the same reason, we shall use s for denoting both an element belonging to S and the possibility distribution on S such that

$$\pi(z) = \begin{cases} 1_V, & \text{if} \quad z = s \\ 0_V, & \text{otherwise.} \end{cases}$$

Similarly, we shall also denote by A both a subset $A \subseteq S$ and the possibility distribution on S such that $\pi(s) = 1_V$ if $s \in A$ and $\pi(s) = 0_V$ otherwise. With this convention, we can consider S as included in $\Pi(S)$.

Now, analogously with the previous Chapter, let us introduce the notion of *possibilistic lotteries*, the qualitative counterpart of the probabilistic lotteries. Given two events A and B, and two values $\lambda,\ \mu \in V$ such that $\max(\lambda, \mu) = 1$, the (possibilistic) binary lottery

$$(\lambda/A, \mu/B),$$

is the prospect of considering that A occurs with plausibility λ, and B occurs with plausibility μ. On the other hand the so-called *Possibilistic mixture*, the qualitative counterpart of the probabilistic mixture, is an operation defined on $\Pi(S)$ that combines two possibility distributions π_1, π_2 with two values $\lambda, \mu \in V$ s.t. $\max(\lambda, \mu) = 1_V$ into a new distribution $M(\pi_1, \pi_2, \lambda, \mu)$, defined as

$$M(\pi_1, \pi_2; \lambda, \mu)(s) = \max(\min(\lambda, \pi_1(s)), \min(\mu, \pi_2(s))). \tag{3.1}$$

In particular, the possibilistic mixture $M(s, y, \lambda, \mu)$ is defined as the possibility distribution on S such that

$$M(s, y; \lambda, \mu)(z) = \begin{cases} \lambda & \text{if} \quad z = s \\ \mu & \text{if} \quad z = y \\ 0_V & \text{otherwise.} \end{cases}$$

Notation 3.2
Analogously to the probabilistic case, any possibility distribution on a finite set may be seen as a compound possibilistic lottery, that is, as a sequence of binary possibilistic lotteries. Hence, from now on, we identify the set $\Pi(S)$ equipped with the possibilistic mixture, with the set of possibilistic lotteries on S with the lottery combination operation. That is, we will identify $M(\pi_1, \pi_2; \lambda, \mu)$ and $(\lambda/\pi_1, \mu/\pi_2)$. Moreover, applying this identification, from now on, we shall sometimes combine the notation of possibilistic mixtures and possibilistic lotteries.

Finally, U will denote a ***finite*** *linearly ordered scale of preference*, with $\sup(U) = 1_U$ and $\inf(U) = 0_U$, while $n_U{:}U \to U$ will denote its *order-reversing involution.*

Notation 3.3
For simplicity reasons we shall omit the reference to the scales in their bottom and top elements, hence 1 and 0 denote both assuming that they are identifiable by the context.

In order to define the qualitative/ordinal utility functions, an assumption of *commensurateness* between the plausibility scale V and the preference scale U has to be made. For the moment, what is basically needed is an *order-reversing mapping* $n{:}V \to U$ such that $n(1) = 0$ and $n(0) = 1$.

Let F be the *fuzzy set of preferred situations*, with U-valued membership function $\mu_F{:}S \to U$.

Notation 3.4
From now on, we identify the membership of a fuzzy set with the fuzzy set.

Dubois and Prade consider the following qualitative utility:

$$ut_F(\pi) = \min_{s \in S} \max(n(\pi(s)), F(s)). \tag{3.2}$$

This criterion was first proposed by Whalen (1984). Observe that (3.2) may also be written as

$$ut_F(\pi) = \min_{s \in S} \max(n_U(\pi^*(s)), F(s))$$

where $\pi^*(s) = n_U \circ n(\pi(s))$ and n_U is the order reversing involution on U. Hence, this utility value $ut_F(\pi)$ coincides with the necessity degree of the fuzzy set of preferred situations F with respect to the possibility distribution π^*. It accounts for a degree of inclusionship of π* into F (more details will be given in Section 5.1). Taking into account that Inuiguchi et al. (1989) show that the necessity of a fuzzy event is a Sugeno integral, we have that ut_F is a Sugeno integral.

Recalling that the well-known Wald maximin criterion suggests that *a decision is evaluated by the value of its worst possible consequence*, we may observe that maximising ut_F generalises Wald's criterion. Indeed, when π is an all or nothing distribution, i.e. when $\pi(S) = \{0, 1\}$, π may be seen as the membership function of a crisp set A, and then we have

$$ut_F(\pi) = \min_{s \in A} F(s).$$

That is, *the worst situation compatible with π is used to assess the utility of the decision underlying π*. Hence, we refer to ut_F as a *pessimistic* or *conservative* criterion.

The following axioms were proposed in (Dubois and Prade, 1995) for a "rational" preference relation $\sqsubseteq$ on $\Pi(S)$ to be represented by a pessimistic qualitative utility (caution: $\pi \sim \pi'$ means $\pi' \sqsubseteq \pi$ and $\pi \sqsubseteq \pi'$):

- $DP1$: $\sqsubseteq$ is a total pre-order (i.e. $\sqsubseteq$ is reflexive, transitive and complete).
- $DP2$: If A is a crisp subset of S, then there is $s \in A$ s.t. $s \sim A$.
- $DP3$ *(uncertainty aversion)*: if $\pi \leq \pi' \ \Rightarrow \ \pi' \sqsubseteq \pi$.
- $DP4$ *(independence)*: $\pi_1 \sim \pi_2 \ \Rightarrow \ M(\pi_1, \pi; \lambda, \mu) \sim M(\pi_2, \pi; \lambda, \mu)$.
- $DP5$ *(reduction of lotteries)*(see Figure 3.1):

$$M(s, M(s, y; \alpha, \beta); \lambda, \mu) \sim M(s, y; max(\lambda, min(\mu, \alpha)), min(\mu, \beta)).$$

- $DP6$ *(continuity)*: $\pi' \sqsubseteq \pi \Rightarrow \exists\, \lambda \in V$ such that $\pi' \sim M(\pi, S; 1, \lambda)$.

Axiom $DP1$ allows us to represent utility on a totally ordered scale.
$DP2$, violated by expected utility, suggests that, contrary to it, the pessimistic utility is not based on the idea of average and repeated decisions, but makes sense for one-shot decisions. $DP2$ expresses that when the agent believes that the state lies in A and decision is put to work, then the state will be some s in A, and the benefit from the decision will indeed be the one in state s. It comes down to rejecting the notion of mean value.

The *uncertainty aversion axiom* states that the less informative π' is, i.e. the more uncertain the situation is, the less preferred π' is: so, the worst state is total ignorance. Because of this axiom, such a preference relation represents a pessimistic vision for decision making, expressing aversion to lack of information. With this perspective, $DP2$ now says that in fact, lottery A is equivalent to the worst situation in A.

The *independence axiom* means that if two distributions are indifferent with respect to decision maker preferences, then we may exchange them in compound lotteries.

Axiom $DP5$ allows us to reduce lotteries to standard ones in the style of possibilistic mixtures.

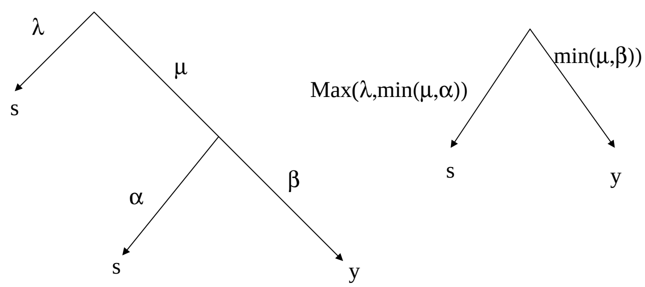

Figure 3.1: Possibilistic Reduction

Finally, the *continuity axiom* establishes that if π is at least as preferred as π', π' is preferentially equivalent to having some uncertainty about π.

The following theorem, to represent such relations by pessimistic qualitative utility functions, is proposed by Dubois and Prade (1995).

Theorem 3.1
Given a preference relation $\sqsubseteq$ on $\Pi(S)$ verifying axioms DP1 - DP6, there exists a fuzzy set F on S and a utility function ut_F from $\Pi(S)$ to a totally ordered set U representing $\sqsubseteq$ such that for each $\pi \in \Pi(S)$, we have that

$$ut_F(\pi) = \min_{s \in S} \max(n(\pi(s)), F(s))$$

where n is an order-reversing function from the possibility scale V to the preference scale U such that $n(0) = 1$ and $n(1) = 0$, where 1 denotes the top elements of U and V and 0 their bottom elements.

Note that

$$ut_F(\pi) = 1 \quad \text{if} \quad \{s \in S |\, \pi(s) > 0\} \subseteq \{s \in S |\, F(s) = 1\}$$

i.e. π has maximum utility if all the more or less possible situations are among the most preferred ones. Also,

$$ut_F(\pi) = 0 \quad \text{if} \quad \{s \in S |\, \pi(s) = 1\} \cap \{s \in S |\, F(s) = 0\} \neq \emptyset$$

i.e. π is the worst if there exists a most plausible situation whose payoff is minimum.

3.2 Possibilistic Qualitative Decision Theory à la Savage

As it was previously mentioned, *EUT* has two axiomatic frameworks: à la Von Neumann-Morgenstern, which works with probabilistic lotteries, linked with acts, and à la Savage, which is expressed directly in terms of acts. Dubois et al. (1997h) propose a possibilistic axiomatics à la Savage. This approach is developed in more detail by Sabbadin (1998a) in his Ph.D. dissertation.

In this approach, they assume a primitive preference relation $\preceq$ on acts. As usual, S represents a finite set of states, while X is the consequences set. The set of decisions will be denoted by D. Before introducing their axiomatic setting, let us introduce some definitions.

Definition 3
Given two decisions d, d' *the* compound act of d and d' w.r.t. $A \subseteq S$ *is defined as*

$$dAd'(s) = \begin{cases} d(s), & \text{if} \quad s \in A \\ d'(s), & \text{if} \quad s \notin A. \end{cases}$$

Let $\pi{:}S \rightarrow V$ a possibility distribution, the plausibility scale V being totally ordered. *Decision Maker*'s preference on consequences are represented by $\mu{:}X \rightarrow U$, U being a finite set linearly ordered. Then, the following qualitative utilities can be defined:

$$\begin{aligned} v_*(d) &= \inf_{s \in S} \max(n(\pi(s)), \mu(d(s))), \\ v^*(d) &= \sup_{s \in S} \min(h(\pi(s)), \mu(d(s))), \end{aligned}$$

with $h{:}V \rightarrow U$ an order-preserving mapping, and $n = n_U \circ h$. Dubois (1986) defines a *qualitative possibility (necessity resp.)* as a set relation that verifies axioms $QP1, QP2$ (see Section 2.2.2) and axiom Π (N respectively) which is a relaxation of the axiom $QP3$,

- Π: $B \trianglelefteq C \Rightarrow \ (B \cup D) \trianglelefteq (C \cup D)$,
- N: $B \trianglelefteq C \Rightarrow \ (B \cap D) \trianglelefteq (C \cap D)$.

Moreover, Dubois (1986) proposes a relaxation of $QP3$ that includes both definitions of qualitative probability and possibility.

- M: $\forall B, C, D$ s.t. $D \cap (B \cup C) = \emptyset,\ B \trianglelefteq C \Rightarrow (B \cup D) \trianglelefteq (C \cup D)$,

includes Π, while its dual

- M': $\forall B, C, D$ s.t. $D \cup (B \cap C) = S,\ B \trianglelefteq C \Rightarrow (B \cap D) \trianglelefteq (C \cap D)$.

includes N.

Savage proves that a relation on acts satisfying $Sav1 - Sav5$ induces a relation on events that is a qualitative probability.

The "utility" functions v_* and v^* do not satisfy Savage's *"sure thing principle"* $(Sav2)$ axiom. Dubois et al. (1997f) observe that this fact results in that $Sav3$ and $Sav4$ are not verified by v_* or v^*, but these functions verify the weaker Savage's axioms they propose.

- $WS2$ *(weak sure thing principle)*: Let $A \subseteq S$, if $d_1Ad \prec d_2Ad$ then $d_1Ad' \preccurlyeq d_2Ad'$.
- $WS3$ *(weak coherence with constant acts)*: If x and y are constant acts, then if y is at least as preferred as x then $xAh \preccurlyeq yAh$.
- $WS4$ *(weak order on events):* If x is preferred to x' and y is preferred to y' then $xAx' \prec yAy'$.

They also propose the following axioms:

- $Pes:\ \forall d, d' \in D,\ \forall A \subseteq S\ d \prec dAd' \Rightarrow d'Ad \preccurlyeq d.$
- $Opt:\ \forall d, d' \in D,\ \forall A \subseteq S\ dAd' \prec d \Rightarrow d \preccurlyeq d'Ad.$
- *RDD (Restricted Disjunctive Dominance)*:

  $$\text{if } g \prec f \text{ and } x \prec f \text{ then } g \vee x \prec f,$$

  with $g \vee x$ the maximum (point-wise) between g and x.

v_* satisfies Pes axiom while v^* verifies Opt.

The following representation theorem for characterising preference relation induced by v_* is proposed by Dubois et al. (1997e).

Theorem 3.2
Let $\preccurlyeq$ be a preference relation over the set of all acts d from S to X, satisfying $Sav1, WS3, Sav5, PES, RDD$. There exists a finite qualitative scale L, a utility function

$$v_* \text{ of the form } v_*(d) = \inf_{s \in S} \max(n(\pi(s)), \mu(d(s)))$$

on X, and a possibility distribution π on S, taking their values on L, such that $f \preccurlyeq f' \iff v_*(f) \leq v_*(f')$, with $\mu{:}X \to L$.

In Dubois et al. (1998e), they consider that uncertainty is modelled by a general monotonic set-function $\sigma : 2^S \to L$, with L a finite linear scale which is applied for measuring both uncertainty and preferences. In this hypothesis, and remaining in à la Savage framework, they characterise the ordering induced in the decisions set by the utility defined in terms of the Sugeno integral with respect to σ.

Chapter 4

Representation of Purely Ordinal Utility Functions

In the previous Chapter we have introduced Dubois and Prade's axiomatic setting to characterise the preference relation induced by a pessimistic qualitative utility which is expressed in terms of the preference on consequences and the "possibilistic" lotteries on S, S being the finite set of situations (Section 3.1).

In this Chapter, we first analyse some shortcomings detected in that proposal. Then, we suggest in Section 4.4 an improvement of the axiomatic characterisation of preference relation induced by a possibilistic pessimistic utility function. We also provide the representation theorem for preference relations satisfying the improved axiomatics. Moreover, in Section 4.5 we introduce the characterisation for optimistic utility functions.

But, before analysing our proposal, first we show in Section 4.2 that some decision problems in which uncertainty is involved may be seen as a problem of ranking possibility distributions on consequences, and we provide some preliminary results in Section 4.3 as well. We end the Chapter showing the behaviour of these criteria in a little toy example.

4.1 Some Remarks on Dubois and Prade's Proposal

Let us briefly recall the proposal given in Section 3.1. The axioms proposed by Dubois and Prade for a preference relation $\sqsubseteq$ on $\Pi(S)$ to be represented by a (pessimistic) qualitative utility are:

- $DP1$: $\sqsubseteq$ is a total pre-order.
- $DP2$: If A is a crisp subset of S then there is $s \in A$ s.t. $s \sim A$.
- $DP3$ *(uncertainty aversion)*: if $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.
- $DP4$*(independence)*: $\pi_1 \sim \pi_2 \Rightarrow M(\pi_1, \pi; \lambda, \mu) \sim M(\pi_2, \pi; \lambda, \mu)$.

- $DP5$*(reduction of lotteries)*:

$$M(s, M(s, y; \alpha, \beta); \lambda, \mu) \sim M(s, y; max(\lambda, min(\mu, \alpha)), min(\mu, \beta)).$$

- $DP6$*(continuity)*: $\pi' \sqsubseteq \pi \Rightarrow \exists \lambda \in V$ such that $\pi' \sim M(\pi, S; 1, \lambda)$.

and their theorem says:

> "Given a preference relation $\sqsubseteq$ on $\Pi(S)$ verifying axioms $DP1 - DP6$, there exists a fuzzy set F on S and a utility function ut_F from $\Pi(S)$ to a totally ordered set U representing $\sqsubseteq$ such that for each $\pi \in \Pi(S)$, we have that
>
> $$ut_F(\pi) = \min_{s \in S} \max(\mathrm{n}(\pi(s)), F(s))$$
>
> where n is an order-reversing function from the possibility scale V to the preference scale U such that $n(0) = 1$ and $n(1) = 0$, where 1 denotes the top elements of U and V and 0 their bottom elements."

In this setting we have identified two possible shortcomings:

- The theorem does not really specify the characterisation of the preference relations induced by

$$ut_F(\pi) = \min_{s \in S} \max(n(\pi(s)), F(s)).$$

- The proof has some problems.

Also, the axiomatic setting turns out to be redundant (see Lemmas 4.2 and 4.3 for more details).

With respect to the proof of the theorem, it starts claiming that the relation induced by ut_F satisfies the axioms. But, there are some hypotheses which are implicitly assumed in the proof that must be explicitly required if we want the preference relation induced by ut_F to satisfy the axiomatic setting, as it is shown in the following example.

Example:
Consider the following sets

$$S = \{\underline{s}, s, \overline{s}\}, V = \{0 < \lambda_1 < \lambda_2 < 1\}$$

and

$$U = \{0 < u_1 < u_2 < 1\}.$$

Let the set of preferred situations F be defined as

$$F(\underline{s}) = 0, F(\overline{s}) = 1, F(s) = u_1,$$

that is, we have $\underline{s} \sqsubset s \sqsubset \overline{s}$. However, for each reversing function n such that $u_1 \notin n(V)$, we have that there is no $\lambda \in V$ s.t. $s \sim (1/\overline{s}, \lambda/S)$ w.r.t. ut_F, i.e.

$$\nexists\, \lambda \text{ s.t. } ut_F(s) = ut_F(1/\overline{s}, \lambda/S).$$

Indeed, $ut_F(s) = F(s) = u_1$, while $ut_F(1/\overline{s}, \lambda/S) = n(\lambda)$. Hence, $DP6$ is not satisfied by the preference relation induced by ut_F. $\diamondsuit$

Let us remark that in the proof they claim the existence of a reversing function n which is also required to be ***bijective***. But, this requirement may be too strong as this other example shows:

Example:
Suppose that $S = \{\underline{s}, \overline{s}\}$ while V is defined as in the previous example. Consider the preference relation $\sqsubseteq$ defined by

$$\underline{s} \sqsubset (1/\overline{s}, \lambda_1/S) \sim (1/\overline{s}, \lambda_2/S) \sqsubset \overline{s},$$

and

$$\underline{s} \sim S \sim (1/\underline{s}, \lambda_1/\overline{s}) \sim (1/\underline{s}, \lambda_2/\overline{s}),$$

and satisfying reflexivity.

This relation $\sqsubseteq$ satisfies the axioms. If $n{:}V \to U$ is a bijective reversing mapping, we have that

$$ut_F(1/\overline{s}, \lambda_1/S) = n(\lambda_1) > n(\lambda_2) = ut_F(1/\overline{s}, \lambda_2/S)$$

i.e.

$$(1/\overline{s}, \lambda_2/S) \sqsubset_{ut_F} (1/\overline{s}, \lambda_1/S),$$

while they are indifferent w.r.t. $\sqsubseteq$. Contradiction. That is, there is no bijective function n such that ut_F may represent the relation. $\diamondsuit$

Nevertheless, Dubois and Prade's intuition with respect to the representation theorem is still valid provided some technical corrections.

4.2 The Possibilistic Decision Framework Specified

A *Decision Maker* may be faced with different cases of incompletely or ill specified decision problems.

Different cases that result in possibility distributions on X are the following:

- *the situation is uncertain*: s_0 is represented by a normalised possibility distribution on S, $\pi_{s_0}{:}S \to V$, representing the belief state about which is the real situation. Then, each decision $d{:}S \to X$ induces a corresponding possibility distribution π_{d,s_0}, on the set of consequences, defined as

$$\pi_{d,s_0}(x) = \max\{\pi_{s_0}(s) | d(s) = x\}, \tag{4.1}$$

  with $\max \emptyset = 0$. $\pi_{d,s_0}(x)$ *represents the plausibility of* x *being the consequence of* d.

  As π_{s_0} is normalised, π_{d,s_0} is normalised as well.

- *the situation is precisely known but the decision is not precisely defined:* in each situation, we do not have a precise consequence but a possibility distribution on the consequences. So, d is modelled by a possibility distribution π_d on the set of consequences.
- *the decision is partially unknown*: we know how the decision resulted in some other situations but not in the actual situation. Thus, we have partial information about decisions by having stored the performance of decisions taken in different past situations. This leads to a case-based decision problem. This point will be developed in Chapter 8, however we advance here that each decision may also be identified with a possibility distribution on consequences.

Therefore, we include these cases in our framework assuming as working hypothesis that *uncertainty may be modelled by possibility distributions on consequences*, that is,

> *For an actual situation s_0, we may identify each decision with a normalised possibility distribution on X, therefore, choosing the "best" decision is equivalent to choosing its associated possibility distribution.*

Hence, in order to select the best decision, we are looking for possibility distributions on consequences that maximise a utility function $\mathcal{U}$ on $\Pi(X)$, i.e. we consider

$$d \preccurlyeq_{s_0} d' \quad \text{iff} \quad \pi_d \sqsubseteq \pi_{d'} \quad \text{iff} \quad \mathcal{U}(\pi_d) \leq \mathcal{U}(\pi_{d'}).$$

From now on, we focus on preference relations in the set of possibility distributions on consequences.

4.3 Some Preliminary Results

Let us recall the context of our work. V will denote a **finite** linear plausibility scale, where $\inf(V) = 0$ and $\sup(V) = 1$, and $\Pi(X)$ will denote *the set of consistent possibility distributions on X over V*, i.e.

$$\Pi(X) = \{\pi : X \to V \,|\, \max_{x \in X} \ \pi(x) = 1\}.$$

We have already introduced qualitative binary lotteries $(\lambda/x, \mu/y)$.[1] More generally using the notation $(\lambda_1/x_1, ..., \lambda_p/x_p)$, with $\lambda_i \in V$ and $\max_i(\lambda_i) = 1$, any consistent possibility distribution π on X can be seen as a multiple consequence qualitative lottery taking $\lambda_i = \pi(x_i)$.

U will denote a **finite** linearly ordered scale of preference (or utility), with $\sup(U) = 1$ and $\inf(U) = 0$ and a preference function u:$X \to U$ that assigns to each consequence of X a preference level of U.

An interesting property of a preference relation $\sqsubseteq$ on $\Pi(X)$ satisfying $DP1, DP2$ and $DP3$ is that the extremal elements of $(X, \sqsubseteq)$ are maximal and minimal elements of $(\Pi(X), \sqsubseteq)$ as well:

[1] Recall, we will identify possibilistic lotteries and mixtures.

Lemma 4.1
If $\sqsubseteq$ verifies axioms DP1, DP2 and DP3, and $\underline{x}$ and $\overline{x}$ are a minimal and a maximal element of X, respectively, then:

- $\underline{x} \sim (1/\overline{x}, 1/\underline{x}) \sim X$.
- $\underline{x}$ *and* $\overline{x}$ *are also the minimal and maximal elements of* $(\Pi(X), \sqsubseteq)$.

Proof:
Let us first prove the equivalences $\underline{x} \sim X \sim (1/\overline{x}, 1/\underline{x})$. $DP1$ guarantees that $\underline{x}$ and $\overline{x}$ exist. By the uncertainty aversion axiom $(DP3)$, it is clear that X is a minimal element of $\Pi(X)$, so it is $X \sqsubseteq \underline{x}$. But, by $DP2$ there exists $x_0 \in X$ such that $x_0 \sim X$. Since $\underline{x}$ is minimal, $\underline{x} \sqsubseteq x_0$, thus it must be $\underline{x} \sim X$.

Furthermore, on $\Pi(X)$ we have $\underline{x} \leq (1/\overline{x}, 1/\underline{x}) \leq X$ (specificity point-wise ordering), and again by $DP3$, $X \sqsubseteq (1/\overline{x}, 1/\underline{x}) \sqsubseteq \underline{x}$, and thus

$$\underline{x} \sim X \sim (1/\overline{x}, 1/\underline{x}).$$

On the other hand, for any $\pi \in \Pi(X)$, since π is normalised, there exists x such that $\pi(x) = 1$. So, we have $x \leq \pi$ and therefore $\pi \sqsubseteq x$, but since $\overline{x}$ is maximal in X, it is $x \sqsubseteq \overline{x}$, and thus $\pi \sqsubseteq \overline{x}$. So, $\overline{x}$ is maximal on $(\Pi(X), \sqsubseteq)$ as well. Moreover, as X is a minimal element of $\Pi(X)$ and $\underline{x} \sim X$, obviously $\underline{x}$ is a minimal element of $\Pi(X) too$.
□

Remark 1
Observe that as a consequence of the possibilistic mixture definition we have that

$$M(x, x; \lambda, \mu) = x \qquad \text{for all } \lambda, \mu \text{ such that } \max(\lambda, \mu) = 1$$

and

$$M(x, X; \lambda, \mu) = M(x, X - \{x\}; 1, \mu) \qquad \text{for all } \lambda, \mu \text{ such that } \max(\lambda, \mu) = 1.$$

Moreover, we have that:

Lemma 4.2

$$M(\pi_1, M(\pi_1, \pi_2; \alpha, \beta); \lambda, \mu) \sim M(\pi_1, \pi_2; max(\lambda, min(\mu, \alpha)), min(\mu, \beta)).$$

always holds.

lema

Proof:

By definition of lotteries, we have that

$$\begin{aligned}
M(\pi_1, M(\pi_1, \pi_2; \alpha, \beta); \lambda, \mu)(x) &= \max\{\min(\pi_1(x), \lambda), \\
&\quad \min(\mu, \max\{\min(\pi_1(x), \alpha), \min(\pi_2(x), \beta)\})\} \\
&= \max[\min(\lambda, \pi_1(x)), \min(\mu, \alpha, \pi_1(x)), \\
&\qquad \min(\mu, \beta, \pi_2(x))] \\
&= \max[\min(\pi_1(x), \max(\lambda, \min(\mu, \alpha))), \\
&\qquad \min(\mu, \beta, \pi_2(x))] \\
&= M(\pi_1, \pi_2; max(\lambda, min(\mu, \alpha)), min(\mu, \beta))(x)
\end{aligned}$$

□

Hence, the axiom on reduction of lotteries $(DP5)$:

$$M(x, M(x, y; \alpha, \beta); \lambda, \mu) \sim M(x, y; max(\lambda, min(\mu, \alpha)), min(\mu, \beta)).,$$

is unnecessary if we take the definition of possibilistic lotteries for granted. The same remark applies to the Von Neumann and Morgenstern's axiomatic setting if the notion of probabilistic mixture is acknowledged (see Herstein and Milnor (1953)).

On the other hand, Axiom $DP2$ is also redundant since it follows from the rest of the axioms. Indeed,

Lemma 4.3
Axioms DP1, DP4 and DP6 imply axiom DP2.

Proof:
Suppose $A = \{x_1, x_2\}$ with $x_1 \sqsubseteq x_2$. By $DP6$ there exists $\lambda \in V$ such that $x_1 \sim (1/x_2, \lambda/X)$, and applying $DP1$, *reduction of lotteries* and $DP4$, we obtain

$$A = (1/x_1, 1/x_2) \sim (1/(1/x_2, \lambda/X), 1/x_2) = (1/x_2, \lambda/X) \sim x_1.$$

The case when A has p elements is an easy generalisation. Indeed, suppose the Lemma is valid if the cardinality of A is p, p being greater than 2. Now, let A be such that $|A| = p + 1$, and let x_1 be one of its minimal elements w.r.t. $\sqsubseteq$. Since $A = (1/x_1, 1/A - \{x_1\})$, by induction hypothesis we have that if x_2 is one of the minimal elements of $A - \{x_1\}$ w.r.t. $\sqsubseteq$, then

$$A \sim (1/x_1, 1/x_2) \sim x_1.$$

□

Another interesting formulation of the continuity of the preference ordering, which will be useful later, is the following one:

- $A4$: For all $\pi \in \Pi(X)$ there exists $\lambda \in V$ such that $\pi \sim (1/\overline{x}, \lambda/\underline{x})$, where $\overline{x}$ and $\underline{x}$ are any maximal and any minimal element of $(X, \sqsubseteq)$ respectively.

Observe that $A4$ will be considered with $DP1$, since $DP1$ guarantees that the maximal elements of $(\Pi(X), \sqsubseteq)$ are equivalent, and the minimal ones are also equivalent to each other.

It can be proved that,

Lemma 4.4
In the context of DP1–DP5 axioms, axiom DP6 is equivalent to A4.

Proof:
$\leftarrow$) Suppose $A4$ holds, and let π, π' be such that $\pi' \sqsubseteq \pi$. We have two cases:

1. $\pi' \sim \pi$. Hence, $\pi' \sim (1/\pi, 0/X)$.

2. $\pi' \sqsubset \pi$. By hypothesis, there exists $\lambda, \lambda' \in V$ such that

$$\pi \sim (1/\overline{x}, \lambda/\underline{x}) \qquad \text{and} \qquad \pi' \sim (1/\overline{x}, \lambda'/\underline{x}).$$

Since $\pi' \sqsubset \pi$, by $DP1$ we have that

$$(1/\overline{x}, \lambda'/\underline{x}) \sqsubset (1/\overline{x}, \lambda/\underline{x}),$$

and by $DP3$, it is $\lambda' > \lambda$. Now, taking into account that $X \sim \underline{x}$, the independence axiom $(DP4)$ and reducing lotteries, we obtain that

$$(1/\pi, \lambda'/X) \sim (1/(1/\overline{x}, \lambda/\underline{x}), \lambda'/\underline{x}) = (1/\overline{x}, \max(\lambda', \lambda)/\underline{x}).$$

Since $\lambda' > \lambda$,

$$(1/\overline{x}, \max(\lambda', \lambda)/\underline{x}) = (1/\overline{x}, \lambda'/\underline{x}) \sim \pi',$$

i.e. $(1/\pi, \lambda'/X) \sim \pi'$. Therefore, $DP6$ also holds.

$\rightarrow$) Suppose now that $DP6$ holds. For any π, we have that $\pi \sqsubseteq \overline{x}$. Then, by hypothesis, there exists λ such that $\pi \sim (1/\overline{x}, \lambda/X)$, and thus $\pi \sim (1/\overline{x}, \lambda/\underline{x})$. This proves that $A4$ also holds. $\square$

Taking into account these results, we propose next an improved set of axioms that characterises pessimistic qualitative utilities providing new proof for the representation theorem, and the corresponding axiomatic setting for an optimistic criterion is given in Section 4.5.

4.4 Representation of Pessimistic Qualitative/Ordinal Utilities

The above discussion has led us to propose this new set of axioms for preference relations on $\Pi(X)$ with the max-min mixture as the internal operation on $\Pi(X)$.

- $A1(structure)$: $\sqsubseteq$ is a total pre-order[2].
- $A2$*(uncertainty aversion)*: if $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.
- $A3$ *(independence)*: $\pi_1 \sim \pi_2 \Rightarrow M(\pi_1, \pi; \lambda, \mu) \sim M(\pi_2, \pi; \lambda, \mu)$.
- $A4$*(continuity)*: $\forall \pi \in \Pi(X) \exists\ \lambda \in V$ such that $\pi \sim M(\overline{x}, \underline{x}; 1, \lambda)$, where $\overline{x}$ and $\underline{x}$ are a maximal and a minimal element of $(X, \sqsubseteq)$ respectively.

Let u:$X \to U$ be a preference function such that $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$, and let h:$V \to U$ be an onto order-preserving function relating both scales V and U.

For any $\pi \in \Pi(X)$, consider the qualitative utility

$$QU^{-}(\pi|u) = \min_{x \in X} \max(n_U(\pi^*(x)), u(x)),$$

where $\pi^*(x) = h(\pi(x))$ and n_U is the reversing involution in U. Notice that $QU^{-}(.|u)$ restricted to X coincides with the preference function u, i.e. $QU^{-}(x|u) = u(x)$, for all $x \in X$. Let us introduce the order-reversing mapping n:$V \to U$ defined as

$$\begin{tikzcd}
X \arrow[r, "\pi"] \arrow[d, "u"'] & V \arrow[d, "h"] \arrow[dl, "n"'] \\
U \arrow[r, leftrightarrow, "n_U"'] & U
\end{tikzcd}$$

Figure 4.1: Diagram of the different mappings

$n(\lambda) = n_U(h(\lambda))$. It verifies $n(0) = 1, n(1) = 0$. Actually, since n_U^2 is the identity in U, the mapping h can also be defined from n, namely $h(\lambda) = n_U(n(\lambda))$ (see Figure.4.1). Using n instead of h, the qualitative utility may be equivalently expressed as:

$$QU^{-}(\pi|u) = \min_{x \in X} \max(n(\pi(x)), u(x)). \tag{4.2}$$

Notation 4.1
For the sake of a simpler notation, we shall write $QU^{-}(\pi)$ instead of $QU^{-}(\pi|u)$ when the mapping u is not relevant for the context.

[2] The reflexivity property involved in this axiom is redundant taking into account $A2$, the reason for remaining here is for the clarity of the presentation.

We will show that the preference ordering on $\Pi(X)$ induced by the qualitative pessimistic utility QU^- satisfies the above set of axioms. First, it is interesting to notice that:

Lemma 4.5
QU^- preserves the possibilistic mixture in the sense that

$$QU^-(M(\pi_1, \pi_2; \lambda, \mu)) = \min\{\max(n(\lambda), QU^-(\pi_1)), \max(n(\mu), QU^-(\pi_2))\}. \quad (4.3)$$

Proof:
By definitions of QU^- and of possibilistic mixtures we have that

$$\begin{aligned}
QU^-(M(\pi_1, \pi_2; \lambda, \mu)) &= \min_{x \in X}(\max(n(M(\pi_1, \pi_2; \lambda, \mu)(x)), u(x))) \\
&= \min_{x \in X}(\max(n((\max(\min(\pi_1, \lambda), \\
&\qquad\qquad \min(\pi_2, \mu)))(x)), u(x))) \\
&= \min_{x \in X}(\max(\min(\max(n(\pi_1(x)), n(\lambda)), \\
&\qquad\qquad \max(n(\pi_2(x)), n(\mu))), u(x))) \\
&= \min_{x \in X}(\min(\max(n(\pi_1(x)), n(\lambda), u(x)), \\
&\qquad\qquad \max(n(\pi_2(x)), n(\mu), u(x)))) \\
&= \min(\min_{x \in X} \max(n(\pi_1(x)), n(\lambda), u(x)), \\
&\qquad \min_{x \in X} \max(n(\pi_2(x)), n(\mu), u(x))) \\
&= \min(\min_{x \in X} \max(n(\lambda), \max(n(\pi_1(x)), u(x))), \\
&\qquad \min_{x \in X} \max(n(\mu), \max(n(\pi_2(x)), u(x)))) \\
&= \min(\max(n(\lambda), \min_{x \in X} \max(n(\pi_1(x)), u(x))), \\
&\qquad \max(n(\mu), \min_{x \in X} \max(n(\pi_2(x)), u(x)))) \\
&= \min(\max(n(\lambda), QU^-(\pi_1)), \max(n(\mu), QU^-(\pi_2)))
\end{aligned}$$

□

Corollary 4.6
$QU^-(\max(\pi_1, \pi_2)) = \min\{QU^-(\pi_1), QU^-(\pi_2)\}$.

Lemma 4.7
Let $\preccurlyeq_{QU^-}$ be the preference ordering on $\Pi(X)$ induced by QU^-, i.e.

$$\pi \preccurlyeq_{QU^-} \pi' \quad \textit{iff} \quad QU^-(\pi) \leq_U QU^-(\pi').$$

Then, $\preccurlyeq_{QU^-}$ verifies axioms A1, A2, A3 and A4.

Proof:
Axiom $A1$ is easily verified, also $A2$ is a consequence of *maximum* and *minimum* being non-decreasing functions, while $A3$ results from the fact that QU^- preserves max-min possibilistic mixtures. Thus, we only check axiom $A4$. We have to prove that

$$\forall \pi \in \Pi(X),\ \exists \lambda \text{ such that } QU^-(\pi) = QU^-(1/\overline{x}, \lambda/\underline{x}),$$

where $\overline{x}, \underline{x}$ are a maximal and a minimal element of X w.r.t. $\preccurlyeq_{QU^-}$.

Since we are assuming $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$, it must be the case that $u(\underline{x}) = 0$ and $u(\overline{x}) = 1$. Thus, by the possibilistic mixture preservation of QU^- we have that

$$\begin{aligned} QU^-(1/\overline{x}, \lambda/\underline{x}) &= \min\{\max(n(1), QU^-(\overline{x})), \max(n(\lambda), QU^-(\underline{x}))\} \\ &= n(\lambda). \end{aligned}$$

Since h is onto, n is onto as well, and it is $u(X) \subseteq U = n(V)$; therefore, for any $\lambda \in n^{-1}(QU^-(\pi))$ we have that

$$QU^-(\pi) = n(\lambda) = QU^-(1/\overline{x}, \lambda/\underline{x}).$$

$\square$

Notation 4.2
For a simpler notation, when it is obvious by the context, we may omit the reference to U *in the relation* $\leq_U$.

Now, we can show that the preference orderings on epistemic states satisfying the axioms proposed can always be represented by a pessimistic qualitative utility of the type of QU^-.

Theorem 4.8 (Representation Theorem of Pessimistic Utility)
A preference relation $\sqsubseteq$ *on* $\Pi(X)$ *satisfies axioms A1,A2, A3 and A4 if, and only if, there exist*

(i) a finite linearly ordered utility scale U *with* $\inf(U) = 0$ *and* $\sup(U) = 1$,

(ii) a preference function $u{:}X \to U$ *such that* $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,

(iii) an onto order-preserving function $h{:}V \to U$,

in such a way that

$$\pi' \sqsubseteq \pi \qquad \textit{iff} \qquad \pi' \preccurlyeq_{QU^-} \pi,$$

where $\preccurlyeq_{QU^-}$ *is the ordering induced on* $\Pi(X)$ *by the qualitative utility* $QU^-(\pi) = \min_{x \in X} \max(n(\pi(x)), u(x))$, *being as usual* $n = n_U \circ h$.

Proof:
The "if" part corresponds to the preceding Lemma. As for the "only if" part, we structure the proof in the following three steps.

- In step (1) we define the utility scale U and an order-preserving (and onto) function h from V to U.
- In step (2) we define a function $QU^-:\Pi(X) \to U$ representing $\sqsubseteq$, i.e. such that

$$QU^-(\pi) \leq QU^-(\pi') \qquad \text{iff} \qquad \pi \sqsubseteq \pi'.$$

- Finally, in step (3) we prove that

$$QU^-(\pi) = \min_{x \in X} \max(n(\pi(x)), u(x)),$$

where $u:X \to U$ is the restriction of QU^- to X.

Now, we develop these steps.

1. First of all, notice that $\sqsubseteq$ stratifies $\Pi(X)$ in a linearly ordered set of classes of equivalently preferred distributions ($\pi' \in [\pi]$ iff $\pi \sim \pi'$). The number of classes is just the number of levels needed to rank the set of distributions. Therefore, we take as utility scale U the quotient set $\Pi(X)/\sim$ together with the natural (linear) order

$$[\pi] \leq [\pi'] \qquad\qquad \text{iff} \qquad\qquad \pi \sqsubseteq \pi'.$$

Denote by 1 and 0 the maximum and minimum elements of $\Pi(X)/\sim$, i.e. of U. By Lemma 4.1, if $\overline{x}$ and $\underline{x}$ are a maximal and minimal elements of $(X, \sqsubseteq)$ respectively, then clearly $[\overline{x}] = 1$ and $[\underline{x}] = 0$.

Let π_λ^- be the possibility distribution corresponding to the qualitative lottery $(1/\overline{x}, \lambda/\underline{x})$, and define the order-reversing function $n:V \to U$ as

$$n(\lambda) = [\pi_\lambda^-].$$

Observe that, since $(1/\overline{x}, 1/\underline{x}) \sim \underline{x}$,

$$n(1) = [(1/\overline{x}, 1/\underline{x})] = [\underline{x}] = 0,$$

also is

$$n(0) = [(1/\overline{x}, 0/\underline{x})] = [\overline{x}] = 1.$$

We verify now that n actually reverses the order. Let $\lambda < \lambda'$, then $\pi_\lambda^- \leq \pi_{\lambda'}^-$, so using $A2$ we have $\pi_{\lambda'}^- \sqsubseteq \pi_\lambda^-$. Then by definition, $[\pi_{\lambda'}^-] \leq [\pi_\lambda^-]$, i.e. $n(\lambda') \leq n(\lambda)$.

Observe that, by construction, n is onto. Indeed, for any $\pi \in \Pi(X)$, $A4$ guarantees that there exists λ s.t. $\pi_\lambda^- \sim \pi$, so $n(\lambda) = [\pi]$.

Let $h = n_U \circ n$, n_U being the reversing involution in U. It is obvious that h satisfies the conditions required.

2. So far we have determined U and h. Now, we define the qualitative function QU^- on $\Pi(X)$ in two steps.

   (a) First, let us define $QU^-(\pi_\lambda^-) = n(\lambda)$.
   It is easy to check that

   $$\pi_\lambda^- \sqsubseteq \pi_{\lambda'}^- \iff QU^-(\pi_\lambda^-) \leq QU^-(\pi_{\lambda'}^-).$$

   Indeed,

   $$\begin{aligned} \pi_\lambda^- \sqsubseteq \pi_{\lambda'}^- \quad &\iff \quad [\pi_\lambda^-] \leq [\pi_{\lambda'}^-] \iff n(\lambda) \leq n(\lambda') \\ &\iff \quad QU^-(\pi_\lambda^-) \leq QU^-(\pi_{\lambda'}^-). \end{aligned}$$

   So, restricted to lotteries of type π_λ^-, QU^- represents $\sqsubseteq$.

   (b) We extend QU^- to any lottery as follows.
   Since for any π, $A4$ guarantees that $\exists\lambda$ s.t. $\pi \sim (1/\overline{x}, \lambda/\underline{x})$, we define

   $$QU^-(\pi) = n(\lambda).$$

   Notice that QU^- is well defined: suppose there exists $\mu \neq \lambda$ such that $\pi \sim (1/\overline{x}, \mu/\underline{x})$. But, since $(1/\overline{x}, \mu/\underline{x}) \sim (1/\overline{x}, \lambda/\underline{x})$ then $[\pi_\lambda^-] = [\pi_\mu^-]$, so $n(\lambda) = n(\mu)$.
   Finally, it is easy to check that QU^- represents $\sqsubseteq$. This is due to the fact that any π is equivalent to some π_λ^-, and by (a) QU^- represents $\sqsubseteq$ over the π_λ^-'s.

3. Now, we define $u{:}X \to U$ as

   $$u(x) =^3 QU^-(x).$$

   Notice that $u(\overline{x}) = 1$ and $u(\underline{x}) = 0$, and thus, $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$. It remains to prove that

   $$QU^-(\pi) = \min_{x\in X} \max(n(\pi(x)), u(x)).$$

   To verify this, we will prove the following equalities:

   - $QU^-(1/x, \lambda/y) = \min(u(x), \max(n(\lambda), u(y)))$.
   Indeed, $A4$ guarantees that $\exists\ \mu, \gamma$ such that $x \sim (1/\overline{x}, \mu/\underline{x})$ and such that $y \sim (1/\overline{x}, \gamma/\underline{x})$ remember that $QU^-(x) = u(x) = n(\mu)$ and $QU^-(y) = u(y) = n(\gamma)$ –, so using $A3$, we have

   $$(1/x, \lambda/y) \sim (1/(1/\overline{x}, \mu/\underline{x}), \lambda/(1/\overline{x}, \gamma/\underline{x})),$$

   and reducing lotteries we obtain

   $$(1/x, \lambda/y) \sim (\max(1, \lambda)/\overline{x}, \max(\mu, \min(\lambda, \gamma))/\underline{x}).$$

[3] Understanding in the righside of the equation x as the singleton distribution.

Therefore,

$$\begin{aligned} QU^-(1/x, \lambda/y) &= n(\max(\mu, \min(\lambda, \gamma))) \\ &= \min(n(\mu), \max(n(\lambda), n(\gamma))) \\ &= \min(u(x), \max(n(\lambda), u(y))). \end{aligned}$$

- $QU^-(\max(\pi_1, \pi_2)) = \min(QU^-(\pi_1), QU^-(\pi_2))$.
By $A4$, $\exists \mu, \gamma$ such that $\pi_1 \sim (1/\overline{x}, \mu/\underline{x})$ and $\pi_2 \sim (1/\overline{x}, \gamma/\underline{x})$.
Then, using $A3$, we have:

$$\max(\pi_1, \pi_2) = (1/\pi_1, 1/\pi_2) \sim (1/(1/\overline{x}, \mu/\underline{x}), 1/(1/\overline{x}, \gamma/\underline{x})),$$

i.e. $\max(\pi_1, \pi_2) \sim (1/\overline{x}, \max(\mu, \gamma)/\underline{x})$.
Therefore, as QU^- represents $\sqsubseteq$,

$$\begin{aligned} QU^-(\max(\pi_1, \pi_2)) &= n(\max(\mu, \gamma)) \\ &= \min(n(\mu), n(\gamma)) \\ &= \min(QU^-(\pi_1), QU^-(\pi_2)). \end{aligned}$$

More generally, we have

$$QU^-(\max_{i=1,\ldots,p}\ \pi_i) = \min_{i=1,\ldots,p}\ QU^-(\pi_i).$$

- $QU^-(\pi) = \min_{i=1,\ldots,p}\ \max(n(\pi(x_i)), u((x_i)))$.
As π is normalised there exists $x_j \in X$ such that $\pi(x_j) = 1$. Without loss of generality, we assume $j = 1$.
Then, let

$$\pi_i = (1/x_1, \pi(x_i)/x_i).$$

Since $\pi = \max_{i=1,\ldots,p}\ \pi_i$, we have:

$$\begin{aligned} QU^-(\pi) &= QU^-(\max_{i=1,\ldots,p} \pi_i) \\ &= \min_{i=1,\ldots,p} QU^-(\pi_i) \\ &= \min_{i=1,\ldots,p} \{\min(u(x_1), \max(n(\pi(x_i)), u(x_i)))\} \\ &=^{4} \min_{i=1,\ldots,p} \max(n(\pi(x_i)), u(x_i)). \end{aligned}$$

This ends the proof of the theorem. $\square$

[4]Note that $\pi(x_1) = 1$, so $u(x_1) = \max(u(x_1), n(\pi(x_1)))$.

4.5 Representation of Optimistic Qualitative/Ordinal Utilities

An ordinal preference function $u{:}X \to U$ can be regarded as describing a preference profile: *the greater* $u(x)$ *is, the more preferred* x *is*, analogously a possibility distribution π on consequences specifies the degree of plausibility of each consequence, i.e. *the greater* $\pi(x)$ *is, the more plausible* x *is*. So, a pessimistic or conservative criterion is to look for distributions which make, at least to some extent, all the bad consequences hardly plausible.

Sometimes this criterion may be too conservative, we may be interested in an optimistic behaviour, like requiring π *to make at least one of the good consequences highly plausible*, at least to some extent. This behaviour is reflected assessing a degree of intersection between the fuzzy sets of possible consequences and the preferred ones (this point will be developed in more detail in Section 5.1). This leads to consider the utility function which is "dual" to QU^-

$$QU^+(\pi|u) = \max_{x \in X} \min(h(\pi(x)), u(x)), \tag{4.4}$$

h being as usual an onto order-preserving mapping between V and U.

Note that $QU^+(\pi|u)$ is the degree of possibility of u with respect to $h \circ \pi$, and when π is an all or nothing distribution, this criterion coincides with the already known maximax criterion proposed by Yager (1979).

Regarding the axiomatic setting, in this new context, we have to change the *uncertainty aversion axiom* $A2$ by a *uncertainty-prone* postulate

- $A2^+$: if $\pi \leq \pi'$ then $\pi \sqsubseteq \pi'$,

and to adequately modify the continuity axiom $A4$ into

- $A4^+$: for all $\pi \in \Pi(X)$, there exists $\lambda \in V$ such that $\pi \sim (\lambda/\overline{x}, 1/\underline{x})$, where $\overline{x}$ and $\underline{x}$ are a maximal and a minimal element of $(X, \sqsubseteq)$.

As in the pessimistic case, we have the following results, whose proofs are analogous to the previous given ones, so they are omitted here.

Lemma 4.9
In the context of the axioms A1, A2$^+$ and A3, the axiom

- *OA4$^+$ (continuity):* $\pi' \sqsubseteq \pi \ \Rightarrow\ \exists\, \lambda \in V$ *such that* $\pi \sim (1/\pi', \lambda/X)$

is equivalent to A4$^+$.

Lemma 4.10
If $\sqsubseteq$ *verifies axioms A1, A2$^+$, A3, and A4$^+$, then* $\sqsubseteq$ *also verifies DP2 axiom*[5]*, that is:*

If A *is a crisp subset of* X *then there is* $x \in A$ *such that* $x \sim A$.

[5] But, now this axiom expresses that A is equivalent to its best consequence.

Lemma 4.11
If $\sqsubseteq$ verifies axioms A1, A2$^+$, A3, and A4$^+$, and $\underline{x}$ and $\overline{x}$ are a minimal and a maximal element of X, respectively, then:

- *the following equivalences holds: $\overline{x} \sim (1/\overline{x}, 1/\underline{x}) \sim X$.*
- *$\underline{x}$ and $\overline{x}$ are the minimal and maximal elements of $(\Pi(X), \sqsubseteq)$ respectively.*

Observe that X is now a maximal element of $(\Pi(X), \sqsubseteq)$, this is a consequence of the optimistic behaviour underlying in $A2^+$. It is also easy to verify that QU^+ preserves mixtures, that is

$$QU^+(\lambda/\pi_1, \mu/\pi_2) = \max\{\min(h(\lambda), QU^+(\pi_1)), \min(h(\mu), QU^+(\pi_2))\}.$$

Now, we verify that the set of axioms $A1, A2^+,\ A3$ and $A4^+$ faithfully characterise the preference orderings induced by an optimistic qualitative utility.

Theorem 4.12 (Representation for Optimistic Utility)
A preference relation $(\Pi(X), \sqsubseteq)$ satisfies axioms A1, A2$^+$, A3 and A4$^+$, if and only if there exist

(i) a finite linearly ordered utility scale U, with $\inf(U) = 0, \sup(U) = 1,$

(ii) a preference function $u{:}X \to U$ such that $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$, and

(iii) an onto order-preserving function $h{:}V \to U$,

in such a way that it holds:

$$\pi' \sqsubseteq \pi \quad \text{iff} \quad \pi' \preccurlyeq_{QU^+} \pi,$$

where $\preccurlyeq_{QU^+}$ is the ordering on $\Pi(X)$ induced by the qualitative utility $QU^+(\pi) = \max_{x\in X} \min(h(\pi(x)), u(x))$.

Proof:
The proof is analogous to the one for pessimistic utility, so we only sketch the proof for the "only if" part.

- For the same reasons as before we choose $U = \Pi(X)/\sim$. Again, if $\underline{x}$ and $\overline{x}$ denote a minimal and a maximal element of $(X, \sqsubseteq)$ respectively, $[\overline{x}]$ and $[\underline{x}]$ will be the 1 and 0 of U.
- We define $h{:}V \to U$ as $h(\lambda) = [(\lambda/\overline{x}, 1/\underline{x})]$. Observe that $h(1) = [(1/\overline{x}, 1/\underline{x})] = [\overline{x}] = 1$, and $h(0) = [(0/\overline{x}, 1/\underline{x})] = [\underline{x}] = 0$. Moreover, due to the uncertainty-prone axiom it is easy to check that h is order-preserving. By $A4^+$, h is onto.

From that, we only sketch the main steps of the proof:

- Define $QU^+(\lambda/\overline{x}, 1/\underline{x}) = h(\lambda)$.
- Let $\pi^+_\lambda = (\lambda/\overline{x}, 1/\underline{x})$. Verify that if $\pi^+_\lambda \sqsubseteq \pi^+_{\lambda'}$ then $QU^+(\pi^+_\lambda) \leq QU^+(\pi^+_{\lambda'})$.

- Extend QU^+ for any π, due to axiom A4$^+$.
- Define $u(x) = QU^+(x)$.
- Verify that $QU^+(1/x, \lambda/y) = \max(u(x), \min(h(\lambda), u(y)))$.
- Verify that $QU^+(\max(\pi_1, \pi_2)) = \max(QU^+(\pi_1), QU^+(\pi_2))$.
- Verify that $QU^+(\pi) = \max_{x \in X}\ \min(h(\pi(x)), u(x))$.
- Verify that $\preccurlyeq_{QU^+}$ agrees with $\sqsubseteq$.

$\square$

In practice, QU^+ is a very optimistic index which can be used for refining the ordering given by QU^-. We will analyse the characterisation of this refinement in Chapter 9.

Finally, we would like to stress that the qualitative utility functions QU^- and QU^+ are indeed "utility" functions in $\Pi(X)$ in the sense that they preserve the preference ordering and the "natural operation" of possibilistic mixture M used to combine possibilistic lotteries or distributions. Indeed, let

$$\phi_{\max} = \{(\alpha, \beta) \in V \times V \mid\ \max(\alpha, \beta) = 1\}.$$

If we consider the possibilistic mixture operation M as the mapping $M{:}\Pi(X) \times \Pi(X) \times \phi_{\max} \to \Pi(X)$ defined as in (3.1), i.e.

$$M(\pi, \pi'; \alpha, \beta)(x) = \max(\min(\lambda, \pi_1(x)), \min(\mu, \pi_2(x))),$$

then by (4.3), we have that

$$QU^-(M(\pi, \pi'; \alpha, \beta)) = UM^-(QU^-(\pi), QU^-(\pi'); \alpha, \beta),$$

where UM^- is the corresponding mixture in the preference scale U, $UM^-{:}U \times U \times \phi_{\max} \to U$, defined by

$$UM^-(\mu, \mu'; \gamma, \delta) = \min(\max(n(\gamma), \mu), \max(n(\delta), \mu')).$$

That is to say, QU^- is a morphism between the structure of possibilistic lotteries and the structure of the qualitative preference scale.

For the optimistic qualitative utility we have analogous results: QU^+ preserves the order and the mixture operation with respect to the operation $UM^+{:}U \times U \times \phi_{\max} \to U$, defined as

$$UM^+(\mu, \mu'; \gamma, \delta) = \max(\min(h(\gamma), \mu), \min(h(\delta), \mu')),$$

in the sense that it holds

$$QU^+(M(\pi, \pi'; \alpha, \beta)) = UM^+(QU^+(\pi), QU^+(\pi'); \alpha, \beta).$$

Remark 2
Note that (4.3) is the median of three terms including $QU^-(\pi_1), QU^-(\pi_2)$*. Indeed,*

- *if* $QU^-(\pi_1) \leq_U QU^-(\pi_2)$*, then*

$$QU^-(\lambda/\pi_1, \mu/\pi_2) = median\{QU^-(\pi_1), QU^-(\pi_2), n(\lambda)\}$$

It behaves like the classical EUT*, changing median by weighted mean. Analogously we have that*

- *if* $QU^+(\pi_1) >_U QU^+(\pi_2)$*, we have that*

$$QU^+(\lambda/\pi_1, \mu/\pi_2) = median\{QU^+(\pi_1), QU^+(\pi_2), h(\lambda)\}$$

4.6 An Example: A Possibilistic View of Savage's Omelette

Finally, let us show the behaviour of QU^- and QU^+ in a little toy example. We take the well-known Savage's omelette example (Savage, 1972) pp. 13–14, already used in (Dubois et al., 1998c) to exemplify the QU^- utility criterion. Here, we develop it further, but first we recall the problem.
The goal of the *DM* is to make a six-egg omelette, already having five eggs in a bowl, so *DM* has to decide what to do with a new egg, that can be either *fresh (F)* or *rotten (R)*. The *DM* can decide on three possible alternatives:

- *to break the egg in the omelette (BIO),*
- *to break it apart in a cup (BAC),*
- *to throw it away (TA).*

The consequences of the alternatives, depending on the state of the egg, are given in Table 4.1. The grades between catch indicate an (reasonable) encoding of the

ACTS /STATES	*fresh egg (F)*	*rotten egg (R)*
break egg in the omelette (BIO)	*a 6 egg omelette (6eO for short)* [1]	*nothing to eat (NE)* [0]
break it apart in a cup (BAC)	*a 6 egg omelette , a cup to wash (6eO-C)* [d]	*a 5 egg omelette, a cup to wash (5eO-C)*[b]
throw it away (TA)	*a 5 egg omelette,one wasted egg (5eO-1se)* [a]	*a 5 egg omelette (5eO)* [c]

Table 4.1: States, acts and consequences in Savage's omelette example.

preferences of consequences, belonging to a totally ordered scale $U = \{0 < a < b < c < d < 1\}$.

Notice that since only two states are present (*Fresh* and *Rotten*), we deal with binary acts. We also assume that plausibility degrees of each state will be measured on the same scale, i.e. we take $V = U$, and thus we also take the commensurateness mapping as $h = identity$, hence $n = n_U$. Assume a possibility distribution on states π:$\{F, R\} \to V$ is given.

Then, every decision $d \in \{BIO, BAC, TA\}$ induces the corresponding possibility distribution π_d:$X \to U$, on the set of consequences

$$X = \{6eO, 6eO - C, 5eO, 5eO - C, 5eO - 1se, NE\},$$

defined as $\pi_d(x) = \max\{\pi(s)|d(s) = x\}$, assuming $\max\ \emptyset = 0$.

