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Functional Definability Issues in Logics Based on Triangular Norms

Enrico Marchioni

Foreword by Lluís Godo

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Foreword by
Lluís Godo
Institut d'Investigació en Intel·ligència Artificial
Consell Superior d'Investigacions Científiques

Volume Author
Enrico Marchioni
Department of Information and Communication Sciences
Open University of Catalonia

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Foreword

The last decade has seen a tremendous development in infinitely valued logics which take the real unit interval as the basic set of truth values. This set is usually endowed with an algebraic structure of residuated lattice defined by a commutative semigroup operation –particularly a (left) continuous t-norm– together with its residuum, which are interpreted as a non-idempotent conjunction and implication operations respectively. These logics, commonly referred to as t-norm based fuzzy logics, are at the core of a new emerging discipline which is named *mathematical fuzzy logic*, after Petr Hájek. Since continuous t-norms are ordinal sums of isomorphic copies of the Lukasiewicz t-norm, the Product t-norm and the minimum operation, the traditional infinitely valued Łukasiewicz and Gödel logics, together with the Product logic, form the core examples for such mathematical fuzzy logics.

One stream of the development in mathematical fuzzy logic is towards enhancing the expressive power of these logics by adding new connectives. Of particular interest are the systems obtained from the addition of an independent involutive negation or by considering a combination of the Łukasiewicz and Product systems (the so-called logic $\text{Ł}\Pi\frac{1}{2}$), which results in a conservative extension of both systems, and that, additionally, has the Gödel logic as a subsystem. Both topics (among others) are addressed by the author of this monograph, which is based on his Ph.D. dissertation.

As for the first topic, the author provides a non-trivial generalization and a systematic treatment of logics with an independent involutive negation. Regarding the second topic, which is the central focus of the monograph, he studies the relationship of $\text{Ł}\Pi\frac{1}{2}$-algebras and ordered fields, and the lattice of subvarieties of $\text{Ł}\Pi\frac{1}{2}$-algebras. He also provides a mutual translation between the equational theory of $\text{Ł}\Pi\frac{1}{2}$ and the universal theory of real closed fields, showing that this translation has polynomial complexity, and thus establishing a polynomial equivalence between these two theories. Then he studies the issue of definability of functions and sets within this logic, showing that many t-norm based fuzzy logics are "contained" in $\text{Ł}\Pi\frac{1}{2}$, establishing decidability and complexity results for them.

Another very interesting line of development in t-norm based fuzzy logics is the possibility to use them to formalize notions which, a priori, seem to be far from the realm of fuzzy logic. One of these issues is uncertain reasoning. It has repeatedly been stressed by many authors that uncertainty (in the sense of belief) and fuzziness are very different notions. A graded logic of uncertainty attaches numbers to logical propositions which do not indicate a degree of truth but a degree of confidence or belief in the truth value of these propositions, while fuzzy logics deal with partial degrees of truth of propositions expressing gradual properties. The latter can be truth-functional while the former cannot. But one can safely understand the degree of belief, probability for instance, in a Boolean proposition φ as the truth degree, not of φ, but of a modal proposition $P\varphi$, read as "φ is probable". This idea has been exploited to define modal theories in different t-norm fuzzy logics to formalize and reason about uncertainty. The goal of this long digression is to properly put in context the final part of the monograph, which is dedicated to the development of a uniform approach for a logical treatment of uncertain reasoning models inside the framework of t-norm based fuzzy logics. Here the author provides a very nice general formalization which encompasses, with original results, many other approaches to this problem existing in the literature.

In summary, I believe that this monograph will attract the attention of many researchers in the field for the many interesting topics addressed, and also for those which are left as challenging open problems. Finally I would also like to highlight the great enthusiasm and effort that the author, Enrico, put from the beginning into this scientific adventure which I had the pleasure to supervise. I hope this will be only the starting point of a brilliant research career.

Bellaterra, October 2007

Lluís Godo
Institut d'Investigació en Intel·ligència Artificial
Consell Superior d'Investigacions Científiques

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Enrico Marchioni
Department of Information and Communication Sciences
Open University of Catalonia

Introduction

The aim of this thesis is to provide an analysis of definability of some classes of functions in the framework of many-valued logics based on triangular norms.