In a vectorial notation, the distributions are as follows:

$$\begin{array}{lcl} \pi_{BIO}(6eO, 6eO - C, 5eO, 5eO - C, 5eO - 1se, NE) & = & (\pi(F), 0, 0, 0, 0, \pi(R)), \\ \pi_{BAC}(6eO, 6eO - C, 5eO, 5eO - C, 5eO - 1se, NE) & = & (0, \pi(F), 0, \pi(R), 0, 0), \\ \pi_{TA}(6eO, 6eO - C, 5eO, 5eO - C, 5eO - 1se, NE) & = & (0, 0, \pi(R), 0, \pi(F), 0), \end{array}$$

In the following we successively consider the different criteria. It is easy to check that under the above hypotheses, and assuming that the distribution is normalised (i.e. $\max(\pi(F), \pi(R)) = 1$), we get the following values for the pessimistic utility QU^-:

$$\begin{array}{lcl} QU^-(\pi_{BIO}) & = & N(F), \\ QU^-(\pi_{BAC}) & = & \min(\max(N(R), d), \max(N(F), b)), \\ QU^-(\pi_{TA}) & = & \min(\max(N(F), c), \max(N(R), a)), \end{array}$$

where $N(F) = 1-\pi(R), N(R) = 1-\pi(F)$ are the necessity values of each state, with $\min(N(F), N(R)) = 0$. Table 4.2 exhibits the best acts according to the pessimistic criterion and depending on the *DM*'s belief about the state of the egg.

$N(F)$	$N(R)$	$QU^-(\pi_{BIO})$	$QU^-(\pi_{BAC})$	$QU^-(\pi_{TA})$	*Best Acts*
1	0	1	d	a	BIO
d, c, b	0	$N(F)$	$N(F)$	a	BIO or BAC
a	0	a	b	a	BAC
0	$0, a$	0	b	a	BAC
0	b	0	b	b	BAC or TA
0	$c, d, 1$	0	b	c	TA

Table 4.2: Pessimistic Qualitative utilities.

One can see that the model recommends decision *BAC* in case of relative ignorance on the egg state, that is when $\max(N(F), N(R))$ is not high enough (less than b), and it advices to act cautiously, breaking the egg in a spare cup, in case of serious doubt. Now, let us consider the optimistic criterion modelled by QU^+. The values are as follows:

$$
\begin{array}{lcl}
QU^+(\pi_{BIO}) & = & \pi(F), \\
QU^+(\pi_{BAC}) & = & \max(\min(\pi(F), d), \min(\pi(R), b)), \\
QU^+(\pi_{TA}) & = & \max(\min(\pi(R), c), \min(\pi(F), a)),
\end{array}
$$

and the best decisions can be found in Table 4.3. As we could expect, this criterion

$N(F)$	$N(R)$	$QU^+(\pi_{BIO})$	$QU^+(\pi_{BAC})$	$QU^+(\pi_{TA})$	*Best Acts*
1	0	$\pi(F)$	d	a	BIO
d, c, b	0	$\pi(F)$	d	$\pi(R)$	BIO
$a, 0$	0	$\pi(F)$	d	c	BIO
0	$1, d$	$\pi(F)$	b	c	TA
0	c	$\pi(F)$	$\pi(F)$	c	TA
0	b	$\pi(F)$	$\pi(F)$	c	TA, BAC, BIO
0	a	$\pi(F)$	$\pi(F)$	c	BAC or BIO

Table 4.3: Optimistic Qualitative utilities.

suggests breaking the egg into the omelette as soon as there is no positive evidence about the egg being rotten, even this is very small. Notice that QU^+ scores each alternative higher than QU^-.

Chapter 5

Generalised Ordinal Utility Functions Based on T-Norms

As it has been mentioned initially in Section 3.1 and in Section 4.5 as well, for modelling a pessimistic behaviour we have been looking for decisions that always gave good results in *all* possible consequences, while for an optimistic one our goal was to find decisions that *at least in one* possible consequences gave good results. Indeed, for example when the distribution is crisp, i.e. for all x, $\pi_d(x) \in \{0, 1\}$, we have that

$$QU^-(\pi_d) = \min_{x \in \pi_d} u(x),$$

that is, π_d is evaluated in terms of the *worst* consequence compatible with π_d, while

$$QU^+(\pi_d) = \max_{x \in \pi_d} u(x),$$

i.e. π_d is evaluated in terms of the *best* possible consequence.
With this objective, the estimation of the pessimistic (optimistic) utility of a decision d was measured in terms of the degree of inclusion (or intersection resp.) of the *fuzzy set of possible consequences for a decision* d, that is, the fuzzy set π_d, into the *fuzzy set of good results* u. In particular, we have that

(i) $supp\ \pi_d \subseteq core\ u$[1] $\Rightarrow QU^-(\pi_d) = 1,$

(ii) $core\ \pi_d \bigcap (supp\ u)^c \neq \emptyset$[2] $\Rightarrow QU^-(\pi_d) = 0,$

(iii) $core\ \pi_d \bigcap core\ u \neq \emptyset \Rightarrow QU^+(\pi_d) = 1,$

(iv) $supp\ \pi_d \subseteq (supp\ u)^c \Rightarrow QU^+(\pi_d) = 0.$

(i) says that if *all* possible consequences of d is a good one, the pessimistic criterion consider d as a “best” decision. (ii) if *there exists* a totally possible consequence of d

[1] If A is fuzzy set on X, $supp\ A = \{x \in X | A(x) > 0\}$,$core\ A = \{x \in X | A(x) = 1\}$.
[2] Recall A^c means the complementary of A.

that is considered bad, the pessimistic criterion consider d as a bad decision. While (iii) says that if *there exists* a totally possible consequence of d which is a good one, the optimistic criterion considers d as a good decision. (iv) if *all* possible consequence of d is considered a bad consequence, the optimistic criterion consider d as a bad decision. Observe that if we have that

$$\text{if } \lambda > 0 \text{ then } n(\lambda) < 1,$$

e.g. if n is injective, then the reciprocals of the first and fourth affirmations are valid. Moreover, if we have that

$$\text{if } \lambda < 1 \text{ then } n(\lambda) > 0,$$

then the reciprocals of the others are true as well.

From alternative definitions of degrees of inclusion and intersection, other utilities are introduced in Section 5.1. These utility functions are based on (finite) conjunctive and implication connectives. In particular, considering a *S-implication*-like defined in terms of t-norms on the uncertainty scale and the reversing mapping linking V and U, we obtain generalised pessimistic qualitative utility functions GQU. While regarding that conjunction is defined in terms of a t-norm on V, generalised optimistic functions are obtained. In the particular case of considering the *t-norm minimum*, QU^- and QU^+ are recovered. But, this is not always the case. Indeed, if other t-norms are chosen, the rankings induced by QU and GQU may be different, as it is shown in the example of Section 5.2. The orderings induced by these generalised qualitative utility are axiomatically characterised in Section 5.3.

5.1 Qualitative Utilities Expressed in Terms of Inclusion and Intersection Degrees

In this Section, we analyse some utility functions that may be defined taking into account that they measure a degree of intersection or inclusion of fuzzy sets. First, we consider the intersection case. We recall usual definitions on $[0, 1]$, and then we extend them to the case of involving two different finite scales V and U. Secondly, we consider two alternative definitions for inclusion degree: a *cardinality-based* or a *"logical"-based* one. Namely, for evaluating the inclusion degree of "$A \subseteq B$":

- one can evaluate the proportion between the fuzzy cardinalities of $A \cap B$ and of A, or
- one can evaluate the truth of the sentence *"all elements of A are elements of B"*, that is, the truth value of

$$(\forall x)(x \in A \Rightarrow x \in B).$$

The problem with the first one is that it may not be applied in problems in which the available information is mainly ordinal. Therefore, we consider different alternatives for applying the "logical" definition involving (mainly) ordinal scales, with this goal we shall analyse different implications operations.

5.1.1 Optimistic Behaviour

Let us first recall two definitions.

Definition 4

- *A* fuzzy conjunction[3] $\wedge$ *is a binary operation* $\wedge$:$[0,1] \times [0,1] \to [0,1]$, $\wedge$ *being commutative, associative, non-decreasing in both variables, also satisfying*

$$(1 \wedge x) = x \qquad \forall x \in [0,1].$$

  $\wedge$ *is also said a* triangular norm *(*t-norm *for short), and we shall also denote it by* $\top$.

- *Given* A *and* B, *two fuzzy sets in* X, *the* degree of intersection of A and B *may be defined as*

$$[A \cap B] = \max_{x \in X}(A(x) \wedge B(x)) \tag{5.1}$$

  with $\wedge$ *a conjunction on [0,1].*

From this definition we may see that if $V = U$ is a subset of $[0,1]$, and choosing $\wedge = minimum$, we have

$$\mathcal{U}^+(d|u) = [\pi_d \cap u] = \max_{x \in X} \min(\pi_d(x), u(x)) = QU^+(\pi_d|u).$$

That is, $QU^+(\pi_d)$ measures the *degree of intersection* between the *set of possibles consequences* and the *set of preferred ones*, as it has been mentioned.

However, the problems in which we are interested in involve two any commensurate finite scales, thus, we are interested in intersection of fuzzy sets whose membership functions may be valued over different scales. Indeed, π_d is V-valuated while u is U-valuated, usually V and U being different.

As a first step, taking into account that in the conjunction definition we may consider that we are only applying ordinal aspects of values on $[0,1]$, we may regard their natural extension to a fuzzy operation from $V \times V$ into V, with V a finite linearly ordered scale. From now on, assuming that we have fuzzy sets defined over V and U, with V and U two finite linearly ordered scales that are commensurate, i.e. there exists an onto order-preserving function h:$V \to U$, we may think of both values of preference and uncertainty as being in the "same" scale (the uncertainty one), although this is not strictly true. So, we may define the conjunction on $V \times U$, in terms of a fuzzy conjunction on V, i.e.

$$(v \wedge u) = h(v \ \wedge_V \ \lambda_u) \tag{5.2}$$

with $\wedge_V$ a conjunction on V and $h(\lambda_u) = u$.
For the sake of a sound definition h is also required to satisfy a *coherence condition w.r.t.* $\top_V$, i.e. h verifies

$$h(\lambda) = h(\mu) \Rightarrow h(\alpha \top_V \lambda) = h(\alpha \top_V \mu) \qquad \forall \alpha, \lambda, \mu \in V.$$

[3]We restrict ourselves to commutative and associative conjunctions.

Notice that, for instance, when h is injective or when $\top_V = \min$, this condition of coherence is satisfied. In particular, when only ordinal information is available and we take $\wedge_V = \min$, we again have

$$\mathcal{U}^+(d|u) = [\pi_d \cap u] = QU^+(\pi_d|u).$$

In the general case, given a conjunction $\wedge_V$ on V, we consider the conjunction induced in $V \times U$ by $\top_V$, so the optimistic generalised utility function take this form,

$$\mathcal{GU}^+(d|u) = \max_{x \in X} h(\pi_d(x) \top_V \lambda_x) \tag{5.3}$$

with $h(\lambda_x) = u(x)$. Obviously, h is involved in $\mathcal{GU}^+(d)$, but we omit h in its notation for simplicity reason. Note that when $\top_V = \min$, then $\mathcal{GU}^+ = \mathcal{U}^+$.

The preference orderings induced by these optimistic generalised utility functions are axiomatised in Section 5.3.

5.1.2 Pessimistic Behaviour

Now, we focus in modelling the degree of inclusion to be applied to evaluate the pessimistic criterion. As it was mentioned we may consider two alternatives, if we are speaking about of two fuzzy sets defined on X over $[0, 1]$, cardinality-based and logical-based definitions. Let us first recall some definitions.

Definition 5

- *Given a fuzzy set* $A{:}X \to [0, 1]$*, its* cardinality *is defined as*

$$|A| = \sum_{x \in X} A(x).$$

- *A* fuzzy implication[4] *is a function* $I{:}[0, 1] \times [0, 1] \to [0, 1]$ *such that* I *is non-increasing with respect to the first argument, while it is non-decreasing with respect to the second one. It also satisfies the following boundary conditions:*

$$I(1, 0) = 0, \;\; I(0, x) = 1 \text{ and } I(x, 1) = 1 \;\forall\, x \in [0, 1].$$

- *A* negation *(Trillas, 1979) is a non-increasing function* $n{:}[0, 1] \to [0, 1]$ *satisfying* $n(0) = 1$*,* $n(1) = 0$*, and* $n(n(a)) \geq a \;\forall\, a \in [0, 1]$*. A* negation *is* strong *if it satisfies that* $n(n(a)) = a$*.*

Hence, the alternatives definitions for an inclusion degree we are led to are:

1. From the "cardinality" point of view:

$$|A \subseteq B|_{card} = \frac{|A \cap B|}{|A|} = \frac{\sum_z (A \cap B)(z)}{\sum_z A(z)} = \frac{\sum_z A(z) \top B(z)}{\sum_z A(z)} \tag{5.4}$$

[4] In (Bouchon-Meunier et al., 1999; Chapter 1), a fuzzy implication is also required to satisfy an exchange condition: $I(x, I(y, z)) = I(y, I(x, z))$.

$\top$ being a t-norm [5].

2. Within the tradition of many valued logic, the evaluation of the degree of truth of the expression $(\forall x)(x \in A \Rightarrow x \in B)$ is defined as

$$|A \subseteq B| = [(\forall x)(x \in A \Rightarrow x \in B)] = \inf_{x \in [0,1]} I(A(x), B(x)),$$

with I a fuzzy implication on [0,1].

In our case if we assume $V = U$, we have that

$$\begin{aligned} \mathcal{U}^-(d|u) &= [\pi_d \subseteq u] \\ &= [(\forall x)(x \in \pi_d \Rightarrow x \in u)] \\ &= \min_{x \in X} I(\pi_d(x), u(x)). \end{aligned}$$

Obviously, the cardinality-based definition require to deal with numerical values, and sometimes we may require more ordinal expressions for the cases of having (mainly) ordinal information available, hence we will focus in the second alternative. But, we have to take into account that we are interested in the degree of inclusion of two fuzzy sets with different valuated sets. So, the first step is to extend this definition. As before, the extension to $U \times U$ of the definition of fuzzy implication is the obvious one, while for speaking about implications on $V \times U$ we propose to consider the "implication" I:$V \times U \to U$,

$$I(v, u) = I_U(h(v), u), \tag{5.5}$$

I_U being an implication on $U \times U$ in the sense of Definition 5.

[5]We would like to remark that if we consider $\top = Product$, then Gilboa and Schmeidler's utility (defined in (2.5)) may be seen as a degree of inclusion too. Indeed, for each decision d, and given the similarity function on situations, Sim, let

$$Sim^d\text{:}\{s|\ (s, d, x) \in M\} \to [0, 1]$$

be the fuzzy set of situations which are similar to s_0 and where decision d was experienced, with

$$Sim^d(s) = Sim(s, s_0).$$

In a similar way, we consider the fuzzy set of preferred situations, that is,

$$G^d\text{:}\{s|\ (s, d, x) \in M\} \to [0, 1],$$

with

$$G^d(s) = u(x).$$

Then, Gilboa and Schmeidler's utility is

$$U_{s_0,M}(d) = \frac{\sum_{(s,d,x)\in M} Sim(s_0, s) \cdot u(x)}{\sum_{(s,d,x)\in M} Sim(s_0, s)} = |Sim^d \subseteq G^d|_{card}.$$

Hence, when we are considering A, B fuzzy sets on X over V and U respectively, we have that

$$[A \subseteq B] = \min_{x \in X} I(A(x), B(x)) = \min_{x \in X} I_U(h(A(x)), B(x)).$$

If we choose $I(v, u) = \max(n_U(h(v)), u)$, n_U being the involution in U, we again obtain that

$$\mathcal{U}^-(d|u) = [\pi_d \subseteq u] = QU^-(\pi_d|u).$$

Below, we propose another model for the fuzzy implication involved in the "logical" definition of degree inclusion taking into account that we may consider available in V and U not only maximum and minimum but also other operators, obtaining therefore their respective utility functions.

By analogy to the usual fuzzy implication on [0,1], some particular fuzzy implications on $V \times U$ may be introduced using t-norms and t-conorms, the three more important groups are:

- *S-Implication*: Given a conorm S on U and the strong negation n_U on U, the*S-implication* associated to them is defined as

$$I_{S,n_U}(v, u) = S(n_U(h(v)), u).$$

- the *residuated implication* with respect to a t-norm $\top_U$ on U is defined as

$$I_{R(\top_U)}(v, u) = \sup\{z \in U |\ h(v) \top_U z \leq u\}.$$

  That is,

$$I_{R(\top_U)}(v, u) = I^U_{R(\top_U)}(h(v), u),$$

  with $I^U_{R(\top_U)}$ the residuated implication on U defined as

$$I^U_{R(\top_U)}(w, u) = \sup\{z \in U |\ w \top_U z \leq u\}.$$

- the *reciprocal implication* with respect to a negation neg_U on U, defined as

$$I_{RR(\top_U)}(v, u) = I^U_{R(\top_U)}(neg_U(u), neg_U(h(v))).$$

We may also consider the following alternative definition:

- the *S-implication-like* defined as

$$I^n_S(v, u) = n(v \top_V z) \tag{5.6}$$

  with $n(z) = u$, $\top_V$ a t-norm on V and n:$V \rightarrow U$ an onto order reversing function.

To guarantee the correctness of the above definition of implication we require n to satisfy the coherence condition with respect to $\top_V$, i.e.

$$n(\lambda) = n(\mu) \Rightarrow n(\alpha \top_V \lambda) = n(\alpha \top_V \mu) \qquad \forall \alpha, \lambda, \mu \in V.$$

Observe that this implication may be seen as a generalisation of an *S-implication*, since when n is injective, then

$$I_S^n(v,u) = n(v \top_V z) = n(v) \bot_{n,\top_V} u,$$

with $\bot_{n,\top_V}$ being the conorm in U defined as

$$(x \bot_{n,\top_V} y) = n(n^{-1}(x) \top_V n^{-1}(y)).$$

That is, $I_S^n(v,u)$ is an *S-implication* w.r.t. the conorm $\bot_{n,\top_V}$.

Next, we analyse the utility functions that emerge from these implications. As the last implication defined include *S-implication*, we restrict the analysis to the residuated, the reciprocal ones and the *S-implication-like*.

1. Consider $I_S^n(v,u)$. As we are interested in a utility function that selects acts such that *all* the possible consequences of the decision are good results, we are looking for

$$\begin{aligned}
\mathcal{GU}^-(d|u) &= [\pi_d \subseteq u] \\
&= \min_{x \in X}(\pi_d(x) \Rightarrow u(x)) \\
&= \min_{x \in X} I_S^n(\pi_d(x), u(x)) \\
&= \min_{x \in X} n(\pi_d(x) \top_V \lambda_x)
\end{aligned}$$

   with $n(\lambda_x) = u(x)$.

   Comparing these utility functions with the pure ordinal ones, we have that, for any decision d,

$$\mathcal{U}^+(d|u) \geq \mathcal{GU}^+(d|u) \geq \mathcal{GU}^-(d|u) \geq \mathcal{U}^-(d|u).$$

   Moreover, if $\mathcal{GU}^+$ and $\mathcal{GU}^-$ are considered in terms of the t-norm $\top_V$ involved, $\mathcal{GU}^-$ is non-increasing with respect to $\top_V$, while $\mathcal{GU}^+$ is non-decreasing. That is, if $\top \leq \top_1$ are t-norms in V, then $\mathcal{GU}^-_\top \geq \mathcal{GU}^-_{\top_1}$ and $\mathcal{GU}^+_\top \leq \mathcal{GU}^+_{\top_1}$.

   Obviously $\mathcal{GU}^-$ coincides with $\mathcal{U}^-$ if the involved t-norm is the minimum. However, the $\mathcal{GU}$ and $\mathcal{U}$ orderings may be different when $\top_V \neq \min$, as it may be verified in the example of the following section (Table 5.3).

2. Consider now the residuated implication

$$I_{R(\top_U)}(v,u) = \sup\{z \in U |\ h(v) \top_U z \leq u\},$$

   and its respective utility

$$\begin{aligned}
U_{I_{R(\top_U)}}(d|u) &= \min_{x \in X} I_{R(\top_U)}(\pi_d(x), u(x)) \\
&= \min_{x \in X} \sup\{z \in U |\ h(\pi_d(x)) \top_U z \leq u(x)\}
\end{aligned}$$

- If $\top_U$ does not have non-trivial zero divisors[6] and $(supp\ (h \circ \pi_d) \cap\ (supp\ u)^c) \neq \emptyset$, then $U_{I_{R(\top_U)}}(d) = 0$.
- $\mathcal{U}^-$ and $U_{I_{R(\top_U)}}$ may induce different rankings. Indeed, for instance:
  - Let $\overline{x}, \underline{x} \in X$ s.t. $u(\overline{x}) = 1$ and $u(\underline{x}) = 0$, let $\lambda,\ \mu \in V,\ \lambda \neq 0 \neq \mu$ and $h(\lambda) \neq h(\mu)$, and consider d and d' s.t.
    $$\pi_d = (1/\overline{x}, \lambda/\underline{x}) \text{ and } \pi_{d'} = (1/\overline{x}, \mu/\underline{x}).$$
    Consider that $\top_U$ does not have non-trivial zero divisors, then $U_{I_{R(\top_U)}}(d) = 0 = U_{I_{R(\top_U)}}(d')$. So, $U_{I_{R(\top_U)}}$ may not distinguish between them, while $\mathcal{U}^-$, may distinguish both because of $\mathcal{U}^-(d) = n(\lambda), \mathcal{U}^-(d) = n(\mu)$.

  - Moreover, although it may be that for all decisions d satisfying
    $$\exists \lambda \in V \text{ s.t. } \pi_d = (1/\overline{x}, \lambda/\underline{x}),$$
    both utilities coincide on their evaluations of these decisions, i.e. $U_{I_{R(\top_U)}}(d) = \mathcal{U}^-(d)$ (for example, it happens when $\top_U$ is *Lukasiewicz t-norm*) however, $\mathcal{U}^-$ is not a refinement of $U_{I_{R(\top_U)}}$.
    Indeed, given y such that $\underline{x} \sqsubset y \sqsubset \overline{x}$, and $\mu \in V$ s.t. $0 < h(\mu) < u(y)$, let d be s.t. $\pi_d = (1/\overline{x}, \mu/y)$. So, we have that $\mathcal{U}^-(d) = \max(n(\mu), u(y)) < 1$, that is,
    $$\pi_d \sqsubset_{QU^-} \overline{x}.$$
    However, $U_{I_{R(\top_U)}}(d) = I_{R(\top_U)}(h(\mu), u(y)) = 1$, that is, π_d and $\overline{x}$ are equivalents for the ordering induced by $U_{I_{R(\top_U)}}$.

3. Given a t-norm $\top_U$ and a negation on U neg_U we consider

$$I_{RR(\top_U)}(v, u) = I^U_{R(\top_U)}(neg_U(u), neg_U(h(v))).$$

Then, the respective utility function is

$$\begin{aligned} \mathcal{U}^{I_{RR(\top_U)}}(d|u) &= \min_{x \in X} I_{RR(\top_U)}(\pi_d(x), u(x)) \\ &= \min_{x \in X} I^U_{R(\top_U)}(neg_U(u(x)), neg_U(\pi_d(x))). \end{aligned}$$

We notice that $\mathcal{U}^{I_{RR(\top_U)}}$ may give results that are considered unsatisfactory in many contexts. For instance, here, the utility value of a decision which is identified with a consequence may be different from the preference value that DM assigns to this consequence. Indeed, let d be s.t. $\pi_d = \{x_0\}$, then

$$\mathcal{U}^{I_{RR(\top_U)}}(d|u) = n_{\top_U}(neg_U(u(x_0))),$$

[6] A t-norm $\top$ in [0,1] has non-trivial zero divisors iff $\exists x, y \in (0, 1]$ s.t. $x \top y = 0$.

where $n_{\top_U}$ is the negation associated to the residuated implication $I^U_{R(\top_U)}$, i.e. $n_{\top_U}(w) = I^U_{R(\top_U)}(w, 0)$. Therefore, if $\top_U$ does not have non-trivial zero divisors, then

$$\mathcal{U}^{I_{RR(\top_U)}}(d|u) = \mathcal{U}^{I_{RR(\top_U)}}(x_0) = \begin{cases} 1, & \text{if } neg_U(u(x_0)) = 0 \\ \\ 0, & \text{otherwise.} \end{cases}$$

That is, $\mathcal{U}^{I_{RR(\top_U)}}(d|u)$ will be different from $u(x_0)$ for almost all possible $u(x_0)$.

- If neg_U is bijective (i.e. $neg_U = n_U$), then $\mathcal{U}^{I_{RR(\top_L)}} = \mathcal{U}^{I_{R(\top_L)}}$. Indeed, if $\top$ is *Lukasiewicz t-norm*, neg_U is bijective, as $I_{RR(\top_U)} = I_{R(\top_U)}$, then $\mathcal{U}^{I_{RR(\top_U)}} = \mathcal{U}^{I_{R(\top_U)}}$.
- If neg_U is not bijective, it may be possible that $\mathcal{U}^{I_{RR(\top_L)}} \neq \mathcal{U}^{I_{R(\top_L)}}$. Indeed, we consider $U = \{0 < u_1 < u_2 < 1\}$, $neg_U(u_1) = 1$, $neg_U(u_2) = u_1$. Let us assume $V = U$, so h is the identity. Let y be such that $u(y) = u_1$, let d be s.t. $\pi_d = (1/\overline{x}, u_2/y)$. Then,

$$\mathcal{U}^{I_{R(\top_L)}}(d) = I_{R(\top_L)}(u_2, u_1) = u_2,$$

while

$$\begin{aligned} \mathcal{U}^{I_{RR(\top_L)}}(d) &= Min\{I_{R(\top_L)}(0,0), I_{R(\top_L)}(neg_U(u_1), neg_U(u_2))\} \\ &= u_1. \end{aligned}$$

Remark 3
As it is mentioned, if neg_U is bijective then $I_{RR(\top_U)} = I_{R(\top_U)}$. Moreover, if we consider now $(v \Rightarrow u) = I^{neg_U}_{S_L}(v, u) = S_L(neg_U(h(v)), u)$ and $\mathcal{U}_{S_\mathcal{L}}$ its respective utility, as we have that $I^{neg_U}_{S_L}(v, u) = I_{R(\top_L)}(v, u)$, that is, the S-implication *based on Lukasiewicz is equal to the respective residuated and reciprocal one, hence the utility functions defined from them are the same.*

Moreover, if we assume that $V = U$, therefore n is bijective, n satisfies coherence and we may consider the generalised utility function $\mathcal{GU}_L$ associated to the Lukasiewicz's t-norm. In this case, we have that $\mathcal{GU}_L$ coincides with the utility functions induced by the $S_L - implication$, $I_{R(\top_L)}$ or the $I_{RR(\top_L)}$.

5.2 An Example: A Safety Decision Problem in a Chemical Plant

To exemplify some of the notions introduced in this Chapter, and that will be continued in other Chapters, we consider the following example.
Chemical plants are potentially dangerous industrial complexes, so they have to foresee emergency plans in case of problems. Assume the chemical plant has three emergency plans:

$$\begin{aligned} EP1 &: \quad emergency\, plan\; 1, \\ EP2 &: \quad emergency\, plan\; 2, \\ EV &: \quad total\; evacuation, \end{aligned}$$

that only may be activated by the head of the Safety Department, depending on his subjective evaluation of the seriousness of possible problems occurring in the plant. Naturally, *total evacuation* means that people would be safe, but the activity in the plant will be interrupted and this means that the plant has loss. The *emergency plan 2* consists of a group of safety measures (like to evacuate a zone of the plant without stopping totally the production) that tries to guarantee the safety of the employees. It has a high cost, but does not stop the production. While *emergency plan 1* means that only local safety measures are taken. Depending on the type of problems occurring in the plant, the situations of the plant may be classified in four modes:

$$\begin{aligned} s_0 &: \quad \textit{normal functioning}, \\ s_1 &: \quad \textit{minor problem}, \\ s_2 &: \quad \textit{major problem}, \\ s_3 &: \quad \textit{very serious problem}. \end{aligned}$$

To survey the functioning of the plant smoke detectors and pressure indicators are distributed throughout different sectors of the plant and connected to alarms to warn about either the existence of fire or broken pipelines. When the alarm system turns on in some sector, plant engineers evaluate the readings of the alarm systems and they forward a report to the head of the Safety Department. He has to undertake one of the following actions:

$$\begin{aligned} d_0 &: \quad \textit{do nothing (DN)}, \\ d_1 &: \quad \textit{activate emergency plan 1}\ (AEP1), \\ d_2 &: \quad \textit{activate emergency plan 2}\ (AEP2), \\ d_3 &: \quad \textit{activate evacuation}\ (AEVA). \end{aligned}$$

Undertaking any of these actions has different consequences depending on which is the actual state of the plant. We describe the consequences from two points of view: *how risky the situation for employees will be after having taken the action* (we will call this situation *post-situation*) and *which is the (economical) cost of the action*. Both issues are measured in a qualitative scale $0 < 1 < 2 < 3$. Their meanings are:

$$\begin{aligned} 0 &: \quad None, \\ 1 &: \quad Small, \\ 2 &: \quad Medium, \\ 3 &: \quad High. \end{aligned}$$

For instance, if decision d_2 is chosen, and it turns out that the actual state was not s_2 but s_1, then there will be no risk after $(Risk = 0)$ but to a higher cost than the required one $(Cost = 2)$. On the other hand, if the actual state were s_3 (a very serious problem) decision d_2 is not enough to completely avoid any risk $(Risk = 1)$ a posteriori. In general, consequences of these actions (the situation after the action has been taken) are given in Table 5.1 where $Risk = i$ stands for risk level i $(i = 0, 1, 2, 3)$ and $Cost = i$

	DN	$AEP1$	$AEP2$	$AEVA$
s_0	$Risk = 0, Cost = 0$	$Risk = 0, Cost = 1$	$Risk = 0, Cost = 2$	$Risk = 0, Cost = 3$
s_1	$Risk = 1, Cost = 0$	$Risk = 0, Cost = 1$	$Risk = 0, Cost = 2$	$Risk = 0, Cost = 3$
s_2	$Risk = 2, Cost = 0$	$Risk = 1, Cost = 1$	$Risk = 0, Cost = 2$	$Risk = 0, Cost = 3$
s_3	$Risk = 3, Cost = 0$	$Risk = 2, Cost = 1$	$Risk = 1, Cost = 2$	$Risk = 0, Cost = 3$

Table 5.1: States, decision and consequences after taking decisions.

for cost level $i(i = 0, 1, 2, 3)$. The *post-situation* is evaluated in terms of two criteria: *personal safety* and *economical expenses*. The final preference evaluation is made assuming that personal safety reasons are considered more important than economical reasons. That is, we rank order the post-situation considering first the level of risk it has and then its cost. Obviously, the smaller the risk is, the most preferred the situation is. For situations with the same level of risk, the smaller the cost, the most preferred the situation is. That is, we consider the following ordering on consequences detailed on Table 5.2, where we take as preference scale $U = \{0 = w_0 < w_1 < ... < w_8 < w_9 = 1\}$.

u	$Cost = 0$	$Cost = 1$	$Cost = 2$	$Cost = 3$
$Risk = 0$	w_9	w_8	w_7	w_6
$Risk = 1$	w_5	w_4	w_3	
$Risk = 2$	w_2	w_1		
$Risk = 3$	w_0			

Table 5.2: Assignment of preference values for each possible consequence.

Qualitative Utility Evaluations: QU^- and QU^+

At a given moment, alarms lights turn on and immediately after the following report arrive to the head of the Department:

> "A problem has been identified in Sector G, most plausibly it is a major problem, but there is still some chance it can actually be a minor problem, or even it might become a very serious problem".

We model the information about the actual state of the chemical plant, provided by the report, with a possibility distributions $\pi_S: S \to V$, where V is a finite uncertainty (plausibility) scale, defined as follows:

$$\pi_S(s_0) = 0,\ \pi_S(s_1) = z_2,\ \pi_S(s_2) = 1,\ \pi_S(s_3) = z_1,$$

with $\{0 < z_1 < z_2 < 1\} \subseteq V$. Thus, π_S is representing that s_2 is a totally plausible state, s_1 and s_3 are somehow plausible and s_0 is not considered plausible at all.

For simplicity reasons we consider that the preference and uncertainty scales are the same, so that $\{z_1, z_2\} \subseteq U$. Then, given the previously mentioned possibility distribution π on the possible states, every decision d_i $(i = 0, 3)$ induces a corresponding possibility lottery (distribution) π_{di}:$X \to U$ on the set of consequences. Here, they are:

$$\begin{aligned}
\pi_{d0} &= (0/(Risk = 0, Cost = 0), z_2/(Risk = 1, Cost = 0), \\
&\quad 1/(Risk = 2, Cost = 0), z_1/(Risk = 3, Cost = 0)); \\
\\
\pi_{d1} &= (z_2/(Risk = 0, Cost = 1), 1/(Risk = 1, Cost = 1), \\
&\quad z_1/(Risk = 2, Cost = 1)); \\
\\
\pi_{d2} &= (1/(Risk = 0, Cost = 2), z_1/(Risk = 1, Cost = 2)); \\
\\
\pi_{d3} &= (1/(Risk = 0, Cost = 3)).
\end{aligned}$$

Now, we evaluate the pessimistic and optimistic criteria under the above hypotheses.

$$\begin{aligned}
QU^-(\pi_{d0}) &= \min[\max(n_V(0), 1), \max(n_V(z_2), w_5), \\
&\qquad \max(n_V(1), w_2), \max(n_V(z_1), 0)] \\
&= \min[\max(n_V(z_2), w_5), w_2, n_V(z_1)] \\
&= \min[w_2, n_V(z_1)]; \\
\\
QU^-(\pi_{d1}) &= \min[w_4, \max(n_V(z_1), w_1)]; \\
\\
QU^-(\pi_{d2}) &= \min[w_7, \max(n_V(z_1), w_3)]; \\
\\
QU^-(\pi_{d3}) &= w_6.
\end{aligned}$$

Independently of the value of z_1, we may see that

$$\pi_{d3} \sqsupset_{QU^-} \pi_{d1} \quad \text{and} \quad \pi_{d3} \sqsupset_{QU^-} \pi_{d0}.$$

That is, d_0 and d_1 are discarded. However, to choose between d_2 and d_3 we have to take into account the value of z_1. Indeed, if $z_1 \leq w_2$, then $\pi_{d2} \sqsupset_{QU^-} \pi_{d3}$, while for z_1 = w_3 we have that $\pi_{d2} \sim_{QU^-} \pi_{d3}$, and for $z_1 > w_3$, the ordering is $\pi_{d3} \sqsupset_{QU^-} \pi_{d2}$.

Analogously, the evaluations for the optimistic criterion are:

$$\begin{aligned}
QU^+(\pi_{d0}) &= \max[\min(z_2, w_5), w_2]; \\
\\
QU^+(\pi_{d1}) &= \max[\min(z_2, w_8), w_4, \min(z_1, w_1)]
\end{aligned}$$

$$
\begin{aligned}
&= \max[\min(z_2, w_8), w_4]; \\
QU^+(\pi_{d2}) &= \max[w_7, \min(z_1, w_3)] = w_7; \\
QU^+(\pi_{d3}) &= w_6.
\end{aligned}
$$

That is, we immediately have that

$$\pi_{d2} \sqsupset_{QU^+} \pi_{d3} \sqsupset_{QU^+} \pi_{d0}.$$

Thus, d_2 (activate plan 2) is preferred to d_3 and d_0. But, to compare d_2 to d_1 we have to take into account the value of z_2. For instance, for $z_2 \geq w_8$, we have that $\pi_{d1} \sqsupset_{QU^+} \pi_{d2}$ and thus d_1 would be preferred to d_2 in that case, while if $z_2 = w_7$, d_2 and d_1 become equally preferable or if $z_2 \leq w_6$, d_2 is preferred to d_1.

Generalised Pessimistic Qualitative Evaluations: GQU^-

Now, let us see how GQU^- evaluates decisions. If we consider an arbitrary t-norm $\top$on V, the values we get are:

$$
\begin{aligned}
GQU^-(\pi_{d0}) &= \min[w_2, n_V(z_1)], \\
GQU^-(\pi_{d1}) &= \min[w_4, n_V(z_1) \bot w_1], \\
GQU^-(\pi_{d2}) &= \min[w_7, n_V(z_1) \bot w_3], \\
GQU^-(\pi_{d3}) &= w_6,
\end{aligned}
$$

where $\bot$ is the dual conorm of $\top$ with respect to the involution n_V. When we choose $\top = minimum$, GQU^- obviously recovers QU^-. Let us consider the case of $\top$ being the so-called *Lukasiewicz t-norm* defined as $w_i \top w_j = w_k$, with $k = \max(0, i + j - 9)$. The corresponding t-conorm $\bot$ turns out to be defined as

$$w_i \bot w_j = \begin{cases} w_{i+j}, & \text{if } 9 \geq i + j \\ w_9, & \text{otherwise.} \end{cases}$$

The choice of *Lukasiewicz t-norm* somehow carries out the implicit assumption that the values in V are equally distributed in the scale, which allows some form of additivity. Hence, it could be argued that this assumption is beyond the pure qualitative approach in which the ordering is what exclusively matters. But this hypothesis on the scale is rather usual and we think it is worth to give room in the model for these, let us say, non pure ordinal or qualitative assumptions.

In Table 5.3 we provide the preference orderings according to both QU^- and GQU^- we get for two particular values of z_1. One can see that for $z_1 = w_3$, the ranking provided by GQU^- seems a refinement of the one by QU^-. However, when $z_1 = w_5$, GQU^- reverses the ordering of QU^- for the decisions d_2 and d_3. In this case, QU^- turns out to be more conservative than GQU^- since it prefers d_3 (evacuation) to d_2 (activate plan 2), while the preference for GQU^- is the opposite.

z_1	*Dist.*	QU^-	GQU^-	*Pref. w.r.t.* QU^-	*Pref. w.r.t.* GQU^-
w_3	π_{d_0} π_{d_1} π_{d_2} π_{d_3}	w_2 w_4 w_6 w_6	w_2 w_4 w_7 w_6	$\pi_{d_2} \sim \pi_{d_3} \sqsupset \pi_{d_1} \sqsupset \pi_{d_0}$	$\pi_{d_2} \sqsupset \pi_{d_3} \sqsupset \pi_{d_1} \sqsupset \pi_{d_0}$
w_5	π_{d_0} π_{d_1} π_{d_2} π_{d_3}	w_2 w_4 w_4 w_6	w_2 w_4 w_7 w_6	$\pi_{d_3} \sqsupset \pi_{d_2} \sim \pi_{d_1} \sqsupset \pi_{d_0}$	$\pi_{d_2} \sqsupset \pi_{d_3} \sqsupset \pi_{d_1} \sqsupset \pi_{d_0}$

Table 5.3: Differences in the rankings by GQU^- and QU^-.

5.3 Representation of Preference Orderings: Extension to Generalised Ordinal Utilities

Now, given a t-norm operation in V, $\top : V \times V \to V$, we are interested in characterising the preference relations on $\Pi(X)$ that are representable by the generalised qualitative utility functions introduced in Section 5.1, which are extensions of the qualitative utilities QU^- and QU^+, that is,

$$\begin{aligned} GQU^-(\pi) &= \min_{x_i \in X} n(\pi(x_i) \top \lambda_i), \\ GQU^+(\pi) &= \max_{x_i \in X} h(\pi(x_i) \top \mu_i), \end{aligned}$$

where $n(\lambda_i) = u(x_i) = h(\mu_i)$, u representing the DM's preferences on consequences, $n = n_U \circ h$, with the onto order-preserving mapping $h{:}V \to U$ being as usual, but further verifying a *coherence condition w.r.t.* $\top$ to guarantee the correctness of the above definition, that is:

$$h(\lambda) = h(\mu) \Rightarrow h(\alpha \top \lambda) = h(\alpha \top \mu), \qquad \forall \alpha, \lambda, \mu \in V.$$

We are especially interested in characterising these utility functions since they may result in different orderings from the associated with QU orderings as it has been shown in the previous example.

The possibilistic mixture operation considered so far to combine possibilistic lotteries has been a max-min combination:

$$(\alpha/\pi_1, \beta/\pi_2) = \max(\min(\alpha, \pi_1), \min(\beta, \pi_2)).$$

Possibilistic mixtures, definable as $\perp$-decomposable[7] consensus functions on, $\perp$ being a t-conorm operation have been studied in (Dubois et al., 1996b). It is shown there that for possibility measures, i.e. $\max$-decomposable measures, an admissible class of mixture operations is obtained by defining

$$M_\top(\pi, \pi'; \alpha, \beta) = \max(\alpha \top \pi, \beta \top \pi') \qquad\qquad \alpha, \beta \in V$$

where $\top$ is any t-norm operation on V and $\max(\alpha, \beta)$ = 1. Thus, a particular case is to take $\top = minimum$, which results in the max-min mixture considered up to now.

[7] A measure $g{:}2^X \to V$ is $\perp$-decomposable if $g(A \cup B) = g(A) \perp g(B)$ when $A \cap B = \emptyset$.

Lemma 5.1
GQU^- and GQU^+ preserve the possibilistic mixture in the sense that it holds

$$\begin{aligned} GQU^-(M_\top(\pi_1, \pi_2; \lambda, \mu)) &= \min(n(\lambda \top \delta_1), n(\mu \top \delta_2)), \\ GQU^+(M_\top(\pi_1, \pi_2; \lambda, \mu)) &= \max(h(\lambda \top \gamma_1), h(\mu \top \gamma_2)), \end{aligned}$$

with $n(\delta_j) = GQU^-(\pi_j),\ h(\gamma_j) = GQU^+(\pi_j)$.

Proof:
As both proofs are analogous, we only include the proof for GQU^-. By definition

$$GQU^-(M_\top(\pi_1, \pi_2; \lambda, \mu)) = \min_{x_i \in X} n(M_\top(\pi_1, \pi_2; \lambda, \mu)(x_i) \top \gamma_i),$$

where $n(\gamma_i) = u(x_i)$. Since

$$\begin{aligned} M_\top(\pi_1, \pi_2; \lambda, \mu)(x_i) \top \gamma_i &= [\max(\lambda \top \pi_1(x_i), \mu \top \pi_2(x_i))] \top \gamma_i \\ &=^{8} \max(\lambda \top \pi_1(x_i) \top \gamma_i, \mu \top \pi_2(x_i) \top \gamma_i), \end{aligned}$$

then

$$\begin{aligned} n((M_\top(\pi_1, \pi_2; \lambda, \mu)(x_i)) \top \gamma_i) &= n(\max(\lambda \top \pi_1(x_i) \top \gamma_i, \mu \top \pi_2(x_i) \top \gamma_i)) \\ &=^{9} \min(n(\lambda \top \pi_1(x_i) \top \gamma_i), n(\mu \top \pi_2(x_i) \top \gamma_i)), \end{aligned}$$

so

$$\begin{aligned} \min_{x_i \in X} n(M_\top(\pi_1, \pi_2; \lambda, \mu)(x_i) \top \gamma_i) &= \min_{x_i \in X} \min(n(\lambda \top \pi_1(x_i) \top \gamma_i), \\ &\qquad n(\mu \top \pi_2(x_i) \top \gamma_i)) \\ &= \min\{\min_{x_i \in X} n(\lambda \top \pi_1(x_i) \top \gamma_i), \\ &\qquad \min_{x_i \in X} n(\mu \top \pi_2(x_i) \top \gamma_i)\}. \end{aligned}$$

Since

$$\begin{aligned} \min_{x_i \in X} n(\lambda \top \pi_1(x_i) \top \gamma_i) &= n(\max_{x_i \in X}(\lambda \top \pi_1(x_i) \top \gamma_i)) \\ &= n(\lambda \top (\max_{x_i \in X}(\pi_1(x_i) \top \gamma_i))), \end{aligned}$$

then

$$\begin{aligned} GQU^-(M_\top(\pi_1, \pi_2; \lambda, \mu)) &= \min\{n(\lambda \top (\max_{x_i \in X} \pi_1(x_i) \top \gamma_i)), \\ &\qquad n(\mu \top (\max_{x_i \in X} \pi_2(x_i) \top \gamma_i))\}. \end{aligned}$$

Since

[8] Because of $\max(\alpha, \beta) \top \gamma = \max(\alpha \top \gamma, \beta \top \gamma)$.

[9] Because we have $n(\max(a, b)) = \min(n(a), n(b))$, since being a reversing ordering mapping *between linear scales* implies to be a reversing morphism.

$$n(\max_{x_i \in X} \pi_j(x_i) \top \gamma_i) = \min_{x_i \in X} n(\pi_j(x_i) \top \gamma_i) = GQU^-(\pi_j) = n(\delta_j),$$

under the coherence hypothesis, we obtain that

$$n(\lambda \top (\max_{x_i \in X} \pi_1(x_i) \top \gamma_i)) = n(\lambda \top \delta_1),$$

and analogously, we have that

$$n(\mu \top (\max_{x_i \in X} \pi_2(x_i) \top \gamma_i)) = n(\mu \top \delta_2).$$

Hence,

$$GQU^-(M_\top(\pi_1, \pi_2; \lambda, \mu)) = \min(n(\lambda \top \delta_1), n(\mu \top \delta_2)),$$

with $n(\delta_j) = GQU^-(\pi_j)$. □

Now, we have that

Lemma 5.2
The reduction of lotteries follows the next rule:

$$M_\top(M_\top(\pi_1, \pi_2; \lambda_1, \lambda_2), M_\top(\pi_1, \pi_2; \mu_1, \mu_2), \alpha, \beta) =$$
$$= \quad M_\top(\pi_1, \pi_2; \max(\alpha \top \lambda_1, \beta \top \mu_1), \max(\alpha \top \lambda_2, \beta \top \mu_2)).$$

Proof:

$$M_\top(M_\top(\pi_1, \pi_2; \lambda_1, \lambda_2), M_\top(\pi_1, \pi_2; \mu_1, \mu_2), \alpha, \beta) =$$

$$= \max[\alpha \top M_\top(\pi_1, \pi_2; \lambda_1, \lambda_2), \beta \top M_\top(\pi_1, \pi_2; \mu_1, \mu_2)]$$

$$= \max[\alpha \top \max(\lambda_1 \top \pi_1, \lambda_2 \top \pi_2), \beta \top \max(\mu_1 \top \pi_1, \mu_2 \top \pi_2)],$$

and since

$$\alpha \top \max(\lambda, \gamma) = \max(\alpha \top \lambda, \alpha \top \gamma) \qquad \forall \alpha, \lambda, \gamma;$$

we obtain that

$$\max[\alpha \top \max(\lambda_1 \top \pi_1, \lambda_2 \top \pi_2), \beta \top \max(\mu_1 \top \pi_1, \mu_2 \top \pi_2)] =$$

$$\begin{aligned}
&= \max[\max(\alpha \top \lambda_1 \top \pi_1, \alpha \top \lambda_2 \top \pi_2), \max(\beta \top \mu_1 \top \pi_1, \beta \top \mu_2 \top \pi_2)] \\
&= \max[\alpha \top \lambda_1 \top \pi_1, \alpha \top \lambda_2 \top \pi_2, \beta \top \mu_1 \top \pi_1, \beta \top \mu_2 \top \pi_2] \\
&= \max[\max(\alpha \top \lambda_1 \top \pi_1, \beta \top \mu_1 \top \pi_1), \max(\alpha \top \lambda_2 \top \pi_2, \beta \top \mu_2 \top \pi_2)] \\
&= \max[\max(\alpha \top \lambda_1, \beta \top \mu_1) \top \pi_1, \max(\alpha \top \lambda_2, \beta \top \mu_2) \top \pi_2] \\
&= M_\top(\pi_1, \pi_2; \max(\alpha \top \lambda_1, \beta \top \mu_1), \max(\alpha \top \lambda_2, \beta \top \mu_2)).
\end{aligned}$$

$\square$

In order to encompass this extended kind of possibilistic mixture operations in the qualitative decision model, we have considered the modified axiom set $AX_\top = \{A1, A2, A3_\top, A4_\top\}$, where

- $A3_\top$*(independence)*: $\pi_1 \sim \pi_2 \Rightarrow M_\top(\pi_1, \pi; \alpha, \beta) \sim M_\top(\pi_2, \pi; \alpha, \beta)$.
- $A4_\top$ $(continuity)$: $\forall \pi \in \Pi(X)\ \exists \lambda \in V$ such that $\pi \sim M_\top(\overline{x}, \underline{x}; 1, \lambda)$, where $\overline{x}$ and $\underline{x}$ are a maximal and a minimal element of $(X, \sqsubseteq)$ respectively.

Now, we introduce some results for this axiomatic setting that are analogous to the results obtained in the previous Chapter.

Lemma 5.3
If $\sqsubseteq$ verifies axioms $A1,\ A2,\ A3_\top$ and $A4_\top$, $\sqsubseteq$ also verifies axiom $DP2$, i.e. if A is a crisp subset of X then there is $x \in A$ such that $x \sim A$.

Proof:
Suppose that $A = \{x_1, x_2\}$, with $x_1 \sqsubseteq x_2$. Let us first suppose that $x_1 \sim x_2$, so

$$A = M_\top(x_1, x_2; 1, 1) \sim M_\top(x_1, x_1; 1, 1) = x_1.$$

If $x_1 \sqsubset x_2$, by $A4_\top$ there exist λ_1 and λ_2 such that

$$x_1 \sim M_\top(\overline{x}, \underline{x}; 1, \lambda_1) \text{ and } x_2 \sim M_\top(\overline{x}, \underline{x}; 1, \lambda_2),$$

as $x_1 \sqsubset x_2$, then by $A2$, $\lambda_1 > \lambda_2$.

Hence, applying $A3_\top$ we obtain:

$$\begin{aligned} A &= M_\top(x_1, x_2, 1, 1) \sim M_\top(M_\top(\overline{x}, \underline{x}; 1, \lambda_1), M_\top(\overline{x}, \underline{x}; 1, \lambda_2), 1, 1) \\ &= M_\top(\overline{x}, \underline{x}; 1, \max(\lambda_1, \lambda_2)) = M_\top(\overline{x}, \underline{x}; 1, \lambda_1) \sim x_1. \end{aligned}$$

Suppose the Lemma is valid if $|A| = p$. Now, let A be such that $|A| = p + 1$, and let x_1 be one of its minimal w.r.t. $\sqsubseteq$.

Since $A = M_\top(x_1, A - \{x_1\}; 1, 1)$, by induction hypothesis we have that if x_2 is one of the minimal elements of $A - \{x_1\}$ w.r.t. $\sqsubseteq$, then

$$A \sim M_\top(x_1, x_2; 1, 1) \sim x_1.$$

$\square$

Lemma 5.4
If $\sqsubseteq$ verifies axioms $A1,\ A2,\ A3_\top$, and $A4_\top$, then, the maximal and minimal elements of X w.r.t. to $\sqsubseteq$ are indeed maximal and minimal elements of $\Pi(X)$ as well.

Moreover, if $\overline{x}$ is a maximal and $\underline{x}$ is a minimal on $(X, \sqsubseteq)$, the following equivalencies holds:

$$\underline{x} \sim X \sim M_{\top}(\overline{x}, \underline{x}; 1, 1).$$

Proof:
We may observe that the proof is "independent" of the definition of the mixture, since we only use that $\underline{x} \leq M_{\top}(\overline{x}, \underline{x}; 1, 1) \leq X$.

Indeed, let us prove first the equivalencies

$$\underline{x} \sim X \sim M_{\top}(\overline{x}, \underline{x}; 1, 1).$$

$A1$ guarantees that $\underline{x}$ and $\overline{x}$ exist. By the *uncertainty aversion* axiom $A2$, it is clear that X is a minimal element of $\Pi(X)$, so it is $X \sqsubseteq \underline{x}$.

But by $DP2$ there exists $x_0 \in X$ such that $x_0 \sim X$, but since $\underline{x}$ is minimal, $\underline{x} \sqsubseteq x_0$, thus it must be $\underline{x} \sim X$.

Furthermore, on $\Pi(X)$ we have $\underline{x} \leq M_{\top}(\overline{x}, \underline{x}; 1, 1) \leq X$, and by $A2$, $X \sqsubseteq M_{\top}(\overline{x}, \underline{x}; 1, 1) \sqsubseteq \underline{x}$, and thus $\underline{x} \sim X \sim M_{\top}(\overline{x}, \underline{x}; 1, 1)$.

On the other hand, for any $\pi \in \Pi(X)$, since π is normalised, there exists x such that $\pi(x) = 1$. So, we have $x \leq \pi$ and therefore $\pi \sqsubseteq x$, but since $\overline{x}$ is maximal of X, it is $x \sqsubseteq \overline{x}$, and thus $\pi \sqsubseteq \overline{x}$. □

For the preference orderings induced by these generalised qualitative utilities we have a representation theorem like in the previous Chapter.

Theorem 5.5
A preference relation $\sqsubseteq$ on $\Pi(X)$, equipped with the mixture operation $M_{\top}$, satisfies the axiom set $AX_{\top}$ if and only if there exist

(i) a finite linearly ordered preference scale U with $\inf(U) = 0$ and $\sup(U) = 1$,

(ii) a preference function $u{:}X \to U$ such that $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,

(iii) an onto order-preserving function $h{:}V \to U$[10], satisfying also

$$h(\lambda) = h(\mu) \Rightarrow h(\alpha \top \lambda) = h(\alpha \top \mu), \qquad \forall \alpha, \lambda, \mu \in V,$$

in such a way that it holds:

$$\pi' \sqsubseteq \pi \qquad \textit{iff} \qquad \pi' \preccurlyeq_{GQU^-} \pi,$$

where $\preccurlyeq_{GQU^-}$ is the ordering on $\Pi(X)$ induced by the qualitative utility $GQU^-(\pi) = \min_{x_i \in X} n(\pi(x_i) \top \lambda_i)$, with $n(\lambda_i) = u(x_i)$ and $n = n_U \circ h$, as usual n_U being the reversing involution in U.

Proof:
$\leftarrow$) Axiom $A1$ is easily verified.

[10]Observe that h also satisfies that such that $h(0) = 0$, $h(1) = 1$, as was observed by a reviewer of one of our papers.

- *A*2*(uncertainty aversion)*: if $\pi \leq \pi' \Rightarrow \pi' \preccurlyeq_{GQU^-} \pi$.
  By definition,
  $$\pi \leq \pi' \Rightarrow \pi(x) \leq \pi'(x) \qquad \forall x.$$
  Since $\top$ is non-decreasing,
  $$(\pi(x_i)\top\lambda_i) \leq (\pi'(x_i)\top\lambda_i) \qquad \forall x_i.$$
  Hence,
  $$\begin{aligned} GQU^-(\pi) &= \min_{x_i \in X} n(\pi(x_i)\top\lambda_i) \\ &\geq \min_{x_i \in X} n(\pi'(x_i)\top\lambda_i) \\ &= GQU^-(\pi'). \end{aligned}$$
  Therefore,
  $$\pi' \preccurlyeq_{GQU^-} \pi.$$

- $A3_\top$*(independence)*:
  $$\begin{aligned} GQU^-(\pi_1) = GQU^-(\pi_2) \quad \Rightarrow \quad & GQU^-(M_\top(\pi_1, \pi'; \alpha, \beta)) = \\ = \quad & GQU^-(M_\top(\pi_2, \pi'; \alpha, \beta)) \end{aligned}$$
  Indeed,
  $$\begin{aligned} GQU^-(M_\top(\pi_1, \pi'; \alpha, \beta)) = \min(n(\alpha\top\lambda_1), n(\beta\top\lambda)), \\ GQU^-(M_\top(\pi_2, \pi'; \alpha, \beta)) = \min(n(\alpha\top\lambda_2), n(\beta\top\lambda)), \end{aligned}$$
  with $GQU^-(\pi_j) = n(\lambda_j)$, and $GQU^-(\pi') = n(\lambda)$.
  By hypothesis, we have that
  $$n(\lambda_1) = GQU^-(\pi_1) = GQU^-(\pi_2) = n(\lambda_2).$$
  As n satisfies the coherence condition w.r.t. $\top$, we obtain that
  $$n(\alpha\top\lambda_1) = n(\alpha\top\lambda_2),$$
  therefore
  $$GQU^-(M_\top(\pi_1, \pi'; \alpha, \beta)) = GQU^-(M_\top(\pi_2, \pi'; \alpha, \beta)).$$

- $A4_\top$: We have to prove that $\forall \pi \in \Pi(X)$, there exists λ such that $GQU^-(\pi) = GQU^-(M_\top(\overline{x}, \underline{x}; 1, \lambda))$, where $\overline{x}$, $\underline{x}$ are a maximal and a minimal elements of $(X, \preccurlyeq_{GQU^-})$.
  Since we are assuming u$^{-1}(1) \neq \emptyset \neq u^{-1}(0)$, it must be the case that $u(\underline{x})$ = 0 and $u(\overline{x}) = 1$, hence

$$GQU^-(M_\top(\overline{x}, \underline{x}; 1, \lambda)) = n(\lambda \top \underline{\lambda}) \qquad \text{with } GQU^-(\underline{x}) = n(\underline{\lambda}) = 0.$$

As $n(1) = 0$, by the coherence condition we have that

$$n(\lambda \top \underline{\lambda}) = n(\lambda \top 1),$$

hence,

$$GQU^-(M_\top(\overline{x}, \underline{x}; 1, \lambda)) = n(\lambda \top \underline{\lambda}) = n(\lambda).$$

Therefore, since $u(X) \subseteq n(V)$, for any $\lambda \in n^{-1}(GQU^-(\pi))$ we have that

$$GQU^-(\pi) = n(\lambda) = GQU^-(M_\top(\overline{x}, \underline{x}; 1, \lambda)).$$

$\rightarrow$) We structure the proof in the following steps:

1. We define the preference scale U and an order-preserving (and onto) function h from V to U.
2. We define the function $GQU^-{:}\Pi(X) \rightarrow U$, for the π_λ^-'s, and then we extend it due to axiom $A4_\top$. GQU^- represents $\sqsubseteq$.
3. Then, we prove that

   $$GQU^-(\pi) = \min_{i=1,\dots,p}\ n(\pi(x_i) \top \lambda_i)$$

   with $n(\lambda_i) = u(x_i)$ where $u{:}X \rightarrow U$ is the restriction of GQU^- on X, and $n = n_U \circ h$.

Now, we develop these steps.

1. As usual, $\sqsubseteq$ stratifies $\Pi(X)$ in a linearly ordered set of classes of equivalently preferred distributions ($\pi' \in [\pi]$ iff $\pi \sim \pi'$). The number of classes is just the number of levels needed to rank order the set of distributions.