Triangular norms (t-norms for short) are binary commutative associative and monotone operations defined over the real unit interval $[0,1]$ having 1 as a neutral element (see [97], and Chapter 1). Left-continuity (i.e. left-continuity as a function on $[0,1]$) guarantees for a t-norm the existence of a unique *residuum*, i.e. a binary operation $\Rightarrow_*$ such that for all $x, y, z \in [0,1]$, $x*y \leq z$ iff $x \leq y \Rightarrow_* z$. T-norms and their residua provide a natural semantic interpretation for many-valued conjunctions and implications.

Many-valued logics have been long studied without relying on the concept of triangular norm. Important proposals (among others[1]) were carried out by Jan Łukasiewicz and Kurt Gödel. Łukasiewicz was the first to introduce a three-valued system in [100, 101] and an infinite-valued system in [102] which was proved to be complete independently by Rose and Rosser in [128] and by Chang in [18, 19].

Kurt Gödel published in 1932 an extremely short paper [66] concerning intuitionistic logic. Gödel introduced an infinite hierarchy of finitely-valued systems: his aim was to show that there is no finitely-valued propositional calculus that is sound and complete for intuitionistic logic. The infinite-valued version of his systems is now known as Gödel logic. This logic was shown to be complete by Dummet in [45].

In [75], Hájek suggested a new approach to many-valued logics as logics associated to t-norms[2]. The idea behind this interpretation consists in the fact that given a left-continuous t-norm $*$ and a propositional language L with set of connectives $\{\&, \rightarrow, \vee, \wedge, \overline{0}, \overline{1}\}$, we can define a $*$-*evaluation* v as a homomorphism from the algebra of formulas of L into the algebra $[0,1]_* = \langle [0,1], *, \Rightarrow_*, \max, \min, 0, 1\rangle$. Each formula is assigned a value from the real unit interval $[0,1]$, & is interpreted as the left-continuous t-norm $*$, and $\rightarrow$ is interpreted as the residuum $\Rightarrow_*$. In this way we can associate to a left-continuous t-norm a set $\mathcal{L}_*$ of formulas, called *the logic of the t-norm* $*$, defined as the set of all formulas φ such that for every $*$-evaluation v, $v(\varphi) = 1$. Similarly, it is

[1] Here we just mention the works of Łukasiewicz and Gödel, being the most relevant to the content of this thesis. See Hájek's monograph [75] for an historical introduction to many-valued logics.

[2] See also Gottwald's monograph [71].

possible to associate a logic $\mathcal{L}_\mathcal{K}$ to a class $\mathcal{K}$ of left-continuous t-norms, defined as the intersection of all $\mathcal{L}_*$ with $* \in \mathcal{K}$.

This new interpretation allowed to look from a different perspective at logics like Łukasiewicz logic and Gödel logic. Indeed, those systems can be regarded as logics associated to two of the fundamental continuous t-norms, i.e.: the Łukasiewicz t-norm $x *_l y = \max(x + y - 1, 0)$, and the minimum t-norm (or Gödel t-norm) $x *_g y = \min(x, y)$, respectively.

Hájek introduced in [75] the Basic Logic BL in order to provide an axiomatization of the tautologies common to all continuous t-norms. BL was proven to be the logic of continuous t-norms and their residua by Hájek in [74] and by Esteva, Cignoli, Godo and Torrens in [24]. Łukasiewicz and Gödel logics were shown to be extensions of BL, along with the Product logic (first introduced by Hájek, Godo and Esteva in [78]) that is the logic of the Product t-norm $x *_\pi y = x \cdot y$.

As mentioned above, left-continuity is a sufficient (and necessary) condition for a t-norm to guarantee the existence of a residual implication. Based on this consideration, Esteva and Godo introduced in [50] the Monoidal T-norm based Logic MTL, that is more general and weaker than BL (see Chapter 2). MTL was conjectured to be the logic of left-continuous t-norms and their residua. This conjecture was proven to be true by Jenei and Montagna in [95].