   Therefore, we take as preference scale U the quotient set $\Pi(X)/\sim$ together with the natural (linear) order

   $$[\pi] \leq [\pi'] \quad \text{iff} \quad \pi \sqsubseteq \pi'.$$

   By Lemma 5.4, again if $\overline{x}$ and $\underline{x}$ denote a maximal and a minimal element of X respectively, $[\overline{x}]$ and $[\underline{x}]$ will be the maximum and minimum elements of $\Pi(X)/\sim$, i.e. of U, and will be denoted by 1 and 0 respectively.

   Now, we denote by π_λ^- the possibility distribution defined as the qualitative lottery $M_\top(\overline{x}, \underline{x}; 1, \lambda)$.

   We define the order-reversing function $n{:}V \rightarrow U$ as $n(\lambda) = [\pi_\lambda^-]$.

   Observe that $n(1) = [M_\top(\overline{x}, \underline{x}; 1, 1)] = [\underline{x}] = 0$ and $n(0) = [M_\top(\overline{x}, \underline{x}; 1, 0)] = [\overline{x}] = 1$.

   By $A2$, n results reversing and it is onto by construction. n results coherent w.r.t. $\top$ because of the reduction property of $M_\top$ and $A3_\top$. As previously, we define now $h = n_U \circ n$. From the properties of n, it is easy to verify that h satisfies the required conditions.

2. So far we have determined U and h. Now, let us define the qualitative function GQU^- on $\Pi(X)$.

   (a) First, define $GQU^-(M_\top(\overline{x}, \underline{x}; 1, \lambda)) = n(\lambda)$.

   (b) It is easy to check that $\pi^-_\lambda \sqsubseteq \pi^-_{\lambda'}$ iff $GQU^-(\pi^-_\lambda) \leq GQU^-(\pi^-_{\lambda'})$. So, restricted to lotteries of type π^-_λ, GQU^- represents $\sqsubseteq$.

   (c) We extend GQU^- to any lottery as follows. For any π, $A4_\top$ guarantees that $\exists\lambda$ such that $\pi \sim M_\top(\overline{x}, \underline{x}; 1, \lambda)$, so we define $GQU^-(\pi) = n(\lambda)$.

   As a consequence of (c) and (b), GQU^- represents $\sqsubseteq$, i.e.

$$\pi \sqsubseteq \pi' \quad \text{iff} \quad GQU^-(\pi) \leq GQU^-(\pi').$$

3. Now, we define $u{:}X \to U$ as $u(x) = GQU^-(x)$, Notice that $u(\overline{x}) = 1$ and $u(\underline{x}) = 0$, and thus, $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$.

   It remains to prove that

$$GQU^-(\pi) = \min_{i=1,\ldots,p}\ n(\pi(x_i) \top \gamma_i)$$

   with $n(\gamma_i) = u(x_i),\ |X| = p$.

   To verify this, we will prove the following equalities:

   - $\forall \pi_1, \pi_2,$

$$GQU^-(M_\top(\pi_1, \pi_2; \alpha, \beta)) = n(\max((\alpha \top \lambda_1), (\beta \top \lambda_2))), \tag{5.7}$$

     with λ_j such that $GQU^-(\pi_j) = n(\lambda_j)$.
     Indeed, $A4_\top$ guarantees that

     $\exists\lambda_1$ s.t. $\pi_1 \sim M_\top(\overline{x}, \underline{x}; 1, \lambda_1)$ and $\exists\lambda_2$ s.t. $\pi_2 \sim M_\top(\overline{x}, \underline{x}; 1, \lambda_2)$,

     remember that $GQU^-(\pi_1) = n(\lambda_1)$ and $GQU^-(\pi_2) = n(\lambda_2)$. So, using the *independence axiom* $A4_\top$,

$$M_\top(\pi_1, \pi_2; \alpha, \beta) \ \sim\ M_\top(M_\top(\overline{x}, \underline{x}, 1, \lambda_1), M_\top(\overline{x}, \underline{x}, 1, \lambda_2); \alpha, \beta),$$

     and by reduction of "lotteries" it reduces to

$$M_\top(\overline{x}, \underline{x}; \max((\alpha \top 1), (\beta \top 1)), \max((\alpha \top \lambda_1), (\beta \top \lambda_2))) \sim$$

$$\sim M_\top(\overline{x}, \underline{x}; \max(\alpha, \beta), \max((\alpha \top \lambda_1), (\beta \top \lambda_2)))$$

$$\sim M_\top(\overline{x}, \underline{x}; 1, \max((\alpha \top \lambda_1), (\beta \top \lambda_2))).$$

     Therefore,

$$GQU^-(M_\top(\pi_1, \pi_2; \alpha, \beta)) = n(\max((\alpha \top \lambda_1), (\beta \top \lambda_2)))$$

with λ_j such that $GQU^-(\pi_j) = n(\lambda_j)$, i.e.

$$GQU^-(M_\top(\pi_1, \pi_2; \alpha, \beta)) = \min(n(\alpha \top \lambda_1), n(\beta \top \lambda_2)).$$

Finally, we verify that (5.7) does not depend on the λ chosen, i.e. if μ is such that $GQU^-(\pi_1) = n(\mu)$, then

$$n(\max((\alpha \top \lambda_1), (\beta \top \lambda_2))) = n(\max((\alpha \top \mu), (\beta \top \lambda_2))).$$

Indeed, as $\pi^-_{\lambda_1} \sim \pi^-_\mu$ then

$$\begin{aligned} M_\top(\overline{x}, \underline{x}; 1, \max((\alpha \top \lambda_1), (\beta \top \lambda_2))) &\sim M_\top(\pi^-_{\lambda_1}, \pi^-_{\lambda_2}; \alpha, \beta) \\ \sim M_\top(\pi^-_\mu, \pi^-_{\lambda_2}; \alpha, \beta) &\sim M_\top(\overline{x}, \underline{x}; 1, \max((\alpha \top \mu), (\beta \top \lambda_2))), \end{aligned}$$

therefore

$$n(\max((\alpha \top \lambda_1), (\beta \top \lambda_2))) = n(\max((\alpha \top \mu), (\beta \top \lambda_2))).$$

In particular, we have that

$$GQU^-(M_\top(x, y; 1, \beta)) = \min(n(1 \top \lambda_1), n(\beta \top \lambda_2))$$

with $u(x) = n(\lambda_1), u(y) = n(\lambda_2)$. So,

$$GQU^-(M_\top(x, y; 1, \beta)) = \min(u(x), n(\beta \top \lambda_2)),$$

with $u(y) = n(\lambda_2)$, and

$$GQU^-(\max(\pi_1, \pi_2)) = \min(GQU^-(\pi_1), GQU^-(\pi_2)).$$

Indeed, as $\max(\pi_1, \pi_2) = M_\top(\pi_1, \pi_2; 1, 1)$, therefore,

$$GQU^-(\max(\pi_1, \pi_2)) = \min(n(\mu_1), n(\mu_2))$$

with $n(\mu_1) = GQU^-(\pi_1), n(\mu_2) = GQU^-(\pi_2)$, so

$$GQU^-(\max(\pi_1, \pi_2)) = \min(GQU^-(\pi_1), GQU^-(\pi_2)).$$

Moreover, we have

$$GQU^-(\max_{i=1,\ldots,p} \pi_i) = \min_{i=1,\ldots,p} GQU^-(\pi_i) \qquad \forall\, \pi_i.$$

- $GQU^-(\pi) = \min_{i=1,\ldots,p}\ n(\pi(x_i) \top \gamma_i)$.
  As π is normalised, there exists $x_j \in X$ such that $\pi(x_j)$ = 1. Without loss of generality, let us assume that $j = 1$. As for each π, $M_\top$ satisfies that

$$M_\top(x_1, x_i; 1, \pi(x_i))(x_k) = \begin{cases} 1, & \text{if} \quad x_k = x_1, \\ \pi(x_i), & \text{if} \quad x_1 \neq x_k = x_i, \\ 0, & \text{otherwise}. \end{cases}$$

Then, choosing

$$\pi_i = M_\top(x_1, x_i; 1, \pi(x_i)),$$

we obtain $\pi = \max_{i=1,\ldots,p}\ \pi_i$, therefore

$$\begin{aligned} GQU^-(\pi) &= GQU^-(\max_{i=1,\ldots,p} M_\top(x_1, x_i; 1, \pi(x_i))) \\ &= \min_{i=1,\ldots,p} GQU^-(M_\top(x_1, x_i, 1, \pi(x_i))) \\ &= \min_{i=1,\ldots,p} [\min(u(x_1), n(\pi(x_i)\top\lambda_i))] \end{aligned}$$

with $u(x_i) = GQU^-(x_i) = n(\lambda_i)$, so

$$GQU^-(\pi) = \min_{i=1,\ldots,p}\ n(\pi(x_i)\top\lambda_i).$$

□

As in the case of purely ordinal information, sometimes these GQU^- functions may result too conservative and we may be interested in more optimistic behaviours. We may model them by

$$GQU^+(\pi) = \max_{x_i \in X} h(\pi(x_i)\top\lambda_i) \tag{5.8}$$

with $h(\lambda_i) = u(x_i)$, $\top$ a t-norm in V, and as usual h being an onto order-preserving mapping that also satisfy coherence w.r.t. $\top$.
For characterising these behaviours, we consider the axiomatic setting $AX_\top^+$ where we replace $A2$ by $A2^+$ and $A4_\top$ by:

- $A4_\top^+$: $\forall \pi \in \Pi(X)\ \exists \lambda \in V$ such that $\pi \sim M_\top(\overline{x}, \underline{x}; \lambda, 1)$, where $\overline{x}$ and $\underline{x}$ are a maximal and a minimal element of $(X, \sqsubseteq)$ respectively.

For this axiomatic setting we have the analogous results of Lemmas 5.3 and 5.4, and of course, the representation theorem:

Theorem 5.6
A preference relation $\sqsubseteq$ on $\Pi(X)$, equipped with the mixture operation $M_\top$, satisfies the axiom set $AX_\top^+$ if and only if there exist

(i) a finite linearly ordered preference scale U with $\inf(U) = 0$ and $\sup(U) = 1$,

(ii) a preference function $u{:}X \to U$ such that $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,

(iii) an onto order-preserving function $h{:}V \to U$, satisfying also

$$h(\lambda) = h(\mu) \Rightarrow h(\alpha\top\lambda) = h(\alpha\top\mu), \qquad \forall \alpha, \lambda, \mu \in V,$$

in such a way that it holds:

$$\pi' \sqsubseteq \pi \qquad \textit{iff} \qquad \pi' \preccurlyeq_{GQU^+} \pi,$$

where $\preccurlyeq_{GQU^+}$ is the ordering on $\Pi(X)$ induced by the qualitative utility $GQU^+(\pi) = \max_{x_i \in X} h(\pi(x_i)\top\lambda_i)$, with $h(\lambda_i) = u(x_i)$.

The proofs are omitted because they are analogues with the "pessimistic" case.

Now, let us show that the axiomatic setting proposed also guarantees the "unicity" of the preference set of values, of the linking mapping h and of the preference function u on consequences. Indeed, we have

Theorem 5.7
Given

(i) two finite linearly ordered preference scales U_1, U_2 *with* $\inf(U_1) = 0_1, \inf(U_2) = 0_2$ *and* $\sup(U_1) = 1_1 \sup(U_2) = 1_2$,

(ii) two preference functions on them, i.e. $u_j{:}X \to U_j$ *such that* $u_j^{-1}(1_j) \neq \emptyset \neq u^{-1}(0_j),\ j = 1, 2,$

(iii) two onto order-preserving functions $h_j{:}V \to U_j$, *satisfying also*

$$h_j(\lambda) = h_j(\mu) \Rightarrow h_j(\alpha \top \lambda) = h_j(\alpha \top \mu), \qquad \forall \alpha, \lambda, \mu \in V, j = 1, 2.$$

in such a way that it holds:

$$\pi' \preccurlyeq_{GQU^-(\cdot|U_1,h_1,u_1)} \pi \qquad \textit{iff} \qquad \pi' \preccurlyeq_{GQU^-(\cdot|U_2,h_2,u_2)} \pi,$$

or

$$\pi' \preccurlyeq_{GQU^+(\cdot|U_1,h_1,u_1)} \pi \qquad \textit{iff} \qquad \pi' \preccurlyeq_{GQU^+(\cdot|U_2,h_2,u_2)} \pi,$$

then

1. U_1 *and* U_2 *are isomorphic.*
2. *If* $U_1 = U_2$, *then* $h_1 = h_2$ *and* $u_1 = u_2$.

Proof:
We assume

$$\pi' \preccurlyeq_{GQU^-(\cdot|U_1,h_1,u_1)} \pi \qquad \text{iff} \qquad \pi' \preccurlyeq_{GQU^-(\cdot|U_2,h_2,u_2)} \pi,$$

the other case being analogous.

1. Suppose $|U_1| = m$, $\sqsubseteq_j$ denotes the relation $\preccurlyeq_{GQU(\cdot|U_j,h_j,u_j)}$. Hence, $\exists \lambda_1, \ldots, \lambda_m \in V$ s.t.

   $$\pi_{\lambda_1} \sqsubset_1 \ldots \sqsubset_1 \pi_{\lambda_m} \iff \pi_{\lambda_1} \sqsubset_2 \ldots \sqsubset_2 \pi_{\lambda_m}.$$

   So, $|U_2| \geq m$. However, if $|U_2| > m$, we have that

   $$\exists \lambda_1, \ldots, \lambda_{m+1} \in V \text{ s.t. } \pi_{\lambda_1} \sqsubset_2 \ldots \sqsubset_2 \pi_{\lambda_{m+1}} \iff \pi_{\lambda_1} \sqsubset_1 \ldots \sqsubset_1 \pi_{\lambda_{m+1}}.$$

   Hence, $|U_1| \geq m + 1$. Contradiction, so $|U_1| = |U_2|$.

2. Now, assuming both scales are the same, say U, we first verify that the linking mapping is unique.

- Suppose $h_1 \neq h_2$, then there exists $\lambda_0 = \inf\{\lambda | h_1(\lambda) \neq h_2(\lambda)\}$. Without loss of generality we may assume $h_1(\lambda_0) > h_2(\lambda_0)$, i.e. $n_1(\lambda_0) < n_2(\lambda_0)$, with $n_i = n_U \circ h_i$. As n_1 is onto, there exists $\mu \in V$ s.t. $n_2(\lambda_0) = n_1(\mu)$, so

  $$n_1(\mu) = n_2(\lambda_0) > n_1(\lambda_0).$$

  Hence, $\pi_\mu \sqsupset_{GQU^-(\cdot|U,h_1,u_1)} \pi_{\lambda_0}$, therefore as by hypothesis both induced orderings are the same, we have that $\pi_\mu \sqsupset_{GQU^-(\cdot|U,h_2,u_2)} \pi_{\lambda_0}$, so

  $$n_2(\mu) > n_2(\lambda_0) = n_1(\mu).$$

  That is, $h_2(\mu) \neq h_1(\mu)$, with $\mu < \lambda_0$. Contradiction with the definition of λ_0. Hence, $h_1 = h_2$.

- Now, denoting by h the linking mapping, we verify that both preference functions are the same. Indeed, given $x \in X$, $u_1(x) \in U$, as $n = n_U \circ h$ is onto, $\exists \lambda \in V$ s.t. $n(\lambda) = u_1(x)$, so $x \sim_1 \pi_\lambda$, with $\sqsubseteq_j$ denoting the relation $\preccurlyeq_{GQU^-(\cdot|U,h,u_j)}$. Hence by hypothesis, we have that $x \sim_2 \pi_\lambda$, i.e. $u_2(x) = n(\lambda) = u_1(x)$, therefore $u_1 = u_2$.

$\square$

Chapter 6

Preference and Uncertainty Measured on Cartesian Product of Linear Scales

So far we have considered that both uncertainty and preferences on consequences are measured on finite linear scales. However, these hypotheses may not be valid in many decision problems. There are certain kinds of decision problems where we are not able to measure uncertainty and/or preferences in such linearly ordered sets, but only in partially ordered ones. For instance, let us comment about some of such possible scenarios:

- When there are several sources of uncertainty, each one being measured in a linear scale, the set of values for uncertainty, $(\overline{V}, \leq_{\overline{V}})$, is a product of scales, that is, $\overline{V} = \Pi_{j=1,\ldots,k} V_j$, each V_j being a finite linearly ordered set.

- In a similar way, we may have that *DM*'s preferences on consequences are only partially ordered. Indeed, a preference relation among consequences is usually modelled by a preference function u:$X \to U$, where U is a finite preference scale, frequently a (numerical or a qualitative) linear scale. However, in many cases, this preference function may be vectorial. Indeed, suppose that consequences are evaluated with respect to k different criteria or attributes, each one represented by a preference function u_j:$X \to U$. Then, the global preference on consequences can be evaluated in terms of the vectorial function $\overline{u}$:$X \to U \times^{k)} \ldots \times U$, with $\overline{u}(x) = (u_1(x), \ldots, u_k(x))$. Considering in $U \times^{k)} \ldots \times U = \overline{U}$ the usual product ordering (Pareto ordering), we are outside of the linear models.

- As it has been mentioned in Section 1.3, once we link the similarity between situations with a possibility distribution on consequences (you may see Section 8.1 for more details), Case-Based Decision may be approached with the qualitative utility functions we have been working. In this case, the distribution is defined over the same set that the similarity function is applied in. Hence, we may have partially ordered uncertainty in case-based decision problems when the degrees of similarity on problems are only partially ordered. For example, consider that each situation is described as a *k-tuple* $s = (s^1, ..., s^k)$. Suppose we are provided with k feature similarity functions, $Sim^j{:}S^j \times S^j \rightarrow E$, that measures the degree of similarity between two *j-features*, where E is a finite linear scale. The global similarity function on situations $Sim{:}S \times S \rightarrow \overline{V}$, can be defined in terms of the k-feature similarity functions as

$$Sim(s, s') = (Sim^1(s^1, s'^1), ..., Sim^k(s^k, s'^k)),$$

with $\overline{V} = E \times \ldots \times E$, $\leq_{\overline{V}}$ being the ordering on $\overline{V}$. Again, if for instance $\leq_{\overline{V}}$ is the Pareto ordering, $(\overline{V}, \leq_{\overline{V}})$ is not a linear lattice.

Hence, we are interested in extending the qualitative decision model to let us make decisions in cases where the *DM*'s preferences on consequences may be only partially ordered or when the uncertainty on the consequences is valued on a non-linear lattice. In order to cope with some of these situations, we propose to extend the model in three steps:

- First, we will consider preferences and/or uncertainty are measured on finite Cartesian product of (finite) linear scales.
- Second, we shall consider both preferences and uncertainty are graded on distributive lattices, in particular when both are non-linear distributive lattices.
- Finally, we consider a particular case of allowing different type of measurement lattices, indeed we measure preferences on a linear one, while uncertainty is measured on a residuated distributive lattice.

In this Chapter, we develop the first extension, the other ones being developed in the next Chapter.

In next Section, we introduce some possible orderings in a finite Cartesian product of linearly ordered sets taking into account the orderings in each scale. Next, we will propose vectorial pessimistic and optimistic qualitative utilities with respect to a vectorial preference function defined over $\overline{U}$, a Cartesian product of preference scales. For these utility functions, we will consider the relations induced by them and by a general "boolean" function g, providing their characterisations. These theorems include the cases of considering the ordering induced by the vectorial functions when we are considering *lexicographic* or *Pareto* orderings in the preference set. Afterwards, assuming that all linear preference scales are the same, we observe some properties of the *weighted-min* and *weighted-max* orderings on the product of scales. In Section 6.3, we analyse the behaviour of these vectorial functions in the example introduced in Section 5.2, but now, we consider a vectorial preference function $\overline{u}$, in terms of the marginal preferences: safety and cost. In Section 6.5, we consider the same example but assuming that two evaluations of the possibility of being in the actual state are provided. Finally in Section 6.4, we analyse the case in which uncertainty is measured on a product of scales taking into account linear or cartesian representation for preferences.

6.1 Some Orderings in Cartesian Products Induced by the Marginal Orderings

Let us recall some possible orderings on a Cartesian product of finite linear scales. Given $\{(E_j, \leq_{E_j})\}_{j=1,\ldots,k}$ a set of finite linear scales, we consider $\overline{E} = \Pi_{j=1,\ldots,k}\ E_j$ the Cartesian product of the E_j's. In $\overline{E}$, different interesting orderings may be considered in terms of the marginal orderings $\leq_{E_j}$. In the following we introduce some of them.

- Possibly the most natural ordering in $\overline{E}$ is the *product ordering*, known as the *Pareto ordering* as well:

  $$\forall\, \overline{e} = (e_1, ..., e_k),\ \overline{e'} = (e'_1, ..., e'_k) \in \overline{E},$$
  $$\overline{e} \leq_{\Pi} \overline{e'} \iff (e_j \leq_{E_j} e'_j \quad \forall j = 1, \ldots, k).$$

  $\leq_{\Pi}$ is only a partial order. Indeed, if there exist i, j such that $e_j <_{E_j} e'_j$ and $e_i >_{E_i} e'_i$, then $\overline{e}$ and $\overline{e'}$ are incomparable with respect to $\leq_{\Pi}$.

- Another alternative option is to use an aggregation operator. That is, if AGG is an aggregation operator from $\overline{E}$ to E (E being a finite linear scale), we define

  $$\overline{e} \leq_{AGG} \overline{e'} \iff AGG(e_1, \ldots, e_k) \leq_E AGG(e'_1, \ldots, e'_k).$$

  $\leq_{AGG}$ is a total preorder. Indeed, as $\leq_E$ is complete, this fact allows us to compare all vectors in $\overline{E}$.
  In the case of all the scales being the same, say E, some particular cases of aggregation orderings are:

  - *min-ordering*:
    $$\overline{e} \leq_{\min} \overline{e'} \iff \min\{e_1, \ldots, e_k\} \leq_E \min\{e'_1, \ldots e'_k\},$$
  - *max-ordering*:
    $$\overline{e} \leq_{\max} \overline{e'} \iff \max\{e_1, \ldots, e_k\} \leq_E \max\{e'_1, \ldots, e'_k\}.$$
  - Moreover, we may consider weighted versions of them, i.e. given a vector of weights $\overline{w} = (w_1, \ldots, w_k) \in E^k$, the *weighted-minimum* is defined as

    $$\overline{e} \leq_{\overline{w}-m} \overline{e'} \iff$$
    $$\min\{\max(w_1, e_1), \ldots, \max(w_k, e_k)\} \leq_E \min\{\max(w_1, e'_1), \ldots, \max(w_k, e'_k)\},$$

    while the *weighted-maximum* is defined as

    $$\overline{e} \leq_{\overline{w}-M} \overline{e'} \iff$$
    $$\max\{\min(w_1, e_1), \ldots, \min(w_k, e_k)\} \leq_E \max\{\min(w_1, e'_1), \ldots, \min(w_k, e'_k)\}.$$

    Note that $\leq_{\min}$ is a weighted minimum with a null vector of weights, while $\leq_{\max}$ is a weighted maximum for the vector whose components are 1's.

Besides, we may rank the vectors in terms of the ordering of one of the components, that is, if $1 \leq r \leq k$ and we consider the vector of weights $w_r = 0$, and $w_j = 1$ otherwise, then

$$\overline{e} \leq_{\overline{w}-m} \overline{e'} \iff e_r \leq_{E_r} e'_r,$$

or in terms of $\leq_{\overline{w}-M}$, if $w_r = 1$, and $w_j = 0$ otherwise,

$$\overline{e} \leq_{\overline{w}-M} \overline{e'} \iff e_r \leq_{E_r} e'_r.$$

- Also, we may consider the *lexicographic ordering*, which acts like a "prioritised" one, in the sense that the smaller the index of the attribute/criterion, the greater is its relevance to determinate the ordering, because a criterion j is only applied if the previous criteria consider the elements equivalent. Indeed, the *lexicographic ordering* is defined as

$$\overline{e} \leq_{LEX} \overline{e'} \iff \exists\, j \leq k \text{ s.t. } \forall\, i < j,\ e_i = e'_i \text{ and } e_j \leq_{E_j} e'_j.$$

  $\leq_{LEX}$ is a total order.

We may consider a generalisation of these orderings. Given a set $\mathcal{R} = \{\sqsubseteq_i\}_{i=1,\ldots,k}$ of binary relations, for each "boolean" mapping $g{:}\{0,1\}^k \times \{0,1\}^k \to \{0,1\}$, let us introduce the following relations:

- if $\sqsubseteq_i \subseteq\ E_i \times E_i$, then the *induced relation by* $\mathcal{R}$ and g is defined as

$$\begin{aligned}\overline{e} \preccurlyeq^g_{\mathcal{R}} \overline{e'} \iff \quad & g((\mu_{\sqsubseteq_1}(e_1,e'_1),\ldots,\mu_{\sqsubseteq_k}(e_k,e'_k)),\\ & (\mu_{\sqsubseteq_1}(e'_1,e_1),\ldots,\mu_{\sqsubseteq_k}(e'_k,e_k))) = 1,\end{aligned}$$

  $\mu_{\sqsubseteq_i}$ being the membership of the preference ordering $\sqsubseteq_i$.

- Analogously, if $\sqsubseteq_i \subseteq\ \overline{E} \times \overline{E}$, then the *induced relation by* $\mathcal{R}$ *and g* is defined as

$$\begin{aligned}\overline{e} \preccurlyeq^g_{\mathcal{R}} \overline{e'} \quad \iff \quad & g(\big(\mu_{\sqsubseteq_1}(\overline{e},\overline{e'}),\ldots,\mu_{\sqsubseteq_k}(\overline{e},\overline{e'})\big),\\ & \big(\mu_{\sqsubseteq_1}(\overline{e'},\overline{e}),\ldots,\mu_{\sqsubseteq_k}(\overline{e'},\overline{e})\big)) = 1.\end{aligned}$$

Remark 4
Note that Pareto and Lexicographic orderings are of the type $\preccurlyeq^g_{\mathcal{R}}$ *. Indeed, if* $g(\overline{x},\overline{y}) = \min_{i=1,\ldots,k}\ x_i$ *and* $\mathcal{R} = \{\leq_{E_i}\}_{i=1,\ldots,k}$ *as usual* $\leq_{E_i}$ *being the linear order in the scale* E_i*, then*

$$\overline{e} \leq_{\Pi} \overline{e'} \iff \overline{e} \leq^g_{\mathcal{R}} \overline{e'}.$$

Analogously, if $g(\overline{x},\overline{y}) = \max_{i=1,\ldots,k}\ z_i$*, with*

$$z_i = \begin{cases} \min(x_1, 1-y_1), & \text{if } i = 1\\ \min(\min_{j=1,\ldots,i-1}\{\min(x_j,y_j)\}, \min(x_i, 1-y_i)), & \text{if } 1 < i < k\\ \min(\min_{j=1,\ldots,k-1}\{\min(x_j,y_j)\}, x_k), & \text{if } i = k,\end{cases}$$

then

$$\overline{e} \leq_{LEX} \overline{e'} \iff \overline{e} \leq^g_{\mathcal{R}} \overline{e'}.$$

6.2 Preferences on Product Scales

The first case we want to analyse is the following one. Assume that *DM* is provided with k criteria of preference on consequences, each one evaluated on a finite linearly ordered set of preference values. That is, the *DM* has a set $\{(U_j, \leq_j)\}_{j=1,\ldots,k}$ of finite linear scales such that $\inf(U_j) = 0_j,\ \sup(U_j) = 1_j$ and each U_j is commensurate with V, as usual V being a finite linear scale. A set of preference functions $u_j{:}X \to U_j$ such that $u_j^{-1}(1_j) \neq \emptyset \neq u_j^{-1}(0_j)$ is also assumed as given.

We consider the global vectorial preference function on consequences $\overline{u}{:}X \to \overline{U}$, where $\overline{U} = \Pi_{j=1,\ldots,z}\, U_j$ is the Cartesian product of the U_j's.

Now, in these conditions, we define the following vectorial qualitative utility functions.

Definition 6
Let $\top$ be a t-norm on V and let the pessimistic generalised qualitative utility functions be defined as usual as

$$GQU^-(\pi|u_j) = \min_{x\in X}\ n_j(\pi(x)\top\lambda_x^j), \qquad j = 1, \ldots, k$$

with $n_j(\lambda_x^j) = u_j(x)$, $n_j = n_{U_j} \circ h_j$, and n_{U_j} being the reversing involution on U_j. The linking mapping $h_j : V \to U_j$ is also required to satisfy coherence with respect to $\top$ for having a good definition of $GQU^-(\cdot|u_j)$. The vectorial pessimistic generalised qualitative utility function *w.r.t. $\overline{u} = (u_1, ..., u_k)$ is defined as*

$$\overline{GQU}^-(\cdot|\overline{u}) = (GQU^-(\cdot|u_1), ..., GQU^-(\cdot|u_k)).$$

Analogously, let the optimistic ones be defined as

$$GQU^+(\pi|u_j) = \max_{x\in X}\ h_j(\pi(x)\top\lambda_x^j), \qquad j = 1, \ldots, k$$

with $h_j(\lambda_x^j) = u_j(x)$. The vectorial optimistic generalised qualitative utility function *w.r.t. $\overline{u}$ is defined as*

$$\overline{GQU}^+(\cdot|\overline{u}) = (GQU^+(\cdot|u_1), ..., GQU^+(\cdot|u_k)).$$

As usual, from these functions we may induce on $\Pi(X)$ the orderings associated with them, that is,

$$\pi \preccurlyeq_{\overline{GQU}^-(\cdot|\overline{u})} \pi' \iff \overline{GQU}^-(\pi|\overline{u}) \leq_{\overline{U}} \overline{GQU}^-(\pi'|\overline{u}),$$

where $\leq_{\overline{U}}$ is the ordering considered on $\overline{U}$, e.g. Pareto, minimum, lexicographic, or one induced by a boolean function.
The dual ordering induced by $\overline{GQU}^+$ is

$$\pi \preccurlyeq_{\overline{GQU}^+(\cdot|\overline{u})} \pi' \iff \overline{GQU}^+(\pi|\overline{u}) \leq_{\overline{U}} \overline{GQU}^+(\pi'|\overline{u}).$$

In particular, we may consider the relation induced by $\overline{GQU}^-$ and a boolean function g. Indeed, for each "boolean" mapping g, we consider the induced relation by GQU^- (or by GQU^+) and g *defined as*

$$\pi \preccurlyeq^g_{\overline{GQU}^-(\cdot|\overline{u})} \pi' \iff \overline{GQU}^-(\pi|\overline{u}) \preccurlyeq^g_{\{\leq_{U_i}\}_{i=1,\ldots,k}} \overline{GQU}^-(\pi'|\overline{u}),$$

that is,

$$\pi \preccurlyeq^{g}_{\overline{GQU}^{-}(\cdot|\overline{u})} \pi' \iff \begin{array}{l} g\left(\left(\mu_{GQU^{-}(\cdot|u_1)}(\pi,\pi'),\ldots,\mu_{GQU^{-}(\cdot|u_k)}(\pi,\pi')\right),\right. \\ \left.\left(\mu_{GQU^{-}(\cdot|u_1)}(\pi',\pi),\ldots,\mu_{GQU^{-}(\cdot|u_k)}(\pi',\pi)\right)\right) = 1 \end{array}$$

$\mu_{GQU^{-}(\cdot|u_i)}$ *being the membership of the preference ordering induced by* $GQU^{-}(\cdot|u_i)$. *Analogously, we may consider the relations induced by the optimistic criterion, i.e.*

$$\pi \preccurlyeq^{g}_{\overline{GQU}^{+}(\cdot|\overline{u})} \pi' \iff \overline{GQU}^{+}(\pi|\overline{u}) \preccurlyeq^{g}_{\{\leq_{U_i}\}_{i=1,\ldots,k}} \overline{GQU}^{+}(\pi'|\overline{u}).$$

Now, we propose a characterisation for these relations.

Axiomatic Setting

Given a boolean function g, let $GAX^{g}_{\top}$ be the following set of axioms for a preference relation $\sqsubseteq$ on $(\Pi(X), M_{\top})$:

- $A0$: There exists a family $\mathcal{R} = \{\sqsubseteq_i\}_{i=1,\ldots,k}$ of orderings such that $\sqsubseteq = \preccurlyeq^{g}_{\mathcal{R}}$, i.e.

$$\pi \sqsubseteq \pi' \iff \begin{array}{l} g((\mu_{\sqsubseteq_1}(\pi,\pi'),\ldots,\mu_{\sqsubseteq_k}(\pi,\pi')), \\ \quad(\mu_{\sqsubseteq_1}(\pi',\pi),\ldots,\mu_{\sqsubseteq_k}(\pi',\pi))) = 1 \end{array}$$

- AxR : Each $\sqsubseteq_i$ satisfies $AX_{\top}$ $i = 1,\ldots,k$

Now, we may also consider the problem from an optimistic view, that is, we consider the axiomatic setting $GAX^{+\,g}_{\top}$, with $A0$ as previous, but now:

- AxR^{+} : $\sqsubseteq_i$ satisfying $AX^{+}_{\top}$ $i = 1,\ldots,k$.

Then, the following theorem is an easy consequence of the representation theorems in the framework of a unique linear preference scale.

Theorem 6.1 (Representation Theorem)
Given a boolean mapping g, a preference relation $\sqsubseteq$ on ($\Pi(X), M_{\top}$) satisfies the axiom set $GAX^{g}_{\top}$ ($GAX^{+\,g}_{\top}$) if and only if there exist:

(i) a set of finite linearly ordered preference scales $\{U_j\}_{j=1,\ldots,k}$, with $\inf(U_j) = 0_j$ and $\sup(U_j) = 1_j$,

(ii) a set $\{u_j : X \to U_j |\, u_j^{-1}(1_j) \neq \emptyset \neq u_j^{-1}(0_j)\}_{j=1,\ldots,k}$ of preference functions,

(iii) a set of onto order-preserving functions $h_j{:}V \to U_j,\ j = 1,\ldots,k$, each h_j also satisfying coherence w.r.t $\top$,

in such a way that it holds:

$$\pi \sqsubseteq \pi' \quad \textit{iff} \quad \pi \preccurlyeq^{g}_{\overline{GQU}^{-}(\cdot|\overline{u})} \pi'.$$

($\pi \sqsubseteq \pi'$ iff $\pi \preccurlyeq^{g}_{\overline{GQU}^{+}(\cdot|\overline{u})} \pi'$ resp.) with $n_j = n_{U_j} \circ h_j$ and considering the vectorial preference function $\overline{u} = (u_1, \ldots, u_k)$.

Proof:
Here, we only verify the pessimistic behaviour, the optimistic case being analogous.
$\rightarrow$) As each relation $\sqsubseteq_j$ satisfies $AX_{\top}$, then the existence of $\{U_j\}_{j=1,\ldots,k}$, $\{u_j\}_{j=1,\ldots,k}$ and $\{h_j\}_{j=1,\ldots,k}$ is guaranteed by the theorem for the linear case (Theorem 5.5). It only remains to verify that the relation induced by $\overline{GQU}^{-}$ and g coincides with $\sqsubseteq$.

By definition, we have that

$$\pi \sqsubseteq \pi' \qquad \text{iff} \qquad \mu_{\sqsubseteq}(\pi, \pi') = 1.$$

Moreover, as $\sqsubseteq_i$ is represented by $GQU^{-}(\cdot|u_i)$, we have that

$$\pi \sqsubseteq_i \pi' \iff GQU^{-}(\pi|u_i) \leq_{U_i} GQU^{-}(\pi'|u_i).$$

That is,

$$\mu_{\sqsubseteq_i}(\pi, \pi') = \mu_{\leq_{U_i}}(GQU^{-}(\pi), GQU^{-}(\pi')) = \mu_{GQU^{-}(\cdot|u_i)}(\pi, \pi').$$

Hence, applying $A0$, we have that

$$\begin{aligned}
\pi \sqsubseteq \pi' \quad &\iff \quad g((\mu_{\sqsubseteq_1}(\pi, \pi'), \ldots, \mu_{\sqsubseteq_k}(\pi, \pi')), \\
&\qquad\quad (\mu_{\sqsubseteq_1}(\pi', \pi), \ldots, \mu_{\sqsubseteq_k}(\pi', \pi))) = 1 \\
&\iff \quad g(\big(\mu_{GQU^{-}(\cdot|u_1)}(\pi, \pi'), \ldots, \mu_{GQU^{-}(\cdot|u_k)}(\pi, \pi')\big), \\
&\qquad\qquad \big(\mu_{GQU^{-}(\cdot|u_1)}(\pi', \pi), \ldots, \mu_{GQU^{-}(\cdot|u_k)}(\pi', \pi)\big)) = 1 \\
&\iff \quad \pi \preccurlyeq^{g}_{\overline{GQU}^{-}(\cdot|\overline{u})} \pi'.
\end{aligned}$$

$\leftarrow$) Now, we verify $A0$. Given $\{U_j\}$, $\{u_j\}$ and $\{h_j\}$, we consider $\sqsubseteq_j$ as the preference relation induced by $GQU^{-}(\cdot|u_j)$. By Theorem 5.5 we have that each $\sqsubseteq_j$ satisfies $AX_{\top}$. Hence,

$$\begin{aligned}
\pi \preccurlyeq^{g}_{\overline{GQU}^{-}(\cdot|\overline{u})} \pi' \iff \quad & g(\big(\mu_{GQU^{-}(\cdot|u_1)}(\pi, \pi'), \ldots, \mu_{GQU^{-}(\cdot|u_k)}(\pi, \pi')\big), \\
& \quad \big(\mu_{GQU^{-}(\cdot|u_1)}(\pi', \pi), \ldots, \mu_{GQU^{-}(\cdot|u_k)}(\pi', \pi)\big)) = 1 \\
\iff \quad & g((\mu_{\sqsubseteq_1}(\pi, \pi'), \ldots, \mu_{\sqsubseteq_k}(\pi, \pi')), \\
& \quad (\mu_{\sqsubseteq_1}(\pi', \pi), \ldots, \mu_{\sqsubseteq_k}(\pi', \pi))) = 1
\end{aligned}$$

$\square$

Remark 5
As it has been mentioned, this theorem includes, as particular cases, the characterisations of the Pareto and the lexicographic orderings.

Preference Functions on the Same Scale

We consider now the particular case in which *all* the preference functions on consequences are evaluated in the *same* scale of preference.

Proposition 6.2
Let $U_1 = \ldots = U_k = U$, all of them with the same ordering on it*, so* $\overline{U} = U^k$. *Then,*

1. (a) *if* $u_{\min}(x) = \min\{u_1(x), \ldots, u_k(x)\}$*, then*

$$\overline{GQU}^-(\pi|\overline{u}) \leq_{\min} \overline{GQU}^-(\pi'|\overline{u}) \iff GQU^-(\pi|u_{\min}) \leq GQU^-(\pi'|u_{\min}).$$

   (b) *Given a vector of weights* $\overline{w} = (w_1, \ldots, w_k) \in U^k$*, if* $u_{\overline{w}-m}(x) = \min\{\max(w_1, u_1(x)), \ldots, \max(w_k, u_k(x))\}$*, then*

$$\overline{GQU}^-(\pi|\overline{u}) \leq_{\overline{w}-m} \overline{GQU}^-(\pi'|\overline{u}) \iff GQU^-(\pi|u_{\overline{w}-m}) \leq GQU^-(\pi'|u_{\overline{w}-m}).$$

2. $\overline{GQU}^-(\pi|\overline{u}) \leq_{\Pi} \overline{GQU}^-(\pi'|\overline{u}) \Rightarrow \overline{GQU}^-(\pi|\overline{u}) \leq_{\min} \overline{GQU}^-(\pi'|\overline{u})$*.*

3. (a) *If* $u_{\max}(x) = \max\{u_1(x), \ldots, u_k(x)\}$*, then*

$$\overline{GQU}^+(\pi|\overline{u}) \leq_{\max} \overline{GQU}^+(\pi'|\overline{u}) \iff GQU^+(\pi|u_{\max}) \leq GQU^+(\pi'|u_{\max}).$$

   (b) *If* $u_{\overline{w}-M}(x) = \max\{\min(w_1, u_1(x)), \ldots, \min(w_k, u_k(x))\}$*, then*

$$\overline{GQU}^+(\pi|\overline{u}) \leq_{\overline{w}-M} \overline{GQU}^+(\pi'|\overline{u}) \iff GQU^+(\pi|u_{\overline{w}-M}) \leq GQU^-(\pi'|u_{\overline{w}-M}).$$

4. $\overline{GQU}^+(\pi|\overline{u}) \leq_{\Pi} \overline{GQU}^+(\pi'|\overline{u}) \Rightarrow \overline{GQU}^-(\pi|\overline{u}) \leq_{\max} \overline{GQU}^-(\pi'|\overline{u})$

Proof:
We only sketch the proofs of 1) and 2), the others being analogue.

1. It is a direct consequence of having

$$\begin{aligned} GQU^-(\pi'|u_{\min}) &= GQU^-(\pi| \min_{j=1,\ldots,k} u_j) \\ &= \min_{j=1,\ldots,k} (GQU^-(\pi|u_j), \end{aligned}$$

   and by the definition of $\leq_{\min}$.
   For the case of the *weighted-minimum*, we also know that

$$\forall j, w_j \in U_j,\ GQU^-(\pi| \max(w_j, u_j)) = \max(w_j, GQU^-(\pi|u_j)).$$

2. By definition of Pareto ordering,

$$\overline{GQU}^{-}(\pi|\overline{u}) \leq_{\Pi} \overline{GQU}^{-}(\pi'|\overline{u}) \iff \forall i,\ GQU^{-}(\pi|u_i) \leq_{U_i} GQU^{-}(\pi'|u_i)$$

and thus we have that

$$\overline{GQU}^{-}(\pi|\overline{u}) \leq_{\Pi} \overline{GQU}^{-}(\pi'|\overline{u}) \text{ implies } \overline{GQU}^{-}(\pi|\overline{u}) \leq_{\min} \overline{GQU}^{-}(\pi'|\overline{u}).$$

$\square$

Remark 6
Let us remark some points with respect to the preceding proposition:

- *In item 1 (a), the proposition guarantees that the order induced in $\Pi(X)$ by the pessimistic vectorial utility function $\overline{GQU}^{-}(\cdot|\overline{u})$ together with the $\leq_{\min}$ ordering in $\overline{U}$, is the same than the order induced by the utility function defined with respect to the function minimum of preferences, i.e. by $GQU^{-}(\cdot|u_{\min})$ with $u_{\min}(x) = \min\{u_1(x), ..., u_k(x)\}$, taking in U its linear ordering.* That is, it is the same to "aggregate" first the preferences with the minimum, and then evaluating with a unidimensional utility function, than evaluating the vectorial utility before aggregating.
Moreover, this property makes clear that the $\leq_{\min}$ ordering satisfies the axiom set $AX_{\top}$ if the set of preference functions u_j:$X \rightarrow U$ not only verifies $\forall j = 1, \ldots, k,\ u_j^{-1}(0) \neq \emptyset$ but $\bigcap_{j=1,\ldots,k} u_j^{-1}(1) \neq \emptyset$ as well.

- *Obviously, the reciprocal of the item 2 is not true, because both orderings may be different since $\leq_{\min}$ is a linear order while $\leq_{\Pi}$ may be an only partial one. Also, both orderings distinguish different in the sense that there are distributions which $\leq_{\min}$ consider them equivalent while $\leq_{\Pi}$ distinguish them. An easy example of this is the following one.*

  Example:
  Suppose $k = 2$, let $x, x' \in X$, s.t. $u_1(x) = u_1(x') < u_2(x) < u_2(x')$. Since

$$\begin{aligned} \overline{GQU}^{-}(x|\overline{u}) &= (u_1(x), u_2(x)) \\ \overline{GQU}^{-}(x'|\overline{u}) &= (u_1(x'), u_2(x')), \end{aligned}$$

  then

$$\overline{GQU}^{-}(x|\overline{u}) <_{\Pi} \overline{GQU}^{-}(x'|\overline{u}),$$

  while

$$\overline{GQU}^{-}(x|\overline{u}) \sim_{\min} \overline{GQU}^{-}(x'|\overline{u}).$$

$\diamondsuit$

- *With respect to item* 3*, analogously with the case of* $minimum$*, it results the same ordering if we max-aggregate first or at the end. Also,* $\leq_{\max}$*-ordering satisfies the axiom set* $AX_{\top}^{+}$ *if the set of preference functions* u_j*:*$X \rightarrow U$ *not only verifies* $\forall\ j = 1, \ldots, k,\ u_j^{-1}(1) \neq \emptyset$*, but* $\bigcap_{j=1,\ldots,k} u_j^{-1}(0) \neq \emptyset$ *as well.*

6.3 An Example: A Safety Decision Problem in a Chemical Plant (Continuation)

To exemplify some of the notions introduced in this Chapter, we consider again the example introduced in Section 5.2. Let us recall the framework. The chemical plant has three emergency plans:

$$\begin{array}{lcl} EP1 & : & emergency\ plan\ 1, \\ EP2 & : & emergency\ plan\ 2, \\ EV & : & total\ evacuation, \end{array}$$

that may be only activated by the head of the Safety Department. Depending on the type of problems, the situations of the plant may be classified in four modes:

$$\begin{array}{lcl} s_0 & : & \textit{normal functioning}, \\ s_1 & : & \textit{minor problem}, \\ s_2 & : & \textit{major problem}, \\ s_3 & : & \textit{very serious problem}. \end{array}$$

The head of the Dept. has to undertake one of the following actions:

$$\begin{array}{lcl} d_0 & : & \textit{do nothing (DN)}, \\ d_1 & : & \textit{activate emergency plan 1}\ (AEP1), \\ d_2 & : & \textit{activate emergency plan 2}\ (AEP2), \\ d_3 & : & \textit{activate evacuation}\ (AEVA), \end{array}$$

whose behaviours are given in Table 5.1.

As it was said, the *post-situation* of the plant is evaluated in terms of two criteria:

- *personal safety* (u_1),
- *economical costs* (u_2).

We take as preference scale for each criterion a linear scale of four values

$$W = \{w_0 = 0 < w_1 < w_2 < w_3 = 1\},$$

the criteria being defined as:

$$u_1(Risk = i, Cost = j) = w_{3-i} \qquad \text{and} \qquad u_2(Risk = i, Cost = j) = w_{3-j}.$$

We take as scale of uncertainty the same linear scale, i.e. $V = U$.

Assume that the received report says:

"A problem has been identified in Building G, likely it is a minor problem, but it is not discarded that either it can finally turn out to be a false alarm or even, in the worst case, it might become a major problem."

This information can be modelled by the possibility distribution on states $\pi_S{:}S \to V$ defined as

$$\pi_S(s_0) = w_1,\ \pi_S(s_1) = 1,\ \pi_S(s_2) = w_2,\ \pi_S(s_3) = 0.$$

Now, for choosing the "best" decision, we have to rank the associated distributions. These distributions are defined as in (4.1), for instance, for declarating that the situation is controlled, that is, to choose do nothing (d_0), its distribution is

$$\pi_{d0}(x) = \sup\{\pi_S(s) |\ d_0(s) = x\}.$$

So, in order to rank decisions we apply the generalised qualitative utility functions to these distributions. We consider the global preference on consequences is given by $\overline{u} = (u_1, u_2)$.
If $\top = minimum$, then we have that:

$$\begin{aligned}
\overline{GQU}^-(\pi_{d0}|\overline{u}) &= (w_1, 1),\\
\overline{GQU}^-(\pi_{d1}|\overline{u}) &= (w_2, w_2),\\
\overline{GQU}^-(\pi_{d2}|\overline{u}) &= (1, w_1),\\
\overline{GQU}^-(\pi_{d3}|\overline{u}) &= (1, 0).
\end{aligned}$$

Hence, only d_3 is discarded if *Pareto ordering* is chosen in $\overline{U} = W \times W$, while d_1 is the most preferred if the *minimum ordering* is considered. However, taking into account that the safety of the persons is involved and it must be prioritised to economical reasons, it is interesting to consider the *lexicographic ordering* considering u_1 first. For this ordering, we have that d_2, *activate emergency plan 2*, is chosen, which responds to giving priority to safety.

6.4 Uncertainty Measured on Product Scales

In this case, we assume the set of values for uncertainty ($\overline{V}, \leq_{\overline{V}}$) is a product of scales, that is, $\overline{V} = \Pi_{j=1,\ldots,k} V_j$, each V_j being a finite linearly ordered set. For instance, this may occurs when there are several sources of uncertainty each one being measured in a linear scale. Although sometimes we might aggregate this information into a linear scale, sometimes it may be interesting not to loose any information and go as far as possible without aggregating.

Hence, we are interested in the special class of *possibility vectorial distributions*, $\Pi{:}X \to \overline{V}$, such that *all their projections are normalised possibility distributions*. That is, if

$$\pi_j{:}X \to V_j \qquad\qquad j = 1, \ldots, k$$

are normalised distributions, then

$$\Pi(x) = (\pi_1(x), \ldots, \pi_k(x))$$

is the product of the normalised distributions. Observe that although Π is consistent, in the sense that $\sup\{\Pi(x)|x \in X\}$ = (1,...,1), Π may result non-normalised.

Let us denote by

$$Vec\Pi(X,\overline{V}) = \{(\pi_1,\ldots,\pi_k)|\ \pi_j \in \Pi(X,V_j),\ j = 1,...,k\},$$

the set of vectorial distributions on $\overline{V}$ whose projections are normalised.

As usual, we consider in this set a mixture operation defined in terms of a t-norm $\top$ in $\overline{V}$.

In order to obtain a mixture operation that satisfies reduction of lotteries, we are interested in t-norms $\top$ in $\overline{V}$ whose projections are join morphisms. By (Baets and Mesiar, 1999; theorem 7.1), $\top$ satisfies this condition if and only there exists a finite family of t-norms $\top_j$ on V_j s.t. $\top = \Pi_{j=1,\ldots,k}\top_j$. From now on, we restrict ourselves to work with t-norms in $\overline{V}$ which are Cartesian products of t-norms in V_j's.

Given a set of t-norms $\{\top_j\}_{j=1,\ldots,k}$, consider the t-norm product of the $\top_j$'s, i.e.

$$\top = \Pi_{j=1,\ldots,k}\top_j.$$

Then, we define the mixture $\overline{M_\top}$ on $Vec\Pi(X,\overline{V})$ as:

$$\overline{M_\top}(\Pi,\Pi';\overline{\alpha},\overline{\beta}) = (\max(\alpha_1\top_1\pi_1,\beta_1\top_1\pi'_1),\ldots,\max(\alpha_k\top_k\pi_k,\beta_k\top_k\pi'_k)),$$

with $\overline{\alpha} = (\alpha_1,\ldots,\alpha_k)$, $\overline{\beta} = (\beta_1,\ldots,\beta_k) \in \overline{V}$ s.t. $\max(\alpha_j,\beta_j) = 1\ \forall j$.

Also, for each t-norm on V_j, we consider $M_{\top_j}$ the mixture induced on $\Pi(X,V_j)$. Observe that $\overline{M_\top}$ satisfies that:

$$\overline{M_\top}(\Pi,\Pi';\overline{\alpha},\overline{\beta}) = (M_{\top_1}(\pi_1,\pi'_1;\alpha_1,\beta_1),...,M_{\top_k}(\pi_k,\pi'_k;\alpha_k,\beta_k)).$$

In $(Vec\Pi(X,\overline{V}),\overline{M_\top})$ we may consider different orderings taking into account that preference on consequences are represented by a linear preference function u or by a vectorial one $\overline{u}$. For each case, we may define a generalised pessimistic or optimistic criterion. Indeed, we may have the following cases:

U **linear.** Given a preference function u:$X \to U$ and a set of onto order-preserving functions h_j:$V_j \to U$, each h_j being coherent w.r.t $\top_j$, we propose to use the following expression[1] for a pessimistic evaluation

$$VGQU^-(\Pi|u) = (GQU^-(\pi_1|u),\ldots,GQU^-(\pi_k|u)),$$

where as usual $GQU^-(\pi_j|u) = \min_{x\in X} n_j(\pi(x)\top_j\ \lambda_x^j)$, with $n_j(\lambda_x^j) = u(x)$. For an optimistic behaviour we propose

$$VGQU^+(\Pi|u) = (GQU^+(\pi_1|u),\ldots,GQU^+(\pi_k|u)),$$

with $GQU^+(\pi_j|u) = \max_{x\in X} h_j(\pi(x)\top_j\ \lambda_x^j)$, where $h_j(\lambda_x^j) = u(x)$.

[1]Actually, we should write $GQU^-(\pi_j|h_j,u)$, however, for the sake of simplicity we omit the h_j's.

U **cartesian product.** Let $U = \Pi_{j=1,k} U_j$, each U_j being a finite linear scale and let $\overline{u} = (u_1, \ldots, u_k)$ be a (vectorial) preference function on U with components $u_j{:}X \to U_j$ such that $u_j^{-1}(1_j^U) \neq \emptyset \neq u_j^{-1}(0_j^U)$. Further we assume each U_j is commensurate with V_j through onto order-preserving functions $h_j{:}V_j \to U_j$ which are coherent w.r.t $\top_j$. Then, we define the following utility functions

$$\overline{VGQU}^-(\Pi|\overline{u}) = (GQU^-(\pi_1|u_1), \ldots, GQU^-(\pi_k|u_k))$$
$$\overline{VGQU}^+(\Pi|\overline{u}) = (GQU^+(\pi_1|u_1), \ldots, GQU^+(\pi_k|u_k)),$$

where $GQU^-(\pi|u_j) = \min_{x \in X} n_j(\pi(x) \ \top_j \ \lambda_x^j)$ and $GQU^+(\pi|u_j) = \max_{x \in X} h_j(\pi(x) \top_j \delta_x^j)$, with $n_j(\lambda_x^j) = h_j(\delta_x^j) = u_j(x)$.

In the following sections we analyse them in some detail.

6.4.1 Linear Preference

Let us consider a particular situation for the first case. We assume that $V_1 = \ldots = V_k = W$, W being a linear scale, and also $\top_1 = \ldots = \top_k$. For this case, for each fixed boolean function g, we have the following representation result.

Theorem 6.3
Let $\sqsubseteq$ a preference relation on $(Vec\Pi(X, W^k), \overline{M_\top})$. Then, it satisfies

- *there exists a preference relation $\sqsubseteq_W$ on $\Pi(X, W)$ such that*

$$\mu_\sqsubseteq(\overline{\Pi}, \overline{\Pi'}) = g(\left(\mu_{\sqsubseteq_W}(\pi_1, \pi_1'), \ldots, \mu_{\sqsubseteq_W}(\pi_k, \pi_k')\right), \left(\mu_{\sqsubseteq_W}(\pi_1', \pi_1), \ldots, \mu_{\sqsubseteq_W}(\pi_k', \pi_k)\right))$$

  with $\Pi = (\pi_1, \ldots, \pi_k), \ \Pi' = (\pi_1', \ldots, \pi_k')$.

- *$\sqsubseteq_W$ satisfies $AX_\top$ ($AX_\top^+$ resp.)*

if and only if there exist:

(i) a finite linearly ordered preference scale U with $\inf(U) = 0$ and $\sup(U) = 1$,

(ii) a preference function $u{:}X \to U$ such that $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,

(iii) an onto order-preserving function $h{:}W \to U$, h being coherent w.r.t $\top$,

in such a way that it holds:

$$\Pi \sqsubseteq \Pi' \iff VGQU^-(\Pi|u) \preccurlyeq^g_{\{\leq_U\}} VGQU^-(\Pi'|u)^2.$$

($\Pi \sqsubseteq \Pi' \iff VGQU^+(\Pi|u) \preccurlyeq^g_{\{\leq_U\}} VGQU^+(\Pi'|u)$ resp.)

Still assuming that all the linear scales in the cartesian product of uncertainty are the same, i.e. $\overline{V} = W^k$, with W linear, we may consider the preference orderings related with *min-ordering* in U^k.

[2]Here, $\preccurlyeq^g_{\{\leq_U\}}$ means that $\mathcal{R} = \{\leq_U\}_{i=1,\ldots,k}$.

Lemma 6.4
$\forall\, \Pi, \Pi' \in Vec\Pi(X,\ W^k)$,

$$VGQU^-(\Pi|u) \leq_{\min} VGQU^-(\Pi'|u) \iff GQU^-(\max\{\pi_1, \ldots, \pi_k\}|u) \leq_U GQU^-(\max\{\pi'_1, \ldots, \pi'_k\}|u)$$

with the distribution $\max\{\pi_1, \ldots, \pi_k\}(x) = \max\{\pi_1(x), \ldots, \pi_k(x)\}$.

Proof:
It is a direct consequence of the definition of the $\leq_{\min}$ ordering and of being

$$GQU^-(\max\{\pi_1, \ldots, \pi_k\}|u) = \min\{GQU^-(\pi_1|u), \ldots, GQU^-(\pi_k|u)\}.$$

$\square$

Notice that we have only considered the special case of having a linear scale of preference and the same scale in the cartesian product where we measure uncertainty. The case of having different scales remains as an open question.

6.4.2 Preferences Measured on Cartesian Products

Now, we consider the case of having a vectorial preference function on consequences over $\overline{U}$.

Axiomatic Setting

Given a boolean function g, let $VGAX^g_\top$ be the following set of axioms for preference relations $\sqsubseteq$ on $(Vec\Pi(X, \overline{V}), \overline{M_\top})$, with $\top$ s.t. $\top = \Pi_{j=1,\ldots,k} \top_j$, each $\top_j$ being a t-norm on V_j:

- *VA0*: There exists a family $\{(\Pi(X, V_i), \sqsubseteq_i)\}_{i=1,\ldots,k}$ of orderings such that

$$\mu_\sqsubseteq(\Pi, \Pi') = g(\left(\mu_{\sqsubseteq_1}(\pi_1, \pi'_1), \ldots, \mu_{\sqsubseteq_k}(\pi_k, \pi'_k)\right), \left(\mu_{\sqsubseteq_1}(\pi'_1, \pi_1), \ldots, \mu_{\sqsubseteq_k}(\pi'_k, \pi_k)\right))$$

- $AxR1$: $\sqsubseteq_i$ satisfies $AX_{\top_i}$ for each $i = 1, ..., k$

For representing the preference relations on $Vec\Pi(X, \overline{V})$ we propose the following theorem.

Theorem 6.5
A preference relation $\sqsubseteq$ *on* $(Vec\Pi(X, \overline{V}), \overline{M_\top})$, *satisfies the axiom set* $VGAX_\top$ *if and only if there exist:*

(i) a set of finite linearly ordered preference scales $\{U_j\}_{j=1,\ldots,k}$ *with* $\inf(U_j) = 0^U_j$ *and* $\sup(U_j) = 1^U_j$,

(ii) a set of preference functions u_j:$X \to U_j$ *such that* $u_j^{-1}(1^U_j) \neq \emptyset \neq u_j^{-1}(0^U_j)$,

(iii) a set of onto order-preserving functions $h_j\colon V_j \to U_j$, each h_j being coherent w.r.t $\top_j$,

in such a way that it holds:

$$\Pi \sqsubseteq \Pi' \iff \overline{VGQU}^-(\Pi|\overline{u}) \preccurlyeq^g_{\{\leq_{U_j}\}_{j=1,\dots,k}} \overline{VGQU}^-(\Pi'|\overline{u}),$$

with

$$\overline{VGQU}^-(\Pi|\overline{u}) = (GQU^-(\pi_1|u_1), ..., GQU^-(\pi_k|u_k)),$$

and $GQU^-(\pi|u_j) = \min_{x\in X} n_j(\pi(x)\top_j\lambda_x^j)$, where $n_j(\lambda_x^j) = u_j(x)$.

The proof of the theorem is straightforward.

As usual for an optimistic behaviour, we consider $VGAX^+_\top$, which is obtained from $VGAX_\top$ replacing $AxR1$ by

- $AxR1^+$: $\sqsubseteq_i$ satisfies $AX^+_{\top_i}$ for each $i = 1, ..., k$

for characterising the preference ordering induced by $\overline{VGQU}^+(\Pi|\overline{u}) = (GQU^+(\pi_1|u_1), ..., GQU^+(\pi_k|u_k))$, $GQU^+(\cdot|u_j)$ being defined as usual.

6.5 Another Framework for the Chemical Plant Example

Now, assume that instead of receiving the report of the plant engineer the head of the Safety Department receives the evaluations of the responsible of control of each system. For each state, two evaluations of the possibility of being in this state are provided. Assume he has the following evaluations:

$$\Pi_S(s_0) = (w_1, w_1),\ \Pi_S(s_1) = (1, 1),\ \Pi_S(s_2) = (w_2, w_1),\ \Pi_S(s_3) = (0, 0).$$

Now, both $\overline{U}$ and $\overline{V}$ are supposed to be equal to $W \times W$, with $W = \{0 = w_0 < w_1 < w_2 < w_3 = 1\}$. We choose the Pareto ordering both in $\overline{U}$ and $\overline{V}$. We are interested in comparing the results of the ranking of distributions with $\overline{VGQU}(\cdot|\overline{u})$ for different t-norms, $\bar{u}$ being defined like as in Section 6.3 and $\bar{h}$ is the identity while the same t-norm is considered in V.

For each decision we have their associated distributions:

$$\begin{aligned}
\Pi_{d0} &= ((w_1, w_1)/(Risk = 0, Cost = 0), (1, 1)/(Risk = 1, Cost = 0),\\
&\quad\ (w_2, w_1)/(Risk = 2, Cost = 0)),\\
\Pi_{d1} &= ((1, 1)/(Risk = 0, Cost = 1), (w_2, w_1)/(Risk = 1, Cost = 1)),\\
\Pi_{d2} &= ((1, 1)/(Risk = 0, Cost = 2)),\\
\Pi_{d3} &= ((1, 1)/(Risk = 0, Cost = 3)),
\end{aligned}$$

and their evaluations are:

$$\begin{aligned}
\overline{VGQU}^{-}(\Pi_{d0}|\overline{u}) &= (\min\{w_2, w_1 \perp w_1\}, 1), \\
\overline{VGQU}^{-}(\Pi_{d1}|\overline{u}) &= (\min\{w_3, w_1 \perp w_2\}, \min\{w_2, w_2 \perp w_2\}), \\
\overline{VGQU}^{-}(\Pi_{d2}|\overline{u}) &= (1, w_1), \\
\overline{VGQU}^{-}(\Pi_{d3}|\overline{u}) &= (1, 0),
\end{aligned}$$

$\perp$ being the dual conorm of $\top$ with respect to the involution in W. Note that d_2 is preferred to d_3 for any t-norm. In order to obtain the utility values for d_0 and d_1, we take two particular t-norms. If we choose $\top = minimum$, we have

$$\begin{aligned}
\overline{VGQU}^{-}(\Pi_{d0}|\overline{u}) &= (w_1, 1), \\
\overline{VGQU}^{-}(\Pi_{d1}|\overline{u}) &= (w_2, w_2).
\end{aligned}$$

So, we have that choosing *minimum* d_0, d_1 and d_2 are incomparable, only d_3 may be discharged. While if we choose *Lukasiewicks t-norm*, we have

$$\begin{aligned}
\overline{VGQU}^{-}(\Pi_{d0}|\overline{u}) &= (w_2, 1), \\
\overline{VGQU}^{-}(\Pi_{d1}|\overline{u}) &= (1, w_2).
\end{aligned}$$

That is, d_1 is preferred to d_2 (d_2 being preferred to d_3), while d_1 and d_0 remains incomparable.

Chapter 7

Utility Functions for Representing Partial Preference Relations

In this Chapter, we consider the remaining extensions mentioned in the introduction of Chapter 6. That is, we consider now the cases in which uncertainty and preferences values belong, in principle, to distributive lattices. Of course, the products of linear scales considered in Chapter 6 are particular types of distributive lattices.

As usual, we are interested in having commensurate valuation sets for uncertainty and preference, this means we require the existence of an onto order-preserving mapping $h{:}V \to U$. But now, we may have incomparable values of uncertainty, and h may be required to treat them in different ways (see Figure 7.1). Indeed, given two incomparable values λ and λ' on V, their respective images may be required to be:

1. **incomparable:** it means that the associated distributions π_λ's are considered incomparable as well. In this case, the requirement will be,

$$\text{if } \lambda <> \lambda' \quad \text{then} \quad h(\lambda) <> h(\lambda').$$

2. **equal:** it means that their associated distributions are considered equivalent with respect to the preference relation. In this case, we have two further alternatives depending on the value that h assigns to $\lambda \vee \lambda'$. Indeed, we have:

   (a) The distribution associated with the supremum of the values is indistinguishable from the associated with λ and λ', i.e.

   $$\text{if } \lambda <> \lambda' \quad \text{then} \quad h(\lambda \vee \lambda') = h(\lambda') = h(\lambda).$$

   In this case, h results a join-morphism.

   (b) The associated distributions π_λ's are again indistinguishable, but they are not indistinguishable with the distribution associated with $\lambda \vee \lambda'$.

   That is,

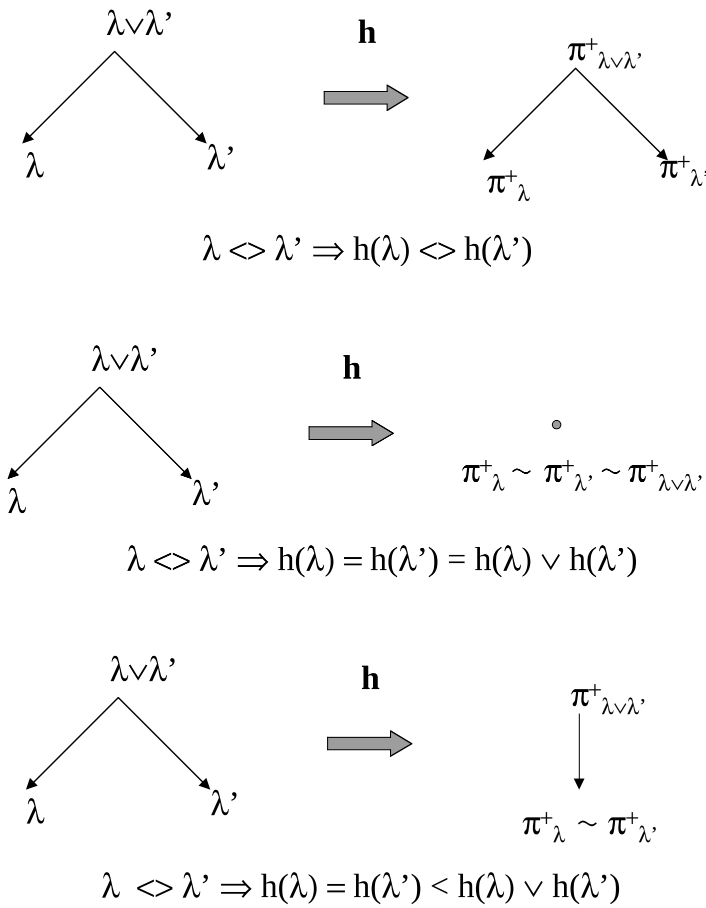

Figure 7.1: Different possible properties for the linking mapping h w.r.t. incomparable values.

$$\text{if } \lambda <> \lambda' \qquad \text{then} \qquad h(\lambda \vee \lambda') > h(\lambda') = h(\lambda).$$

> Now, h is *not* a join-morphism. Observe that in this case the distribution associated with $\lambda \vee \lambda'$ will be less (more) preferred than the associated with λ and λ' if the behaviour is pessimistic (optimistic resp.).

In case 1) incomparability is "preserved", hence if V is a non-linear lattice, so is U. We will analyse this case in detail, taking into account the different operators available in V. In case 2a) incomparability is lost, moreover, it forces U to be linear. We shall deal with the option that considers the three associated distributions as equivalent, the

remaining case being left as a future work[1].
In the next Section, we introduce some necessary background on lattices and some preliminary results that are required through the Chapter. Next, we consider the case of h preserving incomparability. In the first part, we shall only assume available in the lattices the $meet$ and $join$ operations. As usual, we are interested in considering "possibilistic mixtures" (like "max-min" mixtures) on the set of "possibilistic" distributions on V, requiring this operation to satisfy reduction. Because of this, we require the lattices to be distributive. In the second part, we assume available other operations on the lattices, which allows us to consider other alternative mixtures. Again, the requirement of satisfying reduction of lotteries leads us to work with residuated distributive measurement lattices. For both cases, we introduce pessimistic and optimistic criteria for these frameworks and their axiomatic characterisations as well. Finally, in the last Section we consider the case of considering the distribution associated to the supremum of incomparable values, λ, λ', indistinguishable of $\pi_\lambda \sim \pi_{\lambda'}$.

7.1 Some Background on Lattices

Let us recall some definitions and results related with lattices (see, for example, (Davey and Priestley, 1990; Grätzer, 1978) for more details) that we will use in the following.

- A set L with a binary relation on it $\leq$, is an *ordered set*, also called a *partially ordered set*, if for all $x, y, z \in L$, $\leq$ satisfies:

  a) reflexivity: $x \leq x$,

  b) antisymmetry: $x \leq y,\ y \leq x$ imply $x = y$,

  c) transitivity: $x \leq y,\ y \leq z$ imply $x \leq z$.

- Let $(L, \leq)$ be a *partially ordered set*, let be $S \subseteq L$,
  - $x \in S$, x is *an upper bound of* S if $s \leq x\ \forall s \in S$.
  - The set of all upper bounds of S, is denoted by S^u. If S^u has a least element, it is called *least upper bound of S* or *supremum,* also denoted by $\sup\ S$.
  - Analogously, $x \in S,\ x$ is *an lower bound of* S if $s \geq x\ \forall s \in S$, and the set of all lower bounds of S, is denoted by S^l. If S^l has a great element, it is called *greatest lower bound of S* or *infimum* also denoted by $\inf\ S$.

- A non-empty ordered set S is a *join-semilattice* if $\sup\{x, y\} \in S\ \forall x, y \in S$. Analogously, S is a *meet-semilattice* if $\inf\{x, y\} \in S\ \forall x, y \in S$.

- An *ordered set* $(L, \leq)$ is a lattice iff it is a *join-semilattice* and a *meet-semilattice*.

- A lattice $(L, \leq)$ is *bounded* if it has *supremum* (1) and *infimum* (0), in this case we denote it by $(L, \leq, 0, 1)$.

[1] Notice that since h is not join morphism the generalised "utility" functions $GQU(\cdot|h)$ will not preserve mixtures.

- Given a lattice $(L, \leq)$, two binary operations may be defined: *meet*($\wedge$) and *join*($\vee$).

$$x \wedge y = \inf\{x, y\} \quad \text{and} \quad x \vee y = \sup\{x, y\}.$$

- Let $(L_1, \wedge_1, \vee_1)$ and $(L_2, \wedge_2, \vee_2)$ be two lattices. A mapping $f{:}L_1 \to L_2$ is a *lattice homomorphism*, a *homomorphism* for short, if f is join-preserving and meet-preserving, i.e.

$$f(a \vee_1 b) = f(a) \vee_2 f(b) \quad \text{and} \quad f(a \wedge_1 b) = f(a) \wedge_2 f(b).$$

  If f is also onto, it is called *epimorphism*.

- If $(L_1, \wedge_1, \vee_1, 0_1, 1_1)$ and $(L_2, \wedge_2, \vee_2, 0_2, 1_2)$ are bounded lattices, f is a *{0,1}-homomorphism* if it is a homomorphism also satisfying $f(0_1) = 0_2, f(1_1) = 1_2$.

Observe the well known connection between $\vee, \wedge$ and $\leq$: Let L be a lattice and let $a, b \in L$. Then, the following are equivalent:

1. $a \leq b$,
2. $a \vee b = b$,
3. $a \wedge b = a$,

- $(L, \wedge, \vee, n_L, 0, 1)$ will denote a *bounded lattice with a reversing involution*, i.e. L satisfies that $0, 1 \in L$ and $0 \leq x \leq 1 \ \forall x \in L$, and $n_L{:}L \to L$ is a strict decreasing function[2] s.t. $n_L(n_L(x)) = x$.

Proposition 7.1

- *Let* $(L, \wedge, \vee)$ *be a* lattice*, then,* $\wedge$ *and* $\vee$ *are associative, commutative, satisfy idempotency and the absorption laws*[3]*.*

- *If* $(L, \wedge, \vee)$ *is a* finite lattice*, then,* L *is a bounded lattice.*

- *If* $(L, \wedge, \vee, n_L, 0, 1)$ *is a lattice with reversing-involution, then,* n_L *satisfies that:*
  - $n_L(0) = 1$ *and* $n_L(1) = 0$,
  - $n_L(x \wedge y) = n_L(x) \vee n_L(y)$,
  - $n_L(x \vee y) = n_L(x) \wedge n_L(y)$.

Definition 7

Given a partially pre-ordered set $(L, \leq)$*, i.e.* $\leq$ *is reflexive and transitive, the associated* indifference relation $\sim$ *and the* incomparability relation $<>$ *are defined as:*

- $a \sim b \iff (a \leq b \ \ and \ \ b \leq a)$.
- $a <> b \iff (a \nleq b \ \ and \ \ b \nleq a)$.

[2] n_L is bijective.

[3] Idempotency means: $a \vee a = a, a \wedge a = a$, absorption is: $a \vee (a \wedge b) = a, a \wedge (a \vee b) = a$

Now, we introduce a new definition and related results that will be applied in our proposal.

Definition 8
Let $(L, \leq)$ *be a partially pre-ordered set, denote by* $L/\sim$ *the* quotient set w.r.t. $\sim$ *and let* $[a] = \{y \in L | a \sim y\}$.
$(L, \leq)$ *is a* pre-lattice *iff* $(L/\sim, \sqsubseteq)$ *is a lattice, defining* $\sqsubseteq$ *as:*

$$[a] \sqsubseteq [b] \qquad \textit{iff} \qquad a \leq b.$$

As a consequence of the $\sim$ definition, we have that

Proposition 7.2
Let $(L, \leq)$ *be a partially pre-ordered set, then:*

- $\sim$ *is an equivalence relation.*
- *if* $(L, \leq)$ *is totally pre-ordered,* $(L/\sim, \sqsubseteq)$ *is a linearly ordered set.*

Theorem 7.3
$(A, \leq)$ *is a pre-lattice iff it is a partially pre-ordered set, such that satisfies:*

1. *For all* $a, b \in A$ *there exists an unique non-empty subset* $SUP(a, b) \subseteq A$ *s.t.*
   - $SUP(a, b)$ *is an equivalence class of the quotient set* $A/\sim$, *i.e.* $SUP(a, b) \in A/\sim$.
   - $\forall c \in SUP(a, b), a \leq c$ *and* $b \leq c$.
   - *if* $a \leq e$ *and* $b \leq e$, *then, either* $(e \in SUP(a, b))$ *or* $(e >$ [4] $c,\ c \in SUP(a, b))$.
2. *For all* $a, b \in A$ *there exists an unique non-empty subset* $INF(a, b) \subseteq A$ *s.t.*
   - $INF(a, b)$ *is an equivalence class of the quotient set* $A/\sim$, *i.e.* $INF(a, b) \in A/\sim$.
   - *if* $e \leq a$ *and* $e \leq b$, *then either* $(e \in INF(a, b))$ *or* $(c > e\ \ c \in INF(a, b))$.
   - $\forall c \in INF(a, b), c \leq a$ *and* $c \leq b$.

Proof:
$\leftarrow$) We will verify that $(A/\sim, \vee)$ is a joint-semilattice and $(A/\sim, \wedge)$ is a meet-semilattice.

1. First, we verify that $(A/\sim, \vee)$ is a joint-semilattice, with $\vee$ defined as

$$[a] \vee [b] = SUP(a, b). \tag{7.1}$$

[4] $e > c$ iff $c \leq e$ and $e \nleq c$.

Observe that $\vee$ is well defined, i.e.

$$\text{if } \ a \sim a' \text{ then } [a] \vee [b] = [a'] \vee [b].$$

Indeed, if $S_{a,b}$ and $S_{a',b}$ denote an element of $SUP(a,b)$ and $SUP(a',b)$ respectively, we verify now that $S_{a,b} \sim S_{a',b},$ i.e. $SUP(a,b) = SUP(a',b)$.
As

$$S_{a,b} \geq a \sim a' \quad \text{ and } \quad S_{a,b} \geq b,$$

by definition of $SUP(a',b)$, we have that $S_{a,b} \geq S_{a',b}$.
Conversely, since

$$S_{a',b} \geq a' \sim a \quad \text{ and } \quad S_{a',b} \geq b,$$

by definition of $SUP(a,b)$ we have that $S_{a',b} \geq S_{a,b}$, therefore

$$S_{a,b} \sim S_{a',b}.$$

In order to see that $(A/\sim, \vee)$ is a joint-semilattice, we will verify that

- $\vee$ is associative.
  Indeed, by definition of $SUP(c, S_{a,b})$ we have that

  $$S_{c,S_{a,b}} \geq c, \quad S_{c,S_{a,b}} \geq S_{a,b}, \quad S_{a,b} \geq a \quad \text{ and } \quad S_{a,b} \geq b.$$

  So, $S_{c,S_{a,b}} \geq S_{b,c}$ and $S_{c,S_{a,b}} \geq a,$ hence,

  $$S_{c,S_{a,b}} \geq S_{a,S_{b,c}}.$$

  Conversely,

  $$S_{a,S_{b,c}} \geq a, \quad S_{a,S_{b,c}} \geq S_{b,c}, \quad S_{b,c} \geq b \quad \text{ and } \quad S_{b,c} \geq c,$$

  then,

  $$S_{a,S_{b,c}} \geq S_{a,b} \text{ and } S_{a,S_{b,c}} \geq c,$$

  so $S_{a,S_{b,c}} \geq S_{c,S_{a,b}}$, therefore $S_{a,S_{b,c}} \sim S_{c,S_{a,b}}$, i.e.
  $SUP(a, S_{b,c}) = SUP(c, S_{a,b})$.
  Hence,

  $$([a] \vee [b]) \vee [c] = SUP(S_{a,b}, c) = SUP(a, S_{b,c}) = [a] \vee ([b] \vee [c]).$$

- $\vee$ is commutative. It is obvious by definition of SUP.
- $\vee$ satisfies idempotency.
  Indeed, as $a \geq a$, then, $a \geq S_{a,a}$, but by definition of $SUP(a,a)$, $S_{a,a} \geq a$, so $a \sim S_{a,a}$. Therefore,

  $$[a] = [S_{a,a}] = SUP(a,a) = [a] \vee [a].$$

So, $(A/\sim, \vee)$ is a joint-semilattice.

2. We verify that $(A/\sim, \wedge)$ is a meet-semilattice, with $\wedge$ defined as

$$[a] \wedge [b] = INF(a,b).$$

$\wedge$ is well defined, i.e.

$$\text{if } a \sim a' \text{ then } [a] \wedge [b] = [a'] \wedge [b].$$

Indeed, if $I_{a,b}$ and $I_{a',b}$ denotes an element of $INF(a,b)$ and $INF(a',b)$ respectively, we verify now that $I_{a,b} \sim I_{a',b}$, i.e. $INF(a,b) = INF(a',b)$.

As

$$I_{a,b} \leq a \sim a' \text{ and } I_{a,b} \leq b,$$

by definition of $INF(a',b)$, we have that $I_{a,b} \leq I_{a',b}$.

Conversely, since

$$I_{a',b} \leq a' \sim a \quad \text{and} \quad I_{a',b} \leq b,$$

by definition of $INF(a,b)$, we have that $I_{a',b} \leq I_{a,b}$.

Therefore,

$$I_{a,b} \sim I_{a',b}.$$

In order to see that $(A/\sim, \wedge)$ is a meet-semilattice, we will verify that

- $\wedge$ is associative.

  Indeed, by definition of $INF(c, I_{a,b})$ we have that $I_{c,I_{a,b}} \leq c$ and $I_{c,I_{a,b}} \leq I_{a,b}$, and as $I_{a,b} \leq a$ and $I_{a,b} \leq b$, then $I_{c,I_{a,b}} \leq I_{b,c}$ and $I_{c,I_{a,b}} \leq a$, so

  $$I_{c,I_{a,b}} \leq I_{a,I_{b,c}}.$$

  Conversely, $I_{a,I_{b,c}} \leq a$ and $I_{a,I_{b,c}} \leq I_{b,c}$ and $I_{b,c} \leq b, I_{b,c} \leq c$, then $I_{a,I_{b,c}} \leq I_{a,b}$ and $\mathrm{I}_{a,I_{b,c}} \leq c$, so $I_{a,I_{b,c}} \leq I_{c,I_{a,b}}$.

  Therefore,

  $$I_{a,I_{b,c}} \sim I_{c,I_{a,b}}.$$

  So,

  $$([a] \wedge [b]) \wedge [c] = INF(I_{a,b}, c) = INF(a, I_{b,c}) = [a] \wedge ([b] \wedge [c]).$$

- $\wedge$ is commutative. It is obvious by definition of INF.

- $\wedge$ satisfies idempotency.

  As $a \leq a$, then $a \leq I_{a,a}$, but by definition of $INF(a,a)$, $I_{a,a} \leq a$, so $a \sim I_{a,a}$. Therefore,

  $$[a] = [I_{a,a}] = INF(a,a) = [a] \wedge [a].$$

Hence, $(A/\sim, \wedge)$ is a meet-semilattice.

Therefore, $(A/\sim, \wedge, \vee)$ is a lattice.

Note that the order induced from $(A/\sim, \wedge)$, i.e.

$$[a] \leq^{\wedge} [b] \quad \text{iff} \quad [a] \wedge [b] = [a],$$

and the one defined as

$$[a] \sqsubseteq [b] \quad \text{iff} \quad a \leq b,$$

are the same. Indeed,

$$[a] \sqsubseteq [b] \text{ iff } a \leq b \text{ iff } INF(a,b) = [a] \text{ iff } [a] \wedge [b] = [a] \text{ iff } [a] \leq^{\wedge} [b].$$

$\rightarrow$) We verify the existence of $SUP(a,b)$ and $INF(a,b)$. Let $\wedge$ and $\vee$ be induced in $A/\sim$ by the partial order $\sqsubseteq$, and define

$$SUP(a,b) = [a] \vee [b] \quad \text{and} \quad INF(a,b) = [a] \wedge [b].$$

Both sets satisfy the required conditions as it is shown following.

- As $[a] \vee [b]$ ($[a] \wedge [b]$ resp.) is an equivalence class, the elements of $SUP(a,b)$ ($INF(a,b)$ resp.) are indifferent, and obviously if $f \in SUP(a,b)$, then $\forall\, g \sim f, g \in SUP(a,b)$.
- Let $c \in SUP(a,b) = [d]$, we verify that $c \geq a$ and $c \geq b$.

  Indeed, as $c \sim d$, and by definition of $\vee$, $[a] \sqsubseteq [d]$ and $[b] \sqsubseteq [d]$, we have that $a \leq d$ and $b \leq d$, so
  $$c \geq a \text{ and } c \geq b.$$
- It remains to verify that: If $e \geq a$ and $e \geq b$, then,
  $$(e \sim c,\ c \in SUP(a,b)) \text{ or } (e > c,\ c \in SUP(a,b)).$$
  Indeed, as $e \geq a$ and $e \geq b$, we have that $[a] \sqsubseteq [e]$ and $[b] \sqsubseteq [e]$, so
  $$[d] = [a] \vee [b] \sqsubseteq [e],$$
  i.e. $d \leq e$, therefore if $c \in SUP(a,b)$, then, $c \sim d \leq e$.

These sets are unique. Indeed, let $p, p' \in A$. Suppose that $\overline{SUP}(p,p')$ satisfying the conditions exists, denoting by $\overline{S}_{p,p'}$ an element of $\overline{SUP}(p,p')$, we will verify that $\overline{S}_{p,p'} \sim S_{p,p'}$.

As $\overline{S}_{p,p'} \geq p$ and $\overline{S}_{p,p'} \geq p'$, by definition of $SUP(p,p')$, we have that

$$\overline{S}_{p,p'} \geq S_{p,p'}.$$

Conversely, as

$$S_{p,p'} \geq p \text{ and } S_{p,p'} \geq p',$$

then, by definition of $\overline{SUP}(p, p'), S_{p,p'} \geq \overline{S}_{p,p'}$, therefore

$$\overline{S}_{p,p'} \sim S_{p,p'},$$

hence,

$$\overline{SUP}(p, p') = SUP(p, p').$$

Analogously, we may verify that $INF(p, p')$ is unique. $\square$

7.2 Ordinal/Qualitative Utility Functions on Lattices

Now, let us introduce the lattice-based context of an extension of the possibilistic model.

7.2.1 A Possibilistic Context on Lattices

Let $X = \{x_1, ..., x_p\}$ be a finite set of consequences. We will denote by $(V, \vee_V, \wedge_V, 0_V, 1_V, n_V)$ a *finite* distributive *lattice of uncertainty values* with *minimum* 0_V, *maximum* 1_V and a *reversing involution* n_V, $\leq_V$ being the lattice order induced in V.

$(U, \vee_U, \wedge_U, 0_U, 1_U, n_U)$ will be a *finite distributive lattice of preference values* with involution n_U.

Remark 7
In order to simplify notation, we use $\wedge, \vee$ for denoting both operations on V and U, as well as 1 and 0 are used for denoting their minimum and maximum, although they may be different, hoping they may be understood by the context.

We consider the *set of consistent possibility distributions on X over V*,

$$\Pi(X, V) = \{\pi : X \to V | \bigvee_{x \in X} \pi(x) = 1\}.$$

As usual, we define the point-wise order in $(\Pi(X), V)$[5] induced by $\leq_V$

$$\pi \leq \pi' \iff \forall x \in X \;\; \pi(x) \leq_V \pi'(x).$$

For our purposes, we will consider a subset of $\Pi(X)$, the *set of **normalised** possibility distributions*[6], i.e.

$$\Pi^*(X, V) = \{\pi \in \Pi(X) \mid \exists x \; s.t. \; \pi(x) = 1\}. \tag{7.2}$$

As usual, we identify possibilistic lotteries and distributions. Given $x, y \in X, x \neq y$, and $\lambda, \mu \in V$ s.t. $\lambda \vee \mu = 1$, *the qualitative lottery* $(\lambda/x, \mu/y)$ is the consistent possibility distribution on X defined, as usual, as

$$(\lambda/x, \mu/y)(z) = \begin{cases} \lambda, & \text{if} \quad z = x \\ \mu, & \text{if} \quad z = y \\ 0, & \text{otherwise.} \end{cases}$$

[5]For the sake of simplicity, we shall generally omit the reference to the uncertainty set.
[6]When V is a finite linear scale, both $\Pi(X)$ and $\Pi^*(X)$ are the same set.

The Possibilistic Mixture is now an operation defined on $\Pi(X)$ that combines two consistent possibility distributions π_1 and π_2 into a new one, denoted $(\lambda/\pi_1, \mu/\pi_2)$, with $\lambda, \mu \in V$ and $\lambda \vee \mu = 1$, defined as

$$(\lambda/\pi_1, \mu/\pi_2)(x) = (\lambda \wedge \pi_1(x)) \vee (\mu \wedge \pi_2(x)).$$

In order to have a closed operation on $\Pi^*(X)$, the mixture operation is restricted to $\Pi^*(X)$ requiring the scalars to satisfy an additional condition, i.e. if $\pi, \pi' \in \Pi^*(X)$, we consider $(\lambda/\pi, \mu/\pi')$ with $\lambda, \mu \in V$ being $\lambda = 1$ or $\mu = 1$.

Now, as V is distributive, we may verify that reduction of lotteries always holds.

Proposition 7.4
$\forall \lambda_1, \lambda_2, \mu_1, \mu_2 \in V$ *s.t.* $\lambda_1 \vee \lambda_2 = 1,\ \forall \pi \in \Pi(X)$,

$$(\lambda_1/(1/\pi, \mu_1/X), \lambda_2/(1/\pi, \mu_2/X)) = (1/\pi, (\lambda_1 \wedge \mu_1) \vee (\lambda_2 \wedge \mu_2)/X).$$

Proof:
By definition of lotteries, we have that

$$\begin{aligned}(\lambda_1/(1/\pi, \mu_1/X), \lambda_2/(1/\pi, \mu_2/X))(z) &= (\lambda_1 \wedge (\pi(z) \vee \mu_1)) \vee \\ &\qquad (\lambda_2 \wedge (\pi(z) \vee \mu_2)) \\ &=^{7} ((\lambda_1 \wedge \pi(z)) \vee (\lambda_2 \wedge \pi(z))) \vee \\ &\qquad ((\lambda_1 \wedge \mu_1) \vee (\lambda_2 \wedge \mu_2)) \\ &=^{8} \pi(z) \vee ((\lambda_1 \wedge \mu_1) \vee (\lambda_2 \wedge \mu_2)).\end{aligned}$$

Therefore, we have that

$$(\lambda_1/(1/\pi, \mu_1/X), \lambda_2/(1/\pi, \mu_2/X)) = (1/\pi, [(\lambda_1 \wedge \mu_1) \vee (\lambda_2 \wedge \mu_2)]/X).$$

$\square$

Consider $u{:}X \to U$ a preference function that assigns to each consequence of X a preference level of U, requiring V and U to be commensurate, i.e. there exists $h{:}V \to U$ a $\{$*0,1*$\}$*-homomorphism* relating both lattices V and U. Let n be the reversing homomorphism $n{:}V \to U$ defined as $n(\lambda) = n_U(h(\lambda))$. It also verifies $n(0) = 1$, and $n(1) = 0$. For any $\pi \in \Pi^*(X)$, consider the qualitative utility functions:

$$\begin{aligned} QU^-(\pi) &= \bigwedge_{x \in X} (n(\pi(x)) \vee u(x)), \\ QU^+(\pi) &= \bigvee_{x \in X} (h(\pi(x)) \wedge u(x)).^{9} \end{aligned}$$

Now, we will introduce the axioms that characterise the preference relations induced by these functions and some results that we need for the representation theorems.

[7] By distributivity and associativity in V.
[8] Since $\lambda_1 \vee \lambda_2 = 1, (\lambda_1 \wedge \pi) \vee (\lambda_2 \wedge \pi) = \pi$.
[9] Obviously when V an U are linear scales these functions recover the ones introduced in Chapter 4.

Proposition 7.5
If U is a distributive lattice with involution, QU^- and QU^+ preserve the possibilistic mixture in the sense that the following expressions hold:

$$\begin{array}{rcl} QU^-(\lambda/\pi_1,\mu/\pi_2) & = & (n(\lambda)\vee QU^-(\pi_1))\wedge(n(\mu)\vee QU^-(\pi_2)), \\ QU^+(\lambda/\pi_1,\mu/\pi_2) & = & (h(\lambda)\wedge QU^+(\pi_1))\vee(h(\mu)\wedge QU^+(\pi_2)). \end{array}$$

Proof:

$$\begin{array}{rcl} QU^-(\lambda/\pi_1,\mu/\pi_2) & = & \bigwedge\limits_{x\in X}(n((\lambda/\pi_1,\mu/\pi_2)(x))\vee u(x)) \\ & = & \bigwedge\limits_{x\in X}(n(((\pi_1\wedge\lambda)\vee(\pi_2\wedge\mu)))(x)\vee u(x))^{10} \\ & = & \bigwedge\limits_{x\in X}(((n(\pi_1(x))\vee n(\lambda))\wedge \\ & & \quad (n(\pi_2(x))\vee n(\mu)))\vee u(x))^{11} \\ & = & \bigwedge\limits_{x\in X}((\bigvee(n(\pi_1(x)),n(\lambda),u(x)))\wedge \\ & & (\bigvee(n(\pi_2(x)),n(\mu),u(x))))^{12} \\ & = & (\bigwedge\limits_{x\in X}(\bigvee(n(\pi_1(x)),n(\lambda),u(x))))\wedge \\ & & (\bigwedge\limits_{x\in X}(\bigvee(n(\pi_2(x)),n(\mu),u(x))))^{13} \\ & = & (\bigwedge\limits_{x\in X}(n(\lambda)\vee(n(\pi_1(x))\vee u(x))))\wedge \\ & & (\bigwedge\limits_{x\in X}(n(\mu)\vee(n(\pi_2(x))\vee u(x))))^{14} \\ & = & ((n(\lambda)\vee(\bigwedge\limits_{x\in X}(n(\pi_1(x))\vee u(x)))))\wedge \\ & & (n(\mu)\vee(\bigwedge\limits_{x\in X}(n(\pi_2(x))\vee u(x)))) \\ & = & (n(\lambda)\vee QU^-(\pi_1))\wedge(n(\mu)\vee QU^-(\pi_2)). \end{array}$$

Therefore, QU^- preserves the "possibilistic" mixture.
The proof for QU^+ is omitted because of it is analogous to the pessimistic one. □

Now, we have utility functions for making decisions on lattices, in the usual hypotheses that ranking decisions is a problem of ranking normalised possibility distributions.

[10] Since $n = n_U\circ h$, and h is homomorphism, we have that $n(\lambda\vee\lambda') = n(\lambda)\wedge n(\lambda')$ and $n(\lambda\wedge\lambda') = n(\lambda)\vee n(\lambda')$.
[11] Since U is a distributive lattice, $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$.
[12] Associativity of $\wedge$.
[13] Associativity of $\vee$.
[14] Distributivity: $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$.

7.2.2 Characterisations for Ordinal/Qualitative Utility Functions

In this Section, we characterise the orderings induced by these functions as well as the preference relations that are representable by these functions.

Proposition 7.6
Let $(\Pi^*(X), \sqsubseteq)$, *satisfying*

- $AP1(structure)$: $(\Pi^*(X), \sqsubseteq)$ *is a pre-lattice.*
- $A2$ (uncertainty aversion)*: if* $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.

Then,

1. *The* maximal[15] *elements of* $(\Pi^*(X), \sqsubseteq)$ *are equivalent.*
2. *The maximal elements of* $(X, \sqsubseteq)$ *are equivalent, and they are equivalent to the maximal elements of* $(\Pi^*(X), \sqsubseteq)$.

Proof:

1. By $AP1$, $(\Pi^*(X), \sqsubseteq)$ is a finite partial pre-order, then exists at least one maximal element w.r.t. $\sqsubseteq$. Let π_1 and π_2 be maximal elements.
   By $AP1$, exists $SUP(\pi_1, \pi_2)$. Let $\pi \in SUP(\pi_1, \pi_2)$, then
   $$\pi \sqsupseteq \pi_1 \quad \text{and} \quad \pi \sqsupseteq \pi_2,$$
   but as π_1 and π_2 are maximal elements, it must be
   $$\pi_1 \sim \pi \sim \pi_2.$$
2. Let $\overline{x}_M$ be a maximal element of $(X, \sqsubseteq)$. Suppose it is not a maximal element of $(\Pi^*(X), \sqsubseteq)$. Hence, exist $\pi \in (\Pi^*(X), \sqsubseteq)$ s.t. $\overline{x}_M \sqsubset \pi$. As π is normalised, exists $x \in X$ s.t. $\pi(x) = 1$, so by $A2$, we have that as $x \leq \pi$, then $x \sqsupseteq \pi \sqsupset \overline{x}_M$. Contradiction since $\overline{x}_M$ is maximal in $(X, \sqsubseteq)$.
   So, $\overline{x}_M$ is also a maximal element of $(\Pi^*(X), \sqsubseteq)$, and by 1) all maximal elements of $(\Pi^*(X), \sqsubseteq)$ are equivalent, so all maximal elements of $(X, \sqsubseteq)$ are also equivalent.

$\square$

Axiomatic setting

Let ***AXP*** be the following set of axioms on $(\Pi^*(X), \sqsubseteq)$ (as usual, $\pi \sim \pi' \iff \pi \sqsubseteq \pi'$ and $\pi \sqsupseteq \pi'$):

- $AP1$: $(\Pi^*(X), \sqsubseteq)$ is a pre-lattice.

[15] π is a *maximal element* iff $\forall \pi' \in \Pi^*(X), \pi \sqsubseteq \pi' \Rightarrow \pi' \sim \pi$.

- $A2$ *(uncertainty aversion)*: if $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.
- $A3$ *(independence)*: $\pi_1 \sim \pi_2 \Rightarrow (\lambda/\pi_1, \mu/\pi) \sim (\lambda/\pi_2, \mu/\pi)$.

Let $\overline{\pi}$ be a maximal element of $(\Pi^*(X), \sqsubseteq)$ so, for each $\lambda \in V$, we consider $\pi_\lambda^- = (1/\overline{\pi}, \lambda/X)$[16].

- $AP4 : \forall \pi \in \Pi^*(X), \exists \lambda \in V$ s.t. $\pi \sim \pi_\lambda^-$.
- $AP5$: if $\pi_\lambda^- \sqsubseteq \pi_{\lambda'}^- \Rightarrow \pi_{n_V(\lambda)}^- \sqsupseteq \pi_{n_V(\lambda')}^-$.
- $AP6$*(incomparability preservation)*: if $\lambda <> \lambda' \Rightarrow \pi_\lambda^- \sqsubset\sqsupset \pi_{\lambda'}^-$.

$AP1$ says that the quotient set $(\Pi^*(X)/\sim, \sqsubseteq)$ results a lattice. $A2, A3$ and $AP4$ have the analogous meanings to the linear case, while $AP6$ establishes that two incomparable values of uncertainty, λ and λ', lead to two incomparable lotteries. Finally, $AP5$ says that the preference between lotteries with degrees of uncertainty λ and λ' with respect to a maximal $\overline{\pi}$ results reversed when the lotteries are considered with the respective "opposite" values of uncertainty.

Remark 8
If $AP5$ holds then,

$$\pi_\lambda \sim \pi_{\lambda'} \qquad \Rightarrow \qquad \pi_{n_V(\lambda)} \sim \pi_{n_V(\lambda')}.$$

Lemma 7.7
Let $(U, \leq_U, 0, 1, n_U)$ and $(V, \leq_V, 0, 1, n_V)$ be two distributive lattices with involution, $h{:}V \to U$ a epimorphism[17] *and $u{:}X \to U$.*
If $(QU^-)^{-1}(1) \neq \emptyset$ and $(QU^-)^{-1}(0) \neq \emptyset$, then

- *there exists $x \in X$ s.t. $u(x) = 1$ and $\bigwedge_{x \in X} u(x) = 0$.*
- *QU^- is onto.*

lema

Proof:

- Since $(QU^-)^{-1}(1) \neq \emptyset$, there exists $\overline{\pi}$ s.t.

  $$QU^-(\overline{\pi}) = \bigwedge_{x \in X} (n(\overline{\pi}(x)) \vee u(x)) = 1,$$

  then $n(\overline{\pi}(x)) \vee u(x) = 1 \ \forall x \in X$. As $\overline{\pi}$ is normalised there exists $x_1 \in X$ s.t. $\overline{\pi}(x_1) = 1$, hence $1 = n(1) \vee u(x_1)$, so $u(x_1) = 1$. On the other hand,

  $$QU^-(X) = \bigwedge_{x \in X} (n(X(x)) \vee u(x)) = \bigwedge_{x \in X} (0 \vee u(x)) = \bigwedge_{x \in X} u(x).$$

[16] In fact, to be π_λ^- well defined we are assuming that $AP1$ and $A3$ are also required
[17] In fact, in the proof we only require h to be onto and to satisfy $h(0) = 0$ and $h(1) = 1$.

Since $(QU^{-})^{-1}(0) \neq \emptyset$, there exists π s.t. $QU^{-}(\pi) = 0$, and as $QU^{-}(\pi) \geq \bigwedge_{x \in X} u(x)$, we have that

$$\bigwedge_{x \in X} u(x) = 0.$$

- Given $w \in U$, since n is onto there exists $\lambda \in V$ s.t. $n(\lambda) = w$. As we have seen, there exists $x_1 \in X$ s.t. $u(x_1) = 1$, thus $\bigwedge_{x \in X - \{x_1\}} u(x) = 0$. Let π_w be the distribution defined as

$$\pi_w(x) = \begin{cases} 1, & \text{if} \quad x = x_1 \\ \\ \lambda, & \text{otherwise.} \end{cases} \tag{7.3}$$

  Then,

$$\begin{aligned} QU^{-}(\pi_w) &= \bigwedge_{x \in X} (n(\pi_w(x)) \vee u(x)) \\ &= n(\lambda) \vee \left(\bigwedge_{x \in X - \{x_1\}} u(x) \right) \\ &= n(\lambda) \\ &= w. \end{aligned}$$

$\square$

Lemma 7.8
Let h:$V \to U$ be an onto non-decreasing function satisfying that

$$\textit{if } \lambda <> \lambda' \textit{ then } h(\lambda) <> h(\lambda').$$

Then, h is a lattice epimorphism.

Proof:
First, we verify that h also satisfies that

$$h(\lambda) > h(\lambda') \text{ then } \lambda > \lambda'. \tag{7.4}$$

Indeed, suppose that $\lambda' \not< \lambda$, i.e. $\lambda' \geq \lambda$ or $\lambda <> \lambda'$. But,

- if $\lambda <> \lambda'$, then, by hypothesis, $h(\lambda) <>_U h(\lambda')$. Contradiction.
- if $\lambda' \geq \lambda$, as h is non-decreasing, then $h(\lambda') \geq h(\lambda)$. Contradiction.

So, it must be $\lambda > \lambda'$.
Now, we verify that h is distributive w.r.t. $\wedge$ and $\vee$.

- $h(\lambda) \vee h(\lambda') = h(\lambda \vee \lambda')$.

  Indeed, as h is order-preserving we have that $h(\lambda) \vee h(\lambda') \leq h(\lambda \vee \lambda')$.
  As h is onto, we have that there exists $\mu \in V$ s.t. $h(\lambda) \vee h(\lambda') = h(\mu)$, and thus $h(\mu) \geq h(\lambda)$ and $h(\mu) \geq h(\lambda')$.

  – If $h(\lambda) <> h(\lambda')$ then $h(\mu) > h(\lambda)$ and $h(\mu) > h(\lambda')$.
    As h satisfies (7.4), we have that $\mu > \lambda$ and $\mu > \lambda'$, so $\mu \geq \lambda \vee \lambda'$.
    Therefore, $h(\mu) \geq h(\lambda \vee \lambda')$, i.e. $h(\lambda) \vee h(\lambda') \geq h(\lambda \vee \lambda')$.

  – Otherwise, $h(\lambda') \geq h(\lambda)$ or $h(\lambda) \geq h(\lambda')$.
    Suppose that $h(\lambda) \geq h(\lambda')$, then $h(\lambda) \vee h(\lambda') = h(\lambda)$.
    Observe that since $h(\lambda) \geq h(\lambda')$, by hypothesis we have that $\lambda <> \lambda'$ is impossible, so it must be

    $$\lambda \leq \lambda' \quad \text{or} \quad \lambda > \lambda'. \tag{7.5}$$

    Therefore, since

    $$h(\lambda \vee \lambda') = \begin{cases} h(\lambda) & \text{if} \quad \lambda > \lambda' \\ h(\lambda') & \text{if} \quad \lambda \leq \lambda', \end{cases} \tag{7.6}$$

    we have that

    $$h(\lambda) \vee h(\lambda') \geq h(\lambda \vee \lambda').$$

    Analogously, if $h(\lambda') \geq h(\lambda)$ we obtain that $h(\lambda) \vee h(\lambda') \geq h(\lambda \vee \lambda')$.
    Therefore, $h(\lambda) \vee h(\lambda') = h(\lambda \vee \lambda')$.

- In a similar way, we may verify that

  $$h(\lambda \wedge \lambda') = h(\lambda) \wedge h(\lambda').$$

Therefore, h is a lattice epimorphism. $\square$

Finally, let $\preccurlyeq_{QU^-}$ be the preference ordering on $\Pi^*(X)$ induced by QU^-, i.e.

$$\pi \preccurlyeq_{QU^-} \pi' \quad \text{iff} \quad QU^-(\pi) \leq_U QU^-(\pi').$$

In the following, we state that the set of axioms *AXP* characterise these preference orderings.

Theorem 7.9 (Representation Theorem for Pessimistic Utility)
A preference relation $(\Pi^*(X), \sqsubseteq)$ *satisfies axioms* AXP *iff there exist*

(i) a finite distributive utility lattice $(U, \wedge, \vee, n_U, 0, 1)$,

(ii) a preference function $u{:}X \to U$, *s.t.* $u^{-1}(1) \neq \emptyset$ *and* $\bigwedge_{x \in X} u(x) = 0$,

(iii) an onto order-preserving function $h{:}V \to U$ also satisfying

$$\text{if } \lambda <> \lambda' \quad \text{then} \quad h(\lambda) <> h(\lambda'), \tag{7.7}$$

and

$$n_U \circ h \circ n_V = h\text{[18]}, \tag{7.8}$$

in such a way that it holds:

$$\pi' \sqsubseteq \pi \text{ iff } \pi' \preccurlyeq_{QU^-} \pi$$

with $n = n_U \circ h$.

Proof:
$\leftarrow$) We have to verify that the preference ordering on $\Pi^*(X)$ induced by QU^- satisfies the above set of axioms. As $\leq_U$ is a partial order, $\preccurlyeq_{QU^-}$ is reflexive and transitive. By Lemma 7.7, QU^- is onto, so we may define

$$SUP(\pi, \pi') = (QU^-)^{-1}(QU^-(\pi) \vee QU^-(\pi')),$$

and

$$INF(\pi, \pi') = (QU^-)^{-1}(QU^-(\pi) \wedge QU^-(\pi')).$$

Then, by Theorem 7.3, $(\Pi^*(X), \preccurlyeq_{QU^-})$ is a pre-lattice.

$A2$ results from the fact that $\vee$ and $\wedge$ are non-decreasing in U and n is a reversing function. While, $A3$ is a consequence of the fact that QU^- preserves mixtures.

Let us prove now $AP5$: if $\pi^-_{\lambda} \preccurlyeq_{QU^-} \pi^-_{\lambda'} \Rightarrow \pi^-_{n_V(\lambda)} \succcurlyeq_{QU^-} \pi^-_{n_V(\lambda')}$.

Let $\overline{\pi}$ be a maximal element of $\Pi^*(X)$, so $QU^-(\overline{\pi}) = 1$. As QU^- preserves mixtures and $QU^-(X) = 0$, we have that $QU^-(\pi^-_{\lambda}) = n(\lambda)$.

As $n_U \circ n \circ n_V = n$ and n_V and n_U are involutive, then

$$n(\lambda) \leq n(\lambda') \;\Rightarrow\; n(n_V(\lambda)) = n_U(n(\lambda)) \geq n_U(n(\lambda')) = n(n_V(\lambda')).$$

That is, $AP5$ is verified.
$AP6$ is a consequence of the $\preccurlyeq_{QU^-}$ definition and that h satisfies (7.7).

Now, we check $AP4$. Let $\overline{\pi}$ be maximal element of $\Pi^*(X)$ w.r.t. $\preccurlyeq_{QU^-}$. As $QU^-(1/\overline{\pi}, \lambda/X) = n(\lambda)$, then

$$QU^-(\pi) = n(\lambda) = QU^-(1/\overline{\pi}, \lambda/X) \qquad \forall \lambda \in n^{-1}(QU^-(\pi)).$$

$\rightarrow$) The proof is very analogous with the one given for the linear case. We again structure the proof in the following three steps.

1. We define the distributive utility lattice U with involution n_U, and a reversing mapping n from V to U, satisfying if $\lambda <> \lambda'$ then $n(\lambda) <>_U n(\lambda')$, and $n_U \circ n \circ n_V = n$. So, we consider the preserving mapping $h = n_U \circ n$. Hence, h will satisfy (7.8) and (7.7).

   By Lemma 7.8, h is actually a lattice epimorphism.

[18] This means that h is a mophism w.r.t. negations in V and U.

2. A function $QU^{-}:\Pi^*(X) \rightarrow U$ representing $\sqsubseteq$, i.e. such that $QU^{-}(\pi) \leq QU^{-}(\pi')$ iff $\pi \sqsubseteq \pi'$, is defined.

3. Finally, we prove that $QU^{-}(\pi) = \bigwedge_{x\in X}(n(\pi(x)) \vee u(x))$, where $u{:}X \rightarrow U$ is the restriction of QU^{-} on X. u also satisfies that $u^{-1}(1) \neq \emptyset$ and $\bigwedge_{x\in X} u(x) = 0$.

Now, let us develop these steps.

1. We consider on $\Pi^*(X)$ the equivalence relation $\sim$ defined as

$$\pi \sim \pi' \iff \pi \sqsubseteq \pi' \text{ and } \pi' \sqsubseteq \pi.$$

By $AP1, \Pi^*(X)/\sim$ is a lattice. As in the linear case, we take as utility lattice $U = \Pi^*(X)/\sim$. As Theorem 7.3 guarantees the existence of SUP and INF, we define in U the operations $\wedge$ and $\vee$ induced by them, i.e.

$$[\pi] \vee [\pi'] = SUP(\pi, \pi'),$$

and

$$[\pi] \wedge [\pi'] = INF(\pi, \pi').$$

The $\leq_U$ induced from $\vee$ coincides with $\sqsubseteq$. It is not difficult to verify that $[X]$ is minimum of $(U, \leq_U)$, and if $\overline{\pi}$ is a maximal element of $\Pi^*(X)$, $[\overline{\pi}]$ is the maximum on U.

Let $\overline{\pi}$ a maximal element of $\Pi^*(X)$, and for each $\lambda \in V$, let

$$\pi^{-}_{\lambda} = (1/\overline{\pi}, \lambda/X),$$

and let $n{:}V \rightarrow U$ be defined as

$$n(\lambda) = [\pi^{-}_{\lambda}].$$

It is not difficult to see, analogously to the linear case, that n is onto, and that $A2$ guarantees n actually reverses the order. Now, we define n_U from n and n_V. For each $w \in U$, we define

$$n_U(w) = n(n_V(\lambda)),$$

with $\lambda \in V$ s.t. $n(\lambda) = w$. By $AP5$, see Remark 8, n_U is well defined. By $AP6, n$ satisfies

$$\text{if } \lambda <> \lambda' \text{ then } n(\lambda) <> n(\lambda'),$$

and by definition of n_U, we have $n_U \circ n \circ n_V = n$ and $n_U \circ n_U = identity$. Let $h = n_U \circ n$. Then, h satisfies the conditions required.

Hence, as n is a reversing epimorphism, and V is a distributive lattice, so is U.

2. As usual, QU^- can be defined on $\Pi^*(X)$ in two steps. First, we define it on lotteries of type π_λ^-, as $QU^-(\pi_\lambda^-) = n(\lambda)$.

   $AP4$ lets us to extend this definition. Since $\forall \pi\ \exists \lambda$ s.t. $\pi \sim (1/\overline{\pi}, \lambda/X)$, we define $QU^-(\pi) = n(\lambda)$. It is not difficult to verify that QU^- represents $\sqsubseteq$.

3. Consider $u{:}X \to U$ defined as $u(x) = QU^-(x)$.

   It remains to prove that $QU^-(\pi) = \bigwedge_{x\in X}(n(\pi(x)) \vee u(x))$. To verify this, we will prove the following equalities:

   - $QU^-(\lambda_1/\pi_1, \lambda_2/\pi_2) = (n(\lambda_1) \vee QU^-(\pi_1)) \wedge (n(\lambda_2) \vee QU^-(\pi_2))$ with either $\lambda_1 = 1$ or $\lambda_2 = 1$.
     By $AP4, \exists \mu, \gamma$ s.t.
     $$\pi_1 \sim (1/\overline{\pi}, \mu/X) \text{ and } \pi_2 \sim (1/\overline{\pi}, \gamma/X).$$
     By $A3$,
     $$(\lambda_1/\pi_1, \lambda_2/\pi_2) \sim (\lambda_1/(1/\overline{\pi}, \mu/X), \lambda_2/(1/\overline{\pi}, \gamma/X)),$$
     and reducing lotteries we obtain
     $$(\lambda_1/\pi_1, \lambda_2/\pi_2) \sim (1/\overline{\pi}, ((\lambda_1 \wedge \mu) \vee (\lambda_2 \wedge \gamma))/X).$$
     Therefore, as n is a reversing morphism, we have
     $$\begin{aligned} QU^-(\lambda_1/\pi_1, \lambda_2/\pi_2) &= n((\lambda_1 \wedge \mu) \vee (\lambda_2 \wedge \gamma)) \\ &= (n(\lambda_1) \vee n(\mu)) \wedge (n(\lambda_2) \vee n(\gamma)) \\ &= (n(\lambda_1) \vee QU^-(\pi_1)) \wedge (n(\lambda_2) \vee QU^-(\pi_2)). \end{aligned}$$
     Therefore, we have that
     $$QU^-(\pi_1 \vee \pi_2) = QU^-(\pi_1) \wedge QU^-(\pi_2).$$
     More generally, $QU^-(\bigvee_{i=1\dots,p} \pi_i) = \bigwedge_{i=1\dots,p} QU^-(\pi_i)$.
   - $QU^-(\pi) = \bigwedge_{i=1\dots,p} (n(\pi(x_i)) \vee u(x_i))$.
     As $\pi \in \Pi^*(X)$, then $\exists\, x_j \in X$ s.t. $\pi(x_j) = 1$. Without loss of generality assume $j = 1$. Let
     $$\pi_i = (1/x_1, \pi(x_i)/x_i).$$
     Since
     $$\pi = \bigvee_{i=1\dots,p} \pi_i,$$
     we have that
     $$\begin{aligned} QU^-(\pi) &= QU^-\left(\bigvee_{i=1\dots,p} \pi_i\right) \\ &= \bigwedge_{i=1\dots,p} (u(x_1) \wedge (n(\pi(x_i)) \vee u(x_i))) \\ &=\text{[19]} \bigwedge_{i=1\dots,p} (n(\pi(x_i)) \vee u(x_i)). \end{aligned}$$

Finally, as $\overline{\pi}$ is normalised, there exists $x_0 \in X$ s.t. $\overline{\pi}(x_0) = 1$, so $x_0 \leq \overline{\pi}$. Then, by $A2$, $x_0 \sqsupseteq \overline{\pi}$. As QU^- represents $\sqsubseteq$,

$$QU^-(x_0) \geq QU^-(\overline{\pi}) = 1,$$

hence $u(x_0) = 1$, so $u^{-1}(1) \neq \emptyset$. As $QU^-(X) = 0$, and $QU^-(X) = \bigwedge_{x\in X} u(x)$, then $\bigwedge_{x\in X} u(x) = 0$.

This ends the proof. □

As usual, in many situations we may be interested in an optimistic behaviour. With this goal, we consider $\preccurlyeq_{QU^+}$ the preference ordering on $\Pi^*(X)$ induced by QU^+, i.e.

$$\pi \preccurlyeq_{QU^+} \pi' \quad \Longleftrightarrow \quad QU^+(\pi) \leq QU^+(\pi').$$

In order to represent this optimistic preference relation, we have to change the uncertainty aversion axiom $A2$ by the usual uncertainty-prone postulate:

- $A2^+$: if $\pi \leq \pi'$ then $\pi \sqsubseteq \pi'$,

and to modify the axioms involving π^-_λ. Indeed, consider now $\pi^+_\lambda = (\lambda/X, 1/\underline{\pi})$, where $\underline{\pi}$ is a minimal on $(\Pi^*(X), \sqsubseteq)$, we have that

- $AP4^+ : \forall \pi \in \Pi^*(X), \exists \lambda \in V$ such that $\pi \sim \pi^+_\lambda$.
- $AP5^+$: if $\pi^+_\lambda \sqsubseteq \pi^+_{\lambda'} \Rightarrow \pi^+_{n_V(\lambda)} \sqsupseteq \pi^+_{n_V(\lambda')}$.
- $AP6^+$: if $\lambda <> \lambda' \Rightarrow \pi^+_\lambda \sqsubset\sqsupset \pi^+_{\lambda'}$.

Now, the Representation Theorem says:

Theorem 7.10 (Representation Theorem for Optimistic Utility)
A preference relation $\sqsubseteq$ on $\Pi^*(X)$ satisfies axioms set $AXP^+ = \{AP1, A2^+, A3, AP4^+, AP5^+, AP6^+\}$ iff there exist

(i) a finite distributive utility lattice with involution $(U, \vee, \wedge, 0, 1, n_U)$,

(ii) a preference function u:$X \to U$, s.t. $u^{-1}(0) \neq \emptyset$ and $\bigvee_{x\in X} u(x) = 1$,

(iii) an onto order-preserving function h:$V \to U$, s.t. $n_U \circ h \circ n_V = h$, and also satisfying

$$\lambda <> \lambda' \text{ then } h(\lambda) <> h(\lambda'),$$

in such a way that it holds:

$$\pi' \sqsubseteq \pi \quad \Longleftrightarrow \quad \pi' \preccurlyeq_{QU^+} \pi.$$

The proof is very analogous to the one for pessimistic utility, and it will be omitted.