Logics based on left-continuous t-norms have a real-valued semantics, where connectives are interpreted by real-valued functions. Then, given any formula φ in the language of a logic $\mathcal{L}$, we can ask which function or class of functions can be associated to φ by the evaluation v. It is theoretically interesting to investigate and try to characterize classes of functions definable in a certain logic. This is especially important from the point of view of possible applications since we might be interested in using formulas whose interpretation corresponds to certain functions[3]. The investigation of the definability of classes of functions in t-norm based logics will be the central topic of this work.

i. Expansions by Independent Involutive Negations.

The expressive power of t-norm based logics strongly depends on the functions definable from the given t-norm (class of t-norms). However, sometimes we might need to have functions which cannot be obtained by composition of the available operators. In that case, the most direct strategy is to introduce new functions which enhance the expressive power of the logic. The case of triangular conorms (t-conorms for short) is a relevant example. T-conorms are binary commutative associative and monotone operations defined over the real unit interval $[0, 1]$ having 0 as a neutral element (see [97], and Chapter 1). A typical example of a t-conorm is given by the maximum $\max(x, y)$. T-conorms do not play a special role in logics based on t-norms (with the exception of the maximum), since in general they are not definable from other operators. Still, it would be important to have a logic where we can have t-conorms since they are

[3]As an example, we may cite the representation of neural networks by means of Łukasiewicz formulas introduced by Amato, Di Nola and Gerla in [4].

especially important from the point of view of possible applications, and they are a many-valued generalization of the classical Boolean disjunction.

A way of obtaining t-conorms in a t-norm based logic consists in exploiting the presence of a strong negation (see Chapter 1). A strong negation (also called involutive) is a one-variable function defined over the real unit interval that is strictly decreasing, continuous, and symmetric w.r.t. to the diagonal of the unit square. A classical example is the standard negation $n_s(x) = 1 - x$. Given a strong negation n and a t-norm $*$, we can always define a t-conorm $\diamond$ as $x \diamond y = n(n(x) * n(y))$. In t-norm based logics, negations are defined from the residuum as $n(x) = x \Rightarrow 0$, and so they strongly depend on the t-norm or on the class of t-norms. In general, given a left-continuous t-norm, the negation obtained from the residuum is not a strong negation. For instance, neither Product nor Gödel logic has an involutive negation, while in Łukasiewicz logic we immediately have the standard negation n_s. Clearly, this implies that neither in BL nor in MTL the negation will generally behave as an involutive negation.

Hájek proposed in [75, p.274] the following problem:

> "Investigate the extension of Basic Logic with a new negation satisfying the double negation axiom [...], i.e. the logic of continuous t-norms and t-conorms."

A first attempt to solve this problem was given by Esteva, Hájek, Godo and Navara in [51], where the authors expanded logics based on continuous t-norms without zero-divisors (see Chapter 1) by means of an independent involutive negation. However, they did not introduce a general treatment, but they rather dealt with particular logics.

A first general completeness theorem was given by Haniková in [73], but, once again, that results concerned only logics based on continuous t-norms without zero divisors.

Here we will deal with left-continuous t-norms, and our aim will be to give a general method for introducing an involutive negation independent from the t-norm. This will allow to define in a given logic (a class of) t-conorms from the given (class of) t-norm(s).

ii. Definability in the Logic $Ł\Pi\frac{1}{2}$.

$Ł\Pi\frac{1}{2}$ certainly is the most powerful t-norm based logic since it combines the Łukasiewicz logic and the Product logic. The interesting feature of $Ł\Pi\frac{1}{2}$ is that the functions definable in it are strictly related to functions definable in the field of real numbers. Indeed, we will see that formulas of the universal theory of real closed fields can be faithfully translated into equations in $Ł\Pi\frac{1}{2}$. This means that the functions definable in real closed fields can be defined in $Ł\Pi\frac{1}{2}$.

Given this connection with real closed fields, it can be easily seen that functions definable in $Ł\Pi\frac{1}{2}$ are piecewise rational functions, i.e. supremum of fractions of polynomials with rational coefficients. Our aim will be to try to characterize classes of left-continuous t-norms definable by terms in $Ł\Pi\frac{1}{2}$-algebras and consequently by formulas in the logic $\text{Ł}\Pi\frac{1}{2}$. Clearly, only those left-continuous t-norms representable as piecewise rational functions will be definable.