[19] As $\pi(x_1) = 1$, then $u(x_1) = u(x_1) \vee n(\pi(x_1))$.

7.2.3 Generalised Qualitative Utility Functions

Now, we assume available other operators (t-norms) in V. This assumption lets us to consider also other operations on $\Pi^*(X)$. Before analysing this point, let us introduce some notation and some previous facts about residuated lattices that we will use in the following.

Definition 9
Given $(L, \wedge, \vee, 0, 1)$ *a finite lattice, a* t-norm (t-conorm) operation $\top(\bot)$ on L *is a non-decreasing, associative and commutative binary operation on* L *verifying* $\lambda\top 0 = 0$ *and* $\lambda\top 1 = \lambda$ $(\lambda\bot 0 = \lambda \text{and} \lambda\bot 1 = 1, resp.)$ *for all* $\lambda \in L$*. The* residuum of $\top$, $I{:}L \times L \to L$, *is defined as*

$$I(a, c) = \bigvee\{b|\top(a, b) \leq c\}.$$

$(\top, I)$ *is an* adjoin pair *if the following conditions hold:*

1) $(L, \top, 1)$ *is a commutative semigroup with unit element 1.*

2) $\forall a, b, c \in L, (a\top b) \leq c$ *iff* $a \leq I(b, c)$.

$(L, \wedge, \vee, \top, I, 0, 1)$ *is a* residuated lattice *if* $(L, \wedge, \vee, 0, 1)$ *is a lattice and* $(\top, I)$ *is an adjoin pair.*

We will denote by $(V, \wedge_V, \vee_V, 0, 1, n_V, \top)$ a finite distributive lattice of uncertainty values with involution n_V and $\top$ *a t-norm on* V. $(U, \wedge_U, \vee_U, 0, 1, n_U)$ will be a finite distributive lattice of preference values with involution. As before, in the $meet$ and $join$ operators notations we will usually omit the reference to the lattice, assuming that they may be identified by the context.

Theorem 7.11
Let $(L, \wedge, \vee, 0, 1)$ *be a finite lattice, and* $\top$ *a t-norm on* L*. Then,* $\top$ *distributes over the lattice joint operation (that is,* $(a \vee b)\top c = (a\top c) \vee (b\top c)$, $\forall a, b, c \in L$*) iff* $(L, \wedge, \vee, \top, I, 0, 1)$ *is a residuated lattice.*

Proof:
$\to$) Suppose $(a \vee b)\top c = (a\top c) \vee (b\top c)$, $\forall a, b, c \in L$. Hence,

- $(a\top b) \leq c \;\Rightarrow\; a \leq I(b, c)$ by the definition of I
- Let $D = \{d \in L|(b\top d) \leq c\}$, D is closed under supremum. Indeed by distributivity of $\top$ w.r.t. $\vee$, we have that

$$\left(\bigvee_{d\in D} d\right)\top b = \bigvee_{d\in D}(d\top b) \leq \bigvee_{d\in D} c = c,$$

so $\left(\bigvee_{d\in D} d\right) \in D$. Therefore, if

$$a \leq I(b, c) = \bigvee_{d\in D} d$$

then

$$(a\top b) \leq \left(\bigvee_{d\in D} d\right) \top b = \bigvee\{(d\top b)|d \in D\} \leq c.$$

$\leftarrow$) Cf. Lemma 2.3.4 of (Hájek, 1998).

$\square$

Generalised $\vee$-Mixtures and Utilities

We have seen in previous chapters that QU^- and QU^+ are "utility" functions on $\Pi^*(X)$, in the sense that they preserve the preference ordering and the max-min combination of possibilistic mixtures. Now, we analyse the conditions required to guarantee that the generalised utility functions functions preserve a generalised possibilistic mixture. Instead of applying max-min combination of possibility distributions, we consider other mixtures involving t-conorms and t-norms. For each t-norm $\top$ and conorm $\perp$ on V, we will be interested in $\perp - \top$ mixtures that combine two possibility distributions π_1 and π_2 into a new one, denoted $M_{\top,\perp}(\pi_1, \pi_2; \lambda\mu)$, with $\lambda, \mu \in V$ and $\lambda \perp \mu = 1$, defined as:

$$M_{\top,\perp}(\pi_1, \pi_2; \lambda, \mu)(x) = (\lambda \top \pi_1(x)) \perp (\mu \top \pi_2(x)).$$

Remark 9
We require these mixtures to satisfy reduction of lotteries, that is:

$$M_{\top,\perp}(M_{\top,\perp}(\pi_1, \pi_2; \lambda_1, \lambda_2), M_{\top,\perp}(\pi_1, \pi_2; \mu_1, \mu_2); \alpha, \beta) = \\ M_{\top,\perp}(\pi_1, \pi_2; (\alpha\top\lambda_1)\perp(\beta\top\mu_1), (\alpha\top\lambda_2)\perp(\beta\top\mu_2)).$$

Hence, we need that $(a\top c)\perp(b\top c) = c\top(a\perp b)$ be satisfied. Therefore, we have to restrict ourselves to $\vee - \top$ mixtures. Indeed, De Cooman and Kerre prove that if $(L, \leq)$ is a bounded partially ordered set, then if a t-norm $\top$ on $(L, \leq)$ is distributive w.r.t. a conorm $\perp$ in L it implies that

$$(a\top b)\perp a = a, \qquad \forall a, b \in L. \tag{7.9}$$

Moreover, (7.9) implies that $\perp$ satisfies idempotency, and they prove that the only conorm idempotent is $join$ (see (De-Cooman and Kerre, 1993; Propositions 3.5, 3.6 and 3.7) for more details). Besides, by Theorem 7.11 we have to require $(V, \wedge, \vee, \top, I, 0, 1)$ to be a residuated lattice. Henceforth, V will be assumed to be a finite, residuated, and distributive lattice with involution. From now on, $M_\top$ denotes $M_{\top,\vee}$.

So, for each t-norm $\top$ on V, we may consider a generalised $\vee$-$\top$-Possibilistic Mixture. In order to have a closed operation on $\Pi^*(X)$, the scalars λ, μ involved in the mixture operation are also required to satisfy $\lambda = 1$ or $\mu = 1$.

Since now we have in V other operators besides *infimum*, we can consider here another alternative for modelling implication instead of $(v \Rightarrow u) = n(v) \vee u$, namely the *S-implication-like* defined in (5.6), but now with lattices,

$$(v \Rightarrow u) = n(v \top z)$$

with $n(z) = u$, $\top$ a t-norm on V, $n = n_U \circ h$, and $h{:}V \to U$ an onto order-preserving function. $u{:}X \to U$ that assigns to each consequence of X a preference level of U, for a pessimistic behaviour we propose

$$GQU^-(\pi|u) = [\pi \subseteq u] = \bigwedge_{x \in X} n(\pi(x) \top \lambda_x),$$

with λ_x s.t. $n(\lambda_x) = u(x)$. As usual, to guarantee the correctness of the above definition of implication we require h to satisfy the coherence condition w.r.t. $\top$,

$$h(\lambda) = h(\mu) \Rightarrow h(\alpha \top \lambda) = h(\alpha \top \mu) \; \forall \alpha, \lambda, \mu \in V.$$

Like in Chapter 5, notice that, for example, either when $\top = \wedge$ or when h is injective this condition is satisfied. If h is coherent w.r.t. $\top$, so is n.

Instead, for an optimistic behaviour we consider the t-norm as the conjunction, that is we consider

$$GQU^+(\pi|u) = [\pi \cap u] = \bigvee\nolimits_{x \in X} h(\pi(x) \top \mu_x)$$

with μ_x s.t. $u(x) = h(\mu_x)$. Observe that as V is a residuated distributive lattice with involution, if h is join-preserving, then GQU^- and GQU^+ preserves the possibilistic mixture in the sense that:

Lemma 7.12
GQU^- and GQU^+ preserve the possibilistic mixture in the sense that it holds

$$\begin{aligned} GQU^-(M_\top(\pi_1, \pi_2; \lambda, \mu)) &= (n(\lambda \top \delta_1) \wedge n(\mu \top \delta_2)) \\ GQU^+(M_\top(\pi_1, \pi_2; \lambda, \mu)) &= (h(\lambda \top \gamma_1) \vee h(\mu \top \gamma_2)) \end{aligned}$$

with $n(\delta_j) = GQU^-(\pi_j),\ h(\gamma_j) = GQU^+(\pi_j)$.

Proof:
As both proofs are analogous, we only include the proof for GQU^-. By definition

$$GQU^-(M_\top(\pi_1, \pi_2; \lambda, \mu)) = \bigwedge\nolimits_{x_i \in X} n(M_\top(\pi_1, \pi_2; \lambda, \mu)(x_i) \top \gamma_i),$$

where $n(\gamma_i) = u(x_i)$. Since

$$\begin{aligned} M_\top(\pi_1, \pi_2; \lambda, \mu)(x_i) \top \gamma_i &= [(\lambda \top \pi_1(x_i)) \vee (\mu \top \pi_2(x_i))] \top \gamma_i \\ &=^{20} [\lambda \top \pi_1(x_i) \top \gamma_i] \vee [\mu \top \pi_2(x_i) \top \gamma_i], \end{aligned}$$

then

$$\begin{aligned} n((M_\top(\pi_1, \pi_2; \lambda, \mu)(x_i)) \top \gamma_i) &= n([\lambda \top \pi_1(x_i) \top \gamma_i] \vee [\mu \top \pi_2(x_i) \top \gamma_i]) \\ &=^{21} n(\lambda \top \pi_1(x_i) \top \gamma_i) \wedge n(\mu \top \pi_2(x_i) \top \gamma_i), \end{aligned}$$

[20]Because of $(\alpha \vee \beta) \top \gamma = (\alpha \top \gamma) \vee (\beta \top \gamma)$.

so

$$
\begin{aligned}
GQU^{-}(M_{\top}(\pi_1,\pi_2;\lambda,\mu)) &= \bigwedge_{x_i\in X} n(M_{\top}(\pi_1,\pi_2;\lambda,\mu)(x_i)\top\gamma_i) \\
&= \bigwedge_{x_i\in X} (n(\lambda\top\pi_1(x_i)\top\gamma_i)\wedge \\
&\qquad\qquad n(\mu\top\pi_2(x_i)\top\gamma_i)) \\
&= \{\bigwedge_{x_i\in X} n(\lambda\top\pi_1(x_i)\top\gamma_i)\}\wedge \\
&\qquad\qquad \{\bigwedge_{x_i\in X} n(\mu\top\pi_2(x_i)\top\gamma_i)\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\bigwedge_{x_i\in X} n(\lambda\top\pi_1(x_i)\top\gamma_i) &= n\left(\bigvee_{x_i\in X}(\lambda\top\pi_1(x_i)\top\gamma_i)\right) \\
&= n(\lambda\top\left(\bigvee_{x_i\in X}(\pi_1(x_i)\top\gamma_i)\right)),
\end{aligned}
$$

then

$$
\begin{aligned}
GQU^{-}(M_{\top}(\pi_1,\pi_2;\lambda,\mu)) &= \{n(\lambda\top(\bigvee_{x_i\in X}\pi_1(x_i)\top\gamma_i))\}\wedge \\
&\qquad\qquad \{n(\mu\top\left(\bigvee_{x_i\in X}\pi_2(x_i)\top\gamma_i\right))\}.
\end{aligned}
$$

Since

$$n\left(\bigvee_{x_i\in X}\pi_j(x_i)\top\gamma_i\right)=\bigwedge_{x_i\in X} n(\pi_j(x_i)\top\gamma_i)=GQU^{-}(\pi_j)=n(\delta_j),$$

under the coherence hypothesis, we obtain that

$$n(\lambda\top(\bigvee_{x_i\in X}\pi_1(x_i)\top\gamma_i))=n(\lambda\top\delta_1),$$

and analogously, we have that

$$n(\mu\top(\bigvee_{x_i\in X}\pi_2(x_i)\top\gamma_i))=n(\mu\top\delta_2).$$

Hence,

$$GQU^{-}(M_{\top}(\pi_1,\pi_2;\lambda,\mu))=n(\lambda\top\delta_1)\wedge n(\mu\top\delta_2),$$

with $n(\delta_j)=GQU^{-}(\pi_j)$. $\square$

---

[21] Since $n(a\vee b)=n(a)\wedge n(b)$.

Representation of Generalised Qualitative Utilities

In this Section, we propose a set of axioms to characterise the generalised pessimistic and optimistic qualitative utilities for normalised possibility distributions in the present framework of lattice measurements.

Given $(V, \wedge_V, \vee_V, 0, 1, n_V, \top, I)$ a finite distributive residuated lattice of uncertainty values with involution n_V and $\top$ *a t-norm*, we consider the following axiomatic setting.

Axiomatic Setting

Let $AXP_\top$ be the following set of axioms on $(\Pi^*(X,V), \sqsubseteq, M_\top)$,

- $AP1$: $(\Pi^*(X), \sqsubseteq)$ is a pre-lattice.
- $A2$*(uncertainty aversion)*: if $\pi \leq \pi' \ \Rightarrow \ \pi \sqsupseteq \pi'$.
- $A3_\top$*(independence)*: $\pi_1 \sim \pi_2 \ \Rightarrow \ M_\top(\pi_1, \pi; \lambda, \mu) \sim M_\top(\pi_2, \pi; \lambda, \mu)$.

Let $\overline{\pi}$ be a maximal element of $(\Pi^*(X,V), \sqsubseteq, M_\top)$. So, for each $\lambda \in V$, we consider $\pi_\lambda^- = M_\top(\overline{\pi}, X; 1, \lambda)$[22].

- $AP4_\top : \forall \pi \in \Pi^*(X), \exists \lambda \in V$ s.t. $\pi \sim \pi_\lambda^-$.
- $AP5_\top$: if $\pi_\lambda^- \sqsubseteq \pi_{\lambda'}^- \ \Rightarrow \ \pi_{n_V(\lambda)}^- \sqsupseteq \pi_{n_V(\lambda')}^-$.
- $AP6_\top$: if $\lambda <> \lambda' \ \Rightarrow \ \pi_\lambda^- \sqsubset\sqsupset \pi_{\lambda'}^-$.

In order to represent an optimistic preference criterion, we consider now the distribution π_λ^+ defined as $\pi_\lambda^+ = M_\top(X, \underline{\pi}, \lambda, 1)$, where $\underline{\pi}$ is minimal of $(\Pi^*(X), \sqsubseteq)$, and we have to change the uncertainty aversion axiom $A2$ by the uncertainty-prone postulate:

- $A2^+$: if $\pi \leq \pi'$ then $\pi \sqsubseteq \pi'$,

and to modify the axioms involving the lottery π_λ^- by the axioms related with π_λ^+, that is, we have:

- $AP4_\top^+ : \forall \pi \in \Pi^*(X), \exists \lambda \in V$ s.t. $\pi \sim \pi_\lambda^+$,
- $AP5_\top^+$: if $\pi_\lambda^+ \sqsubseteq \pi_{\lambda'}^+ \ \Rightarrow \ \pi_{n_V(\lambda)}^+ \sqsupseteq \pi_{n_V(\lambda')}^+$.
- $AP6_\top^+$: if $\lambda <> \lambda' \ \Rightarrow \ \pi_\lambda^+ \sqsubset\sqsupset \pi_{\lambda'}^+$.

Lemma 7.13
Let $(U, \wedge_U, \vee_U, 0, 1, n_U)$ *a distributive lattice with involution and* $(V, \wedge, \vee, \top, I, 0, 1, n_V)$ *a residuated distributive lattice with involution,* $h{:}V \to U$ *an onto join-preserving mapping satisfying coherence w.r.t.* $\top$*, and* $u{:}X \to U$*. If* $(GQU^-)^{-1}(1) \neq \emptyset$ *and* $(GQU^-)^{-1}(0) \neq \emptyset$ *(if* $(GQU^+)^{-1}(1) \neq \emptyset$ *and* $(GQU^+)^{-1}(0)$ *resp.), then*

[22] As usual, to be π_λ^- well defined we are assuming that $AP1$ and $A3$ are required.

a) there exists $x \in X$ *s.t.* $u(x) = 1$ *and* $\bigwedge_{x\in X} u(x) = 0$ *(there exists* $x \in X$ *s.t.* $u(x) = 0$ *and* $\bigvee_{x\in X} u(x) = 1$*, resp.).*

b) GQU^- *is onto (*GQU^+ *is onto, resp).*

Proof:
We only provide the proof related with the pessimistic criterion, being the other very analogous.

- Since $(GQU^-)^{-1}(1) \neq \emptyset$, there exists $\overline{\pi}$ s.t.

$$GQU^-(\overline{\pi}) = \bigwedge_{x\in X} n(\overline{\pi}(x)\top\lambda_x) = 1,$$

  with $n(\lambda_x) = u(x)$. Then, $n(\overline{\pi}(x)\top\lambda_x) = 1\ \forall x \in X$. As $\overline{\pi}$ is normalised there exists $x_1 \in X$ s.t. $\overline{\pi}(x_1) = 1$, hence $1 = n(1\top\lambda_{x_1}) = n(\lambda_{x_1}) = u(x_1)$.

- On the other hand, since $(GQU^-)^{-1}(0) \neq \emptyset$, there exists π s.t. $GQU^-(\pi) = 0$, and as $\pi \leq 1$, then $n(\pi(x)\top\lambda_x) \geq n(1\top\lambda_x) = u(x)$. So,

$$0 = GQU^-(\pi) \geq \bigwedge_{x\in X} u(x),$$

  therefore we have that

$$\bigwedge_{x\in X} u(x) = 0.$$

- Given $w \in U$, since n is onto there exists $\lambda \in V$ s.t. $n(\lambda) = w$. As we have seen, there exists $x_1 \in X$ s.t. $u(x_1) = 1$, thus $\bigwedge_{x\in X-\{x_1\}} u(x) = 0$. Let π_w be the distribution defined as

$$\pi_w(x) = \begin{cases} 1 & \text{if} \quad x = x_1 \\ \\ \lambda & \text{otherwise.} \end{cases} \tag{7.10}$$

  Then,

$$\begin{aligned} GQU^-(\pi_w) &= \bigwedge_{x\in X} (n(\pi_w(x)\top\lambda_x)) \\ &= n(1\top\lambda_{x_1}) \wedge \left(\bigwedge_{x\in X-\{x_1\}} n(\lambda\top\lambda_x) \right) \\ &= \bigwedge_{x\in X-\{x_1\}} n(\lambda\top\lambda_x) \\ &= n\left(\bigvee_{x\in X-\{x_1\}} (\lambda\top\lambda_x) \right) \\ &= n\left(\lambda\top \left(\bigvee_{x\in X-\{x_1\}} \lambda_x \right) \right). \end{aligned}$$

Recalling that $n(1) = 0 = \bigwedge_{x \in X - \{x_1\}} u(x) = \bigwedge_{x \in X - \{x_1\}} n(\lambda_x) = n(\bigvee_{x \in X - \{x_1\}} \lambda_x)$, and by coherence condition we have that

$$GQU^{-}(\pi_w) = n(\lambda \top 1) = n(\lambda) = w.$$

$\square$

The Representation Theorem comes next.

Theorem 7.14 (Representation for Pessimistic/Optimistic Utility)
A preference relation $(\Pi^*(X), \sqsubseteq, M_\top)$ satisfies axioms $AXP_\top$ ($AXP_\top^+$ resp.) iff there exist

(i) a utility finite distributive lattice with involution $(U, \wedge, \vee, n_U, 0, 1)$,

(ii) a preference function $u{:}X \to U$, s.t. $u^{-1}(1) \neq \emptyset$ and $\bigwedge_{x \in X} u(x) = 0$, (s.t. $u^{-1}(0) \neq \emptyset$ and $\bigvee_{x \in X} u(x) = 1$, resp.)

(iii) an onto join-preserving mapping $h{:}V \to U$, satisfying coherence w.r.t. $\top$, and also satisfying

$$\text{if } \lambda <> \lambda' \text{ then } h(\lambda) <> h(\lambda'),$$

and $n_U \circ h \circ n_V = h$,

in such a way that it holds:

$$\pi' \sqsubseteq \pi \iff GQU^{-}(\pi'|u) \leq_U GQU^{-}(\pi|u).$$

($\pi' \sqsubseteq \pi \iff GQU^{+}(\pi'|u) \leq_U GQU^{+}(\pi|u)$ resp.)

Proof:
$\leftarrow$) We have to verify that the preference ordering on $\Pi^*(X)$ induced by GQU^- satisfies the above set of axioms. As $\leq_U$ is a partial order, $\preccurlyeq_{GQU^-}$ is reflexive and transitive. By Lemma 7.7, GQU^- is onto, so we may define

$$SUP(\pi, \pi') = (GQU^-)^{-1}(GQU^-(\pi) \vee GQU^-(\pi')),$$

and

$$INF(\pi, \pi') = (GQU^-)^{-1}(GQU^-(\pi) \wedge GQU^-(\pi')).$$

Then, by Theorem 7.3,$(\Pi^*(X), \preccurlyeq_{GQU^-})$ is a pre-lattice.

$A2$ results from the fact that $\top$ and $\wedge$ are non-decreasing in U and n is a reversing function. While, $A3_\top$ is a consequence of the fact that GQU^- preserves mixtures.

Let us prove now $AP5_\top$: if $\pi^-_\lambda \preccurlyeq_{GQU^-} \pi^-_{\lambda'} \Rightarrow \pi^-_{n_V(\lambda)} \succcurlyeq_{GQU^-} \pi^-_{n_V(\lambda')}$.

Let $\overline{\pi}$ be a maximal element of $\Pi^*(X)$, so $GQU^-(\overline{\pi}) = 1$. As GQU^- preserves mixtures and $GQU^-(X) = \bigwedge_{x \in X} n(X(x) \top \lambda_x) = 0$, we have that $GQU^-(\pi^-_\lambda) =$

$n(1\top\delta_1) \wedge n(\lambda\top\delta_2)$, with $n(\delta_1) = GQU^-(\overline{\pi}) = 1,\ n(\delta_2) = GQU^-(X) = 0$. So, by coherence condition,

$$GQU^-(\pi_\lambda^-) = n(\lambda\top\delta_2) = n(\lambda\top 1) = n(\lambda).$$

As $n_U \circ n \circ n_V = n$, and n_V and n_U are involutive, then

$$n(\lambda) \leq n(\lambda') \ \Rightarrow\ n(n_V(\lambda)) = n_U(n(\lambda)) \geq n_U(n(\lambda')) = n(n_V(\lambda')).$$

That is, $AP5_\top$ is verified.
$AP6_\top$ is a consequence of the $\preccurlyeq_{GQU^-}$ definition and that h satisfies (7.7).

Now, we check $AP4_\top$. Let $\overline{\pi}$ be maximal element of $\Pi^*(X)$ w.r.t. $\preccurlyeq_{GQU^-}$. As $GQU^-(\pi_\lambda^-) = n(\lambda)$, then

$$GQU^-(\pi) = n(\lambda) = GQU^-(\pi_\lambda^-) \qquad \forall\lambda \in n^{-1}(GQU^-(\pi)).$$

$\rightarrow$) We structure the proof in the following three steps.

1. We define a finite distributive utility lattice U with involution n_U, and a reversing mapping n from V to U, satisfying if $\lambda <> \lambda'$ then $n(\lambda) <>_U n(\lambda')$, and $n_U \circ n \circ n_V = n$. So, we consider the preserving mapping $h = n_U \circ n$. Hence, h will satisfy (7.8) and (7.7).

   By Lemma 7.8, h is actually a lattice epimorphism.

2. A function GQU^-:$\Pi^*(X) \rightarrow U$ representing $\sqsubseteq$, i.e. such that $GQU^-(\pi) \leq GQU^-(\pi')$ iff $\pi \sqsubseteq \pi'$, is defined.

3. Finally, we prove that $GQU^-(\pi) = \bigwedge_{x\in X}(n(\pi(x)\top\lambda_x))$, where u:$X \rightarrow U$ is the restriction of GQU^- on X. u also satisfies that $u^{-1}(1) \neq \emptyset$ and $\bigwedge_{x\in X} u(x) = 0$.

Now, let us develop these steps.

1. We consider on $\Pi^*(X)$ the equivalence relation $\sim$ defined as

   $$\pi \sim \pi' \iff \pi \sqsubseteq \pi' \text{ and } \pi' \sqsubseteq \pi.$$

   By $AP1, \Pi^*(X)/\sim$ is a lattice. We take as utility lattice $U = \Pi^*(X)/\sim$. As Theorem 7.3guarantees the existence of SUP and INF, we define in U the operations $\wedge$ and $\vee$ induced by them, i.e.

   $$[\pi] \vee [\pi'] = SUP(\pi, \pi'),$$

   and

   $$[\pi] \wedge [\pi'] = INF(\pi, \pi').$$

   The $\leq_U$ induced from $\vee$ (or $\wedge$) coincides with $\sqsubseteq$. It is not difficult to verify that $[X]$ is minimum on $(U, \leq_U)$, and if $\overline{\pi}$ is a maximal element of $\Pi^*(X)$, $[\overline{\pi}]$ is the maximum on U.

Let $\overline{\pi}$ a maximal element of $\Pi^*(X)$, and for each $\lambda \in V$, let

$$\pi^-_\lambda = (1/\overline{\pi}, \lambda/X),$$

and let $n{:}V \rightarrow U$ be defined as

$$n(\lambda) = [\pi^-_\lambda].$$

It is not difficult to see that n is onto, and that $A2$ guarantees n actually reverses the order. Now, we define n_U from n and n_V. For each $w \in U$, we define

$$n_U(w) = n(n_V(\lambda)),$$

with $\lambda \in V$ s.t. $n(\lambda) = w$. By $AP5_\top$, $\pi^-_\lambda \sim \pi^-_{\lambda'}$ implies $\pi^-_{n_V(\lambda)} \sim \pi^-_{n_V(\lambda')}$, hence n_U is well defined. By $AP6_\top, n$ satisfies

$$\text{if } \lambda <> \lambda' \text{ then } n(\lambda) <> n(\lambda'),$$

and by definition of n_U, we have $n_U \circ n \circ n_V = n$ and $n_U \circ n_U = identity$. Let $h = n_U \circ n$. Then, h satisfies the conditions required.

Hence, as n is a reversing epimorphism, and V is a distributive lattice, so is U.

2. GQU^- can be defined on $\Pi^*(X)$ in two steps. First, we define it on lotteries of type π^-_λ, as $GQU^-(\pi^-_\lambda) = n(\lambda)$.

   $AP4_\top$ lets us to extend this definition. Since $\forall\pi\ \exists\lambda$ s.t. $\pi \sim \pi^-_\lambda$ we define $GQU^-(\pi) = n(\lambda)$. It is not difficult to verify that GQU^- represents $\sqsubseteq$.

3. Consider $u{:}X \rightarrow U$ defined as $u(x) = GQU^-(x)$.

   It remains to prove that $GQU^-(\pi) = \bigwedge_{x\in X} n(\pi(x)\top\lambda_x)$. To verify this, we will prove the following equalities:

   - $GQU^-(M_\top(\pi_1, \pi_2, \lambda_1, \lambda_2)) = (n(\lambda_1\top\delta_1)) \wedge (n(\lambda_2\top\delta_2))$
     with $n(\delta_j) = GQU^-(\pi_j)$, j=1,2, and either $\lambda_1 = 1$ or $\lambda_2 = 1$.
     By $AP4_\top, \exists\mu, \gamma$ s.t.

     $$\pi_1 \sim \pi^-_\mu \text{ and } \pi_2 \sim \pi^-_\gamma.$$

     By $A3_\top$,
     $$M_\top(\pi_1, \pi_2; \lambda_1, \lambda_2) \sim M_\top(\pi^-_\mu, \pi^-_\gamma; \lambda_1, \lambda_2)$$

     and reducing lotteries we obtain

     $$M_\top(\pi_1, \pi_2; \lambda_1, \lambda_2) \sim M_\top(\overline{\pi}, X; 1, ((\lambda_1\top\mu) \vee (\lambda_2\top\gamma))).$$

     Therefore, as n is a reversing morphism, we have

     $$\begin{aligned} GQU^-(M_\top(\pi_1, \pi_2; \lambda_1, \lambda_2)) &= n((\lambda_1\top\mu) \vee (\lambda_2\top\gamma)) \\ &= n(\lambda_1\top\mu) \wedge (n(\lambda_2\top\gamma)). \end{aligned}$$

Hence, by coherence, we have that

$$GQU^-(M_\top(\pi_1,\pi_2;\lambda_1,\lambda_2)) \quad = \quad n(\lambda_1 \top \delta_1) \wedge n(\lambda_2 \top \delta_2).$$

As a consequence, we have that

$$GQU^-(\pi_1 \vee \pi_2) = GQU^-(\pi_1) \wedge GQU^-(\pi_2).$$

More generally, $GQU^-(\bigvee_{i=1\ldots,p} \pi_i) = \bigwedge_{i=1\ldots,p} GQU^-(\pi_i)$.

- $GQU^-(\pi) = \bigwedge_{i=1\ldots,p} (n(\pi(x_i)\top\lambda_{x_i}))$.
  As $\pi \in \Pi^*(X)$, then $\exists\, x_j \in X$ s.t. $\pi(x_j) = 1$. Without loss of generality assume $j = 1$. Let
  $$\pi_i = M_\top(x_1, x_i, 1, \pi(x_i)).$$
  Since
  $$\pi = \bigvee_{i=1\ldots,p} \pi_i,$$
  we have that
  $$\begin{aligned} GQU^-(\pi) \quad &= \quad GQU^-\left(\bigvee_{i=1,\ldots,p} \pi_i\right) \\ &= \quad \bigwedge_{i=1,\ldots,p} (u(x_1) \wedge (n(\pi(x_i)\top\lambda_{x_i}))) \\ &=^{23} \quad \bigwedge_{i=1,\ldots,p} n(\pi(x_i)\top\lambda_{x_i}) \end{aligned}$$

Finally, as $\overline{\pi}$ is normalised, there exists $x_0 \in X$ s.t. $\overline{\pi}(x_0) = 1$, so $x_0 \leq \overline{\pi}$. Then, by $A2$, $x_0 \sqsupseteq \overline{\pi}$. As GQU^- represents $\sqsubseteq$,

$$GQU^-(x_0) \geq GQU^-(\overline{\pi}) = 1,$$

hence $u(x_0) = 1$, so $u^{-1}(1) \neq \emptyset$. As $GQU^-(X) = 0$, and $GQU^-(X) = \bigwedge_{x\in X} u(x)$, then $\bigwedge_{x\in X} u(x) = 0$.

This ends the proof for the pessimistic criterion, the optimistic one is very similar.
□

Remark 10
As h is onto and non-decreasing, if V is linear, so is U (i.e. If U is non-linear, then V is non-linear as well). Moreover, as a consequence of the condition "if $\lambda <> \lambda'$ then $h(\lambda) <> h(\lambda')$"*, if V is non-linear so is U. Hence, for the case that the linking mapping h is a non-decreasing function also satisfying (7.7), V and U are either both linear lattices or both non-linear lattices. That is, the cases analysed in the previous Chapter of having a linear scale of uncertainty and a partial order on the cartesian product of preferences, or having a linear scale of preferences and a partial order on the cartesian product of uncertainty are not covered by Theorem 7.14.*

[23] As $\pi(x_1) = 1$, then $u(x_1) = n(\pi(x_1)\top\lambda_{x_1})$.

7.3 The Particular Case of Allowing Different Types of Measurement Lattices

In the introduction of this Chapter we announced that there exist decision making problems in which incomparability may not be preserved by the mapping linking V and U. In this Section, we analyse these cases. Let U be a finite linear scale, and let $(V, \wedge, \vee, \top, I, 0, 1, n_V)$ be a residuated distributive lattice with involution[24], $h{:}V \to U$ is an onto join-preserving mapping satisfying coherence w.r.t. $\top$, and $u{:}X \to U$. Under these hypotheses, let us consider:

$$GQU_L^-(\pi|u) = \min_{x \in X} n(\pi(x) \top \lambda_x),$$

with λ_x s.t. $n(\lambda_x) = u(x)$, and

$$GQU_L^+(\pi|u) = \max_{x \in X} h(\pi(x) \top \mu_x)$$

μ_x being s.t. $u(x) = h(\mu_x)$. As usual GQU_L^- and GQU_L^+ preserve the possibilistic mixture in the sense that the following expressions hold,

$$\begin{array}{rcl} GQU_L^-(M_\top(\pi_1, \pi_2; \lambda, \mu)|u)(x) & = & \min\{n(\lambda \top \delta_1), n(\mu \top \delta_2)\}, \\ GQU_L^+(M_\top(\pi_1, \pi_2; \lambda, \mu)|u)(x) & = & \max\{h(\lambda \top \gamma_1),\ h(\mu \top \gamma_2)\}, \end{array}$$

with $n(\delta_j) = GQU_L^-(\pi_j|u)$, and $h(\gamma_j) = GQU_L^+(\pi_j|u)$, for $j = 1, 2$.

We consider as usual the set of distributions $\Pi^*(X, V)$ with the mixture operation $M_\top$. We want to characterise the orderings induced by the GQU_L^- and GQU_L^+ functions. With this goal, we consider the following axiomatic setting $BXP_\top = \{A1, A2, A3_\top, AP4_\top, AP6eq_\top\}$, with

- $AP6eq_\top$: if $\lambda <> \lambda' \ \Rightarrow\ \pi_\lambda^- \sim \pi_{(\lambda \vee \lambda')}^-$.

where $\pi_\lambda^- = M_\top(\overline{\pi}, X; 1, \lambda)$, with $\overline{\pi}$ being a maximal[25] element of $(\Pi^*(X, V), \sqsubseteq)$.

Observe that since $<>$ is symmetric we have that $\lambda <> \lambda' \ \Rightarrow\ \pi_\lambda^- \sim \pi_{\lambda'}^-$.

$AP6eq_\top$ establishes that two incomparable values of uncertainty, λ and λ', lead to two indistinguishable lotteries, the lottery associated with their supremum being indistinguishable with them as well.

For an optimistic behaviour we consider the axiom set $BXP_\top^+ = \{A1, A2^+, A3_\top, AP4_\top^+, AP6eq_\top^+\}$, with

- $AP6eq_\top^+$: if $\lambda <> \lambda' \ \Rightarrow\ \pi_\lambda^+ \sim \pi_{(\lambda \vee \lambda')}^+$.

where $\pi_\lambda^+ = M_\top(X, \underline{\pi}; \lambda, 1)$, with $\underline{\pi}$ a minimal element of $(\Pi^*(X, V), \sqsubseteq, M_\top)$.

[24] In Section 6.4.1 it has been mentioned that we have only considered there the special case of having a linear scale of preference and the same scale in the cartesian product where we measure uncertainty. The case of having different scales remains an open question. Here, we provide a first answer.

[25] In fact, to be π_λ^- well defined we are assuming that $A1$ and $A3_\top$ are required.

Theorem 7.15 (Representation Theorem)
A preference relation $(\Pi^*(X,V), \sqsubseteq)$ satisfies axioms $BXP_\top$ ($BXP_\top^+$ resp.) iff there exist

(i) a finite linear utility scale U,

(ii) a preference function $u{:}X \to U$, s.t. $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,

(iii) an onto join-preserving mapping $h{:}V \to U$, satisfying coherence w.r.t. $\top$, and also satisfying

$$\text{if } \lambda <> \lambda' \text{ then } h(\lambda \vee \lambda') = h(\lambda'), \tag{7.11}$$

in such a way that it holds:

$$\pi \sqsubseteq \pi' \iff \pi \preccurlyeq_{GQU_L^-(\cdot|u)} \pi',$$

($\pi \sqsubseteq \pi' \iff \pi \preccurlyeq_{GQU_L^+(\cdot|u)} \pi'$ resp.) with $n = n_U \circ h$.

Proof:
We consider the pessimistic case, the optimistic one being analogous.
$\leftarrow$) We verify that the preference ordering on $\Pi^*(X)$ induced by GQU_L^- satisfies the above set of axioms. As $\leq_U$ is a linear order, so is $\preccurlyeq_{GQU_L^-}$. As usual, $A2$ results from the fact that $supremum$ and $infimum$ are non-decreasing in U and n is a reversing function. While, $A3_\top$ is a consequence of the fact that GQU_L^- preserves mixtures.

$AP6eq_\top$ is a consequence of the definition of $\preccurlyeq_{GQU_L^-}$ and that h satisfies (7.11).

We check $AP4_\top$. Let $\overline{\pi}$ be maximum element of $\Pi^*(X)$ w.r.t. $\preccurlyeq_{GQU_L-}$. As $GQU_L^-(\pi_\lambda^-) = n(\lambda)$, then

$$GQU_L^-(\pi) = n(\lambda) = GQU_L^-(\pi_\lambda^-) \qquad \forall \lambda \in n^{-1}(GQU_L^-(\pi)).$$

$\rightarrow$) The proof is again very analogous with the one given for the linear case. As usual, we structure the proof in the following three steps.

1. We define the finite linear utility scale $U = \Pi^*(X)/\sim$ with the ordering induced by $\sqsubseteq$. $n{:}V \to U$ is defined as

   $$n(\lambda) = [\pi_\lambda^-],$$

   with $\pi_\lambda^- = M_\top(\overline{\pi}, X; 1, \lambda)$, $\overline{\pi}$ being the maximum element of $(\Pi^*(X,V), \sqsubseteq, M_\top)$. By $A2$, n is a reversing ordering mapping, while $AP4_\top$ guarantees it is onto. By $AP6eq_\top$ and n being reversing ordering, we have that

   $$n(\lambda \vee \lambda') = n(\lambda) \wedge n(\lambda').$$

   As usual, n results coherent w.r.t. $\top$ because of the reduction property of $M_\top$ and $A3_\top$. So, we consider the onto join-preserving mapping $h = n_U \circ n$. Hence, h will satisfy (7.11) and coherence w.r.t. $\top$.

2. Again, GQU_L^- may be defined on $\Pi^*(X)$ in two steps. First, we define it on lotteries of type π_λ^-, as $GQU_L^-(\pi_\lambda^-) = n(\lambda)$.

   $AP4_\top$ lets us to extend this definition. Since $\forall\pi\ \exists\lambda$ s.t. $\pi \sim \pi_\lambda^-$, we define $GQU_L^-(\pi) = n(\lambda)$. It is not difficult to verify that GQU_L^- represents $\sqsubseteq$.

   Consider $u{:}X \to U$ defined as $u(x) = GQU_L^-(x)$.

3. We will prove that

$$GQU_L^-(\pi) = \min_{i=1,\dots,p}\ n(\pi(x_i)\top\gamma_i)$$

   with $n(\gamma_i) = u(x_i)$.

   To verify this, we will prove the following equalities:

   - $\forall\pi_1, \pi_2$,

$$GQU_L^-(M_\top(\pi_1,\pi_2;\alpha,\beta)) = n((\alpha\top\lambda_1)\vee(\beta\top\lambda_2)), \tag{7.12}$$

     with λ_j such that $GQU_L^-(\pi_j) = n(\lambda_j)$.
     Indeed, $A4_\top$ guarantees that $\exists\lambda_1$ s.t. $\pi_1 \sim M_\top(\overline{\pi}, X; 1, \lambda_1)$ and $\exists\lambda_2$ s.t. $\pi_2 \sim M_\top(\overline{\pi}, X; 1, \lambda_2)$, remember that $GQU_L^-(\pi_1) = n(\lambda_1)$ and $GQU_L^-(\pi_2) = n(\lambda_2)$. So, using the independence axiom $A3_\top$,

$$M_\top(\pi_1,\pi_2;\alpha,\beta) \sim M_\top(M_\top(\overline{\pi}, X, 1, \lambda_1), M_\top(\overline{\pi}, X; 1, \lambda_2); \alpha, \beta),$$

     and by reduction of "lotteries" it reduces to

$$\begin{aligned} &M_\top(\overline{\pi}, X; ((\alpha\top 1)\vee(\beta\top 1)), ((\alpha\top\lambda_1)\vee(\beta\top\lambda_2))) \sim \\ &\qquad \sim M_\top(\overline{\pi}, X; (\alpha\vee\beta), ((\alpha\top\lambda_1)\vee(\beta\top\lambda_2))) \\ &\qquad \sim M_\top(\overline{\pi}, X; 1, ((\alpha\top\lambda_1)\vee(\beta\top\lambda_2))). \end{aligned}$$

     Therefore,

$$GQU_L^-(M_\top(\pi_1,\pi_2;\alpha,\beta)) = n((\alpha\top\lambda_1)\vee(\beta\top\lambda_2))$$

     with λ_j such that $GQU_L^-(\pi_j) = n(\lambda_j)$, i.e.

$$GQU_L^-(M_\top(\pi_1,\pi_2;\alpha,\beta)) = \min(n(\alpha\top\lambda_1), n(\beta\top\lambda_2)).$$

     Finally, we verify that (7.12) does not depend on the λ chosen, i.e. if μ is such that $GQU_L^-(\pi_1) = n(\mu)$, then

$$n((\alpha\top\lambda_1)\vee(\beta\top\lambda_2)) = n((\alpha\top\mu)\vee(\beta\top\lambda_2)).$$

     Indeed, as $\pi_{\lambda_1}^- \sim \pi_\mu^-$ then

$$\begin{aligned} M_\top(\overline{\pi}, X; 1, (\alpha\top\lambda_1)\vee(\beta\top\lambda_2)) &\sim M_\top(\pi_{\lambda_1}^-, \pi_{\lambda_2}^-; \alpha, \beta) \\ &\sim M_\top(\pi_\mu^-, \pi_{\lambda_2}^-; \alpha, \beta) \\ &\sim M_\top(\overline{\pi}, X; 1, (\alpha\top\mu)\vee(\beta\top\lambda_2)), \end{aligned}$$

     therefore

$$n((\alpha\top\lambda_1)\vee(\beta\top\lambda_2)) = n((\alpha\top\mu)\vee(\beta\top\lambda_2)).$$

In particular, we have that

$$GQU_L^-(M_\top(x,y;1,\beta)) = \min(n(1\top\lambda_1), n(\beta\top\lambda_2))$$

with $u(x)=n(\lambda_1), u(y)=n(\lambda_2)$. So,

$$GQU_L^-(M_\top(x,y;1,\beta)) = \min(u(x), n(\beta\top\lambda_2)),$$

with $u(y)=n(\lambda_2)$, and

$$GQU_L^-(\pi_1\vee\pi_2) = \min(GQU_L^-(\pi_1), GQU_L^-(\pi_2)).$$

Indeed, as $\pi_1\vee\pi_2 = M_\top(\pi_1,\pi_2,1,1)$, therefore,

$$GQU_L^-(\pi_1\vee\pi_2) = \min(n(\mu_1), n(\mu_2))$$

with $n(\mu_1)=GQU_L^-(\pi_1), n(\mu_2)=GQU_L^-(\pi_2)$, so

$$GQU_L^-(\pi_1\vee\pi_2) = \min(GQU_L^-(\pi_1), GQU_L^-(\pi_2)).$$

Moreover, we have

$$GQU_L^-\left(\bigvee_{i=1,\ldots,p}\pi_i\right) = \min_{i=1,\ldots,p} GQU_L^-(\pi_i) \qquad \forall\,\pi_i.$$

- $GQU_L^-(\pi) = \min_{i=1,\ldots,p}\ n(\pi(x_i)\top\gamma_i)$.
  As π is normalised, there exists $x_j\in X$ such that $\pi(x_j)=1$. Without loss of generality, let us assume that $j=1$. As for each π, $M_\top$ satisfies that

  $$M_\top(x_1,x_i;1,\pi(x_i))(x_k) = \begin{cases} 1, & \text{if} \quad x_k = x_1, \\ \pi(x_i), & \text{if} \quad x_1 \neq x_k = x_i, \\ 0, & \text{otherwise}. \end{cases}$$

  Then, choosing

  $$\pi_i = M_\top(x_1,x_i;1,\pi(x_i)),$$

  we obtain $\pi = \bigvee_{i=1,\ldots,p}\ \pi_i$, therefore

  $$\begin{aligned} GQU_L^-(\pi) &= GQU_L^-\left(\bigvee_{i=1,\ldots,p} M_\top(x_1,x_i;1,\pi(x_i))\right) \\ &= \min_{i=1,\ldots,p} GQU_L^-(M_\top(x_1,x_i,1,\pi(x_i))) \\ &= \min_{i=1,\ldots,p} [\min(u(x_1), n(\pi(x_i)\top\lambda_i))] \end{aligned}$$

  with $u(x_i) = GQU_L^-(x_i) = n(\lambda_i)$, so

  $$GQU_L^-(\pi) = \min_{i=1,\ldots,p}\ n(\pi(x_i)\top\lambda_i).$$

$\square$

Chapter 8

An Extended Model Allowing Partially Inconsistent Belief States: Application to Possibilistic Case-Based Decision Theory

The decision models described so far obviously rely on a possibilistic representation of the belief states. Such a representation, i.e. a possibility distribution, can be made explicit for instance if (uncertain) generic knowledge and information is available under the form of a possibilistic knowledge base (Dubois et al., 1997g). But, suppose that the available information about the consequences of decisions appears in the form of already experienced instances of decision problem cases. A *decision problem case* is an account of a previous situation where a decision was made, and the actual consequence of that decision was recorded. A *decision problem case* can be thus formalised as a 3-tuple *(situation-description, decision, consequence)*. The idea of the so called *"Case-Based Decision Theory"* is to select a decision that gave good results in the past in situations similar to the current one.

For example, it is possible, and probably more realistic, to present the omelette story of Savage of Section 4.6 as a case-based decision problem. The memory would consist of descriptions of eggs broken in the past by the agent, the decisions made about those eggs and the outcomes (described in Table 4.1). Descriptions could be done in terms of attributes like colour, the smell, weight of the egg, etc. The decision made about a new egg for a new omelette could then be based on the resemblance between the present egg and the past ones. If the egg looks fresh (e.g. it is similar to the descriptions of past fresh eggs), then, *Break the egg In the Omelette (BIO)*, if the the egg looks rotten, then, *Throw it Away (TA),* if the egg is only mildly fresh but not clearly rotten, or it is a new type of egg not encountered in the past, then, for instance, *Break it Apart in a Cup*

(BAC).

In such a framework, as it has been mentioned in Section 2.3, Gilboa and Schmeidler (1995) have proposed a case-based decision model where the decision-maker, in face of a new situation s_0, is supposed to choose a decision d which maximises a counterpart of classical expected utility. Namely,

$$U_{s_0,M}(d) = \sum_{(s,d,x)\in M} Sim(s_0, s) \cdot u(x)$$

where Sim is a non-negative function which estimates the similarity between situations and the current situation s_0 and u provides a numerical preference for each consequence x.

Dubois and Prade (1997d) propose another approach to case-based decision, based on possibility and necessity measures. Instead of averaging the preference of consequences obtained in similar situations, weighted by similarity degrees, they propose to look for decisions that always gave good results in similar experienced situations.

In the next Section, a link is established between Dubois and Prade's Case-based and Qualitative Decision models, by estimating *how plausible x is a consequence of a decision d, in the current situation s_0, in terms of the extent to which s_0 is similar to situations in which x was experienced after taking the decision d.* So again, a decision or action d can be identified with a possibility distribution on consequences.

This link between similarity on situations and possibility distributions on consequences allows us to apply the possibilistic qualitative criteria described in the previous Chapters to case-based decision problems. However, working with case-based decision we face with problems in which non-normalised possibility distribution are involved. Non-normalisation problems may also appear in *QDT* when different sources of information about the actual situation are available and they are partially conflicting. Namely, in such a case, if a min-based aggregation of the corresponding possibility distributions is performed to merge them into a single one, then, we can come up with a non-normalised distribution as soon as their cores are disjoint, i.e. when the distributions are mutually inconsistent to some extent. But even under these hypotheses of partial inconsistency, one may be interested in making rational decisions.

In order to allow a proper handling of non-normalised distributions, in Section 8.2 we extend the basic model and provide corresponding characterisations of the orderings induced by suitably modified utility functions. Then, we shall be ready to return in Section 8.3 to the case based decision problem, applying these utility functions. In Section 8.4 we analyse the example of the safety problem in the chemical plant from a case-based decision problem view, while in Section 8.5 we consider the case of non-normalised distributions in a lattice measurement framework. In Section, 8.6 we extend the model in another direction to take into account the performance of “similar” acts for evaluating the utility of a decision d. This extension again leads us to deal with possibility distributions on consequences, hence we may approach this type of problem with the qualitative utility functions analysed in the previous Chapters.

8.1 Possibilistic Case-Based Decision Theory

Dubois and Prade (1997d) propose an approach to case-based decision based on possibility and necessity measures. Instead of averaging the utility of consequences obtained in similar situations, they propose to look for decisions that always gave good results in similar experienced situations. As in Gilboa and Schmeidler (1995)'s proposal, they assume a given memory of cases M and a "similarity"[1] function Sim:$S \times S \rightarrow [0, 1]$ that measures the degree of similarity between two situations, and a preference function u:$X \rightarrow [0, 1]$ representing preferences on consequences. They propose the following utility function

$$U^{-}_{s_0,M}(d|u) = \min_{(s,d,x)\in M} (Sim(s, s_0) \Rightarrow u(x)),$$

where $\Rightarrow$ is chosen as $(x \Rightarrow y) = N(x)\perp y$ with $\perp$ a conorm and N an involutive negation in the real interval [0,1]. If only ordinal interpretations are meaningful, $\perp$ is taken as $maximum$, so

$$U^{-}_{s_0,M}(d|u) = \min_{(s,d,x)\in M} \max(N(Sim(s, s_0)), u(x)).$$

The interpretation of this criterion is very natural if we think of it in terms of fuzzy set inclusionship (see Section 5.1 for more details). Indeed, let us respectively denote by Sim^d and G^d the *fuzzy set of situations which are similar to* s_0 *and where* d *was already experienced* and the *fuzzy set of situations where decision* d *led to good results* respectively, with membership functions $Sim^d(s) = Sim(s, s_0)$ and $G^d(s) = u(x)$, if $(s, d, x) \in M$. Then, the above criterion of maximising $U^{-}_{s_0,M}$ looks for decisions d such that, in *all situations where* d *was previously experienced, it led to good results.*

Indeed, if

$$\{s|\ (s, d, x) \in M, Sim(s, s_0) > 0\} \subseteq \{s|(s, d, x) \in M,\ u(x) = 1\},$$

then $U^{-}_{s_0,M}(d) = 1$, and

$U^{-}_{s_0,M}(d) = 0$ as soon as $\exists s$ s.t. $Sim(s, s_0) = 1, (s, d, x) \in M$ and $u(x) = 0$.

Actually, $U^{-}_{s_0,M}(d)$ is a rather drastic criterion since it requires that in *all* the situations similar to s_0, d led to good results.

A more "optimistic" behaviour consists in selecting decisions which *led to a good result in* at least one *situation similar to* s_0. They model it using the dual criterion

$$U^{+}_{s_0,M}(d) = \max_{(s,d,x)\in M} min(Sim(s, s_0), u(x)).$$

Thus, $U^{+}_{s_0,M}(d)$ is maximal as soon as there exists a case corresponding to a situation completely similar to s_0 where d led to an excellent result.

The pessimistic and optimistic decision rules differ from the Gilboa-Schmeidler rule in that they do not assume that results obtained in past experiences accumulate

[1] Actually, we are speaking about a fuzzy proximity relation on S, i.e *Sim* is a symmetric and reflexive relation.

and, particularly, compensate. For instance, in the omelette example, using Gilboa-Schmeidler rule, a few bad experiences with a certain kind of egg very similar to the current one can be fully counterbalanced by sufficiently many half-fresh eggs of similar appearance. The pessimistic criterion suggests mistrusting these eggs and the optimistic one only partially tolerates them.

Observe that if the fuzzy set Sim^d is normalised, then,

$$U^+_{s_0,M}(d) \geq U^-_{s_0,M}(d)$$

as it is expected.

It is obvious the close relationship between these criteria and the ones described in the previous Chapters. Actually, one can represent the *Case-Based Reasoning Principle* stated in (Dubois et al., 1997b) saying that for each $(s,d,x) \in M$,

> "the more similar s_0 is to s, the more plausible x is a consequence for s_0 under decision d",

by the following inequality

$$\pi_{d,s_0}(x) \geq \max\{Sim(s_0,s) |\ (s,d,x) \in M\},$$

where $\pi_{d,s_0}: X \rightarrow V$ is the *possibility distribution* representing the *plausibility of x being the consequence of d at s_0*. For computational reasons (using a kind of minimum specificity principle (Dubois and Prade, 1987)) we can just take the equality above and let

$$\pi_{d,s_0}(x) = \max\{Sim(s_0,s)|(s,d,x) \in M\}^2,$$

and so, a decision or act d at the new situation s_0 can be identified with the possibility distribution π_{d,s_0}. Taking $U = V \subset [0,1]$, it can be shown that

$$U^-_{s_0,M}(d|u) = QU^-(\pi_{d,s_0}|u) = \min_{x \in X} \max(N(\pi_{d,s_0}(x)), u(x)),$$

$$U^+_{s_0,M}(d|u) = QU^+(\pi_{d,s_0}|u) = \max_{x \in X} \min(\pi_{d,s_0}(x), u(x)).$$

We have, however, to be very cautious if we want to apply this qualitative decision model: *nothing prevents the distributions π_{d,s_0} from being non-normalised*. And this may have undesirable consequences, such as the fact that the pessimistic utility $U^-_{s_0,M}(d)$ may be higher than the optimistic utility $U^+_{s_0,M}(d)$. For example, when

$$\max_{(s,d,x)\in M} Sim(s,s_0) < 1,$$

it means that decision d has been never experienced on a situation completely similar to s_0. In particular, when

$$\{s \mid (s,d,x) \in M, Sim(s,s_0) > 0\} = \emptyset,$$

we have $U^-_{s_0,M}(d) = 1$ which is non-satisfactory.

In order to avoid these shortcomings, for distributions defined on [0,1], Dubois et al. (1997b) suggest the following modifications. Consider

[2] By convention we take $\max\ \emptyset = 0$.

$$h_{Sim}(s_0) = \max\{Sim(s, s_0) \,|(s, d, x) \in M\},$$

Sim* a renormalisation[3] of Sim and $U^{-*}_{s_0,M}(U^{+*}_{s_0,M}$ resp.) the result of considering $U^{-}_{s_0,M}$ ($U^{+}_{s_0,M}$) with the similarity Sim* instead of Sim,

$$\begin{aligned} U^{-}_{s_0,M}(d) &= \min(h_{Sim}(s_0),\, U^{-*}_{s_0,M}(d)), \\ U^{+}_{s_0,M}(d) &= \max(1 - h_{Sim}(s_0), U^{+*}_{s_0,M}(d)). \end{aligned}$$

Analogously, for each V and U, we propose to modify our previous definitions and let

$$U^{-}_{s_0,M}(d) = \underline{QU}^{-}(\pi_{d,s_0}),$$

where π_{d,s_0} is the distribution associated to Sim and M, and

$$\underline{QU}^{-}(\pi_{d,s_0}) = \min(\mathcal{H}(\pi_{d,s_0}), QU^{-}(\mathcal{N}(\pi_{d,s_0}))) \qquad (8.1)$$

where $\mathcal{H}(\pi)$ is the *height of the distribution* π, $\mathcal{H}(\pi) = \max_{x \in X}\ \pi(x)$, and $\mathcal{N}(\pi_{d,s_0})$ is a normalised version of π_{d,s_0} defined as

$$\mathcal{N}(\pi_{d,s_0})(x) = \begin{cases} 1, & \text{if } \ \pi_{d,s_0}(x) = \mathcal{H}(\pi_{d,s_0}) \\ \\ \pi_{d,s_0}(x), & \text{otherwise.} \end{cases}$$

Notice that when $\mathcal{H}(\pi_{d,s_0}) = 1$, the original expression is retrieved. The rationale behind this expression is that our willingness to apply decision d in s_0 is upper bounded by the existence of situations completely similar to s_0 where decision d was experienced. Moreover, π_{d,s_0} is renormalised in order to obtain a meaningful degree of inclusion. Thus, equation (8.1) corresponds to the expression of the compound condition:

> “there exist situations similar to s_0 where decision d was applied and the situations which are the most similar to s_0 are among the situations where decision d led to good results”.

Note that the similarity is no longer estimated in an absolute manner, but in a relative way, hence the normalisation. Clearly, it would be also natural that the optimistic evaluation be all the greater as the decision d was never applied to situations similar to s_0 in the past (indeed, in this case, the optimistic *Decision Maker* is prone to experiencing new decisions on new situations he never met).

8.2 Representation of Possibilistic Utilities for Non-Normalised Distributions

In Possibilistic Logic (Dubois et al., 1994), non-normalised possibility distributions account for partially inconsistent belief states. Indeed, if $\pi{:}S \to V$ is such that $\pi(s) < 1$

[3] There are several forms of defining the renormalisation of a fuzzy set A, they suggest e.g. $A^*(z) = \frac{A(z)}{\max_z A(z)}$.

for all $s \in S$, it means that there is no situation which is fully plausible. The consistency degree of π is measured by the *height of the distribution*, $\mathcal{H}(\pi) = \max_{s \in S} \ \pi(s)$, whereas how far $\mathcal{H}(\pi)$ is from 1, measured as $n_V(\mathcal{H}(\pi))$, provides an estimate of how inconsistent the belief state is. Notice that in the case not dealt in our framework of V being the real unit interval [0, 1], the inconsistency degree is usually $1 - \mathcal{H}(\pi)$.

In this Section, we extend the possibilistic decision model described through the previous Chapters in order to take into account, not only fully consistent belief states, but also those which are partially inconsistent. The idea is to adapt the solutions presented in the previous Section, which basically consist of suitably transforming the non-normalised distributions into normalised ones and then applying the original model. However, the transformation is not simply a normalisation, the inconsistency degree is also taken into account to endow the possibility distribution with a uniform level of uncertainty. Hence, we could say that, in doing the transformation, inconsistency is exchanged for uncertainty (you may see the details in the next Subsections).

8.2.1 The Pure Ordinal Case

Here we consider as the working set of possibilistic lotteries the *set* Π^{ex}*(X) of non-necessarily normalised distributions on* X with values on a finite linear uncertainty scale V, keeping the same definition of possibilistic mixture of (3.1), i.e.

$$(\lambda/\pi_1, \mu/\pi_2)(x) = \max\{\min(\lambda, \pi_1(x)), \min(\mu, \pi_2(x))\},$$

with $\max(\lambda, \mu) = 1$. Thus, the reduction property

$$(\lambda/\pi_1, \mu/(\alpha/\pi_1, \beta/\pi_2)) = (\max(\lambda, \min(\mu, \alpha))/\pi_1, \min(\mu, \beta)/\pi_2)$$

still holds.

Now, in the usual linear setting, i.e. with finite linear uncertainty and preference scales V and U, we extend the utility functionals QU^- and QU^+ to evaluate non-normalised distributions of $\Pi^{ex}(X)$ as well, reflecting the solution proposed at the end of the previous Section. Given an onto order-preserving mapping h:$V \to U$ and u:$X \to U$ as usual, we define for any $\pi \in \Pi^{ex}(X)$:

$$\underline{QU}^-(\pi|u) = \min\{QU^-(\mathcal{N}(\pi)|u), n \circ n_V(\mathcal{H}(\pi))\}$$

$$\underline{QU}^+(\pi|u) = \max\{QU^+(\mathcal{N}(\pi)|u), h \circ n_V(\mathcal{H}(\pi))\}.$$

From these definitions, it is obvious that, for all $\pi \in \Pi^{ex}(X)$, we have $\underline{QU}^+(\pi) \geq \underline{QU}^-(\pi)$, in particular, if $\pi \equiv 0$, $\underline{QU}^-$ (π) = 0 and $\underline{QU}^+(\pi) = 1$. Moreover, $\underline{QU}^-$ ($\underline{QU}^+$ resp.) is an extension of QU^- (of QU^+ resp.) since, if π is normalised, $\mathcal{H}(\pi)$ = 1, and $n \circ n_V(1) = 1$ and $h \circ n_V(1) = 0$, and thus $\underline{QU}^-$ and QU^- ($\underline{QU}^+$ and QU^+ resp.) collapse on $\Pi(X)$. As before, when clear from the context, we will omit the preference function u from $\underline{QU}^-$ and $\underline{QU}^+$ for the sake of a simpler notation.

Notice[4] that, instead of introducing the modifying factor $\mathcal{H}(\pi_{d,s_0})$ into the final step of the utility computations, one could already introduce this factor in the normalisation of the distributions by considering

$$\mathcal{N}'(\pi_{d,s_0}) = \max(\mathcal{H}(\pi_{d,s_0}), \mathcal{N}(\pi_{d,s_0}))$$

and then just write, for instance, $\underline{QU}^-(\pi|u) = QU^-(\mathcal{N}'(\pi)|u)$. We shall however stick to the usual notion of (ordinal) normalisation and explicitly deal with the factors in spite of a bit heavier notation.

In order to characterise the preference orderings $\sqsubseteq$ induced in $\Pi^{ex}(X)$ by $\underline{QU}^-$ and $\underline{QU}^+$, we need to extend the axiom sets AX and AX^+ respectively, defined on $\Pi(X)$, with the following additional axiom:

- $A7$: for all $\pi \in \Pi^{ex}(X), \pi \sim (1/\mathcal{N}(\pi), n_V(\mathcal{H}(\pi))/X)$.[5]

The intuitive idea behind axiom $A7$ is that, as already pointed out, we make a non-normalised possibilistic lottery π indifferent to the corresponding normalised lottery $\mathcal{N}(\pi)$, provided that it is modified by a uniform uncertainty level corresponding to the inconsistency degree of π, i.e. from a decision point of view, π is made equivalent to π^*, where $\pi^*(x) = \max(\mathcal{N}(\pi)(x), n_V(\mathcal{H}(\pi)))$. In other words, according to Possibility Theory, the statement "it is certain that π represents the belief state" is understood as "it is $\mathcal{H}(\pi)$-certain that $\mathcal{N}(\pi)$ represents the belief state". Obviously, if π is an already normalised distribution, $\mathcal{N}(\pi) = \pi, \mathcal{H}(\pi) = 1$, and both statements are exactly the same.

Now, let us prove the following representation theorem.

Theorem 8.1 (Representation Theorem)
A preference relation $\sqsubseteq$ on $\Pi^{ex}(X)$ satisfies axiom set $AX^{ex} = AX + A7$ (resp. $AX^{+ex} = AX^+ + A7$) if, and only if, there exist

(i) a linearly ordered and finite preference scale U with $\inf(U) = 0$ and $\sup(U) = 1$,

(ii) a preference function $u{:}X \to U$ such that $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$, and

(iii) an onto order-preserving mapping $h{:}V \to U$,

in such a way that it holds:
for each $\pi \in \Pi^{ex}(X)$,

$$\pi' \sqsubseteq \pi \quad \textit{iff} \quad \underline{QU}^-(\pi'|u) \sqsubseteq \underline{QU}^-(\pi|u),$$

($\pi' \sqsubseteq \pi$ iff $\underline{QU}^+(\pi'|u) \sqsubseteq \underline{QU}^+(\pi|u)$ resp.) where, as usual, $n = n_U \circ h$.

[4]This remark was made by a referee of one of our publications.

[5]Let $\sqsubseteq$ be a preference relation on $\Pi^{ex}(X)$. We will denote by $\sqsubseteq'$ its restriction to $\Pi^*(X)$, the *set of **normalised** possibility distributions*, and by $\sim$ and $\sim'$ the corresponding indifference relations. We say that $\sqsubseteq$ on $\Pi^{ex}(X)$ satisfies axiom set $AX^{ex} = AX \cup \{A7\}$ ($AX^{ex,+} = AX^+ \cup \{A7\}$ resp.) if and only if its restriction to $\Pi^*(X)$, satisfies AX (AX^+ resp.) and $\sqsubseteq$ also satisfies $A7$. Following, for a simpler notation, we use $\sqsubseteq$ for denoting this relation and its restriction too, understanding that they may be distinguished by the context.

Proof:
We only prove the theorem for the pessimistic criterion, the proof for the optimistic criterion being very similar.

$\leftarrow$) We have to prove that, given a preference function $u{:}X \to U$ verifying (ii), and an onto order-preserving mapping $h{:}V \to U$, the ordering on possibility distributions of $\Pi^{ex}(X)$ induced by the utility evaluation $\underline{QU}^-$ satisfies the axioms of AX^{ex}. Since $\underline{QU}^-$ coincides with QU^- on $\Pi(X)$, all axioms from AX are automatically satisfied by the theorem for the linear normalised case (Theorem 4.8). Thus, it only remains to verify that $A7$ also holds. According to (ii), there is $\underline{x}$ such that $u(\underline{x}) = 0$, and thus $\underline{QU}^-(X) = 0$. But since QU^- preserves possibilistic mixtures, we have for all $\pi \in \overline{\Pi^{ex}}(X)$,

$$\begin{aligned} QU^-(1/\mathcal{N}(\pi), n_V(\mathcal{H}(\pi))/X) &= \min(\max(n(1), QU^-(\mathcal{N}(\pi))), \\ &\qquad \max(n(n_V(\mathcal{H}(\pi))), QU^-(X))) \\ &= \min(QU^-(\mathcal{N}(\pi)), n \circ n_V(\mathcal{H}(\pi))) \\ &= \underline{QU}^-(\pi). \end{aligned}$$

Thus, π is equivalent to $(1/\mathcal{N}(\pi), n_V(\mathcal{H}(\pi))/X)$ w.r.t. to the ordering induced by $\underline{QU}^-$.

$\rightarrow$) Let us assume now that we have an ordering ($\Pi^{ex}(X)$, $\sqsubseteq$) satisfying the axioms of AX^{ex}. In particular, $\sqsubseteq$ satisfies all AX axioms, hence, applying Theorem 4.8 again, we can suppose the existence of U, $u{:}X \to U$ and $h{:}V \to U$ satisfying (i), (ii) and (iii), and such that the corresponding utility QU^- represents $\sqsubseteq$ on $\Pi(X)$, i.e. for all normalised π, we have that $\pi' \sqsubseteq \pi$ iff $QU^-(\pi'|u) \sqsubseteq QU^-(\pi|u)$. Axiom $A7$ guarantees that, for any π, $\pi \sim (1/\mathcal{N}(\pi), n_V(\mathcal{H}(\pi))/X)$. Since $QU^-(X) = 0$, and $(1/\mathcal{N}(\pi), n_V(\mathcal{H}(\pi))/X)$ is a normalised distribution, we define

$$\begin{aligned} \underline{QU}^-(\pi) &= QU^-(1/\mathcal{N}(\pi), n_V(\mathcal{H}(\pi))/X) \\ &= \min(QU^-(\mathcal{N}(\pi)), n \circ n_V(\mathcal{H}(\pi))). \end{aligned}$$

Now, we have to verify that $\underline{QU}^-$ represents $\sqsubseteq$, i.e. that for each π, $\pi' \in \Pi^{ex}(X)$ the following equivalence holds

$$\pi' \sqsubseteq \pi \quad \text{iff} \quad \underline{QU}^-(\pi') \sqsubseteq \underline{QU}^-(\pi).$$

Indeed, by the continuity axiom $A4$, there exist λ and λ' such that $(1/\mathcal{N}(\pi), n_V(\mathcal{H}(\pi))/X) \sim (1/\overline{x}, \lambda/\underline{x})$ and $(1/\mathcal{N}(\pi'), n_V(\mathcal{H}(\pi'))/X) \sim (1/\overline{x}, \lambda'/x)$, where $\overline{x}$ and $\underline{x}$ denote a maximal and a minimal element of $(X, \sqsubseteq)$ respectively. Therefore,

$$\pi' \sqsubseteq \pi \quad \text{iff} \quad (1/, \lambda'/\underline{x}) \sqsubseteq (1/\overline{x}, \lambda/\underline{x}),$$

and we have that:

- since QU^- represents $\sqsubseteq$ on $\Pi(X)$, $(1/\overline{x}, \lambda'/\underline{x}) \sqsubseteq (1/\overline{x}, \lambda/\underline{x})$ iff $QU^-(1/\overline{x}, \lambda'/\underline{x}) \leq QU^-(1/\overline{x}, \lambda/\underline{x})$,

- $\underline{QU}^{-}(\pi) = QU^{-}(1/\mathcal{N}(\pi), n_V(\mathcal{H}(\pi))/\underline{x}) = QU^{-}(1/\overline{x}, \lambda/\underline{x})$,
- $\underline{QU}^{-}(\pi') = QU^{-}(1/\mathcal{N}(\pi'), n_V(\mathcal{H}(\pi'))/\underline{x}) = QU^{-}(1/\overline{x}, \lambda'/\underline{x})$.

Hence, we finally have

$$\pi' \sqsubseteq \pi \quad \text{iff} \quad \underline{QU}^{-}(\pi') \leq \underline{QU}^{-}(\pi),$$

that is, $\underline{QU}^{-}$ represents $\sqsubseteq$. □

8.2.2 The Case of Max - $\top$ Possibilistic Mixtures

Given a t-norm $\top$ on V, let us consider now, in the set of possibility distributions $\Pi^{ex}(X)$, the generalised $\max - \top$ mixtures introduced in Section 5.3

$$M_{\top}(\pi, \pi'; \alpha, \beta) = \max(\alpha \top \pi, \beta \top \pi'),$$

with $\max(\alpha, \beta) = 1$. In this general setting, in order to correctly deal with non-normalised distributions, we extend the utility evaluations GQU^{-} and GQU^{+} in an analogous way to the previous subsection:

$$\underline{GQU}^{-}(\pi|u) = \min\{GQU^{-}(\mathcal{N}(\pi)|u), n \circ n_V(\mathcal{H}(\pi))\},$$

$$\underline{GQU}^{+}(\pi|u) = \max\{GQU^{+}(\mathcal{N}(\pi)|u), h \circ n_V(\mathcal{H}(\pi))\}.$$

In
a very mimetic way, we consider the axiom sets $AX^{ex}_{\top} = \{A1, A2, A3_{\top}, A4_{\top}, A7_{\top}\}$, and $AX^{+ex}_{\top} = \{A1, A2^{+}, A3_{\top}, A4^{+}_{\top}, A7_{\top}\}$ where the new axiom $A7_{\top}$ is the suitable adaptation of previous axiom $A7$ for the present type of mixtures.

- $A7_{\top}$: For all $\pi \in \Pi^{ex}(X), \pi \sim M_{\top}(\mathcal{N}(\pi), X; 1, n_V(\mathcal{H}(\pi)))$.

The corresponding representation theorem comes next.

Theorem 8.2 (Representation Theorem)
A preference relation $\sqsubseteq$ on $\Pi^{ex}(X)$, equipped with a mixture operation $M_{\top}$, satisfies the axioms $AX^{ex}_{\top} = \{A1, A2, A3_{\top}, A4_{\top}, A7_{\top}\}$ (resp. $AX^{+ex}_{\top} = \{A1, A2^{+}, A3_{\top}, A4^{+}_{\top}, A7_{\top}\}$) if and only if there exist

(i) a linearly ordered and finite preference scale U with $\inf(U) = 0$ and $\sup(U) = 1$,

(ii) a preference function $u: X \to U$ such that $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,

(iii) an onto order-preserving mapping $h: V \to U$ satisfying coherence w.r.t. $\top$,

in such a way that it holds

$$\pi' \sqsubseteq \pi \quad \text{iff} \quad \underline{GQU}^{-}(\pi'|u) \sqsubseteq \underline{GQU}^{-}(\pi|u),$$

($\pi' \sqsubseteq \pi$ iff $\underline{GQU}^{+}(\pi'|u) \sqsubseteq \underline{GQU}^{+}(\pi|u)$ respectively), where, as usual, we take $n = n_U \circ h$.

Proof:
The proof is very similar to the case $\top = minimum$ of previous subsection, so we shall only pay attention to main differences for the pessimistic utility.

$\leftarrow$) By Theorem 5.5, it only remains to verify axiom $A7_\top$. Taking into account that $\underline{GQU}^-$ coincides with GQU^- on $\Pi(X)$, and that GQU^- preserves generalised mixtures, we have

$$GQU^-(M_\top(\mathcal{N}(\pi), X; 1, n_V(\mathcal{H}(\pi)))) = \min\{n(1\top\delta_1), n(n_V(\mathcal{H}(\pi))\top\delta_2)\}$$

where $n(\delta_1) = GQU^-(\mathcal{N}(\pi))$ and $n(\delta_2) = GQU^-(X) = 0$. But, according to the coherence condition, we have that $n(\delta_2) = 0 = n(1)$ implies $n(n_V(\mathcal{H}(\pi))\top\delta_2) = n(n_V(\mathcal{H}(\pi)))$, so we actually have

$$\begin{aligned} GQU^-(M_\top(\mathcal{N}(\pi), X; 1, n_V(h(\pi)))) &= \min\{GQU^-(\mathcal{N}(\pi)), n \circ n_V(\mathcal{H}(\pi))\} \\ &= \underline{GQU}^-(\pi). \end{aligned}$$

Hence, axiom $A7_\top$ is satisfied.

$\rightarrow$) Since $\sqsubseteq$ satisfies $AX_\top$, we may establish the existence of U, $u{:}X \rightarrow U$ and $h{:}V \rightarrow U$ satisfying (i), (ii) and (iii), such that $GQU^-(\pi) = \min_{x_i \in X} n(\pi(x_i)\top\lambda_i)$, where $n(\lambda_i) = u(x_i)$, represents $\sqsubseteq$ on $\Pi(X)$. In particular, GQU^- so defined preserves mixtures and verifies $GQU^-(X) = 0$. Axiom $A7_\top$, $\pi \sim M_\top(\mathcal{N}(\pi), X; 1, n_V(\mathcal{H}(\pi)))$, allows us to define, for each $\pi \in \Pi^{ex}(X)$,

$$\begin{aligned} \underline{GQU}^-(\pi) &= GQU^-(M_\top(\mathcal{N}(\pi), X; 1, n_V(\mathcal{H}(\pi)))) \\ &= \min\{GQU^-(\mathcal{N}(\pi)), n \circ n_V(\mathcal{H}(\pi))\}. \end{aligned}$$

Finally, one can easily check that $\underline{GQU}^-$ represents $\sqsubseteq$ on $\Pi^{ex}(X)$ using the fact that GQU^- already represents $\sqsubseteq$ on $\Pi(X)$, together with axioms $A7_\top$ and $A4_\top$. $\square$

Remark 11
Instead of using the involution n_V in the definition of the mappings $\underline{GQU}^-$ and $\underline{GQU}^+$, one could simply use a more general function $F{:}V \rightarrow V$ s.t. $F(1) = 0$, and define the pessimistic and optimistic utilities as

$$\begin{aligned} \underline{GQU}^-_F(\pi) &= \min\{GQU^-(\mathcal{N}(\pi)), h_F(\mathcal{H}(\pi))\} \\ \underline{GQU}^+_F(\pi) &= \max\{GQU^+(\mathcal{N}(\pi)), n_F(\mathcal{H}(\pi))\} \end{aligned} \tag{8.2}$$

where $h_F = n_U \circ h \circ F$ and $n_F = h \circ F$.

In that case, given such a function F, it is not difficult to show that Theorem 8.2 is still valid provided that we replace axiom $A7_\top$ by an analogous one:

- $A7F_\top : \forall \pi \in \Pi^{ex}(X),\ \pi \sim M_\top(\mathcal{N}(\pi), X; 1, F(\mathcal{H}(\pi)))$,

and $\underline{GQU}$ by $\underline{GQU}_F$.

8.3 Back to Case-Based Decision

Again, using the link between similarity on situations and possibility distributions on consequences, we just propose here to apply the generalised qualitative utility functions $\underline{GQU}^-$ and $\underline{GQU}^+$ for case-based decision problems.

So, if we are interested in acts d such that in all the situations similar to s_0, d led to good results, we are looking for decisions maximising the function

$$GU^-_{F,s_0}(d) = \underline{GQU}^-_F(\pi_{d,s_0}) = \min\{h_F(\mathcal{H}(\pi_{d,s_0})), GQU^-(\mathcal{N}(\pi_{d,s_0}))\}$$

while if we are looking for decisions which gave a good result in a similar situation we may want to maximise

$$GU^+_{F,s_0}(d) = \underline{GQU}^+_F(\pi_{d,s_0}) = \max\{n_F(\mathcal{H}(\pi_{d,s_0})), GQU^+(\mathcal{N}(\pi_{d,s_0}))\}.$$

Finally, let us remark that $GQU^-(\mathcal{N}(\pi_{d,s_0}))$ can still be regarded as a degree of inclusion [*Sim*$^{*d} \subseteq G^d$] of the *normalised fuzzy set of situations similar to* s_0, Sim^{*d}, into the *fuzzy set of situations in which* d *led to good results*, if we define

$$[Sim^{*d} \subseteq G^d] = \min_{s:(s,d,x)\in M} \left(Sim^{*d}(s) \Rightarrow G^d(s)\right).$$

In this expression, $\Rightarrow: V \times U \to U$ is a many-valued implication-like operation of the type "not (a and not b)", interpreting the "and" as it was mentioned in Chapter 5 by a t-norm $\top$ on V and, because of the different domains involved (V and U) it has to be formally expressed as

$$a \Rightarrow \beta = n(\alpha \top \gamma),$$

where $n(\gamma) = \beta$. Analogously, $GQU^+(\mathcal{N}(\pi_{d,s_0}))$ is still a degree of intersection $[Sim^{*d} \subseteq G^d]$ provided that we define

$$[Sim^{*d} \subseteq G^d] = \max_{s:(s,d,x)\in M}(Sim^{*d}(s) \otimes G^d(s))$$

where $\otimes$ is a t-norm-like operation defined as $\alpha \otimes \beta = h(\alpha) \top_U \beta$, where $\top_U$ is a transform by h of the t-norm $\top$ (defined on V) into U.

8.4 A Case-based Decision View of the Safety Decision Problem in a Chemical Plant

To exemplify some of the notions introduced in this Chapter, let us return to the safety problem in the chemical plant introduced in Section 5.2.

So far we have assumed that, in order to take a decision in front of a problem in the plant, the head of the Dept. had available a report, under the form of a possibility distribution, about the actual state of the plant. Now, assume the following situation: *the alarms turn on but, for some strange reason, the head of the Dept. does not receive any report about the emergency state of the plant, and he is only provided with the readings of the two alarm systems (fire and pipeline pressure).*

The possible values for the readings of each system are

- e_0 = *normal,*

- e_1 =*small problem,*

- e_2 = *big problem,*

- e_3 = *danger*

This time, the readings he gets are:

$$system_1 = \textit{big problem } (e_2) \qquad system_2 = \textit{normal } (e_0).$$

Nevertheless, he had recorded past experienced problems and for each of those problems he stored triples of the form ***(state-description, action, consequence)***, where state-descriptions consist of pairs $(evaluation - system_1,\ \ evaluation - system_2)$, where $system_1$ refers to the fire alarm system and $system_2$ refers to the pressure pipelines alarm system.

We shall apply the model for case-based decision previously described. To do that, consider the similarity evaluation between situation-description tuples defined as:

$$Sim((e_i, e_k), (e_j, e_t)) = \min(S(e_i, e_j), \max(n(\alpha), S(e_k, e_t)))$$

with $\alpha \in V$, and S the similarity on system evaluations defined in Table 8.1.

S	e_0	e_1	e_2	e_3
e_0	1	w_6	w_4	0
e_1	w_6	1	w_7	w_5
e_2	w_4	w_7	1	w_8
e_3	0	w_5	w_8	1

Table 8.1: Similarity on alarm system evaluations $S(e_i, e_j)$.

Notice that the global similarity is computed as a weighted-min aggregation of the marginal similarities (which are the same), all of them taking values in the common scale U. A value $\alpha < 1$ denotes a partial reliability on the alarm system 2. The available memory M of previously experienced cases is given in Table 8.2.

According to the model, the Decision Maker has to rank the induced possibility distributions by the current case $c_0 = (e_2, e_0)$ and the above similarity function *Sim*, which are defined as follows

$cases$	$evaluation\ sensor1$	$evaluation\ sensor2$	$decision$	consequence
c_1	e_0	e_1	d_2	*(risk=0,cost=2)*
c_2	e_1	e_0	d_2	*(risk=0,cost=2)*
c_3	e_2	e_1	d_1	*(risk=1,cost=1)*
c_4	e_1	e_2	d_1	*(risk=0,cost=1)*
c_5	e_2	e_3	d_3	*(risk=0,cost=3)*
c_6	e_1	e_3	d_3	*(risk=0,cost=3)*

Table 8.2: Memory of cases.

$$
\begin{aligned}
\pi_{d0} &= 0; \\
\\
\pi_{d1} &= (Sim((e_2, e_0), (e_2, e_1))/(Risk = 1, Cost = 1), \\
&\quad\ Sim((e_2, e_0), (e_1, e_2))/(Risk = 0, Cost = 1)) \\
&= (\max(n(\alpha), w_6)/(Risk = 1, Cost = 1), \\
&\quad\ \max(w_4, \min(w_7, n(\alpha))/(Risk = 0, Cost = 1))); \\
\\
\pi_{d2} &= (\max(Sim((e_2, e_0), (e_0, e_1)), Sim((e_2, e_0), (e_1, e_0)))/(Risk = 0, Cost = 2)) \\
&= (\max(w_4, w_7)/(Risk = 0, Cost = 2)) \\
&= (w_7/(Risk = 0, Cost = 2)); \\
\\
\pi_{d3} &= (\max(Sim((e_2, e_0), (e_2, e_3)), Sim((e_2, e_0), (e_1, e_3)))/(Risk = 0, Cost = 3)) \\
&= (\max(n(\alpha), \min(n(\alpha), w_7))/(Risk = 0, Cost = 3)) \\
&= (n(\alpha)/(Risk = 0, Cost = 3)).
\end{aligned}
$$

Observe that *if we do not pay attention to the fact that these distributions are non-normalised* and we rank them in terms of QU^-, we get:

$$
\begin{aligned}
&QU^-(\pi_{d0}) = 1, \\
&QU^-(\pi_{d1}) = w_4, \\
&QU^-(\pi_{d2}) = w_7, \\
&QU^-(\pi_{d3}) = \max(\alpha, w_6).
\end{aligned}
$$

That is, for each $\alpha \neq 1$, we have that d_0 *(do nothing) is ranked as the best, in spite of the fact that the alarm system 1 warns about a big problem, and that personal safety is the most important criteria*. Moreover, in the case $\alpha = 1$, it is equally supported either to do nothing or to evacuate, one may be too dangerous while the other may result too drastic. However, if decisions are ranked taking into account that the distributions involved are non-normalised we have that:

$$
\begin{aligned}
&\underline{QU}^-(\pi_{d0}) = \min\{0, QU^-(\mathcal{N}(\pi_{d0}))\} = \min\{0, 0\} = 0, \\
&\underline{QU}^-(\pi_{d1}) = \min\{\max(n(\alpha), w6), QU^-(\mathcal{N}(\pi_{d1}))\}, \\
&\underline{QU}^-(\pi_{d2}) = \min\{w_7, QU^-(\mathcal{N}(\pi_{d2}))\} = \min\{w_7, w_7\} = w_7, \\
&\underline{QU}^-(\pi_{d3}) = \min\{n(\alpha), QU^-(\mathcal{N}(\pi_{d3}))\}.
\end{aligned}
$$

Hence, if $\alpha < 1$, $\underline{QU}^-(\pi_{d3}) = \min(n(\alpha), w_6)$, and $\underline{QU}^-(\pi_{d3}) = 0$ otherwise. Moreover, since $QU^-(\mathcal{N}(\pi_{d1})) \leq w_4$, we have that $\underline{QU}^-(\pi_{d1}) \leq w_4$. Therefore, d_2 is the best decision, which is coherent with the fact of having one alert of a major problem and giving preference to personal safety.

8.5 An Extension of the Model for Partially Inconsistent Belief States Using Uncertainty and Preference Lattices

Throughout these sections we have assumed that plausibility and preferences are evaluated on (finite) linear scales. However, as already claimed, sometimes we may face decision problems where the *Decision Maker*'s preferences may be only partially elicited, or in case-based decision problems where a complete global similarity between cases is not available but only partially specified. Along this line, we have proposed in Chapter 7 an extension of the axiomatic model where both preferences and uncertainty are measured on distributive lattices that are commensurate. Now, this proposal is extended to also include belief states that may be partially inconsistent.

As is in the linear case, there are some decision problems in which the distributions involved are non-normalised. Hence, we will consider other functions that let us work with these distributions.

First, let us introduce the concepts of $normalization$ and *height of a distribution* in the context of lattices. Define $\mathcal{H}$, the *height of a distribution,* π:$X \rightarrow V$, where $(V, \vee, \wedge, 0, 1)$ is a lattice, as

$$\mathcal{H}(\pi) = \bigvee_{x \in X} \pi(x),$$

and for each distribution we consider the *subset of consequences with maximal plausibility*

$$X_\pi = \{x \in X \mid \forall\, y \in X \;\; \pi(y) \not> \pi(x)\}.$$

We define $\mathcal{N}(\pi)$, the *normalisation of* π, as the normalised distribution

$$\mathcal{N}(\pi)(x) = \begin{cases} 1, & \text{if} \quad x \in X_\pi \\ \\ \pi(x), & \text{otherwise.} \end{cases}$$

Analogously, we extend the set of possibilistic lotteries to the set $\Pi^{ex}(X)$ *of non-necessarily normalised distributions on* V. Hence, first we need to extend the concept of possibilistic mixture PME on $\Pi^{ex}(X)$ to combine π_1 and π_2 with $(\lambda, \mu) \in \Phi_\vee$, with

$$\Phi_\vee = \{(\lambda, \mu) \in V \times V \mid \lambda \vee \mu = 1\},$$

i.e. PME:$\Pi^{ex}(X) \times \Pi^{ex}(X) \times \Phi_\vee \rightarrow \Pi^{ex}(X)$, and we define

$$PME(\pi_1, \pi_2, \lambda, \mu)(x) = (\lambda/\pi_1, \mu/\pi_2)(x) = (\lambda \wedge \pi_1(x)) \vee (\mu \wedge \pi_2(x)).$$

Given a function $F{:}V \to V$, such that $F(1) = 0$, now we may consider the qualitative (or ordinal) utility functions on $\Pi^{ex}(X)$, corresponding to those considered previously:

$$\underline{QU}^-_F(\pi) = QU^-(\mathcal{N}(\pi)) \wedge n(F(\mathcal{H}(\pi))),$$
$$\underline{QU}^+_F(\pi) = QU^+(\mathcal{N}(\pi)) \vee h(F(\mathcal{H}(\pi))).$$

Let $\sqsubseteq_F$ be a preference relation on $\Pi^{ex}(X)$. We will denote by $\sqsubseteq$ its restriction to $\Pi^*(X)$, the *set of **normalised** possibility distributions*, and by $\sim_F$ and $\sim$ the corresponding indifference relations.

In order to characterise the preference orderings induced by the utilities $\underline{QU}^-_F$ and $\underline{QU}^+_F$, we extend the axiom sets AXP and AXP^+, defined on $(\Pi^*(X), \sqsubseteq)$, with the axiom:

- $A7PF : \forall \pi \in\ \Pi^{ex}(X),\ \pi \sim_F (1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X).$

We say that a preference relation $\sqsubseteq_F$ on $\Pi^{ex}(X)$ satisfies axiom set $AXPN = AXP \cup \{A7PF\}$ ($AXPN^+ = AXP^+ \cup \{A7PF\}$ resp.) if and only if its restriction to $\Pi^*(X)$, satisfies AXP (AXP^+ resp.) and $\sqsubseteq_F$ also satisfies $A7PF$.

Theorem 8.3
Given a function $F{:}V \to V$, such that $F(1) = 0$, then a preference relation $\sqsubseteq_F$ on $\Pi^{ex}(X)$ satisfies axiom set $AXPN$ ($AXPN^+$ resp.) iff there exist

(i) a finite distributive utility lattice with involution $(U, \vee, \wedge, 0, 1, n_U)$,

(ii) a preference function $u{:}X \to U$, s.t. $u^{-1}(1) \neq \emptyset$ and $\bigwedge_{x \in X} u(x) = 0$ ($u^{-1}(0) \neq \emptyset$ and $\bigvee_{x \in X} u(x) = 1$ resp.),

(iii) an onto order-preserving function $h{:}V \to U$ s.t. $n_U \circ h \circ n_V = h$, h also satisfying

$$\text{if } \lambda <> \lambda' \text{ then } h(\lambda) <> h(\lambda'),$$

in such a way that it holds:

$$\pi' \sqsubseteq_F \pi \iff \underline{QU}^-_F(\pi') \leq \underline{QU}^-_F(\pi),$$

($\pi' \sqsubseteq_F \pi \iff \underline{QU}^+_F(\pi') \leq \underline{QU}^+_F(\pi)$ resp.), with $n = n_U \circ h$.

Proof:
Since the proofs for pessimistic and optimistic criteria are very similar, we only provide the pessimistic one.
$\leftarrow$) Consider now the utility function QU^- defined in terms of h and u. Axioms AXP are verified because $\underline{QU}^-_F$ restricted to $\Pi^*(X)$ is equal to QU^- since $F(1) = 0$, and by Theorem 7.9, we have that the ordering induced by QU^- in $\Pi^*(X)$ satisfies

AXP. Now, we verify $A7PF$. Since QU^- preserves mixtures because U is distributive, $A7PF$ verifies trivially. Indeed by definition of QU_F^- and as

$$QU^-(X) = \bigwedge_{x \in X} u(x) = 0,$$

we have that

$$\underline{QU}_F^-(\pi) = QU^-(\mathcal{N}(\pi)) \wedge n(F(\mathcal{H}(\pi))) = QU^-(1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X).$$

$\rightarrow$) Since $\sqsubseteq$, the restriction of $\sqsubseteq_F$ to $\Pi^*(X)$, satisfies axioms AXP, we may apply Theorem 7.9. So, we have determined the existence of U, h and u satisfying the conditions such that QU^- represents $\sqsubseteq$, with

$$QU^-(\pi) = \bigwedge_{x \in X} (n(\pi(x)) \vee u(x)).$$

Since $A7PF$ guarantees that

$$\pi \sim_F (1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X),$$

we define

$$\underline{QU}_F^-(\pi) = QU^-(1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X).$$

Now, we verify that $\underline{QU}_F$ represents $\sqsubseteq_F$, i.e.

$$\pi' \sqsubseteq_F \pi \iff \underline{QU}_F^-(\pi') \leq \underline{QU}_F^-(\pi).$$

By $A7PF$ and $A6$ we have that there exist λ, λ' such that $\pi \sim_F \pi_\lambda^-$, $\pi' \sim_F \pi_{\lambda'}^-$, so

$$\underline{QU}_F^-(\pi) = \underline{QU}_F^-(\pi_\lambda^-),$$
$$\underline{QU}_F^-(\pi') = \underline{QU}_F^-(\pi_{\lambda'}^-).$$

As $\pi' \sqsubseteq_F \pi \iff \pi_{\lambda'}^- \sqsubseteq_F \pi_\lambda^-$ and as QU^- represents $\sqsubseteq$ we have that $QU^-(\pi_{\lambda'}^-) \leq QU^-(\pi_\lambda^-)$.

Then, recalling that QU^- coincides with $\underline{QU}_F^-$ on $\Pi^*(X)$, we obtain that $\pi' \sqsubseteq_F \pi \iff \underline{QU}_F^-(\pi') \leq \underline{QU}_F^-(\pi)$.

It remains to prove that $\underline{QU}_F^-(\pi) = QU^-(\mathcal{N}(\pi)) \wedge n(F(\mathcal{H}(\pi)))$. Since QU^- preserves mixtures, $QU^-(X) = 0$ and $A7PF$ guarantees that $\pi \sim_F (1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X)$, we finally have that

$$\underline{QU}_F^-(\pi) = QU^-(1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X) = QU^-(\mathcal{N}(\pi)) \wedge n(F(\mathcal{H}(\pi))).$$

$\square$

Generalised Utilities

As usual, we may consider that there are available in V more operators, and this fact let us consider other utility functions. Now, we introduce the corresponding extension of our previous proposal for generalised qualitative utility functions GQU^- and GQU^+. We propose the qualitative (or ordinal) utility functions on $\Pi^{ex}(X)$,[6]

[6] Take into account that now we are considering distributions on lattices.

$$\begin{aligned}\underline{GQU}_F{}^{-}(\pi|u) &= GQU^{-}(\mathcal{N}(\pi)|u) \wedge n(F(\mathcal{H}(\pi)))\\ \underline{GQU}^{+}_F(\pi|u) &= GQU^{+}(\mathcal{N}(\pi)|u) \vee h(F(\mathcal{H}(\pi))).\end{aligned} \tag{8.3}$$

where the necessary additional axiom is:

- $A7F_\top :\ \forall \pi \in \Pi^{ex}(X),\ \pi \sim\ M_\top(\mathcal{N}(\pi), X; 1, F(\mathcal{H}(\pi)))$.

The representation theorem is analogous to the previous case and is omitted.

8.6 Similarity between Acts for Possibilistic Case-Based Decision Theory

Many economical decision problems such as whether or not to "Offer to sell at price p" for a specific value p, would likely be affected by the results of previous offers to sell with different but close values of p. We would like to let the *Decision Maker* evaluate a new decision taking into account the performance of other "similar" acts he has experienced.

Gilboa and Schmeidler (1996) made a proposal along this line, they also claimed that while a straightforward application of *CBDT* to economical models with an infinite set of acts may result in counter-intuitive and unrealistic predictions, the introduction of a similarity also involving acts may improve these predictions.

We will analyse, in the *finite* possibilistic context, a model to evaluate utilities on each decision taking into account the performance of others acts, i.e. to deal with cases in which the evaluation of an act may also depend on past performance of the acts, maybe different but "similar" acts. Therefore, we shall consider a global similarity function over problem-act pairs. The difference with the approach analysed in Section 8.1 is that for evaluating a decision now we are also interested in the behaviour of "similar" acts in previous "similar" situations.

Given a situation s and an act d, we will refer to the pair (s, d) as a decision-case.

Our proposal is to estimate to what extent a consequence x can be considered plausible of being the consequence of s_0 by d, in terms of what extent the current decision-case (s_0, d) is similar to previous decision-cases (s, d') in which x was experienced. That is, for each case (s, d', x) in a memory M, a principle stating that

> "The more similar are the decision-cases (s_0, d) and (s, d'), the more possible x is the consequence of d in s_0".

is assumed.

Considering D the set of available decisions, we assume a similarity relation $GlSim$ available on the decision-case set, i.e. a function $GlSim$:$(S \times D)^2 \to V$ that measures the degree of similarity between two pairs *(situation, decision)*.

Therefore, according to this principle, analogously to Section 8.1, we propose to consider the following utility function:

$$\mathcal{U}^{-}_{s_0,M}(d|u) = \min_{(s,d',x)\in M} \left(GlSim((s_0, d), (s, d')) \Rightarrow u(x)\right).$$

As already seen, this corresponds with a view of the degree of inclusion of the fuzzy set of the similar decision-cases to (s_0, d) into the fuzzy set of good consequences experienced. That is, we are considering

$$GlG : \{(s, d')|(s, d', x) \in M\} \to U$$

the fuzzy set of decision-cases that obtained good results, whose membership is $GlG(s, d') = u(x)$.[7]

For each d, let

$$GlSim^d : \{(s, d') \mid (s, d', x) \in M\} \to V$$

be the fuzzy set of decision-cases which are similar to (s_0, d), defined as $GlSim^d(s, d') = GlSim((s_0, d), (s, d'))$. Hence, the above expression for $\mathcal{U}^-_{s_0,M}(d|u)$ may be rewritten as the following degree of inclusion:

$$\mathcal{U}^-_{s_0,M}(d|u) = [GlSim^d \subseteq GlG].$$

We may apply here the alternative implications analysed in Section 5.1, obtaining their respective utility functions. Analogously, we may consider the intersection of the fuzzy set to reflect a more optimistic behaviour:

$$\mathcal{U}^+_{s_0,M}(d|u) = [GlSim^d \cap GlG].$$

[7] GlG is well defined because we are assuming a minimal deterministic memory, i.e. for each situation we only retain in the memory the case with the best consequence for any decision.

Chapter 9

Further Results: Ordering Refinements and Weaker Commensurability Conditions

In this Chapter, we introduce the last results obtained in the on going work. The first concerns to the refinement orderings problem[1] when ranking distributions. Indeed, in some problems it may be not enough to rank distribution taking into account only one criterion, for example the pessimistic criterion, and we may be interested in refining the ranking with another criterion, e.g. the optimistic one.

The second topic is related with an issue that has been of our interest since the beginning, the commensurability hypothesis between the preference and the uncertainty sets. Up to now, we have assumed the existence of an **onto** order-preserving mapping linking both sets. This fact forced to restrict ourselves to work with problems in which the uncertainty set has a greater cardinality than the preference one. Here, we propose to weaken this hypothesis requiring h to be only an order-preserving mapping satisfying $h(\max\ V) = \max\ U$ and $h(\min\ V) = \min\ U$.

9.1 Some Possible Refinements

We may consider different qualitative utility functionals for ranking decisions, among them of course we have the pessimistic and optimistic criteria QU^- and QU^+ and their generalised versions GQU^- and GQU^+ introduced in Chapters 4, 5, and 7. However, in some decision problems it may be interesting to consider some refinements of these orderings. In this Section, we summarise our first results in this issue.

Among different possible refinements we may consider the following ones:

[1]This work was begun during a Short-Term Scientific Mission of the author within the frame of COST Action 15, Many-valued Logics for Computer Science Applications, at the Institut de Rechèrche en Informatique de Toulouse (IRIT) with Dr. Henri Prade.

1. A first approach is to use the optimistic criterion to refine the pessimistic one, i.e.

$$\begin{aligned}\pi \sqsubseteq_0 \pi' \quad \Longleftrightarrow \quad & \{\{GQU^-(\pi|u) <_U GQU^-(\pi'|u)\} \text{ or} \\ & \{[GQU^-(\pi|u) = GQU^-(\pi'|u)] \wedge \\ & [GQU^+(\pi|u) \leq_U GQU^+(\pi'|u)]\}\},\end{aligned}$$

where we are considering that both generalised utility functions are defined in the same lattice U and with the same preference function u. But sometimes we may have different lattices and preference functions for both criteria, hence in such a situation the refinement would be defined as:

$$\begin{aligned}\pi \sqsubseteq_1 \pi' \quad \Longleftrightarrow \quad & \{\{GQU^-(\pi|u^-) <_{U^-} GQU^-(\pi'|u^-)\} \text{ or} \\ & \{[GQU^-(\pi|u^-) = GQU^-(\pi'|u^-)] \wedge \\ & [GQU^+(\pi|u^+) \leq_{U^+} GQU^+(\pi'|u^+)]\}\}.\end{aligned}$$

2. In some cases, we may be interested in considering the problem of evaluating a distribution π by applying two different criteria to π, depending on the type of consequences. Indeed, suppose for instance that the consequences involved in the safety decision problem may be classified in two groups: consequences *involving the safety of persons* and another group of consequences *related to economic costs*. In this case, we may be interested in being conservative with respect to consequences of the first set, while a more optimistic criterion may be applied on the second set. That is, given a subset[2] I of X we consider

$$\pi \sqsubseteq_2 \pi' \iff Ut(\pi) \leq_U Ut(\pi'),$$

with

$$Ut(\pi) = \min(GQU^+_{I^c}(\pi|u^+), GQU^-_I(\pi|u^-)),^3$$

where

$$GQU^-_I(\pi|u^-) = \underline{GQU}^-_F(\pi \wedge I|u^-)$$

and

$$GQU^+_{I^c}(\pi|u^+) = \underline{GQU}^+_F(\pi \wedge I^c|u^+),$$

where $\pi \wedge I$ denotes the intersection of the distributions, i.e. the distribution, non-necessarily normalised, defined as

$$(\pi \wedge I)(x) = \inf(\pi(x), I(x)).$$

$\pi \wedge I$ may be seen as the conditioning of π by the event I. As we will apply the same set I for all distributions π, we will call GQU_I the generalised utility function conditioned by I. That is,

$$\pi \sqsubseteq_2 \pi' \iff (GQU^+_{I^c}(\pi|u^+), GQU^-_I(\pi|u^-)) \leq_{\min} (GQU^+_{I^c}(\pi'|u^+), GQU^-_I(\pi'|u^-)).$$

[2] Analogously, if we are interested in a V-fuzzy set I on X.

[3] As usual, I^c denotes the complement of I with respect to X.

3. Sometimes we may be interested in refining in a *lexicographic* style *ordering* considering these priority levels: first $\leq_{GQU^-(\cdot|u^-)}$, then, $\leq_{GQU^+(\cdot|u^+)}$ and finally $\leq_{GQU_I^-(\cdot|u^-)}$. That is,

$$\begin{aligned}\pi \sqsubseteq_3 \pi' \quad \Longleftrightarrow \quad & \{\{GQU^-(\pi|u^-) <_{U^-} GQU^-(\pi'|u^-)\} \text{ or} \\ & \{GQU^-(\pi|u^-) = GQU^-(\pi'|u^-) \ \wedge \\ & \ GQU^+(\pi|u^+) <_{U^+} GQU^+(\pi'|u^+)\} \ \text{ or} \\ & \{GQU^-(\pi|u^-) = GQU^-(\pi'|u^-) \ \wedge \\ & \ GQU^+(\pi|u^+) = GQU^+(\pi'|u^+) \ \wedge \\ & \ GQU_I^-(\pi|u^-) \leq_{U^-} GQU_I^-(\pi'|u^-)\}\},\end{aligned}$$

4. or, analogously, considering $\leq_{GQU_I^+(\cdot|u^+)}$ instead of $\leq_{GQU_I^-(\cdot|u^-)}$:

$$\begin{aligned}\pi \sqsubseteq_4 \pi' \quad \Longleftrightarrow \quad & \{\{GQU^-(\pi|u^-) <_{U^-} GQU^-(\pi'|u^-)\} \ \text{ or} \\ & \{GQU^-(\pi|u^-) = GQU^-(\pi'|u^-) \ \wedge \\ & \ GQU^+(\pi|u^+) <_{U^+} GQU^+(\pi'|u^+)\} \ \text{ or} \\ & \{GQU^-(\pi|u^-) = GQU^-(\pi'|u^-) \ \wedge \\ & \ GQU^+(\pi|u^+) = GQU^+(\pi'|u^+) \ \wedge \\ & \ GQU_I^+(\pi|u^+) \leq_{U^+} GQU_I^+(\pi'|u^+)\}\}.\end{aligned}$$

Let us show a little example about how these rankings may classify distributions.

Example:
Let $X = \{\underline{x}, x_1, x_2, \overline{x}\}$ and its subset $I = \{\overline{x}, x_1\}$. We consider $U^- = U^+ = V = \{0 < \mu < \lambda < 1\}$, and the distributions:

$$\pi = (1/\overline{x}, 1/x_1, \lambda/\underline{x}),$$

and

$$\pi' = (1/\overline{x}, 1/x_2, \lambda/\underline{x}).$$

We assume both preference functions are the same, say u, with $u(\underline{x}) = 0,\ u(x_1) = \mu,\ u(x_2) = \lambda$ and $u(\overline{x}) = 1$. So,

$$QU^-(\pi) = QU^-(\pi') = n(\lambda) \qquad \text{and} \qquad QU^+(\pi) = QU^+(\pi') = 1.$$

That is, both distributions are indistinguishable w.r.t. the pessimistic and optimistic criteria. Moreover, QU_I^+ cannot distinguish both distributions. However, other rankings can do it. Indeed,

$$QU_I^-(\pi) = u(x_1), \text{while} \qquad QU_I^-(\pi') = 1,$$

and

$$QU_{I^c}^+(\pi) = \max\{QU^+(\mathcal{N}(\pi \wedge I^c)), h \circ n_V(\lambda)\} = h(\mu) = \mu,$$

while

$$QU^{+}_{I^c}(\pi') = u(x_2) = \lambda.$$

Moreover,

$$Ut(\pi) = \mu \text{ and } Ut(\pi') = \lambda.$$

$\diamondsuit$

Remark 12
We might wonder if the GQU_I rankings induced by subsets of the same cardinality coincide. This is not true. Indeed, given proper subsets of X with the same cardinality, we can show that the orderings induced by GQU conditioned by these subsets may be different.
Given $Y_1 \subset X,\ Y_2 \subset X$, *s.t.* $|Y_1| = |Y_2|$,

$$GQU^{-}_{Y1}(\pi) > GQU^{-}_{Y1}(\pi') \nrightarrow GQU^{-}_{Y2}(\pi) > GQU^{-}_{Y2}(\pi').$$

Indeed, suppose, $X = \{x_1 \sqsubset \ldots \sqsubset x_5\}$, *consider the "crisp" distributions*

$$\pi = \{x_1, x_3, x_4\}, \pi' = \{x_1, x_2, x_5\},$$

and the sets

$$Y_1 = X - \{x_1, x_3\} \text{ and } Y_2 = X - \{x_1, x_2\}.$$

So, we have that

$$QU^{-}_{Y1}(\pi) > QU^{-}_{Y1}(\pi'),$$

while

$$QU^{-}_{Y2}(\pi) < QU^{-}_{Y2}(\pi').$$

That is, the rankings conditioned by Y_1 and Y_2 are different.

There are several other refinements, for example, other refinements orderings based in ordinal information are: *discrimax* and *leximin*. If $\overline{x} = (x_1, \ldots, x_n), \overline{y} = (y_1, \ldots, y_n)$,, considering the set $D(\overline{x}, \overline{y}) = \{i | x_i \neq y_i\}$, we have that

$$\overline{x} \geq_{discrimax} \overline{y} \iff \max_{i \in D(\overline{x}, \overline{y})} x_i \geq \max_{i \in D(\overline{x}, \overline{y})} y_i,$$

while

$$\overline{x} \geq_{leximin} \overline{y} \iff \overline{x*} \geq_{lex} \overline{y*},$$

where $\overline{x*}, \overline{y*}$ are increasing reordering of $\overline{x}$ and $\overline{y}$ (for more details you may see (Dubois et al., 1996a; Moulin, 1988)).

9.1.1 Axiomatic Characterisation of some Refinement Orderings

Here, we provide the axiomatic characterisation of some refinements of the orderings involving the generalised qualitative criteria. In particular, we characterise the refinement orderings $\sqsubseteq_1, \sqsubseteq_3$ and $\sqsubseteq_4$ previously introduced. First, let us introduce some definitions analogous to the ones introduced in Chapter 6. Given a finite set $\mathcal{R} = \{\sqsubseteq_i\}_{i=1,\dots,k}$ of binary relations on sets $\{E_i\}_{i=1,\dots,k}$ respectively, for each "boolean" mapping $g{:}\{0,1\}^k \times \{0,1\}^k \to \{0,1\}$, the following relation may be considered:

$$\begin{aligned}\overline{e} \preccurlyeq^g_{\mathcal{R}} \overline{e'} \quad \Longleftrightarrow \quad & g((\mu_{\sqsubseteq_1}(e_1, e'_1), \dots, \mu_{\sqsubseteq_k}(e_k, e'_k))\,, \\ & (\mu_{\sqsubseteq_1}(e'_1, e_1), \dots, \mu_{\sqsubseteq_k}(e'_k, e_k))) = 1\end{aligned}$$

where $\overline{e} = (e_1, \dots, e_k)$, $\overline{e'} = (e'_1, \dots, e'_k)$, and $\mu_{\sqsubseteq_i}$ is the membership of the preference ordering $\sqsubseteq_i$.

Recall (see Section 6.1) that *Pareto* and *lexicographic orderings* are particular types of the relations $\preccurlyeq^g_{\mathcal{R}}$.

Consider $(V, \wedge, \vee, \top, I, 0, 1, n_V)$ a finite distributive residuated lattice with involution for uncertainty and two utility finite distributive lattices with involution $(U^-, \wedge^-, \vee^-, n_{U^-}, 0, 1)$, $(U^+, \wedge^+, \vee^+, n_{U^+}, 0, 1)$, both lattices being commensurate with V, i.e. there exist two onto order-preserving functions $h^-{:}V \to U^-$, $h^+{:}V \to U^+$, both h's also satisfying coherence w.r.t. $\top$, and let $u^-{:}X \to U^-$, $u^+{:}X \to U^+$ be two preference functions representing preferences on consequences on these lattices such that $(u^-)^{-1}(1) \neq \emptyset \neq (u^+)^{-1}(0)$, $\bigwedge_{x\in X}(u^-)(x) = 0$ and $\bigvee_{x\in X}(u^+)(x) = 1$. Then, we can consider the following "utility" functional:

$$\overline{RGQU}^{-,+}(\cdot|(u^-, u^+)(h^-, h^+)) = (\underline{GQU}^-(\cdot|u^-, h^-), \underline{GQU}^+(\cdot|u^+, h^+)),$$

where $\underline{GQU}^-(\cdot|u^-, h^-)$ is the generalised pessimistic utility function defined in terms of u^-, h^- (and the involution in $(U^-, \leq^-)$) and the t-norm $\top$ in V, and $\underline{GQU}^+(\cdot|u^+, h^+)$ is the optimistic one expressed in terms of u^+, h^+ and $\top$.

Notation 9.1
For the sake of a simpler notation, we shall write $\overline{RGQU}^{-,+}(\cdot|(u^-, u^+))$ *instead of* $\overline{RGQU}^{-,+}(\cdot|(u^-, u^+)(h^-, h^+))$ *when the mapping* h *involved in the* $\underline{GQU}$ *function has in its notation the same sign that* u. *The same rule is applied to* $\underline{GQU}$*, in the sense that instead of writing, for instance,* $\underline{GQU}^-(\cdot|u^-, h^-)$ *we will write* $\underline{GQU}^-(\cdot|u^-)$.

Under these hypotheses, and given a boolean function g, we may consider the orderings[4] induced by g and $\overline{RGQU}^{-,+}(\cdot|(u^-, u^+))$ defined as

[4]It is obvious that not for all g we obtain an ordering, however for decision making we are interested in those that result in orderings.

$$\pi \preccurlyeq^{g}_{\{u^-,u^+\}} \pi' \iff \overline{RGQU^{-,+}}(\pi|(u^-,u^+)) \preccurlyeq^{g}_{\{\leq^-,\leq^+\}} \overline{RGQU^{-,+}}(\pi'|(u^-,u^+)).$$

That is,

$$\pi \preccurlyeq^{g}_{\{u^-,u^+\}} \pi' \iff$$

$$g\Big(\Big(\mu_{\leq^-}(\underline{GQU}^-(\pi|u^-),\underline{GQU}^-(\pi'|u^-)),\mu_{\leq^+}(\underline{GQU}^+(\pi|u^+),\underline{GQU}^+(\pi'|u^+))\Big),$$
$$\Big(\mu_{\leq^-}(\underline{GQU}^-(\pi'|u^-),\underline{GQU}^-(\pi|u^-)),\mu_{\leq^+}(\underline{GQU}^+(\pi'|u^+),\underline{GQU}^+(\pi|u^+))\Big)\Big) = 1.$$

Remark 13
In particular, if we choose g for lexicographic ordering, we have that the refinement orderings $\sqsubseteq_1, \sqsubseteq_3$ and $\sqsubseteq_4$ proposed in the beginning of the Chapter are obtained. For example, if we take,

$$g(\overline{x},\overline{y}) = \max(\min(x_1, 1-y_1), \min(x_1, y_1, x_2)),$$

and $\sqsubseteq_1 = \preccurlyeq_{GQU^-(\cdot|u^-)}$, $\sqsubseteq_2 = \preccurlyeq_{GQU^+(\cdot|u^+)}$*; we have that*

$$\begin{aligned}\pi \sqsubseteq_1 \pi' \iff \quad & \{\{GQU^-(\pi|u^-) < GQU^-(\pi'|u^-)\} \vee \\ & \{GQU^-(\pi|u^-) = GQU^-(\pi'|u^-) \text{ and} \\ & GQU^+(\pi|u^+) \leq GQU^+(\pi'|u^+)\}\}.\end{aligned}$$

As a first approach for characterising these orderings, we propose the following set of axioms, $RAX^g_\top$, for a preference relation $\sqsubseteq$ on $(\Pi^{ex}(X), M_\top)$:

- $GA0$: There exists a set $\mathcal{R} = \{\sqsubseteq^-, \sqsubseteq^+\}$ of orderings such that $\sqsubseteq = \preccurlyeq^g_{\mathcal{R}}$, i.e.

  $$\pi \sqsubseteq \pi' \iff g\left(\left(\mu_{\sqsubseteq^-}(\pi,\pi'), \mu_{\sqsubseteq^+}(\pi,\pi')\right), \left(\mu_{\sqsubseteq^-}(\pi',\pi), \mu_{\sqsubseteq^+}(\pi',\pi)\right)\right) = 1$$

- $AxGroup$: $\sqsubseteq^-$ satisfies $AXPN_\top$, while $\sqsubseteq^+$ satisfies $AXPN^+_\top$.

Then, the following theorem is a consequence of the representation Theorem 7.14.

Theorem 9.1 (Representation Theorem)
Given a boolean mapping g, a preference relation $\sqsubseteq$ on $(\Pi^{ex}(X), M_\top)$, satisfies the axiom set $RAX^g_\top$ if and only if there exist:

(i) two utility finite distributive lattices with involution $(U^-, \wedge^-, \vee^-, n_{U^-}, 0, 1)$ and $(U^+, \wedge^+, \vee^+, n_{U^+}, 0, 1)$,

(ii) two preference functions $u^-:X \to (U^-, \leq^-)$, $u^+:X \to (U^+, \leq^+)$ such that $(u^-)^{-1}(1) \neq \emptyset \neq (u^+)^{-1}(0)$, $\bigwedge_{x\in X}(u^-)(x) = 0$ and $\bigvee_{x\in X}(u^+)(x) = 1$.

(iii) two onto join-preserving mappings $h^-:V \to U^-$, $h^+:V \to U^+$, both satisfying coherence w.r.t $\top$, also satisfying

$$\text{if } \lambda <> \lambda' \text{ then } h^-(\lambda) <> h^-(\lambda'),$$

$n_{U^-} \circ h^- \circ n_V = h^-$, $\;n_{U^+} \circ h^+ \circ n_V = h^+$, *and*

$$\text{if } \lambda <> \lambda' \text{ then } h^+(\lambda) <> h^+(\lambda'),$$

in such a way that it holds:

$$\pi \sqsubseteq \pi' \qquad \textit{iff} \qquad \pi \preccurlyeq^{g}_{\{u^-,u^+\}} \pi'.$$

The vectorial function of utility inducing $\preccurlyeq^{g}_{\{u^-,u^+\}}$ *being*

$$\overline{RGQU}^{-,+}(\cdot|(u^-,u^+)) = (\underline{GQU}^-(\cdot|u^-,h^-), \underline{GQU}^+(\cdot|u^+,h^+)),$$

with $n = n_{U^-} \circ h^-$.