Why will this analysis turn out to be fundamental? The reason is simple. $\text{Ł}\Pi\frac{1}{2}$ is well-known to be decidable and in PSPACE. If we take a logic of a certain (class of) t-norm(s) that is (are) definable in $\text{Ł}\Pi\frac{1}{2}$, then this logic can be directly interpreted in $\text{Ł}\Pi\frac{1}{2}$, and can inherit those properties (modulo a polynomial-time translation). In particular, a logic associated to a definable t-norm would immediately turn out to be decidable and in PSPACE, while a logic of a class of definable t-norms would be decidable. It would be then interesting to answer the following question: are the main t-norm based logics also logics of certain classes of definable t-norms? For instance, is BL the logic of definable continuous t-norms? Is MTL the logic of definable left-continuous t-norms? A positive answer to those questions would show that those logics do enjoy (some of) the above mentioned properties. $\text{Ł}\Pi\frac{1}{2}$ may therefore be viewed as a super-expressive t-norm based logic that covers many important t-norm based logics and consequently ensures their decidability and inclusion in PSPACE.

iii. Application to the Representation of Uncertainty.

Besides the theoretical interest from the logical, algebraic and functional point of view, one might ask if the investigation concerning functional definability might shed some light on other fields or have any application. In other words: how useful can be the results on definability? A first answer to that question will be given by the above mentioned transmission of decidability and computational properties from $\text{Ł}\Pi\frac{1}{2}$ to logics of definable t-norms.

Furthermore, we will also show an interesting application of the results on definability concerning the representation of uncertainty. Indeed, we will show how the expressive power given by the possibility of defining several real-valued functions in t-norm based logics will make them a general and powerful framework for representing measures of uncertainty.

Measures of uncertainty aim at formalizing the strength of our beliefs in the occurrence of some events by assigning to those events a degree of uncertainty. From the mathematical point of view a measure of uncertainty is a real-valued function that gives an event a value from the real unit interval $[0, 1]$. A well-known example is given by probability measures which try to capture our degree of confidence in the occurrence of events by real-valued assessments. Esteva, Hájek, and Godo proposed in [77, 67] a new interpretation of measures of uncertainty in the framework of t-norm based logics. Given a sentence as "The proposition φ is plausible (probable, believable)", its degrees of truth can be interpreted as the degree of uncertainty of the proposition φ. Indeed, the higher is our degree of confidence in φ, the higher the degree of truth of the above sentence will result. In some sense, the predicate "is plausible (believable, probable)" can be regarded as a modal operator over the proposition φ. Then, given a measure of uncertainty μ, we can define modal many-valued formulas $\kappa(\varphi)$, whose interpretation is given by a real number corresponding to the degree of uncertainty assigned to φ under μ. Furthermore, we can translate the peculiar axioms governing the behavior of an uncertainty measure into formulas of a certain t-norm based logic, depending on the operations we need to represent. An adequate analysis of functional definability will allow an adequate choice among several possible logics.

Previous particular results concerning the representation of measures of uncertainty were presented in several works. We can mention the treatment of probability measures, necessity measures and belief functions proposed by Esteva, Hájek, and Godo in [77, 75, 67], [77], and [68], respectively; the treatment of conditional probability proposed by the present author and Godo in [106, 69, 70]; the treatment of (generalized) conditional possibility and necessity given by the present author in [104, 105]; and finally the treatment of simple and conditional non-standard probability given by Flaminio and Montagna in [59].

Here, our aim will be to give a general and comprehensive treatment of the representation of measures of uncertainty. In particular, we will show how it is possible to represent classes of measures such as probabilities, lower and upper probabilities, possibilities and necessities. We will deal with both conditional and unconditional measures. Important properties of the functions of t-norm based logics will then be useful in order to prove relevant features of the classes of measures represented.

iv. Structure of the Work and Contributions.

This work is divided in three parts. Part I is devoted to providing background notions concerning t-norms and logics based on t-norms that will be needed in the following chapters. Part II focuses on functional definability and contains the main contributions of this thesis. Part III deals with applications and develops from a general point of view the representation of measures of uncertainty based on functional definability.

Chapter 1. We introduce the basic properties of triangular norms and negation functions. We also review the fundamentals of construction methods such as ordinal sum, rotation, annihilation and rotation-annihilation.

Chapter 2. In this chapter we focus on the fundamental algebraic and logical properties of the Monoidal T-norm based Logic MTL and its main schematic extensions. We review the basic results concerning several kinds of completeness, expansions by means of rational truth constants and the Delta connective, and combination of different t-norms. Finally we recall the essential properties of first-order expansions.

Chapter 3. We provide a general treatment concerning the expansion of members of the family of t-norm based logics by means of an independent involutive negation. We establish the basic requirements in order to obtain completeness w.r.t. classes of algebras of the related varieties, like linearly ordered algebras and algebras over the real unit interval, both for the propositional and the first-order case. This generalizes previous works by Esteva, Godo, Hájek and Navara, [51] and by Haniková [73] that dealt only with expansions for logics based on continuous t-norms without zero-divisors, and gives a (more general) solution to an open problem proposed by Hájek in [75].

Chapter 4. We show that, given an ordered field between the field of rational numbers and the field of real algebraic numbers, Boolean combinations of polynomial equations and inequalities with rational coefficients for such a field can be translated into equations over the related $\text{Ł}\Pi\frac{1}{2}$-algebra. In particular, the universal theory of Real Closed Fields is shown to be interpretable in the equational theory of $\text{Ł}\Pi\frac{1}{2}$-algebras.

An immediate generalization of a result proven by Montagna in [111] shows that the $\text{Ł}\Pi\frac{1}{2}$-algebra associated to any real closed field generates the whole variety of $\text{Ł}\Pi\frac{1}{2}$-algebras. In particular so does the $\text{Ł}\Pi\frac{1}{2}$-algebra $\mathbb{A}\text{Ł}\Pi\frac{1}{2}$ whose lattice reduct is the unit interval of the real algebraic numbers, and consequently the logic $\text{Ł}\Pi\frac{1}{2}$ is finitely strongly standard complete w.r.t. evaluations over the real algebraic numbers. We show that $\mathbb{A}\text{Ł}\Pi\frac{1}{2}$ is the smallest subalgebra of the $\text{Ł}\Pi\frac{1}{2}$-algebra over the real unit interval generating the whole variety.

We answer an open problem raised by Montagna in [111], and show that the lattice of subvarieties of $\text{Ł}\Pi\frac{1}{2}$-algebras has the cardinality of the continuum.

We characterize sets definable by $\text{Ł}\Pi\frac{1}{2}$-terms, and show that they are sets defined by Boolean combinations of polynomial equations and inequalities with rational coefficients (called $\mathbb{Q}$-semialgebraic sets).

We show that there is a polynomial-time reduction of the theory of real closed fields to $\text{Ł}\Pi\frac{1}{2}$. This means that the universal theory of real closed fields and the equational theory of $\text{Ł}\Pi\frac{1}{2}$ are both in PSPACE and that they are strictly linked from the computational point of view.

Chapter 5. We deal with definability of (left-continuous) t-norms both in the first-order theory of real closed fields and in the equational logic of $\text{Ł}\Pi\frac{1}{2}$-algebras. We begin by giving negative results concerning left-continuous t-norms. In particular, we show that left-continuous t-norms with a set of infinite isolated discontinuity points, and with a dense set of discontinuity points are not definable. We provide a complete characterization of definable continuous t-norms, proving that a continuous t-norm is definable iff it is representable as a finite ordinal sum. We also give a complete characterization of definability of weak nilpotent minimum t-norms, showing that they are definable iff the induced negation has a finite number of discontinuity points. Moreover, we also show that the class of definable left-continuous t-norms is closed under constructions like Annihilation, Rotation and Rotation-Annihilation (under certain conditions).

We show that the logics MTL, SMTL, IMTL, WNM, BL, and SBL are finitely strongly standard complete w.r.t. to the related classes of t-norms definable in $\text{Ł}\Pi\frac{1}{2}$.

We show that every logic that is complete w.r.t. a left-continuous t-norm definable in $\text{Ł}\Pi\frac{1}{2}$ is in PSPACE. Moreover, we prove that every finitely axiomatizable logic that is complete w.r.t. a class of definable left-continuous t-norms is decidable.