Proof:
$\rightarrow$) Since relation $\sqsubseteq^-$ satisfies $AXPN_{\mathsf{T}}$ and $\sqsubseteq^+$ satisfies $AXPN^+_{\mathsf{T}}$, then, the existence of $\{(U^-,\leq^-),(U^+,\leq^+)\}$[5], $\{u^+,u^-\}$ and $\{h^-,h^+\}$ is guaranteed by the Theorem analogous to Theorem 8.3. So, it only remains to verify that the relation induced by $\overline{RGQU}^{-,+}$ and g coincides with $\sqsubseteq$.
As $\sqsubseteq^-$ and $\sqsubseteq^+$ are represented by $\underline{GQU}^-(\cdot|u^-,h^-)$ and $\underline{GQU}^+(\cdot|u^+,h^+)$ respectively, we have that

$$\pi \sqsubseteq^- \pi' \iff \underline{GQU}^-(\pi|u^-,h^-) \leq^- \underline{GQU}^-(\pi'|u^-,h^-),$$

and

$$\pi \sqsubseteq^+ \pi' \iff \underline{GQU}^+(\pi|u^+,h^+) \leq^+ \underline{GQU}^+(\pi'|u^+,h^+).$$

That is,

$$\mu_{\sqsubseteq^-}(\pi,\pi') = \mu_{\leq^-}(\underline{GQU}^-(\pi|u^-), \underline{GQU}^-(\pi'|u^-))$$

and

$$\mu_{\sqsubseteq^+}(\pi,\pi') = \mu_{\leq^+}(\underline{GQU}^+(\pi|u^+), \underline{GQU}^+(\pi'|u^+)).$$

Hence, applying $GA0$, we have that

$$\begin{aligned}
\pi \sqsubseteq \pi' \quad &\iff \quad g\big(\big(\mu_{\sqsubseteq^-}(\pi,\pi'), \mu_{\sqsubseteq^+}(\pi,\pi')\big)\,,\\
&\qquad \big(\mu_{\sqsubseteq^-}(\pi',\pi), \mu_{\sqsubseteq^+}(\pi',\pi)\big)\big) = 1\\
&\iff \quad g((\mu_{\leq^-}(\underline{GQU}^-(\pi|u^-), \underline{GQU}^-(\pi'|u^-)),\\
&\qquad \mu_{\leq^+}(\underline{GQU}^+(\pi|u^+), \underline{GQU}^+(\pi'|u^+))),\\
&\qquad (\mu_{\leq^-}(\underline{GQU}^-(\pi'|u^-), \underline{GQU}^-(\pi|u^-)),\\
&\qquad \mu_{\leq^+}(\underline{GQU}^+(\pi'|u^+), \underline{GQU}^+(\pi|u^+)))) = 1\\
&\iff \quad \pi \preccurlyeq^{g}_{\{u^-,u^+\}} \pi'.
\end{aligned}$$

$\leftarrow$) Given $\{(U^-,\leq^-),(U^+,\leq^+)\}, \{u^+,u^-\}$ and $\{h^-,h^+\}$, we consider $\sqsubseteq^-$ and $\sqsubseteq^+$ as the preference relations induced by $\underline{GQU}^-(\cdot|u^-)$ and $\underline{GQU}^+(\cdot|u^+)$ respectively. By the Theorem analogous to Theorem 8.3, we have that $\sqsubseteq^-$ satisfies $AXPN_{\mathsf{T}}$ and $\sqsubseteq^+$ satisfies $AXPN^+_{\mathsf{T}}$. That is, $AxGroup$ is verified.
Taking into account the definition of $\preccurlyeq^{g}_{\{u^-,u^+\}}$ and the fact that

$$\mu_{\underline{GQU}^-(\cdot|u^-)}(\pi,\pi') = \mu_{\leq^-}(\underline{GQU}^-(\pi|u^-), \underline{GQU}^-(\pi'|u^-)),$$

[5] $\leq$ is the order induced in the lattice by the meet or joint operation of the lattice.

and

$$\mu_{\underline{GQU}^+(\cdot|u^+)}(\pi,\pi') = \mu_{\leq^+}(\underline{GQU}^+(\pi|u^+), \underline{GQU}^+(\pi'|u^+)),$$

we have that

$$\pi \preccurlyeq^g_{\{u^-,u^+\}} \pi' \iff g((\mu_{\underline{GQU}^-(\cdot|u^-)}(\pi,\pi'), \mu_{\underline{GQU}^+(\cdot|u^+)}(\pi,\pi')), (\mu_{\underline{GQU}^-(\cdot|u^-)}(\pi',\pi), \mu_{\underline{GQU}^+(\cdot|u^+)}(\pi',\pi))) = 1$$

That is, $GA0$ is verified. □

9.1.2 A First Approach for Characterising Refinements Orderings Applying the Same Preference Function on Consequences

Now, we focus in the refinement orderings that apply the same preference function on consequences. As a first approach for characterising these orderings, we propose the following set of axioms, $MRAX^g_\top$, for a preference relation $\sqsubseteq$ on $(\Pi^*(X), M_\top)$ induced by the linear orders $\sqsubseteq^-$ and $\sqsubseteq^+$:

- $GA0$: There exists a set $\mathcal{R} = \{\sqsubseteq^-, \sqsubseteq^+\}$ of orderings such that $\sqsubseteq = \preccurlyeq^g_{\mathcal{R}}$, i.e.

$$\pi \sqsubseteq \pi' \iff g\left(\left(\mu_{\sqsubseteq^-}(\pi,\pi'), \mu_{\sqsubseteq^+}(\pi,\pi')\right), \left(\mu_{\sqsubseteq^-}(\pi',\pi), \mu_{\sqsubseteq^+}(\pi',\pi)\right)\right) = 1$$

- $AxGroup'$: $\sqsubseteq^-$ satisfies $AX^-_\top*$, while $\sqsubseteq^+$ satisfies $AX^+_\top*$[6].
- $AxCompl$
  1. $x \sqsubseteq^- y \iff x \sqsubseteq^+ y$.
  2. Let $\overline{x}, \underline{x}$ be a maximal and a minimal element of $(X, \sqsubseteq^-) = (X, \sqsubseteq^+)$, denote $\pi^-_\lambda = M_\top(\overline{x}, \underline{x}, 1, \lambda), \pi^+_\lambda = M_\top(\overline{x}, \underline{x}, \lambda, 1)$. Then,
  $$\pi^-_\lambda \sqsubset^- \pi^-_\mu \iff \pi^+_\lambda \sqsupset^+ \pi^+_\mu.$$
  3. $|X/\sim^-| = |\Pi(X)/\sim^-|$

Observe that as consequence of axiom $AxCompl1$, we have that $|X/\sim^-| = |X/\sim^+|$.

Before considering the characterisations of these orderings, let us introduce some necessary results:

Proposition 9.2

1. *Consider two finite lattices U, U', $b : U \to U'$ a lattice isomorphism, a preference mapping $u : X \to U$, and an onto linking mapping $h : V \to U$, satisfying coherence.*
   If $u' = b \circ u$ and $h' = b \circ h$, the orderings induced by GQU w.r.t. U', h', u' and w.r.t. U, h, u, are the same.

[6] $AX^-_\top*$ and $AX^+_\top*$ are the same axiom sets as $AX^-_\top$ and $AX^+_\top$ respectively but now considering the distributions on $\Pi^*(X, V)$ with the mixture operation defined with the supremum and the infimum instead of the maximum and teh minimum

2. *Given a finite linear scale W, and two onto mappings $m : X \to U$, $m' : A \to W$, such that they represent the same ordering in A, i.e.*

$$m(x) < m(y) \iff m'(x) < m'(y), \forall x, y \in A$$

then $m = m'$.

Proof:

1. We consider the optimistic criterion, being the pessimistic one very analogous. We have that

$$GQU^{+}(\pi|U', h', u') = \bigvee_{x \in X}' h'(\pi(x) \top \lambda'_x)$$

with $h'(\lambda'_x) = u'(x)$. Moreover, since $u' = b \circ u$, $h' = b \circ h$, we have that $(b \circ h)(\lambda'_x) = h'(\lambda'_x) = u'(x) = (b \circ u)(x)$, that is, $h(\lambda'_x) = u(x)$, hence

$$GQU^{+}(\pi|U, h, u) = \bigvee_{x \in X} h(\pi(x) \top \lambda'_x).$$

Therefore, as b is an isomorphism and

$$\begin{aligned} GQU^{+}(\pi|U', h', u') &= \bigvee_{x \in X}' h'(\pi(x) \top \lambda'_x) \\ &= \bigvee_{x \in X}' (b \circ h)(\pi(x) \top \lambda'_x) \\ &= b\left(\bigvee_{x \in X} h(\pi(x) \top \lambda'_x)\right) \\ &= b(GQU^{+}(\pi|U, h, u)), \end{aligned}$$

both orderings are the same.

2. Indeed, consider $(A, \sqsubseteq)$, with $x \sqsubseteq y \iff m(x) \leq m(y) (\iff m'(x) \leq m'(y))$. Suppose that $m \neq m'$, hence $Z = \{x | m(x) \neq m'(x)\} \neq \emptyset$. Let x_0 be the minimum, w.r.t $\sqsubseteq$, of Z. Without loss of generality we may assume, that $m'(x_0) > m(x_0)$, as m' is onto there exists $x_1 \in A$ s.t. $m'(x_1) = m(x_0) < m'(x_0)$. That is, $x_1 \sqsubset x_0$.
By hypotheses, $m'(x_1) < m'(x_0) \iff m(x_1) < m(x_0)$, so, we have that $m(x_1) < m(x_0) = m'(x_1)$, that is, $m(x_1) \neq m'(x_1)$, hence $x_1 \in W$. Contradiction because $x_1 \sqsubset x_0$, and x_0 is the infimum of Z. Hence, $m = m'$.

$\square$

	u	u'
x_1	a	c
x_2	b	b
x_3	c	a

Table 9.1: u and u' definitions.

Notice that 2) is not true if U is non-linear. Indeed, consider $U = \{a, b, c\}$ s.t. $a < b, c < b$, $X = \{x_1, x_2, x_3\}$, and u, u' defined in Table 9.1, u, u' satisfy that $u(x) < u(y) \iff u'(x) < u'(y)$ and they are different mappings.

Then, the following theorem is a consequence of the representation Theorem for the linear case and the previous proposition.

Theorem 9.3 (Representation Theorem)
Given a boolean mapping g, a preference relation $\sqsubseteq$ on $(\Pi^*(X), M_\top)$, satisfies the axiom set $MRAX^g_\top$ if and only if there exist:

(i) a finite linear scale of utility U

(ii) an onto preference function $u{:}X \to U$,

(iii) an onto order-preserving mapping $h{:}V \to U$, satisfying coherence w.r.t $\top$,

in such a way that it holds:

$$\pi \sqsubseteq \pi' \qquad \textit{iff} \qquad \pi \preccurlyeq^g_{\{u,u\}} \pi'.$$

The vectorial function of utility inducing $\preccurlyeq^g_{\{u,u\}}$ being

$$\overline{RGQU}^{-,+}(\cdot|(u,u)) = (GQU^-(\cdot|u,h), GQU^+(\cdot|u,h)),$$

with $n = n_U \circ h$.

Proof:
$\rightarrow$) As usual, $\sqsubseteq^+$ stratifies $\Pi(X)$ in a linearly ordered set of classes of equivalently preferred distributions ($\pi' \in [\pi]$ iff $\pi \sim \pi'$). The number of classes is just the number of levels needed to rank order the set of distributions.

Therefore, we take as preference scale $(U^+, \leq^+)$ the quotient set $\Pi(X)/\sim^+$ together with the natural (linear) order

$$[\pi]^+ \leq^+ [\pi']^+ \quad \text{ iff } \quad \pi \sqsubseteq^+ \pi'.$$

Again, as usual we define the order-preserving function $h^+{:}V \to U^+$ as $h^+(\lambda) = [\pi^+_\lambda]$, while we define $GQU^+(M_\top(\overline{x}, \underline{x}; \lambda, 1)) = h^+(\lambda)$, and we extend it due to axiom $A4^\pm_\top$. While $u^+{:}X \to U^+$ is defined as $u^+(x) = GQU^+(x)$. It is known that $GQU^+(\pi) = \max_{i=1,\ldots,p} h^+(\pi(x_i) \top \lambda^+_i)$ and that GQU^+ represents $\sqsubseteq^+$.

Analogously we defined U^-, u^-, h^-, s.t. $GQU^-(\cdot|U^-, u^-, h^-)$ represents $\sqsubseteq^-$
Now, we verify that $GQU^+(\cdot|U^-, u^-, h^-)$ also represents $\sqsubseteq^+$.

Indeed, as by $AxCompl2$ we have that

$$\pi_\lambda^+ \sqsubset^+ \pi_\mu^+ \iff \pi_\lambda^- \sqsupset^- \pi_\mu^-,$$

then U^-, U^+ are isomorphic. Let $b : U^- \to U^+$ be an isomorphism. Moreover,

$$\begin{aligned} h^+(\lambda) <^+ h^+(\mu) &\iff \pi_\lambda^+ \sqsubset^+ \pi_\mu^+ \\ &\iff \pi_\lambda^- \sqsupset^- \pi_\mu^- \\ &\iff h^-(\lambda) <^- h^-(\mu) \\ &\iff b \circ h^-(\lambda) <^+ b \circ h^-(\mu). \end{aligned}$$

Hence, by Proposition 9.2, $h^+ = b \circ h^-$. Analogously, as by $AxCompl1$ we have that u^+, u^- represent the same ordering, and by $AxCompl3,$ both mappings are onto, again by Proposition 9.2, we have that $u^+ = b \circ u^-$. Therefore,

$$\preccurlyeq_{GQU^+(\cdot|U^-,u^-,h^-)} = \preccurlyeq_{GQU^+(\cdot|U^+,u^+,h^+)} .$$

Hence, we define $U = U^-, h = h^-, u = u^-$.

So, it only remains to verify that the relation induced by $\overline{RGQU}^{-,+}$ and g coincides with $\sqsubseteq$.

As $\sqsubseteq^-$ and $\sqsubseteq^+$ are represented by $GQU^-(\cdot|u,h)$ and $GQU^+(\cdot|u,h)$ respectively, we have that

$$\pi \sqsubseteq^- \pi' \iff GQU^-(\pi|u,h) \leq GQU^-(\pi'|u,h),$$

and

$$\pi \sqsubseteq^+ \pi' \iff GQU^+(\pi|u,h) \leq GQU^+(\pi'|u,h).$$

That is,

$$\mu_{\sqsubseteq^-}(\pi,\pi') = \mu_{\leq}(GQU^-(\pi|u), GQU^-(\pi'|u))$$

and

$$\mu_{\sqsubseteq^+}(\pi,\pi') = \mu_{\leq}(GQU^+(\pi|u), GQU^+(\pi'|u)).$$

Hence, applying $GA0,$ we have that

$$\begin{aligned} \pi \sqsubseteq \pi' \quad &\iff \quad g\big(\big(\mu_{\sqsubseteq^-}(\pi,\pi'), \mu_{\sqsubseteq^+}(\pi,\pi')\big)\,, \\ &\qquad\quad \big(\mu_{\sqsubseteq^-}(\pi',\pi), \mu_{\sqsubseteq^+}(\pi',\pi)\big)\big) = 1 \\ &\iff \quad g((\mu_{\leq}(GQU^-(\pi|u), GQU^-(\pi'|u)), \\ &\qquad\qquad \mu_{\leq}(GQU^+(\pi|u), GQU^+(\pi'|u))), \\ &\qquad\qquad (\mu_{\leq}(GQU^-(\pi'|u), GQU^-(\pi|u)), \\ &\qquad\qquad \mu_{\leq}(GQU^+(\pi'|u), GQU^+(\pi|u)))) = 1. \\ &\iff \quad \pi \preccurlyeq^g_{\{u,u\}} \pi'. \end{aligned}$$

$\leftarrow$) Given $(U, \leq)$ u and h, we consider $\sqsubseteq^-$ and $\sqsubseteq^+$ as the preference relations induced by $GQU^-(\cdot|u)$ and $GQU^+(\cdot|u)$ respectively. By an analogous to the Theorem 5.5 considering supremum and infimum instead of maximum and minimum, we have that $\sqsubseteq^-$ satisfies $AX^*_\top$ and $\sqsubseteq^+$ satisfies $AX^{+*}_\top$. That is, $AxGroup'$ is verified. Taking into account the definition of $\preccurlyeq^g_{\{u,u\}}$ and the fact that

$$\mu_{GQU^-(\cdot|u)}(\pi,\pi') = \mu_\leq(GQU^-(\pi|u), GQU^-(\pi'|u)),$$

and

$$\mu_{GQU^+(\cdot|u)}(\pi,\pi') = \mu_\leq(GQU^+(\pi|u), GQU^+(\pi'|u)),$$

we have that

$$\begin{aligned}\pi \preccurlyeq^g_{\{u,u\}} \pi' \quad \Longleftrightarrow \quad & g((\mu_{GQU^-(\cdot|u)}(\pi,\pi'), \mu_{GQU^+(\cdot|u)}(\pi,\pi')),\\ & (\mu_{GQU^-(\cdot|u)}(\pi',\pi), \mu_{GQU^+(\cdot|u)}(\pi',\pi))) = 1\end{aligned}$$

That is, $GA0$ is verified. *AxCompl*, verifies trivially. □

9.2 A First Approach with a Weaker Commensurability Hypothesis

In the models developed up to now, we have been assuming an hypothesis of *commensurateness* between the plausibility set V and the preference set U in order to define the criteria for ranking possibility distributions. Actually, in Section 4.4, it is assumed the existence of an order-preserving mapping $h{:}V \to U$ such that $h(1) = 1$ and $h(0) = 0$ to define the qualitative utility functions. However, to characterise the orderings, h is also required to be onto (Lemma 4.7 and Theorem 4.12).
Now, we are interested in characterising the orderings resulting when h is *not* required to be onto. This weakening of the commensurability hypothesis will allow us to deal with other types of problems, in particular, those in which the cardinality of the preference valuation set may be greater than the cardinality of the uncertainty valuation set.

9.2.1 A New Working Framework

Let us define the framework for this section. V will denote a finite linear plausibility scale, where $\inf(V) = 0$ and $\sup(V) = 1$, and $\Pi(X)$ will denote *the set of consistent possibility distributions on X over V*, i.e.

$$\Pi(X) = \{\pi : X \to V \,|\, \max_{x\in X} \ \pi(x) = 1\}.$$

U will denote a finite linearly ordered scale of preference (or utility), with $\sup(U) = 1$ and $\inf(U) = 0$. As usual, we assume as working hypothesis the existence of a preference function representing *Decision Maker*'s preference on consequences, i.e. there exists a function $u{:}X \to U$ that assigns to each consequence of X a preference level of U such that $u(x) \leq u(y)$ if and only if y is at least as preferred as x.

Let $h{:}V \to U$ *be an order-preserving function* relating both scales V and U such that $h(0) = 0,\ h(1) = 1$. In such a framework, *also assuming that* h *is onto*, we have been considering the preference relations induced by the utility functions

$$QU^-(\pi|u) = \min_{x \in X} \max(n(\pi(x)), u(x)),$$

where $n = n_U \circ h$, n_U is the reversing-involution in U, and

$$QU^+(\pi|u) = \max_{x \in X} \min(h(\pi(x)), u(x)).$$

Notation 9.2
As usual, for the sake of a simpler notation, we shall write $QU^-(\pi)$ *instead of* $QU^-(\pi|u)$ *when the mapping* u *is not relevant for the context. In fact, these utility functions* also depend on the mapping h linking both scales. *With the goal of simplicity, we will omit it and will use the notation of* QU *to refer a utility involving an onto* h *and* QU_W *for the case of not requiring* h *this onto condition.*

9.2.2 Qualitative Utility Functions with a Weaker Assumption of Commensurability

Let us remark that the great difference with the cases analysed previously in Chapter 4 and with the work of (Dubois et al., 1997e) is that now h is ***not*** required to be onto.

Given $h : V \to U$, for any $\pi \in \Pi(X)$, consider the qualitative utility functions

$$QU_W^-(\pi|u) = \min_{x \in X} \max(n(\pi(x)), u(x))$$

where $n = n_U \circ h$, n_U being the reversing involution in U, and

$$QU_W^+(\pi|u) = \max_{x \in X} \min(h(\pi(x)), u(x)).$$

Notice that $QU_W^-(\cdot|u)$ and $QU_W^-(\cdot|u)$, restricted to X, coincide with the preference function u, i.e. $QU_W^-(x|u) = u(x) = QU_W^+(x|u)$, for all $x \in X$. As usual, since n_U^2 is the identity in U, the mapping h can also be defined from n, namely $h(\lambda) = n_U(n(\lambda))$.

It is interesting to notice that these functions still preserves the possibilistic mixture:

Lemma 9.4
QU_W^- *and* QU_W^+ *preserve the possibilistic mixture in the sense that*

$$QU_W^-(\lambda/\pi_1, \mu/\pi_2) = \min\{\max(n(\lambda), QU_W^-(\pi_1)), \max(n(\mu), QU_W^-(\pi_2))\},$$

and

$$QU_W^+(\lambda/\pi_1, \mu/\pi_2) = \max\{\min(h(\lambda), QU_W^+(\pi_1)), \min(h(\mu), QU_W^+(\pi_2))\}.$$

We omit the proof since it is easy to verify that in the proof of Lemma 4.5 we do not apply the fact of h being onto.

Corollary 9.5
The following properties remain true for QU_W^- and QU_W^+:

1. $QU_W^-(\max(\pi_1, \pi_2)) = \min\{QU_W^-(\pi_1), QU_W^-(\pi_2)\}$.
2. *if* $QU_W^-(\pi_1) \leq QU_W^-(\pi_2)$, *then*
$$QU_W^-(\lambda/\pi_1, \mu/\pi_2) = median\{QU_W^-(\pi_1), QU_W^-(\pi_2), n(\lambda)\}.$$
3. *if* $QU_W^-(\pi_1) > QU_W^-(\pi_2)$ *then*
$$QU_W^-(\lambda/\pi_1, \mu/\pi_2) = median\{QU_W^-(\pi_1), QU_W^-(\pi_2), n(\mu)\}.$$

The fact of allowing h to be a *non*-onto mapping results in that the continuity axiom $A4$ may be not true. Indeed, if we consider $V = \{0, 1\}$, $U = \{0 < w < 1\}$ and $X = \{\underline{x},\ x_1,\ \overline{x}\}$, with $u(\underline{x}) = 0,\ u(x_1) = w,\ u(\overline{x}) = 1$, it is obvious that $QU_W^-(\pi) = \min_{x \in \pi}\ u(x)$. That is, the ordering induced by QU_W^- coincides with the *maximin* criterion while the ordering induced by QU_W^+ coincides with the *maximax* one. Observe that if $\pi = x_1$, there does not exist $\lambda \in V$ such that $\pi \sim (1/\overline{x}, \lambda/\underline{x})$.

Now, let us introduce the axiomatic setting we propose for characterising the ordering induced by these pessimistic qualitative utility functions.

9.2.3 Axiomatic Setting Proposed

The above discussion has led us to propose this new set of axioms AXM for preference relations on $\Pi(X)$ with the max-min mixture as the internal operation on $\Pi(X)$.

- $A1(structure)$: $\sqsubseteq$ is a total pre-order .
- $A2(uncertainty\ aversion)$: if $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.
- $A3(independence)$: $\pi_1 \sim \pi_2 \Rightarrow (\lambda/\pi_1, \mu/\pi) \sim (\lambda/\pi_2, \mu/\pi)$.

Let $\overline{x}$ and $\underline{x}$ be a maximal and a minimal of $(X, \sqsubseteq)$ respectively. We denote by π_λ^- the lottery $(1/\overline{x}, \lambda/\underline{x})$.

- $A4C$ (*relaxed continuity*): There exists a subset[6] $X_{NM} \subseteq X$ such that all maximal elements of $(X, \sqsubseteq)$ and all minimal elements of $(X, \sqsubseteq)$ are in the complement of X_{NM}, and such that
$$(\forall \pi \in \Pi(X)) \text{ either } (\exists \lambda \in V \text{ s.t. } \pi \sim \pi_\lambda^-) \text{ or } (\exists x \in X_{NM} \text{ s.t. } \pi \sim x).$$
- $AxMix$:
  1. if $x, y \in X_{NM},\ \beta \in V$, then,
$$(1/x, \beta/y) \sim \begin{cases} x & \text{if } (x \sqsubseteq y) \text{ or } (x \sqsubset \pi_\beta^-) \\ \pi_\beta^- & \text{if } y \sqsubset \pi_\beta^- \sqsubset x \\ y & \text{if } \pi_\beta^- \sqsubset y \sqsubset x, \end{cases}$$

[6]Observe that $X_{NM} = \emptyset$ is possible, and then axiom $A4$ (see Section 4.4) is recovered.

2. if $x \in X_{NM}$, then,

$$(1/\pi^-_\lambda, \beta/x) \sim \begin{cases} \pi^-_\lambda & \text{if } (\pi^-_\lambda \sqsubset x) \text{ or } (\pi^-_\lambda \sqsubseteq \pi^-_\beta) \\ \pi^-_\beta & \text{if } x \sqsubset \pi^-_\beta \sqsubset \pi^-_\lambda \\ x & \text{if } \pi^-_\beta \sqsubset x \sqsubset \pi^-_\lambda. \end{cases}$$

The underlying idea in $A4C$ is to relax the continuity of the preference. Now, we may say that there exists a subset on X such that either the distributions are preferentially equivalent to individual consequences on this set, or, the distributions are preferentially equivalent to having a λ level of uncertainty with respect to $\overline{x}$.

Remark 14
Let us consider the simplest scale of uncertainty, $V = \{0, 1\}$, *that is, consequences can be either fully possible or fully impossible. This is a very particular case since for any preference scale* U, *the only requirement to be fulfilled by a mapping* $h{:}V \to U$ *is that* $h(0) = 0$ *and* $h(1) = 1$. *In this framework* $\Pi(X)$ *is just the power set* 2^X *and the resulting utility functionals are*

$$QU^-_W(A|u) = \min_{x \in A} u(x),$$

$$QU^+_W(A|u) = \max_{x \in A} u(x),$$

leading to the well-known maximin and maximax decision models.

Now, it is very easy to check that, in order to fully characterise a preference relation on 2^X *induced by these* QU^-_W *and* QU^+_W, *the above axioms simplify to these ones:*

- $A1$: $\sqsubseteq$ *is a total preorder,*
- $A2$: *if* $A \subseteq B$ *then* $B \sqsubseteq A$,
- $A3$: *if* $A \sim B$ *then* $A \cup C \sim B \cup C$,
- $A4C$: *for all* $A \subseteq X$, *there exists* $x \in X$ *such that* $A \sim x$,
- $AxMix$: *if* $x \sqsubseteq y$ *then* $\{x, y\} \sim x$.

Actually, in this setting axiom $A2$ *is redundant since it is a logical consequence of the remaining axioms. Moreover, as we are working as usual with a finite set* X, $A4C$ *is a consequence of* $AxMix$.

The axiomatic frameworks à la Savage of these maximax and maximin criteria are provided in (Brafman and Tennenholtz, 1996; Brafman and Tennenholtz, 1997).

Some Auxiliary Results

Now, we introduce some results that will be applied to characterise the pessimistic orderings.

Lemma 9.6
Axioms A1, A2, A3 , A4C and AxMix imply

$Ax2$: If A is a crisp subset of X then there exists $x \in A$ s.t. $x \sim A$.

Proof:
Suppose $A = \{x_1, x_2\}$ with $x_1 \sqsubseteq x_2$. Note that $A = (1/x_1, 1/x_2)$. If $x_1 \sim x_2$, then, $A \sim x_1$. Now, we assume $x_1 \sqsubset x_2$.

By $A4C$, there are four alternatives for x_1, x_2:

1. $\exists\, \mu, \lambda$ s.t. $x_1 \sim (1/\overline{x}, \lambda/\underline{x})$ and $x_2 \sim (1/\overline{x}, \mu/\underline{x})$.
2. $\exists\, x, y \in X_{NM}$ s.t. $x_1 \sim x$ and $x_2 \sim y$.
3. $\exists\, \lambda \in V, x \in X_{NM}$ s.t. $x_1 \sim x$ and $x_2 \sim \pi_\lambda^-$.
4. $\exists\, \lambda \in V, x \in X_{NM}$ s.t. $x_1 \sim \pi_\lambda^-$ and $x_2 \sim x$.

Now, we analyse them:

1. By $A2$, as $x_1 \sqsubset x_2$, then, $\lambda > \mu$. Applying reduction of lotteries, we have that
$$A \sim \pi^-_{\max(\lambda,\mu)} \sim (1/\overline{x}, \lambda/\underline{x}) \sim x_1.$$
2. As $A \sim (1/x, 1/y)$ and $x \sqsubset y$ by $AxMix1$, we have that
$$A \sim x \sim x_1$$
3. Since $A \sim (1/x, 1/(1/\overline{x}, \lambda/\underline{x}))$, applying $AxMix2$ we have that
$$A \sim x \sim x_1.$$
4. Finally, $A \sim (1/\pi_\lambda^-, 1/x)$ and by $AxMix2$, it results
$$A \sim \pi_\lambda^- \sim x_1.$$

Therefore, if $x_1 \sqsubset x_2$, it holds that $A \sim x_1$.
The case when A has p elements is an easy generalisation. Indeed, suppose the Lemma is valid if $|A|$ = p. Now, let A be such that $|A| = p+1$, and let x_1 be one of its minimal elements w.r.t. $\sqsubseteq$.

Since $A = (1/x_1, 1/A - \{x_1\})$, by induction hypothesis we have that if x_2 is one of the minimal elements of $A - \{x_1\}$ w.r.t. $\sqsubseteq$, then,
$$A \sim (1/x_1, 1/x_2) \sim x_1.$$
$\square$

An interesting property of a preference relation $\sqsubseteq$ on $\Pi(X)$ satisfying $A1, Ax2$ and $A2$ is that the extremal elements of $(X, \sqsubseteq)$ are maximal and minimal elements of $(\Pi(X), \sqsubseteq)$ as well. Indeed, recall that we have proved Lemma 4.1:

> If $\sqsubseteq$ verifies axioms $A1$, $Ax2$ and $A2$, and $\underline{x}$, $\overline{x}$ are a minimal and a maximal element of X, respectively, then:
>
> - $\underline{x} \sim \pi_1^- \sim X$.
> - $\underline{x}$ and $\overline{x}$ are also the minimal and maximal elements of $(\Pi(X), \sqsubseteq)$.

9.2.4 Representation of Pessimistic Qualitative/Ordinal Utilities

Next, we show that the preference ordering on $\Pi(X)$ induced by the qualitative pessimistic utility QU^-_W satisfies the above set of axioms.

Lemma 9.7
Let $\preccurlyeq_{QU^-_W}$ *be the preference ordering on* $\Pi(X)$ *induced by* QU^-_W*, i.e.*

$$\pi \preccurlyeq_{QU^-_W} \pi' \quad \textit{iff} \quad QU^-_W(\pi) \leq QU^-_W(\pi').$$

Then, $\preccurlyeq_{QU^-_W}$ *verifies axioms set* AXM.

Proof:
Axiom $A1$ is easily verified, also $A2$ is a consequence of maximum and minimum being non-decreasing functions, while $A3$ results from the fact that QU^-_W preserves max-min possibilistic mixtures.
Thus, we only check axioms $A4C$ and $AxMix$. If h is onto, $X_{NM} = \emptyset$, and $A4C$ reduces to $A4$, hence, we are in the case detailed in Section 4.4.

Now, we consider the case of h being non-onto. Let

$$X_{NM} = (\{x | \, u(x) \in n(V)\})^c.$$

As $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$ and $h(0) = 0$ and $h(1) = 1$, if x is a maximal or a minimal element of $(X, \preccurlyeq_{QU^-_W})$, then $x \notin X_{NM}$.

With respect to $A4C$, we have to prove that if $\overline{x}, \underline{x}$ are a maximal and a minimal element of $(X, \preccurlyeq_{QU^-_W})$, for any distribution π in $\Pi(X)$ we have either

$$(\exists \lambda \text{ s.t. } QU^-_W(\pi) = QU^-_W(1/\overline{x}, \lambda/\underline{x}))$$

or

$$(\exists x \in X_{NM} \text{ s.t. } QU^-_W(\pi) = QU^-_W(x)).$$

By definition of QU^-_W, for each π, we have that exists $x_0 \in X$ s.t. $QU^-_W(\pi) = \max(n(\pi(x_0)), u(x_0))$.
Hence,

- if $QU^-_W(\pi) = n(\pi(x_0))$, then, taking $\lambda = \pi(x_0)$(obviously λ is in V), we have that $QU^-_W(\pi) = QU^-_W(1/\overline{x}, \lambda/\underline{x})$.
- Otherwise, $QU^-_W(\pi) = u(x_0)$. In this case, there are two alternatives, either $u(x_0) \in n(V)$ or not. In the first option, we have that there exists $\lambda \in V$ s.t. $QU^-_W(\pi) = u(x_0) = n(\lambda) = QU^-_W(1/\overline{x}, \lambda/\underline{x})$. While in the second option, we have that $u(x_0) \in X_{NM}$, and $QU^-_W(\pi) = u(x_0) = QU^-_W(x_0)$.

Finally, is not difficult to verify $AxMix$ taking into account Lemma 9.4. □

Now, we can show that the preference orderings satisfying the axioms proposed can always be represented by a pessimistic qualitative utility of the type of QU^-_W.

Theorem 9.8 (Representation Theorem of Pessimistic Utility)
A preference relation $\sqsubseteq$ on $\Pi(X)$ satisfies axiom set AXM if, and only if, there exist

(i) a finite linearly ordered utility scale U with $\inf(U) = 0$ and $\sup(U) = 1$,

(ii) a preference function $u{:}X \to U$ such that $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,

(iii) an order-preserving[7] function $h{:}V \to U$ such that $h(0) = 0$ and $h(1) = 1$,

in such a way that

$$\pi' \sqsubseteq \pi \qquad \textit{iff} \qquad \pi' \preccurlyeq_{QU_W^-} \pi,$$

where $\preccurlyeq_{QU_W^-}$ is the ordering in $\Pi(X)$ induced by the qualitative utility $QU_W^-(\pi) = \min_{x\in X} \max(n(\pi(x)), u(x))$, being as usual $n = n_U \circ h$.

Proof:
The "if" part corresponds to the preceding Lemma. As for the "only if" part, we go on structuring the proof, analogously to our previous approaches, in the following three steps:

- In step (1) we define the utility scale U and an order-preserving function h from V to U.
- In step (2) we define a function $QU_W^-{:}\Pi(X) \to U$ representing $\sqsubseteq$, i.e. such that

  $$QU_W^-(\pi) \leq QU_W^-(\pi') \qquad \text{iff} \qquad \pi \sqsubseteq \pi'.$$

- Finally in step (3) we prove that

  $$QU_W^-(\pi) = \min_{x\in X} \max(n(\pi(x)), u(x)),$$

  where $u{:}X \to U$ is the restriction of QU_W^- to X and $n = n_U \circ h$, n_U being the reversing involution on U.

Now, we develop these steps.

1. As usual, $\sqsubseteq$ stratifies $\Pi(X)$ in a linearly ordered set of classes of equivalently preferred distributions ($\pi' \in [\pi]$ iff $\pi \sim \pi'$). The number of classes is just the number of levels needed to rank the set of distributions. Therefore, we take as utility scale U the quotient set $\Pi(X)/\sim$ together with the natural (linear) order

   $$[\pi] \leq [\pi'] \qquad \text{iff} \qquad \pi \sqsubseteq \pi'.$$

   Denote by 1 and 0 the maximum and minimum elements of $\Pi(X)/\sim$, i.e. of U. As Lemma 4.1 still holds, $\overline{x}$ and $\underline{x}$ are a maximal and minimal elements of $(X, \sqsubseteq)$ respectively, then $[\overline{x}] = 1$ and $[\underline{x}] = 0$.

   Let π_λ^- be the possibility distribution corresponding to the qualitative lottery $(1/\overline{x}, \lambda/\underline{x})$ and define the order-reversing function $n{:}V \to U$ as

[7]Note that h is not required to be onto.

$$n(\lambda) = [\pi_\lambda^-].$$

Observe that, since $(1/\overline{x}, 1/\underline{x}) \sim \underline{x}$, we have

$$n(1) = [(1/\overline{x}, 1/\underline{x})] = [\underline{x}] = 0,$$

and

$$n(0) = [(1/\overline{x}, 0/\underline{x})] = [\overline{x}] = 1.$$

$A2$ guarantees that n reverses the order.

Let $h = n_U \circ n$, n_U being the reversing involution in U. It is obvious that h satisfies the conditions required.

2. Now, we define the qualitative function QU_W^- on $\Pi(X)$ in three steps.

   (a) First, let us define $QU_W^-(1/\overline{x}, \lambda/\underline{x}) = n(\lambda)$.
   It is easy to check that

   $$\pi_\lambda^- \sqsubseteq \pi_{\lambda'}^- \iff QU_W^-(\pi_\lambda^-) \leq QU_W^-(\pi_{\lambda'}^-).$$

   (b) Secondly, let us define for each $x \in X_{NM}$, $QU_W^-(x) = [x]$. Analogously, it is easy to verify that, restricted to distributions of type x, QU_W^- represents $\sqsubseteq$.

   (c) We extend QU_W^- to any lottery as follows.
   Since for any π, $A4C$ guarantees that either $(\exists \lambda \text{ s.t. } \pi \sim \pi_\lambda^-)$ or $(\exists x \in X_{NM} \text{ s.t. } \pi \sim x)$, we define

   $$QU_W^-(\pi) = \begin{cases} n(\lambda) & \text{if } \exists\, \lambda \text{ s.t. } \pi \sim \pi_\lambda^- \\ [x] & \text{if } \exists x \in X_{NM} \text{ s.t. } \pi \sim x. \end{cases}$$

   Notice that QU_W^- is well defined: by $A4C$ it is not possible to have $\lambda \in V$ and $x \in X_{NM}$ s.t. $\pi \sim (1/\overline{x}, \lambda/\underline{x})$ and $\pi \sim x$. However, one of these cases may happen:

   - $\exists x, x' \in X_{NM}$, s.t. $\pi \sim x$ and $\pi \sim x'$, or
   - there exists $\mu \neq \lambda$ such that $\pi \sim \pi_\mu^-$ and $\pi \sim \pi_\lambda^-$.

   But, since $x' \sim \pi \sim x$, we have that $x' \sim x$, therefore they are in the same equivalence class, and $QU_W^-(\pi) = [x] = [x']$. In the other case, since $\pi_\mu^- \sim \pi_\lambda^-$ then $[\pi_\lambda^-] = [\pi_\mu^-]$, so $n(\lambda) = n(\mu)$.

   Finally, it is not difficult to verify that QU_W^- represents $\sqsubseteq$. This is due to the fact that any π is equivalent to some π_λ^- or to some $x \in X_{NM}$ and QU_W^- represents $\sqsubseteq$ over the π_λ^-'s and over the x's in X_{NM}.

3. Now, we define u:$X \to U$ as

$$u(x) = QU^-_W(x).$$

Notice that $u(\overline{x}) = 1$ and $u(\underline{x}) = 0$, and thus, $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$.

It remains to prove that $QU^-_W(\pi) = \min_{x\in X}\ \max(n(\pi(x)), u(x))$.
With this goal, we will prove the following equalities:

- $QU^-_W(1/\pi_1, \beta/\pi_2) = \min(QU^-_W(\pi_1), \max(n(\beta), QU^-_W(\pi_2)))$.
  By $A4C$, there are several alternatives for π_1, π_2 :
  (a) $\exists\mu, \lambda$ s.t. $\pi_1 \sim (1/\overline{x}, \lambda/\underline{x})$ and $\pi_2 \sim \pi^-_\mu$.
  (b) $\exists x, y \in X_{NM}$ s.t. $\pi_1 \sim x$ and $\pi_2 \sim y$,
  (c) $\exists\lambda \in V, x \in X_{NM}$ s.t. $\pi_1 \sim x$ and $\pi_2 \sim \pi^-_\lambda$,
  (d) $\exists\lambda \in V, x \in X_{NM}$ s.t. $\pi_1 \sim \pi^-_\lambda$ and $\pi_2 \sim x$.

  Now, we analyse them:

  (a) By $A3$,
  $$(1/\pi_1, \beta/\pi_2) \sim (1/\pi^-_\lambda, \beta/(1/\overline{x}, \mu/\underline{x})),$$
  and reducing lotteries we obtain
  $$(1/\pi_1, \beta/\pi_2) \sim (1/\overline{x}, \max(\lambda, \min(\mu, \beta))/\underline{x}).$$
  Therefore,
  $$\begin{aligned} QU^-_W(1/\pi_1, \beta/\pi_2) &= n(\max(\lambda, \min(\mu, \beta))) \\ &= \min(n(\lambda), \max(n(\mu), n(\beta))) \\ &= \min(QU^-_W(\pi_1), \max(n(\beta), QU^-_W(\pi_2))). \end{aligned}$$

  (b) Again by $A3$,
  $$(1/\pi_1, \beta/\pi_2) \sim (1/x, \beta/y).$$
  Now, taking into account $AxMix$, we have that
  $$(1/x, \beta/y) \sim \begin{cases} x & \text{if } (x \sqsubseteq y) \text{ or } (\ x \sqsubset \pi^-_\beta) \\ \pi^-_\beta & \text{if } y \sqsubset \pi^-_\beta \sqsubset x \\ y & \text{if } \pi^-_\beta \sqsubset y \sqsubset x. \end{cases}$$
  So,
  $$QU^-_W(1/x, \beta/y) = \begin{cases} u(x) & \text{if } (x \sqsubseteq y) \text{ or } (x \sqsubset \pi^-_\beta) \\ n(\beta) & \text{if } y \sqsubset \pi^-_\beta \sqsubset x \\ u(y) & \text{if } \pi^-_\beta \sqsubset y \sqsubset x. \end{cases}$$
  That is,
  $$QU^-_W(1/\pi_1, \beta/\pi_2) = \min(QU^-_W(\pi_1), \max(n(\beta), QU^-_W(\pi_2))).$$

(c) Now,

$$(1/\pi_1, \beta/\pi_2) \sim (1/x, \beta/\pi_\lambda^-) \sim (1/x, 1/\pi^-_{\min(\lambda,\beta)})$$

and by $AxMix$, we have that

$$(1/x, 1/\pi^-_{\min(\lambda,\beta)}) \sim \begin{cases} \pi^-_{\min(\lambda,\beta)}, & \text{if } (\pi^-_{\min(\lambda,\beta)} \sqsubset x) \text{ or} \\ & \quad (\pi^-_{\min(\lambda,\beta)} \sim X) \\ x, & \text{if } X \sqsubset x \sqsubset \pi^-_{\min(\lambda,\beta)}. \end{cases}$$

So,

$$\begin{aligned} QU_W^-(1/\pi_1, \beta/\pi_2) &= \min(u(x), n(\min(\lambda,\beta))) \\ &= \min(QU_W^-(\pi_1), \max(n(\beta), QU_W^-(\pi_2))). \end{aligned}$$

(d) Analogously, if $\pi_1 \sim (1/\overline{x}, \lambda/\underline{x})$ and $\pi_2 \sim x$, then,

$$(1/\pi_1, \beta/\pi_2) \sim (1/\pi_\lambda^-, \beta/x),$$

so,

$$(1/\pi_1, \beta/\pi_2) \sim \begin{cases} \pi_\lambda^- & \text{if } (\pi_\lambda^- \sqsubset x) \text{ or } (\pi_\lambda^- \sqsubseteq \pi_\beta^-) \\ \pi_\beta^- & \text{if } x \sqsubset \pi_\beta^- \sqsubset \pi_\lambda^- \\ x & \text{if } \pi_\beta^- \sqsubset x \sqsubset \pi_\lambda^-. \end{cases}$$

Hence,

$$QU_W^-(1/\pi_1, \beta/\pi_2) = \min(QU_W^-(\pi_1), \max(n(\beta), QU_W^-(\pi_2))).$$

In particular, we have that

$$QU_W^-(\max(\pi_1, \pi_2)) = \min(QU_W^-(\pi_1), QU_W^-(\pi_2)).$$

This may be easy generalised to

$$QU_W^-(\max_{i=1,\ldots,p} \pi_i) = \min_{i=1,\ldots,p} QU_W^-(\pi_i).$$

- Now, we verify

$$QU_W^-(\pi) = \min_{i=1,\ldots,p} \max(n(\pi(x_i)), u(x_i)).$$

As π is normalised there exists $x_j \in X$ such that $\pi(x_j) = 1$. Without loss of generality we assume $j = 1$.
Then, let

$$\pi_i = (1/x_1, \pi(x_i)/x_i).$$

Since $\pi = \max_{i=1,\ldots,p} \ \pi_i$, we have:

$$\begin{aligned} QU^-_W(\pi) &= QU^-_W(\max_{i=1,\ldots,p} \ \pi_i) \\ &= \min_{i=1,\ldots,p} \ QU^-(\pi_i) \\ &= \min_{i=1,\ldots,p} \{\min(u(x_1), \max(n(\pi(x_i)), u(x_i)))\} \\ &=^8 \min_{i=1,\ldots,p} \ \max(n(\pi(x_i)), u(x_i)). \end{aligned}$$

This ends the proof of the theorem. □

9.2.5 Representation of Optimistic Qualitative/Ordinal Utilities

For modelling an optimistic behaviour of the Decision Maker, we consider the axiom set $AXM^+ = \{A1, A2^+, A3, A4C^+, \ AxMix^+\}$, with $\pi^+_\lambda = (\lambda/\overline{x}, 1/\underline{x})$ where as usual $\overline{x}$ and $\underline{x}$ are a maximal and a minimal element of $(X, \ \sqsubseteq)$ respectively, with

- $A2^+$: if $\pi \leq \pi'$ then $\pi \sqsubseteq \pi'$,
- $A4C^+$: There exists a subset[9] $X_{NM} \subseteq X$, such that all maximal elements of $(X, \sqsubseteq)$ and all minimal elements of $(X, \sqsubseteq)$ are in its complement, such that

$$\forall \pi \in \Pi(X) \text{ either } \ (\exists \lambda \in V \ \text{ s.t. } \ \pi \sim \pi^+_\lambda) \text{ or } \ (\exists x \in X_{NM} \ \text{ s.t. } \ \pi \sim x).$$

- $AxMix^+$:

1. if $x, y \in X_{NM}, \ \beta \in V$ then,

$$(1/x, \beta/y) \sim \begin{cases} x & \text{if } (x \sqsupseteq y) \text{ or } (\ x \sqsupset \pi^+_\beta) \\ \pi^+_\beta & \text{if } y \sqsupset \pi^+_\beta \sqsupset x \\ y & \text{if } \pi^+_\beta \sqsupset y \sqsupset x, \end{cases}$$

2. if $x \in X_{NM}$ then

$$(1/\pi^+_\lambda, \beta/x) \sim \begin{cases} \pi^+_\lambda & \text{if } (\pi^+_\lambda \sqsupset x) \text{ or } (\pi^+_\lambda \sqsupseteq \pi^+_\beta) \\ \pi^+_\beta & \text{if } x \sqsupset \pi^+_\beta \sqsupset \pi^+_\lambda \\ x & \text{if } \pi^+_\beta \sqsupset x \sqsupset \pi^+_\lambda. \end{cases}$$

As in the pessimistic case, we have the following results, whose proofs are analogous to the previous ones, so they are omitted here.

Lemma 9.9

1. *Axioms* $A1, A2^+, A3, A4C^+$ *and* $AxMix^+$ *imply*

[8] Note that $\pi(x_1) = 1$, so $u(x_1) = \max(u(x_1), n(\pi(x_1)))$.

[9] Observe that $X_{NM} = \emptyset$ is possible, and then, axiom $A4^+$ is recovered.

$Ax2$: If A is a crisp subset of X then there is $x \in A$ s.t. $x \sim A$.

2. *We still have the Lemma 4.11:*

   If $\sqsubseteq$ verifies axioms $A1,\ A2^+$, and $Ax2$, and $\underline{x}$ and $\overline{x}$ are a minimal and a maximal element of X, respectively, then:

   - *the following equivalences hold: $\overline{x} \sim (1/\overline{x}, 1/\underline{x}) \sim X$.*
   - *$\underline{x}$ and $\overline{x}$ are the minimal and maximal elements of $(\Pi(X), \sqsubseteq)$ respectively.*

Lemma 9.10
Let $\preccurlyeq_{QU^+_W}$ be the preference ordering on $\Pi(X)$ induced by QU^+_W, i.e.

$$\pi \preccurlyeq_{QU^+_W} \pi' \quad \text{iff} \quad QU^+_W(\pi) \leq QU^+_W(\pi').$$

Then, $\preccurlyeq_{QU^+_W}$ verifies the axioms set AXM^+.

The respective Representation Theorem is:

Theorem 9.11 (Representation Theorem of Optimistic Utility)
A preference relation $\sqsubseteq$ on $\Pi(X)$ satisfies axiom set AXM^+ if, and only if, there exist

(i) a finite linearly ordered utility scale U with $\inf(U) = 0$ and $\sup(U) = 1$,

(ii) a preference function $u{:}X \to U$ such that $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,

(iii) an order-preserving function $h{:}V \to U$ such that $h(0) = 0$ and $h(1) = 1$,

in such a way that

$$\pi' \sqsubseteq \pi \qquad \text{iff} \qquad \pi' \preccurlyeq_{QU^+_W} \pi,$$

where $\preccurlyeq_{QU^+_W}$ is the ordering in $\Pi(X)$ induced by the qualitative utility $QU^+_W(\pi) = \max_{x \in X} \min(h(\pi(x)), u(x))$.

9.2.6 Utilities for Non-Normalised Distributions

Now, we consider as the working set of possibilistic lotteries the set $\Pi^{ex}(X)$ of non-necessarily normalised distributions on X with values on the finite uncertainty scale V, keeping the usual definition of possibilistic mixture.

We extend the utility functionals QU^-_W and QU^+_W to evaluate non-normalised distributions of $\Pi^{ex}(X)$ as well. Given an order-preserving mapping $h{:}V \to U$, s.t. $h(0) = 0$ and $h(1) = 1$, and $F{:}V \to V$ s.t. $F(1) = 0$, we define, for any $\pi \in \Pi^{ex}(X)$:

$$\underline{QU}^-_W(\pi|u) = \min\{QU^-_W(\mathcal{N}(\pi)|u), n \circ F(\mathcal{H}(\pi))\},$$

$$\underline{QU}^+_W(\pi|u) = \max\{QU^+_W(\mathcal{N}(\pi)|u), h \circ F(\mathcal{H}(\pi))\}.$$

From these definitions, it is obvious that, for all $\pi \in \Pi^{ex}(X)$, we have $\underline{QU}^+_W(\pi) \geq \underline{QU}^-_W(\pi)$, in particular, if $\pi \equiv 0$, $\underline{QU}^-_W$ (π) = 0 and $\underline{QU}^+_W(\pi)$ = 1. Moreover, $\underline{QU}^-_W$ ($\underline{QU}^+_W$ resp.) is an extension of QU^-_W (of QU^+_W resp.) since, if π is normalised, $\mathcal{H}(\pi)$ = 1, and $n \circ F(1) = 1$ and $h \circ F(1) = 0$, and thus $\underline{QU}^-_W$ and QU^-_W ($\underline{QU}^+_W$ and QU^+_W resp.) coincide on $\Pi(X)$.

In order to characterise the preference orderings $\sqsubseteq$ induced in $\Pi^{ex}(X)$ by $\underline{QU}^-_W$ and $\underline{QU}^+_W$, we need to extend the axiom sets AXM and AXM^+ respectively, defined on $\Pi(X)$, with the usual additional axiom:

- $A7F$: for all $\pi \in \Pi^{ex}(X), \pi \sim (1/\mathcal{N}(\pi),\ F(\mathcal{H}(\pi))/X)$.

Now, let us prove the following representation theorem.

Theorem 9.12 (Representation Theorem)
A preference relation $\sqsubseteq$ on $\Pi^{ex}(X)$ satisfies axiom set $AXM^{ex} = AXM + A7F$ (resp. $AXM^{+ex} = AXM^+ + A7F$) if, and only if, there exist

(i) a linearly ordered and finite preference scale U with $\inf(U) = 0$ and $\sup(U) = 1$,

(ii) a preference function $u{:}X \rightarrow U$ such that $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$, and

(iii) an order-preserving mapping $h{:}V \rightarrow U$, $h(0) = 0$ and $h(1) = 1$,

in such a way that it holds, for each $\pi \in \Pi^{ex}(X)$,

$$\pi' \sqsubseteq \pi \quad \textit{iff} \quad \underline{QU}^-_W(\pi'|u) \leq \underline{QU}^-_W(\pi|u),$$

($\pi' \sqsubseteq \pi$ iff $\underline{QU}^+_W(\pi'|u) \leq \underline{QU}^+_W(\pi|u)$ respectively) where, as usual, $n = n_U \circ h$.

Proof:
We only prove the theorem for the pessimistic criterion, the proof for the optimistic criterion being very similar.

$\leftarrow$) We have to prove that, given U, a preference function $u{:}X \rightarrow V$, and an order-preserving mapping $h{:}V \rightarrow U$, verifying (i),(ii) and (iii), the ordering on possibility distributions of $\Pi^{ex}(X)$ induced by the utility evaluation $\underline{QU}^-_W$ satisfies the axioms of AXM^{ex}. Since $\underline{QU}^-_W$ coincides with QU^-_W on $\Pi(X)$, all axioms from AXM are automatically satisfied by Theorem 9.8. Thus, it only remains to verify that $A7F$ also holds. According to (ii), there is $\underline{x}$ such that $u(\underline{x}) = 0$, and thus $\underline{QU}^-_W(X)$ = 0. Since QU^-_W preserves possibilistic mixtures, we have for all $\pi \in \Pi^{ex}(X)$,

$$\begin{aligned} QU^-_W(1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X) &= \min(\max(n(1), QU^-_W(\mathcal{N}(\pi))), \\ &\qquad \max(n(F(\mathcal{H}(\pi))), QU^-_W(X))) \\ &= \min(QU^-_W(\mathcal{N}(\pi)), n \circ F(\mathcal{H}(\pi))) \\ &= \underline{QU}^-_W(\pi). \end{aligned}$$

Thus, π is equivalent to $(1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X)$ w.r.t. to the ordering induced by $\underline{QU}^-_W$.

$\rightarrow$) Let us assume now that we have an ordering $(\Pi^{ex}(X), \sqsubseteq)$ satisfying the axioms of AXM^{ex}. In particular, $\sqsubseteq$ satisfies all AXM axioms, hence, applying Theorem 9.8 again, we can suppose the existence of U, $u{:}X \rightarrow U$ and $h{:}V \rightarrow U$ satisfying (i), (ii) and (iii), and such that the corresponding utility QU^-_W represents $\sqsubseteq$ on $\Pi(X)$, i.e. for all normalised π, we have that $\pi' \sqsubseteq \pi$ iff $QU^-_W(\pi'|u) \leq QU^-_W(\pi|u)$. Axiom $A7F$ guarantees that, for any π, $\pi \sim (1/\mathcal{N}(\pi),\ F(\mathcal{H}(\pi))/X)$. Since $QU^-_W(X) = 0$, and $(1/\mathcal{N}(\pi),\ F(\mathcal{H}(\pi))/X)$ is a normalised distribution, we define

$$\begin{aligned}\underline{QU^-_W}(\pi) &= QU^-_W(1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X) \\ &= \min(QU^-_W(\mathcal{N}(\pi)), n \circ F(\mathcal{H}(\pi))).\end{aligned}$$

Now, we have to verify that $\underline{QU^-_W}$ represents $\sqsubseteq$, i.e. that for each $\pi,\ \pi' \in \Pi^{ex}(X)$ the following equivalence holds

$$\pi' \sqsubseteq \pi \qquad \text{iff} \qquad \underline{QU^-_W}(\pi') \leq \underline{QU^-_W}(\pi).$$

Indeed, by axiom $A7F$, $\pi \sim (1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X)$ and $\pi' \sim (1/\mathcal{N}(\pi'), F(\mathcal{H}(\pi'))/X)$, so we have that

$$\pi' \sqsubseteq \pi \iff \pi' \sim (1/\mathcal{N}(\pi'), F(\mathcal{H}(\pi'))/X) \sqsubseteq (1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X),$$

and since QU^-_W represents $\sqsubseteq$ on normalised distributions, we have that

$$\pi' \sqsubseteq \pi \iff QU^-_W(1/\mathcal{N}(\pi'), F(\mathcal{H}(\pi'))/X) \leq QU^-_W(1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X).$$

As QU^-_W preserves mixtures we have that

$$\pi' \sqsubseteq \pi \iff \min(QU^-_W(\mathcal{N}(\pi')), n \circ F(\mathcal{H}(\pi'))) \leq \min(QU^-_W(\mathcal{N}(\pi)), n \circ F(\mathcal{H}(\pi))).$$

That is,

$$\pi' \sqsubseteq \pi \qquad \text{iff} \qquad \underline{QU^-_W}(\pi') \leq \underline{QU^-_W}(\pi).$$

$\square$

Remark 15
We have considered other alternatives for characterising the ordering induced by QU^-_W*, in particular these ones:*

1. *The set of axioms* $\{\ A1,\ A2,\ A3, A4L, Ax2\}$ *with*
   - $A4L$: $\forall \pi \in \Pi(X)\ \exists x_0 \in X\ \exists\ \lambda \in V$ s.t. $\pi \sim (1/\overline{x}, \lambda/x_0)$.
2. *The set* $\{A1,\ A2,\ A3, A4L, Ax2,$ A-Monotony$\}$, *with*
   - A-Monotony*: if* $\pi_1 \sqsubseteq \pi_2$ *then* $(1/\pi, \lambda/\pi_1) \sqsubseteq (1/\pi, \lambda/\pi_2)$.

However, they do not characterise it as the following examples show.

Example:
Consider the following examples:

1. Let $X = \{\underline{x} \sqsubset x \sqsubset \overline{x}\}, X_{NM} = \{x\}$, $V = \{0 < \beta < 1\}$, and consider the relation
$$\underline{x} \sqsubset x \sqsubset \pi_\beta \sqsubset \overline{x},$$
also satisfying
$$x \sim (1/\overline{x}, \beta/x) \sim (\mu/\overline{x}, 1/x) \qquad \forall \mu \in V.$$
All other distributions are taken equivalent to $\underline{x}$.
This relation does not satisfy $AxMix2$, since although $x \sqsubset \pi_\beta^- \sqsubset \overline{x}$, instead of being $(1/\overline{x}, \beta/x) \sim \pi_\beta^-$ we have $(1/\overline{x}, \beta/x) \sqsubset \pi_\beta^-$.
That means that having a relation satisfying $A1 - A3,\ A4L$ and $Ax2$ is not enough for having a relation that is representable by QU_W^-, since of course QU_W^- satisfies $AxMix$.

2. Let $X = \{\underline{x} \sqsubset x \sqsubset \overline{x}\},\ X_{NM} = \{x\}$, $V = \{0 < \beta < 1\}$, and consider the relation
$$\underline{x} \sqsubset x \sqsubset \pi_\beta \sqsubset \overline{x},$$
also satisfying
$$\overline{x} \sim (1/\overline{x}, \beta/x),$$
and
$$x \sim (\mu/\overline{x}, 1/x) \qquad \forall \mu \in V$$
All other distributions are taken equivalent to $\underline{x}$.
This relation does not satisfy $AxMix2$, since although $x \sqsubset \pi_\beta^- \sqsubset \overline{x}$, instead of being $(1/\overline{x}, \beta/x) \sim \pi_\beta^-$ we have $(1/\overline{x}, \beta/x) \sqsupset \pi_\beta^-$.
Again, this shows that having a relation satisfying $A1 - A3$, *A-Monotony*, $A4L$ and $Ax2$ is not enough for having a relation representable by QU_W^-.

$\diamondsuit$

Chapter 10

Possible Applications of the Possibilistic Decision Model

In this Chapter, we analyse two possible applications of the qualitative/ordinal models we have been working with. Indeed, we show that these models may be applied to solve problems of making decisions in the context of two of the projects in which the Institut d'Investigació en Intel.ligència Artificial (IIIA-CSIC) was involved: *Co-Habited Mixed-Reality Information Spaces project (COMRIS)* and *FishMarket*[1]. In the case of *COMRIS* we propose an approach to solve a particular decision problem in it, while in *FishMarket* we revise an approach already proposed by other IIIA researchers.

10.1 *Co-Habited Mixed-Reality Information Spaces Project*

Big conferences bring different ways for interacting: people talk about the results obtained, show demos, want to meet people with the same interests, etc; moreover, the same person may has different roles during the event like being an invited talker and looking for partners for an european project.

Usually there are a lot of available information, events and possible activities on different topics, making the organisation for optimising the participation a non-trivial work.

The *Co-Habited Mixed-Reality Information Spaces project (COMRIS)* (deVelde, 1997) propose an approach for integrating software and human agents moving in virtual and real spaces closely related (see Figure 10.1 (Plaza et al., 1998)).

COMRIS chooses for experimentation a conference center as their framework.

> "In the mixed-reality conference center real and virtual conference activities are going on in parallel. Each participant wears its personal

[1] For more details you may see http://www.iiia.csic.es/Projects/comris/ and http://www.iiia.csic.es/Projects/fishmarket/ respectively.

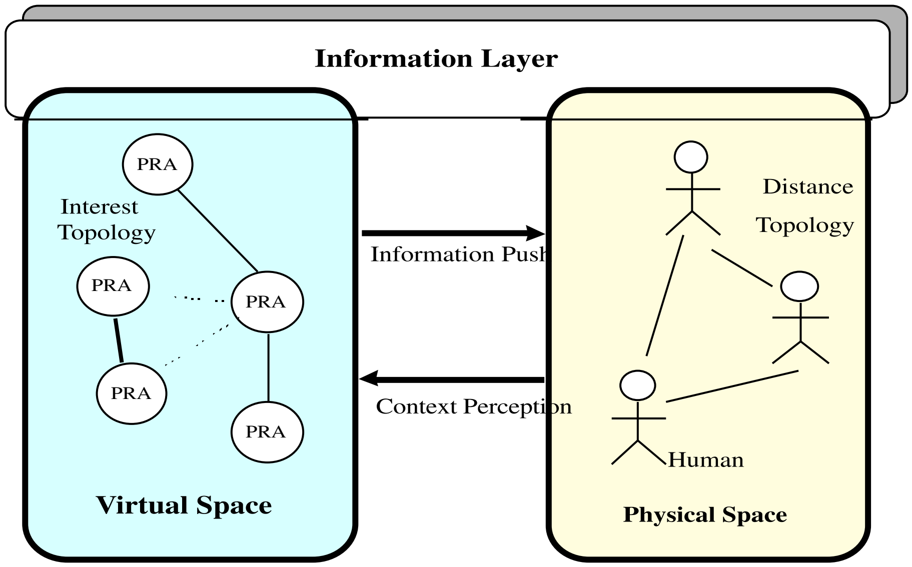

Figure 10.1: A description of the virtual interest-based space and the physical proximity-based space of *COMRIS*.

> assistant, an electronic badge and ear-phone device, wirelessly hooked into an Intranet. This personal assistant - the COMRIS parrot - realises a bidirectional link between the real and virtual spaces. It observes what is going on around its host (whereabouts, activities, other people around), and it informs its host about potentially useful encounters, ongoing demonstrations that may be worthwhile attending, and so on. This information is gathered by several personal representatives, the software agents that participate on behalf of a real person in the virtual conference. Each of these has the purpose to represent, defend and further a particular interest or objective of the real participant, including those interests that this participant is not explicitly attending to."

The *COMRIS* project studies the synergy of these two spaces, and their relationship. Its goal is to help the user in optimising the user's participation in terms of his interests while attending to the conference. With this goal they propose (Plaza et al., 1998):

> "To develop software agents inhabiting the virtual space that take up some specific activities on behalf of some interest of an attendant in the conference. Specifically, a *Personal Representative Agent (PRA)* is an agent inhabiting the virtual space that is in charge of advancing some particular interest of a conference attendant by searching for information and talking to other software agents."

Next, we analyse the application of the possibilistic decision making model in the context of the *COMRIS* Project.

10.1.1 The Framework

For each user, we have two different type of agents:

- *Personal Representative Agents* (*PRAs* for short), each one pursuing a different interest for a same user. They search information at the virtual space for some particular interests, for example, one of them may be in charge of looking for appointments with people who may know about vacancies in their laboratories while other is instructed to look for activities related with the topic *CBR*. The collection of the possible actions in which the *PRA* may participate, in order to achieve user interests, is provided by the conference organisation, for instance, meeting people, attending a demo, etc. The *PRA* chooses its "best" proposal in terms of the knowledge about user preferences and the *context information* (i.e. the physical situation and the activity of the user and of other attendants) it has. It will try so send this information to the user, but its communication with him is not direct, since a user may have several *PRAs* that would try to compete for his attention. Each *PRA* sends its information to a *Personal Assistant agent*.

- *Personal Assistant (PA)* agents coordinate the proposals presented by all the *PRAs* of the users. Each user has only one *PA* that evaluates all proposals in terms of the contextual information it has. That is, it "solves" the problem of competition, in the sense that it decides which *PRA* will be listened by the user.

Each *PRA* presents its most relevant *proposal* among one of the following:

- an *appointment* with a person (*app*),
- a *proximity alert* of a person or event of interest for the user (*pro*),
- a proposal of *receiving propaganda* (*rp*) related with events like demonstrations, future conferences, etc.,
- a *commitment reminder* of an event that will happen soon (*rem*) and to which the user has promised to be present, for example, it may remember the user that he has soon a meeting;

together with a *estimation of the relevance degree of the proposal*:

- *great importance* (*gi*),
- *moderate importance* (*mi*),
- *doubtful importance* (*di*),
- *null*.

In fact, a *PRA* not only has to provide a relevance of the proposal but an argumentation of it as well. However, this point is out of the scope of our work.
For more details of the project you may consult the URL http://www.iiia.csic.es/Projects/comris/ or (Plaza et al., 1998).

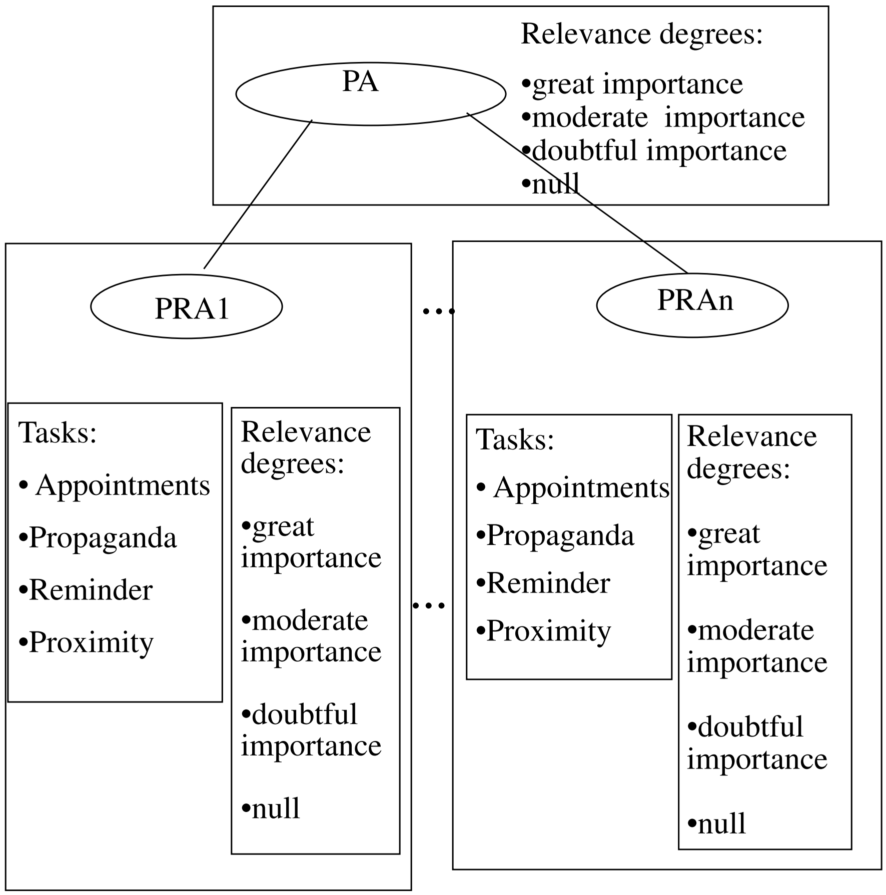

Figure 10.2: Comris Framework

10.1.2 Our Proposal

As it is mentioned, the *PA*'s goal is to choose, in the current context, one of the received proposals to send it to the user, but previously the *PA* has to assign *its own evaluation of relevance* to the proposal. On the other hand, the goal of each *PRA* is to make a proposal to the *PA* based in the result/proposal of each task (the set of available tasks

being {appointment, proximity, propaganda, reminder}), taking into account the local context[2] information available it has. An assignment of the proposal relevance has to be made as well.
In this framework, the available information is of qualitative nature rather than numerical. *Possibilistic Decision Theory* is specially suited for this framework since it can be based only on ordinal scales of uncertainty and preference. Besides, the feasibility of working with partial orders may be useful in this context, because sometimes giving a total global preference may result very difficult for the user.

Moreover, is it feasible to have available a memory of cases summarising the behaviour of the *PA* and *PRAs* in previous experienced situations. This, leads us to propose that:

- *PA* may be supported in looking for its goal by*Possibilistic Case-Based Decision Theory (PCBDT).*
- Analogously, *PCBDT* may be applied for giving support to each *PRA* for making its decisions.

Following, we focus in the behaviour of the *PA*.

PA's Decision Making Problem

We assume as available a memory of cases for helping the *PA*. Consider cases given by the following 4-tuple:

$$c_{PA} = (vs, \textit{proximity-context,winner,user-feedback}),$$

where

- $vs = ((d_1, rel_1), ..., (d_n, rel_n))$, with (d_i, rel_i) describing the proposal d_i made by the PRA_i and the importance, rel_i, that the PRA_i assigned to its proposal, n being the number of *PRAs* the user has.
- *proximity-context* is a 3-tuple $(\textit{user-loc}, \textit{user-neigh}, \textit{user-activ})$ representing the information that *PA* has about the actual context of the user. Where *user-loc* gives information about the place in which the user is (e.g. *hall, meeting point, demo-room5*, etc.), *user-neigh* is a list of the keywords in common that the user and the participants that are "near" the user have. Finally, *user-activ* provides information about the type of activity in which user is involved (e.g. *session, social event, appointment,* etc.).[3]
- *winner* is a pair *(PA-proposal, PA-eval-rel)*, where *PA-proposal* is one of the d_i received, which the *PA* preferred, while *PA-eval-rel* is the own evaluation of the relevance that PA assigns to *PA-proposal*.

[2]This context information although in some sense is more "partial" than the one managed by its *PA*, however, may result more complete in the sense that not only include context information about his owner but the one provided by *PRAs* of other persons as well.

[3]As it is said, we assume that there may exist different levels of information with respect to this topic, the *PA* having the most complete one, and each *PRA* has a partial view of it.

- Finally, *user-feedback* is a pair (z_1, z_2) reflecting the user opinion. Its first component v_1 is user's evaluation on *PA*'s proposal, while the second one v_2 is his evaluation of the relevance *PA* has assigned to it.

For applying *PCBDT*, also a *similarity* function defined on the set of pairs *(vs,proximity-context)* has to be available, as well as the *user's general preferences*. The latter is referred to his main or priority goals. For example, although he may be more interested in the keyword *Decision Theory* than in *CBR*, however, if his first goal is to obtain a fellowship, the user might prefer an appointment for a possible fellowship related to *CBR* to a invited talk about *Decision Theory*. With respect to the *similarity* on pairs *(vs,proximity-context)*, it may be given either explicitly (i.e. directly from the user) or it may be evaluated in terms of marginal similarity functions corresponding to tasks, labels of relevance, etc, and then, for instance, performing a weighted aggregation where the weights may depend on the *user general preferences.* That is,we can propose the following expression:

$$\begin{aligned} SIM\,((vs_0, cont_0), (vs_1, cont_1)) = \quad & GAGG(S_{st}(vs_0, vs_1), S_{cont}(cont_0, cont_1), \\ & w_{st}, w_{cont}) \end{aligned}$$

where *GAGG* is an aggregation operator and and w_{st} and w_{cont} are the weights related with S_{st} and S_{cont} respectively, and

$$\begin{aligned} S_{st}(vs_0, vs_1) \quad = \quad & AGG(S_{task}(d_1^0, d_1^1), \ldots, S_{task}(d_n^0, d_n^1), \\ & S_{rel}(rel_1^0, rel_1^1), \ldots, S_{rel}(rel_1^0, rel_1^1), w_{task}, w_{rel}) \end{aligned}$$

with $vs_k = ((d_1^k, rel_1^k), ..., (d_n^k, rel_n^k))$, and S_{task}, S_{rel} and S_{cont} are the marginal similarity functions defined on task proposals, labels of relevance and proximity contexts respectively and w_{task} and w_{rel} are the weights related with S_{task} and S_{rel} respectively, and AGG is an aggregation operator.

Example:
As a matter of example, we consider a simplified perspective of the problems involved in this project. For instance, we may assume *user-feedback* is measured on $U = E \times E$, with $E = \{0 < \lambda < \mu < 1\}$, and n_E being the reversing involution on E. The set of labels for *user-activ* is $\{private, social,$ ***public-active***$,$ ***public-passive***$\}$, while for *user-loc* is $\{$*working-room, social-room, private-room*$\}$.

The similarity function S_{task} on tasks defined over E, is described in Table 10.1, while the similarity on labels of relevance, S_{rel}, is provided in Table 10.2.

Now, we consider the similarity function on contexts defined as:

$$\begin{aligned} S_{cont}(cont_0, cont_1) \quad = \quad & \min(\hat{S}_{cont}((\textit{user-loc}_0, \textit{user-act}_0), \\ & (\textit{user-loc}_1, \textit{user-act}_1)), S_E(u_{kw}(L_0), u_{kw}(L_1))), \end{aligned}$$

where $\hat{S}_{cont}$ is the similarity function on pairs $(\textit{user-loc}, \textit{user-act})$, while S_E is the similarity on E, provided in Table 10.3, and $u_{kw}(L)$ summarises the user preference

S_{task}	app	pro	rem	rp
app	1	μ	λ	0
pro	μ	1	λ	0
rem	λ	λ	1	0
rp	0	0	0	1

Table 10.1: Similarity between tasks.

S_{rel}	gi	mi	di	$null$
gi	1	μ	λ	0
mi	μ	1	λ	0
di	λ	λ	1	0
$null$	0	0	0	1

Table 10.2: Similarity between relevance labels.

S_E	0	λ	μ	1
0	1	μ	λ	0
λ	μ	1	μ	λ
μ	λ	μ	1	μ
1	0	λ	μ	1

Table 10.3: Similarity on E.

with respect to the keywords involved in the list L (list of keywords of interest for the user's neighbours).

Now, we assume that memory of cases provides us directly with $u_{kw}(L)$ instead of L.

The aggregation operator can be defined, for example, as

$$GAGG(x, y; w_1, w_2) = (n_E(w_1) \vee x) \wedge (n_E(w_2) \vee y).$$

and

$$\begin{aligned} AGG(\overline{x}, \overline{y}; w_1, w_2) &= (n_E(w_1) \vee \left(\bigwedge_{i=1\ldots n} x_i \right)) \wedge \\ &\quad (n_E(w_2) \vee \left(\bigwedge_{i=1\ldots n} y_i \right)). \end{aligned}$$

Consider the current situation-context described as:

$$(vs_0, cont_0) = (((app1, mi), (rem2, mi), (rem3, di)), (work-room, \mu, social)),$$

and suppose there are 3 *PRA*s. Hence, the similarity on states is:

$$S_{st}(vs_0, vs_i) = (n_E(w_{task}) \vee \bigwedge_{j=1,\ldots,3} S_{task}(d_j^0, d_j^i)) \wedge$$

$$(n_E(w_{rel}) \vee \bigwedge_{j=1,\ldots,3} S_{rel}(rel_j^0, rel_j^i)).$$

The subset of cases of the memory M related with the current situation, that is, cases in which PA has proposed an $app1, rem2$ or $rem3$ with some relevance level, is described in Table 10.4.

	vs	$prox-cont$	$winner$	$us-feed$
c_1	$((app1, gi), (pro2, mi), (rem3, gi))$	$(soc-room, 1, publ-pass)$	$(rem3, gi)$	$(1, 1)$
c_2	$((rp1, mi), (rem2, gi), (pro3, di))$	$(work-room, \mu, publ-pass)$	$(rem2, mi)$	$(1, \mu)$
c_3	$((app1, di), (rem2, mi), (rem3, mi))$	$(soc-room, \lambda, social)$	$(rem2, mi)$	$(1, \mu)$
c_4	$((app1, mi), (pro2, mi), (rem3, di))$	$(soc-room, \mu, social)$	$(app1, di)$	$(1, \lambda)$
c_5	$((app1, mi), (rem2, di), (rp3, di))$	$(work-room, \mu, social)$	$(app1, gi)$	$(1, \mu)$
c_6	$((app1, di), (rem2, mi), (rem3, di))$	$(work-room, \mu, social)$	$(app1, gi)$	$(1, \mu)$
c_7	$((rem1, di), (pro2, mi), (rem3, di))$	$(work-room, \mu, social)$	$(rem3, gi)$	(λ, λ)
c_8	$((pro1, mi), (app2, mi), (rem3, di))$	$(private-room, \mu, social)$	$(rem3, gi)$	(μ, λ)
c_9	$((app1, gi), (app2, gi), (rem3, di))$	$(work-room, \mu, social)$	$(rem3, gi)$	$(0, 0)$

Table 10.4: The memory of cases M.

Hence, for each *PA*'s available decision d^4, we define the associated distribution as usual, i.e.

$$\pi_{d,(vs_0,cont_0)}(x) = \bigvee \{SIM((vs_0, cont_0), (vs, cont)) |\, ((vs, cont), d, x) \in M\}.$$

Notice that for defining these distributions it is necessary to know the similarity $\hat{S}_{cont}$ on pairs (*user-loc*, *user-act*), at least for some particular pairs. Table 10.5 provide these similarity values.

$\hat{S}_{cont}$	$(work-room, social)$
$(work-room, pub-pass)$	λ
$(soc-room, social)$	μ
$(work-room, social)$	1
$(private-room, social)$	μ
$(work-room, pub-pass)$	λ

Table 10.5: Some values of the similarity $\hat{S}_{cont}$.

Now, we consider some of the associated distributions:

- for *d=(app1,gi)*,

$$\pi_{d,(vs_0,cont_0)}(x) = \begin{cases} SIM((vs_0, cont_0), (vs_5, cont_5)) \vee \\ SIM((vs_0, cont_0), (vs_6, cont_6)), & \text{if } x = (1, \mu) \\ \\ 0, & \text{otherwise}, \end{cases}$$

[4] Recall that since PA has to choose between the received proposal, the possible decisions are $(app1, rel), (rem2, rel)$ and $(rem3, rel)$, where rel is the degree of relevance that PA assigns to the proposal.

- for *d=(app1,di)*,

$$\pi_{d,(vs_0,cont_0)}(x) = \begin{cases} SIM((vs_0, cont_0),(vs_4, cont_4)), & \text{if } x = (1,\lambda) \\ 0, & \text{otherwise}, \end{cases}$$

- if *d=(rem3,gi)*,

$$\pi_{d,(vs_0,cont_0)}(x) = \begin{cases} SIM((vs_0, cont_0),(vs_1, cont_1)), & \text{if } x = (1,1) \\ SIM((vs_0, cont_0),(vs_7, cont_7)), & \text{if } x = (\lambda,\lambda) \\ SIM((vs_0, cont_0),(vs_8, cont_8)), & \text{if } x = (\mu,\lambda) \\ SIM((vs_0, cont_0),(vs_9, cont_9)), & \text{if } x = (0,0) \\ 0, & \text{otherwise} \end{cases}$$

- for *d=(rem2,mi)*,

$$\pi_{d,(vs_0,cont_0)}(x) = \begin{cases} SIM((vs_0, cont_0),(vs_2, cont_2))\vee & \\ SIM((vs_0, cont_0),(vs_3, cont_3)), & \text{if } x = (1,\mu) \\ & \\ 0, & \text{otherwise}. \end{cases}$$

Hence, once we are provided with, or have choosen, the values of the weights $w_{task}, w_{rel}, w_{cont}$ and w_{st}, we are ready to rank the distributions.

As several of these distributions may be non-normalised, we apply $\underline{\overline{GQU}}^{+}_{F}$ and $\underline{\overline{GQU}}^{-}_{F}$,[5] where we consider $F = n_V$. In U we may consider different orderings like Pareto, minimum, lexicographic, etc.. So, we would consider for each d the values

$$\begin{aligned} U^{-}_{F,\ (vs_0,cont_0)}(d) &= \underline{\overline{GQU}}^{-}_{F}(\pi_{d,(vs_0,cont_0)}) \\ &= n \circ n_V(\mathcal{H}(\pi_{d,\ (vs_0,cont_0)})) \wedge \overline{GQU}^{-}(\mathcal{N}(\pi_{d,(vs_0,cont_0)})), \end{aligned}$$

and

$$U^{+}_{F,(vs_0,cont_0)}(d) = \overline{GQU}^{+}(\mathcal{N}(\pi_{d,(vs_0,cont_0)})) \vee (h \circ n_V)(\mathcal{H}(\pi_{d,(vs_0,cont_0)})),$$

where these values are obtained taking into account the ordering chosen in U. For example, the distributions associated to *PA*'s proposals not made before like *(rem3, di),(rem3, null),(rem3, di), (rem2, di),(rem2,null),(rem2,gi), (app1,mi)* or *(app1,null)*, are null. Hence, their utilities are 0_U and 1_U w.r.t. pessimistic and optimistic criteria respectively.

$\diamondsuit$

[5]In fact, we have not provided in Chapter 8 the extension for non-normalised distributions for the utility functions introduced in Chapter 6, but it may be done analogously.

***PRA*'s Decision Making Problem**

Now, we focus on the behaviour of each *PRA*, which is the main interest of the IIIA *COMRIS* team. *PRA* has to make a proposal to the *PA* based in the results/proposal of each task, taking into account the available local context information it has. The relevance of its proposal has to be assigned as well.

As in the case of PA, we think *PCBDT* may provide support for this problem if we assume we have a *memory of cases* storing the performance of proposals made in the past by the *PRA*, and the ones made by others *PRAs*, together with the final *PA* proposal.

Indeed, a *PRA*-case may be represented as the 4-tuple:

C_{PRA}= *(vs, partial-context, PRA-task-prop, PA-answer)*

with:

- *vs* is defined as previously, i.e. $vs = ((d_1, rel_1), ..., (d_n, rel_n))$.
- *partial-context* is a variable describing the actual context *taking into account the information that the* PRA *has*.
- *PRA-task-prop* is a 4-tuple descriptor, *(app-result, proximity-result, propaganda-result, reminder-result)*, each component representing the "best" task-proposal. Observe that the winner task, i.e. the task that PRA proposed, is included (with its degree of relevance) in *vs*. Indeed, if we are working with the PRA_j, the winner task is d_j.
- *PA-answer* is a pair *(win?,PA-relevance)* representing the feedback that *PA* may provide its *PRA*, *win?* tells wether this *PRA* was or not the winner, and *PA-relevance* is the relevance assigned by *PA* to the proposal (this wants to reflect that for example the relevance function of the *PRA* may be modified for next time taking into account the *PA's* answer, since *PA* has more information).

itemize

The possibility distributions associated to each decision are defined as usual, then, they are ranked applying the generalised utility functions for non-normalised distributions as usual.

Finally, let us introduce, some comments on PRA's Tasks. So far, we have assumed that each *PRA* has the results of each task, now we are interested in analysing a bit more this point, that is, having a local context information, some knowledge about user preferences with respect to the activity he/she is interested, which may be the best proposal for a task. As an example, we consider the appointment task. Its goal is to look for the more interesting appointment in terms of the available information it has about the preferences of the user and the other participants of the conference.

The available information in this moment specifies the actual situation as

$$s = \{s_i | i \in I\},$$

with I a finite set, and

$$s_i = (reg, kw, TA, g, \textit{partial-context}_{app}),$$

with:

Task	$Characterisation\ of\ its\ result$
Appointment	$(reg, kw, TA, g, partial - context_{app})$
Reminder	$(deadline,$ *distance-from*$, TA, kw, partial - context_{rem})$
Proximity	$(reg\ or\ event, kw, partial - context_{pro})$
Propaganda	$(kw,$ *way-of*$, TA, g)$

Table 10.6: Results of the Different Tasks

- *reg*: is the identifier of the person, for example, the registration number each participant has.
- *kw*: is a (or a set of) keyword(s) in which the user is interested.
- *TA*: stands for a type of activity, (for example grants, future projects, etc.). This wants to represent that although the user may be interested in an appointment related with a certain *kw*, it is not the same interest for example for a person who gave an invited talk related with this topic or for a person who is selling books of this issue.
- g stands for the group to which the person belongs (we may have a classification taking into account for example the organisation of the person pertains).
- $partial - context_{app}$, as usual, it summarises the information of context related with this task, in this case, the appointment one.

As it is mentioned, the goal of the appointment task is to choose the best ranked s_j. The ranking has to take into account user's preferences with respect to *kw* and *TA*, i.e. $u = f(kw, TA)$. However, other facts have to be taken into account, for example, it may be the case that the preferences are also expressed in terms of g.

Another point to consider is the number of persons related with *kw* and *TA* that are available as well as whether they are near the user (which may be known by the $partial - context_{app}$), and of course the *user-activ* has to be taken into account, mainly if the activity proposed is a forthcoming event.

As a conclusion, we may say that this is a first analysis and several points need to be considered with more detail. However, it already allows us to propose some answers to the decision making problems involved in the project. Of course, we are interested in following this work to improve our proposal and to face some issues not yet worked.

10.2 *FishMarket*: A Possibilistic Based Strategy for Bidding

Electronic commerce is currently an increasing area of interest, there are many research works related with this matter in the broad sense of it. In particular, there is a considerable number of electronic auction houses (as you may see in the URL http://fullcoverage.yahoo.com/Full_Coverage/Business/Online_Auctions/, for instance, http://www.auctionline.com or http://www.onsale.com, etc.). Taking into account the actual development of internet, and in particular of electronic commerce, we think that this is an interesting topic.

In auction houses, different bidding protocols may be applied, for example the *Downward Bidding Protocol* (*DBP* also known as *Dutch Bidding Protocol*) or the *English Bidding Protocol*.

The *FishMarket* project is mainly concerned with communicational aspects of multi-agent systems (see http://www.iiia.csic.es/ Projects/fishmarket/ for more details). To test these ideas, Rodríguez-Aguilar et al. (1998) propose a multi-agent test-bed, *FM96.5*[6], which is an electronic auction house that allows the definition and evaluation of some experimental trading scenarios, in particular the *FishMarket* one with a *Dutch Bidding Protocol.* In this context, a very interesting issue is to model buyer's strategies to bid. The goal is to model a buyer's strategy to make a bid, trying to maximise the tournament evaluation function, taking into account that the strategies of other buyers is unknown. To bid in a such environment means to decide a price to offer taking into account all the available information like goods that will be auctioned and their expected resale prices, other buyers in the buyers' room as well, etc. This information has to be handled with some restrictions, the behaviour of other buyers may be approximated but not precisely predicted, deliberations are time-bounded, etc. That is, the buyer has to bid in an uncertain environment, i.e. he has to face a decision problem under uncertainty. Garcia et al. (1998b) made a first proposal in this line applying the possibilistic qualitative decision model.

Although in this moment the problem is only attacked in terms of tournaments rather than in actual market situations, the analysis is interesting. It is a problem with a lot of information and so with many possible sources of uncertainty as well.

Of course, there are many possible approaches for modelling the strategy of buyer's bidding, moreover, inside the model there are many alternatives available. The knowledge the agent has about the other agents' strategies is usually incomplete, if we assume that the knowledge the agent has is reduced to a memory of previous market situations and their results, and to general information about the market, *Possibilistic Case-Based Decision Theory* may be useful.

In the following, we describe the *FishMarket* environment and the restrictions in which the problem of bidding will be attacked. In Section 10.2.2, we introduce Garcia et al. (1998b,1998,1998a)'s proposal. In a first analysis of their proposal, we realise that the implementation of the model has some drawbacks. In Section 10.2.3, we make some remarks about them, like for instance that there are some specification problems with the referential sets, and that they do *not* take into account that the possibility distributions involved are probably non-normalised. This latter point may have unsatisfactory results as it has been mentioned before in this dissertation. In order to solve the issue of possible non-normalised distributions, we propose to use the generalised utility functions we have described in Chapter 8. Finally, we also include some remarks about some points that, although are non-directly related with our framework, may result interesting to develop in the future from the application point of view.

[6]Currently, it is available a new version FM100, which may be download at http://www.iiia.csic.es/Projects/fishmarket/agents2000/FM100/index.html.

10.2.1 Background: The *FishMarket* Environment

The definition of a tournament involves a set of descriptor parameters, for example, the time between prices, decrement or increment in the price, goods that will be auctioned, etc..

In order to characterise the elements of *FishMarket* as a tournament scenario, Garcia et al. (1998b) first introduce the notion of *Tournament Descriptor*. A Tournament Descriptor is described as the 6-tuple

$$\mathcal{T} = \langle \Delta_{price}, \mathcal{B}, \mathcal{S}, \overline{Cr}, \mu, E \rangle,$$

Δ_{price} being the decrement of price between two consecutive quotations; $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a finite set of identifiers of all[7]the participating buyers, and $\mathcal{S}$ for the participating sellers; $\overline{Cr}$ is a vector which components are the initial endowment of each buyer at the beginning of each auction; $\mu \in \mathcal{M}$ is the tournament mode where $\mathcal{M} = \{$random, automatic, one auction, fish market, $\ldots\}$ is the set of possible tournament modes. Finally, E is the buyers' evaluation function.

The *FishMarket* uses a specific *Downward-Bidding Protocol (DBP)*, which is implemented in *FM96.5*, as follows:

Step 1 The auctioneer chooses a good out of a lot of goods that is sorted according to the order in which sellers deliver their goods to the sellers' admitter.

Step 2 With a chosen good g, the auctioneer opens[8] a bidding round by quoting offers downward from the good's starting price, previously fixed by the sellers' admitter, as long as these price quotations are above a reserve price previously set by the seller.

Step 3 For each price called by the auctioneer, several situations might arise during the open round in an interval of time previously fixed:

- *Multiple bids*: Several buyers submit their bids at the current price. In this case, a collision comes about, the good is not sold to any buyer, and the auctioneer restarts the round at a higher price. Nevertheless, the auctioneer tracks whether a given number of successive collisions is reached (*Cmax*), in order to avoid an infinite collision loop. This loop is broken by randomly selecting one buyer out of the set of colliding bidders.[9]

- *One bid*: Only one buyer submits a bid at the current price. The good is sold to this buyer whenever his credit can support his bid. Whenever there is an unsupported bid the round is restarted by the auctioneer at a higher price, the unsuccessful bidder is punished with a fine, and he is expelled out of the auction room unless such fine is paid off.

[7]In fact, they forget to include in this set b_0 the buyer agent which is being modelled.

[8]We assume that a condition that is checked by the auctioneer is whether there is any buyer with credit higher than the reserve price.

[9]Other option for assigning the good to a buyer may be considered.

- *No bids*: No buyer submits a bid at the current price. If the reserve price has not been reached yet, the auctioneer quotes a new price which is obtained by decreasing the current price according to the price step. If the reserve price is reached, the auctioneer declares the good withdrawn (i.e. the good is returned to its owner) and closes the round.

Step 4 The first three steps repeat until there are no more goods left.

For describing the *FishMarket* environment these additional parameters are involved:

Ps Since a *Dutch Bidding Protocol* is assumed, the price is decreasing . Ps represents the decrement of price between two consecutive offers shouted out by the auctioneer.

to is the delay between consecutive offers.

tr Delay between the end of a round and the beginning of the next round.

$Cmax$ Maximum number of successive collisions. The auctioneer randomly chooses one buyer out of the set of bidders when the maximum number of successive collisions is reached.

Sf This coefficient, *Sanction factor*, is utilised by the buyers' manager to calculate the amount of the sanction to be imposed on buyers submitting unsupported bids.

Pi Price increment determines how the new offer is calculated by the auctioneer from the current offer when either a collision, a fine or an expulsion occurs.

$\overline{Cr}$ As it is said, it is a vector which establishes the available credit of each buyer. At the beginning of each auction of the tournament all them are provided with the same credit.

For example, for the "Agent Mediated Electronic Commerce III Trading Agents' Tournament", they are initialised (for more details http://www.iiia.csic.es/Projects/fishmarket/agents2000/tourdesc.html) as it is shown in Table 10.7.

$Parameter$	$InitialValue$
Ps	$50EUR$
to	$500ms$
tr	$4000ms$
$Cmax$	3
Sf	25%
Pi	25%

Table 10.7: Initialisation of the Parameters.

While $\overline{Cr}$, that is, the buyers' credits initial value, is assigned in terms on the number of participants, usually they assign each buyer an initial credit on EUR that results of dividing 70,000 by the total number of buyers.

Available Information for Buyers

All the buyers that are in the auction room are provided with general information of the goods that will be auctioned before the tournament begin. They are informed of the types of goods (i.e. cod, prawns, etc.) that will participate in the auction as well as the number of boxes of each type of good, and the upper and lower bounds for the starting and resales prices. Indeed, up to this moment all these numbers are generated by uniform distributions on different intervals. At the beginning of the tournament, buyers are only informed on these intervals, not on the values on which the distributions results (see Table 10.8). But in the beginning of each round, a more precisely information is

$good$	$number\ of\ boxes$	$starting\ price$	$resale\ price$
cod	$U[1..15]$	$U[1200..2000]$	$U[1500..3000]$
$tunafish$	$U[1..15]$	$U[800..1500]$	$U[1200..2500]$
$prawns$	$U[1..15]$	$U[4000..5000]$	$U[4500..9000]$
$halibut$	$U[1..15]$	$U[1000..2000]$	$U[1500..3500]$
$haddock$	$U[1..15]$	$U[2000..3000]$	$U[2200..4000]$

Table 10.8: Previous information available

given. That is, the number of boxes of each good is precisely known as well as the starting price and the resale one.

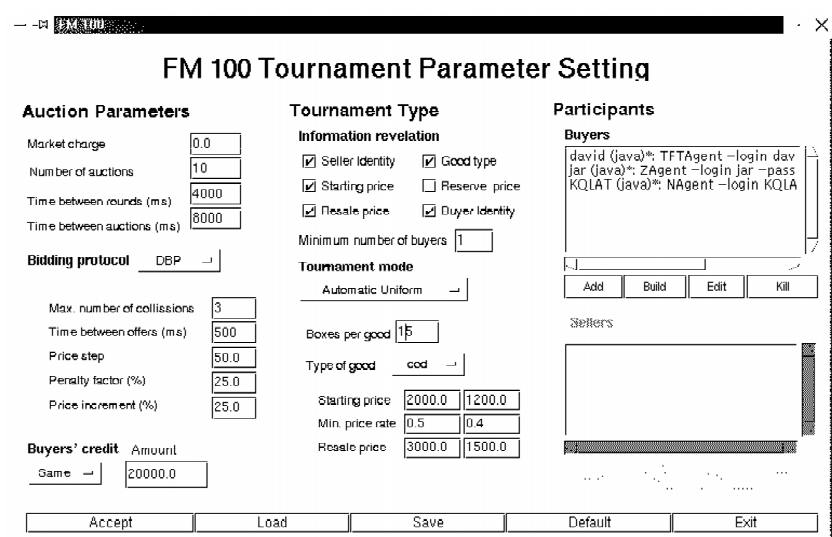

Figure 10.3: The Parameter Setting that buyers see.

Determining the evaluation of Buyers

There are many different possible functions for evaluating the behaviour of the buyer agents. The one proposed in http://www.iiia.csic.es/ Projects/fishmarket/ is

$$E(b) = \sum_{k=1}^{z} \ln(k+1) B_k(b) \tag{10.1}$$

b being a buyer, $B_k(b)$ stands for the accumulated benefit [10] of buyer b during auction k, and z is the number of auctions of the tournament.

They argue that this evaluation tends to favour buyers learning in order to improve their strategy.

10.2.2 Previous Proposal: Building a Possibilistic-Based Strategy for *FishMarket*

We are in a decision problem, where our buyer agent has to take a decision, i.e. to choose a bid among a set of available alternatives taking into account its preferences on the set of possible consequences in terms of maximising its utility. The winner is determined as the the buyer maximising (10.1). The buyer has to take into account not only its benefits but other buyers' benefits as well. The agent has to choose a bid for each round of each auction of the tournament.

Garcia et al. (1998b) affirm that:

> " Due to the nature of the domain faced by the agent, we must demand that such bidding strategy balances the agent's short-term benefits with its long-term benefits in order to succeed in long-run tournaments."

They structure their proposal in three steps:

- They apply interpolation to obtain a first subset of possible bids.
- Fuzzy Rules are applied for improving the global behaviour.
- Possibilistic Case-based Decision Model is applied on this subset of bids to came up with a single bid.

First of all, let us introduce the definitions of the problem they suggest.

The Decision Problem

For each round the agent has to choose a bid between the allowed ones. A memory of cases M summarising the behaviour of market in previous situations of (past and the current) tournaments is assumed, hence the idea is to apply Possibilistic Case-Based Decision Theory to choose a bid. The first requirement is, obviously, the identification of the variables involved in the problem. Garcia et al. (1998a) propose to consider

[10] The benefit is the difference between the resale price and the paid price.

the following ones. The modelled buyer agent will be denoted by b_0, while the market situation at round r, of the auction a will be specified as:

$$s = (r, a, \tau, g, p_\alpha, p_{rsl}, \overline{Cr}, \overline{E}, R),$$

with τ being the type of the good g to be auctioned, p_α is its starting price, p_{rsl} is its resale price. As it is mentioned, $\overline{Cr}$ is the vector of buyers' credits and $\overline{E}$ is the vector of scores (E_i is the score of buyer b_i in terms of the evaluation function E). Finally, R is the number of remaining rounds to end auction a.

The set of possible decisions D for a round r, that is, the set of bids that the agent b_0 may do in a market situation s_0, is initially defined by them as:

$$\mathcal{D} = \{bid(p) \mid p = p_\alpha - m.\Delta_{price}, m \in \mathbb{N}, p_{rsv} \leq p \leq \overline{Cr}(b_0)\}, \tag{10.2}$$

where $bid(p)$ means that the agent submits a bid at price p, Δ_{price} being the decrement in the price (also denoted by Ps) and p_{rsv} the reserve price. At each round, if the reserve price is not reached, one of the possible buyers acquires the good. For each round, the set of possible consequences is defined as the set

$$X = \{win(b_i, p) \mid i = 0, \ldots, n\, ; p \in [p_{rsv} + \Delta_{price}, p_\alpha]\}, \tag{10.3}$$

where $x = win(b_i, p)$ means that buyer b_i wins the round by submitting a bid at price p. As it is mentioned, a memory of cases M summarising the behaviour of market is assumed. They consider the following cases:

$$c = (s, b, p_s)$$

with s the market situation previously defined, b the buyer who bought the good at a price p_s.

Let us summarise the different stages they proposed:

- *Interpolation*: To apply directly the possibilistic case-based model to this set D might be too slow for this type of problem, hence the idea is to reduce the set of potential bids according to the general trend of the market. This is the goal of the interpolation stage. They assume a principle establishing:

  > "*Similar market situations* usually lead to *similar sale prices* of the good".

  The idea is to take advantage of the interpolation mechanism implicit in the fuzzy case-based reasoning model proposed in (Dubois et al., 1997b). That is, for each case $(s, p) \in M$[11] gradual fuzzy rule (you may see Dubois and Prade (1996c) for the semantics of fuzzy gradual rules)

  > " If Σ is $\tilde{s}$ then Υ is $\tilde{p}$ ",

[11] They omit the reference to the buyer arguing they are only interested in the situation and in the sale price.

where $\tilde{s}$ is the fuzzy set of situations similar to s, and $\tilde{p}$ is the fuzzy set of prices similar to p; Σ and Υ are variables defined on situations and prices respectively. This leads them to define the following fuzzy set of possible bids:

$$pbid(p') = I(\tilde{s}(s_0), \tilde{p}(p')),$$

with I a residuated implication. As a memory of cases M is assumed as given, and similarity functions $\mathcal{T}$ on prices and situations $\mathcal{S}$ are assumed as well, they consider:

$$pbid(p') = \min_{(s,p)\in M} I(\mathcal{S}(s, s_0), \mathcal{T}(p, p')).$$

Finally, they propose to restrict the set of bids to $\hat{B}_\alpha$, the α-cut of $pbid$ ($\alpha > 0$), i.e.

$$\hat{B}_\alpha = \{p' \mid pbid(p') \geq \alpha\}.$$

- *Fuzzy Rules:* Garcia et al. (1998a) argue that for modelling the rational behaviour of buyers in particular situations which may not be sufficiently described by the cases in the memory M, they consider the following set of fuzzy rules:

  if [$\mathcal{C}(b_i)$ is *high*] and [*R* is *very_short*] and [$E(b_i)$ is *low*]
  then ΔBid_{b_i} is *very_positive*,

  if [$\mathcal{C}(b_i)$ is *medium*] and [*R* is *very_short*] and [$E(b_i)$ is *low*]
  then ΔBid_{b_i} is *slightly_positive*

- *Possibilistic Case-Based Decision Theory*: As it was mentioned, in *PCBDT* instead of ranking decisions, possibility distributions on consequences are ranked. Hence, it is necessary to obtain the possibility distributions associated to each decision, in this case, to each bid that the buyer b_0 may make, for the current market situation s_0. Garcia et al. (1998a) define first the distributions in terms of the similarities on situations and prices. Indeed, they assume the principle:

  > "the *more similar* is (s_0, p_0) to (s, p), the *more possible* b_i will be the winner in s_0 (paying a price p)"

  Hence, for each consequence $win(b_i, p_0)$ they consider that for each $(s, b_i, p) \in M$, they have that

  $$\pi_{s_0}(win(b_i, p_0)) \geq \tilde{s}(s_0) \otimes \tilde{p}(p_0)$$

  with $\tilde{s}$ the fuzzy set of situations similar to s and $\tilde{p}$ the fuzzy set of prices similar to p[12] and $\otimes$ is a t-norm on $[0, 1]$. Hence, they propose for each $b_i \neq b_0$ and for

[12] Both sets are defined in terms of similarity functions from situations and prices respectively over [0,1].

all $win(b_i, p_0) \in X$:

$$\pi_{s_0}(win(b_i, p_0)) = \max_{\{(s,b_i,p)\in M | p \leq p_0\}} \tilde{s}(s_0) \otimes \tilde{p}(p_0).$$

From these distributions, for each participating buyer $b_i \neq b_0$, they propose an initial fuzzy set $Bid^0_{b_i}$ of the possible winner bids

$$Bid^0_{b_i}(p) = \pi_{s_0}(win(b_i, p))$$

with p such that $win(b_i, p) \in X$.

Following, they modify these sets by the fuzzy rules previously mentioned, that is,

$$Bid^{\omega}_{b_i} = Bid^0_{b_i} \oplus \Delta Bid_{b_i},$$

where $\oplus$ denotes fuzzy addition, i.e.

$$Bid^{\omega}_{b_i}(p) = \max\{\min\{Bid^0_{b_i}(p_1), \Delta Bid_{b_i}(p_2)\} \mid p = p_1 + p_2\},$$

and ΔBid_{b_i} is the fuzzy set representing the expected variation on the observed bidding strategy of other buyers. Now, they define the possibility distribution associated to each bid p_d as:

– each $b_i \neq b_0$

$$\pi_{s_0,p_d}(win(b_i, p)) = \begin{cases} Bid^{\omega}_{b_i}(p), & \text{if } p_\alpha \geq p \geq p_d \\ 0, & \text{otherwise} \end{cases}$$

– for b_0, they retrieve those cases such that the sale price was not greater than p_d, i.e. a subset of the memory $M_{p_d} = \{(s, b_i, p) \in M \mid p < p_d, b_i \neq b_0\}$. Then,

$$\pi_{s_0,p_d}(win(b_0, p)) = \begin{cases} \max_{(s,b_i,p')\in M_{p_d}} Bid^{\omega}_{b_i}(p'), & \text{if } p = p_d \\ 0, & \text{otherwise} \end{cases}$$

Finally, they rank decisions applying $QU^-(|u)$ and $QU^+(|u)$, u being the preference functions on consequences $x = win(b_i, p)$. Several functions u may be considered, with this goal, they introduce one arguing that it models an agent that is conservative when it is winning and becomes aggressive when it is handing back. The preference function is defined in terms of a scoring function f, and a linear scaling function r over $[0, 1]$. Where f is defined as:

$$f(b_i, s_0, p) = \begin{cases} k \cdot t, & \text{if } k \leq 0 \\ k \cdot t^{-1}, & \text{otherwise,} \end{cases}$$

with

$$k = (\max_{j \neq i} E(b_j)) - E(b_i),$$

and

$$t = (R-1)/(\max(\overline{Cr}(b_i) - p, 1) \cdot (p_{rsl} - p)).$$

They assume that $p_{rsl} - p \geq 0$, that is nobody pay more than the resale price, and no buyers make unsupported bids, i.e. $\overline{Cr}(b_i) - p \geq 0$.[13]. They mention that k " accounts for the position of buyer b_i with respect to the other buyers in the ranking of scores", and the first factor involved in t estimates the cost of winning the round, while obviously $(p_{rsl} - p)$ is the benefit of the buyer agent. Finally, they define

$$u(win(b_i, p)) = \begin{cases} r(f(b_0, s_0, p)), & \text{if } i = 0 \\ r(-f(b_i, s_0, p)), & \text{otherwise} \end{cases} \tag{10.4}$$

where r is a normalisation linear scaling function.

10.2.3 Comments on the Proposal

In a first analysis we realise about the following drawbacks of the proposal:

- D and X are not well defined, and it seems that the involved measurement sets may be not finite.

- The problem may involve non-normalised distributions and this fact is not taken into account in the proposal.

itemize Next, we give more details about these points, and we introduce some general comments on the proposal.

Some Problems Detected

- The definitions of D (10.2) and X (10.3) may result confuse. They are expressed in terms of the reserve price, however, *the buyer agents have not information about it.* Thus, both sets are not well defined.
  There is another upper bound for possible decisions that could be taken into account: the *resale price*. Since the evaluation function takes into account the benefits of the agents in terms of the difference between the paid price and the resale price p_{rsl}, the bids greater or equal than p_{rsl} must be discarded as feasible bids for our buyer.

  Obviously a buyer may submit a bid greater than his available credit, however he could not win because his bid will be discarded. This fact allows us to restrict the values of p in the set of consequences X.
  A little remark is that taking into account (10.3) X seems a non-finite set, but it is easy to see that it if we assume that $\Delta_{price} \in \mathbb{N}$, X is finite as soon as we consider:

$$X = \{win(b_i, p) | i = 0, \ldots, n\,; \Delta_{price} \leq p = p_\alpha - m.\Delta_{price} \leq \overline{Cr}(b_i), m \in \mathbb{N} \cup \{0\}\}.$$

[13]However, it seems that these hypotheses may be too strong, since in some tournaments it is the case that some buyers do not satisfy these conditions.

while for the initial decision set D we propose:

$$D = \{bid(p) \mid p < p_{rsl}, \Delta_{price} \leq p = p_\alpha - m.\Delta_{price} \leq \overline{Cr}(b_0), m \in \mathbb{N} \cup \{0\}\}.$$

- The proposed preference function u is not well defined since in the case that it only remains one round to finish an auction, that is, when $R = 1$, then, $t = 0$. Hence, if b_i is a buyer that is not winning in this moment, i.e. $(\max_{j\neq i} \overline{E}(b_j)) - \overline{E}(b_i) > 0,\ f(b_i, s_0, p)$ is not well defined for each p.
  We wonder how this function works when the auction begins, in particular which values takes during the rounds of the the first auction (which value takes k?)?
  It is not clear for us the meaning of r in (10.4), since it seems it is not only a linear function to scaling f but it may exchange the order in the ranking.
  We think that the function should consider that the case of a buyer (in particular, if it is currently in a better position in the evaluation ranking w.r.t. our agent) paying a price greater than the resale one, i.e. b_j s.t. $win(b_j, p)$ with $p > p_{rsl}$. This is a case that benefits for our agent since that agent has loss if he pays this amount.
  We consider that this preference function u has to be analysed with more detail, [14] but it may be interesting to take into account other facts as well.

- In *PQDT* we may face in with non-normalised distributions. This point has not been taken into account in Garcia et al.'s proposal. Indeed, the possibility distribution π_{s_0} may be non-normalised, then, the distributions π_{s_0,p_d} may be non-normalised as well.

  In this dissertation we have analysed the drawback of applying the QU utility functions to non-normalised distributions, to avoid it, we propose to apply the generalised utilities for non-normalised distributions introduced in Chapter 8.

Some General Comments

- In the proposal, some fuzzy rules are suggested to improve the heuristic in order to reduce the number of decisions to be evaluated. They argue that they attempt to model the rational behaviour of *buyers* in particular situations.
  We are not convinced about applying rules to model the behaviour of the other agents, however, we agree in the convenience of applying fuzzy rules, but we are thinking in rules "directly" related with the behaviour of the buyer agent b_0. As an example, we may consider rules like:

  - if [$pot-benefit$ is $high$] and [R is $short$], then [p is $nearly-to-min-\{p_\alpha, \overline{Cr}(b_0)\}$].
  - if [$R = 1$] $p = \overline{Cr}(b_0)$.

[14]In particular, if we adequate it to a finite set, and U and V as well, we will be able of characterising the behaviour of the agent we are modelling as well.

that may result useful. Another option for proposing rules is to take into account the available credit that the other buyers have in this round.

- We suggest that a first analysis, before starting the auction, may be to determine which are the more potential profitable rounds to participate. It might be done in terms of a possibility distribution evaluating the potential benefits margin expressed as the expected difference between the initial sale price and the expected resale one.

- In the suggested algorithm for *DBP*, in Step 3, it is analysed the situations that may occur during the round: multiple bids, one bid, no bid.

  In the case of only one bid, if the buyer has not enough credit, the round is restarted at a higher price. May be this is the usual procedure in the actual market, but it seems this results in a disadvantage for other buyers, why at a higher price?, why not restart the round at the price in which was stopped?

- It seems that the credit of the buyers is not controlled when the round begins. Suppose that the reserve price of the good is higher than the credit of each possible buyer, why not to declare the good withdrawn?

We are interested both in deepening the analysis of their current proposal and in the necessary improvements for adapting it to actual auction houses.

Chapter 11

Conclusions and Future Work

In *Decision under Uncertainty* it is usually the case that the available information is of qualitative nature rather than numerical. *Possibilistic Qualitative Decision Theory* is specially suited for this framework since it can be based only on ordinal scales of uncertainty and preference.
In this proposal, our aim has been to develop some extensions to the initial proposal of Dubois and Prade (1995) for making decision under uncertainty in a framework analogous to vonNeumann and Morgenstern (1944) assuming that uncertainty is of possibilistic nature. The initial working hypotheses were:

- To deal with individuals' preferences.
- To assume rationality hypothesis, i.e. *DM* will try to maximise his benefit.
- To deal with one-shot decision problems.
- To assume the feasibility of representing *DM*'s preference relation on consequences by a preference function u on them. But, instead of choosing u as a real-valued-function as it is usual, we consider that it is defined over a *finite* linearly ordered set U.
- The sets of decisions, of consequences $X,$ and of situations S are finite.
- Uncertainty, assumed of being of possibilistic nature, is measured on a *finite* linearly ordered set V.
- The valuation sets for measuring uncertainty and preferences are assumed to be commensurate, that is, there exists an onto order-preserving mapping h linking them.
- A *decision or act* d on S is represented by a function $d : S \rightarrow X$ which provides the consequence of the decision in each situation. Hence, each decision is identified with a possibility distribution on consequences. Therefore, choosing decisions amounts to ranking possibility distributions on consequences.

The original proposal by Dubois and Prade deals with normalised distributions considering the max-min possibilistic mixture as its internal operation, in the sense that the qualitative utility functions they propose not only preserve the ordering but the possibilistic mixture as well.
In this context, the extensions we have proposed are:

- Besides max-min mixtures of possibility distributions, we have considered other mixtures involving t-norms $\top$ on V. We have axiomatically characterised the behaviour of the generalised qualitative utility functions that preserve these possibilistic mixtures. Namely, in the same context but requiring h to further verify a coherence condition w.r.t. $\top$, we have defined the pessimistic (optimistic) generalised qualitative utility as:

$$\forall \pi \in \Pi(X), \qquad GQU^{-}(\pi|u) = \min_{x_i \in X} \, n(\pi(x_i) \top \lambda_i),$$

  with $n(\lambda_i) = u(x_i), \; n_U$ being the reversing involution in U, and $n = n_U \circ h$. The dual optimistic evaluation is defined as

$$\forall \pi \in \Pi(X), \qquad GQU^{+}(\pi|u) = \max_{x_i \in X} \, h(\pi(x_i) \top \gamma_i),$$

  where $h(\gamma_i) = u(x_i)$.
  These utilities may result in different rankings than the ones induced by the qualitative criteria introduced by Dubois and Prade.

- We have considered partially ordered uncertainty and preference measurement sets. There are certain kinds of decision problems where we are not able to measure uncertainty and/or preferences in linearly ordered scales, but only in partially ordered ones. For example, preference on consequences may be given in terms of a vectorial function over a product of linear scales if preference is expressed in terms of a set of criteria. To deal with these types of problems, we have provided different generalised utility functions for these cases taking into account the available operations in the set of uncertainty values V. We have also been working with different (finite) lattice structures where to measure preferences and uncertainty. Again, we have supplied the respective utility functions for working in these structures and the characterisations of the preference relations that are representable by them.

- We have considered the applications of the possibilistic decision models for case-based decision problems. We have proposed to estimate to what extent a consequence x can be considered plausible, in a current situation s_0 after taking a decision d, in terms of the extent to which the current situation s_0 is similar to situations in which x was experienced after taking the decision d. This amounts to assume, for each case (s, d, x) in a memory M, a principle stating that

  > "The more similar s_0 is to s, the more plausible x is a consequence of d at s_0".

According to this principle, one can derive the possibility distribution associated to each decision. Thus, the utility of a decision can be estimated in terms of its associated distribution.
Besides, we have shown that the utility of a decision may be evaluated also taking into account the previous behaviours of other similar decisions.

- In Possibilistic Case-Based Decision Theory or in Decision Making problems involving several sources of information, we may be faced with non-normalised possibilistic distributions. We have extended the model to deal with these types of problems.

- We have also proposed an approach to weaken the commensurability hypothesis, non-requiring h to be onto. We have provided the characterisations of these resulting orderings for finite linear scales.

- Sometimes it may be not enough to rank distributions taking into account, for example, the pessimistic criterion, and it is interesting to refine it by another one, for example by optimistic one. We have analysed the characterisations of some refinements involving the generalised qualitative criteria we have proposed.

The proposed extensions provide us with possibilistic qualitative models of broader applicability. These decision models may be useful for a large range of applications in different areas, from Medicine to Economy.

Future Work

We have provided several extensions to the model, however, it is also true that there are still several extensions and improvements of Possibilistic Qualitative Decision Theory to be developed, extensions that will become interesting not only from a theoretic point of view, but also in order to provide a better decision theoretic support to many real problems as well. Let us summarise some of them:

- *Commensurability:* This hypothesis has been a point for interest of some researchers (see for example (Fargier and Perny, 1999) in à la Savage framework). In particular, the onto condition involved in the commensurability mapping forces us to restrict our work to problems in which the uncertainty set has an equal or greater cardinality than preference one. We have already proposed to weaken this hypothesis, by non-requiring the commensurability mapping h to be onto, but we have restricted to linear scales and to work with max-min mixtures. Hence, it will be interesting to extend our analysis of weakening commensurability to distributive lattices. Moreover, it will be also interesting to analyse the behaviour of other utility functions involving t-norms on V. This problem is more complicated since the onto condition is also required to guarantee the good definition of the utility functions.

- *Refinement Orderings:* This point may result specially interesting since in many applications refinements of orderings are necessary. We are interested in deepening the analyses on the characterisations of some refinements involving the generalised qualitative criteria we have proposed. A related topic is conditional preferences. Sabbadin (1998a) has worked with them in the Savage framework, and it may be interesting to see how conditional preferences can be introduced in our framework.

- *Frameworks:* There are a number of algebraic structures (e.g. interval orders, semiorders or distributive lattices without requiring their maximal elements to be equivalent) that are being applied by other researchers, in other contexts, for evaluating preferences. We want to analyse the feasibility of measuring uncertainty and/or preference in these more general structures.
  There are two frameworks that may also result interesting from the characterisations point of view. Indeed, as it has been mentioned, Godo and Torra (1998a) propose a method for aggregating qualitative information weighted with natural numbers, by mean of qualitative weighted means involving t-norms on the set of values. Their characterisations have not been provided yet. (Dubois et al., 2000b) propose a family of mixtures that combines probabilistic and possibilistic mixtures via a threshold, also suggesting hybrid utility functions for this framework. We are interested in the behaviour of these utilities.
  Another point is to consider non-finite structures for representing uncertainty and preferences.

- *Dynamic Decision Problems*: There are some works studying the problem of adapting these possibilistic qualitative decision models to dynamic problems (Pereira et al., 1997; Fargier et al., 1996). We are interested in analysing them from the axiomatic setting point of view.

- *Applications:* As it is obvious, up to now, we have been mainly involved in the representational issues of these possibilistic decision models, however, as we are also interested in applying the models, we hope that in our future works we will be involved in other actual decision making problems. In particular we are interested in following with the analysis of the the decision problems involved in both projects we have been working on.

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