Chapter 6. We give a general treatment for the representation of uncertainty in the framework of t-norm based logics. We will establish general conditions for a logic of uncertainty to be complete, and provide a general discussion about such a representation. We will deal with both simple measures and conditional measures of uncertainty. Previous results concerning the representation of probability, possibility and necessity measures (see [77, 75]) will be a direct consequence of our approach. Moreover, we will also provide a logical treatment of both lower and upper conditional probabilities.

We show that for logics with rational truth constant it is possible to define suitable theories whose consistency is tantamount to the coherence of the related assessment of uncertainty. Given certain functional properties of the logics, such assessments can be showed to be compact.

Chapter 7. We lay out a list of open problems related to the content of this work which deserve further investigation.

Appendix A. We will review the basic properties of uninorms (i.e. a generalization of both t-norms and t-conorms) and of their logics. We will extend the previous results concerning definability to several classes of left-continuous conjunctive uninorms. Furthermore, we will prove completeness of the logics UML and BUL w.r.t. the related classes of definable left-continuous conjunctive uninorms.

Appendix B. In this appendix we briefly review some basic algebraic notions used in the text.

v. Publications.

Many of the results contained in this work appeared or will appear in the following international publications.

The general treatment of the addition of an independent involutive negation is basically contained in the following papers:

- Flaminio T. and Marchioni E.: T-norm based logics with an independent involutive negation. *Fuzzy Sets and Systems*, Vol. 157, Issue 24, 3125–3144, 2006.

- Flaminio T. and Marchioni E.: Extending the Monoidal T-norm based Logic with an independent involutive negation. In *Proceedings of the 4th EUSFLAT Conference*, Barcelona (Spain), 860–865, 2005.

All the results about $\L\Pi\frac{1}{2}$ contained in Chapter 4 and Chapter 5 can be found in:

- Marchioni E. and Montagna F.: On triangular norms and uninorms definable in $\L\Pi\frac{1}{2}$. *International Journal of Approximate Reasoning*, in print, 2007.

- Marchioni E.: Ordered fields and $Ł\Pi\frac{1}{2}$-algebras. Submitted.

- Marchioni E. and Montagna F.: Complexity and definability issues in $Ł\Pi\frac{1}{2}$. *Journal of Logic and Computation*, Vol. 17, Number 2, 311–331, 2007.

- Marchioni E. and Montagna F.: A note on definability in $Ł\Pi\frac{1}{2}$. In *Proceedings of the 11th IPMU International Conference*, Paris (France), 1588–1595, 2006.

The general treatment for the representation of uncertainty measures in the framework of t-norm based logics is contained in the following works:

- Godo L. and Marchioni E.: Theories of uncertainty as modal theories in t-norm based logics. In preparation.

- Marchioni E.: Uncertainty as a modality over t-norm based logics. In *New Dimensions in Fuzzy Logic and Related Technologies*, Proceedings of the 5th EUSFLAT Conference, Ostrava (Czech Republic), 169–176, 2007.

Specific results concerning conditional probability and possibility can be found in:

- Marchioni E.: Possibilistic conditioning framed in fuzzy logics. *International Journal of Approximate Reasoning*, Vol. 43, Issue 2, 133–165, 2006.

- Godo L. and Marchioni E.: Reasoning about coherent conditional probability in a fuzzy logic setting. *Logic Journal of the IGPL*, Vol. 14, Number 3, 457–481, 2006.

- Marchioni E.: A logical treatment of possibilistic conditioning. In *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, Lecture Notes in Artificial Intelligence 3571, 701–713, Springer-Verlag, Berlin-Heidelberg, 2005.

- Godo L. and Marchioni E.: Reasoning about coherent conditional probability in the fuzzy logic FCP(ŁΠ). In *Proceedings of the Workshop on Conditionals, Information and Inference*, Ulm (Germany), 1–16, 2004.

- Marchioni E. and Godo L.: A logic for reasoning about coherent conditional probability: A modal fuzzy logic approach. In *Logics in Artificial Intelligence*, Lecture Notes in Artificial Intelligence 3229, 213–225, Springer-Verlag, Berlin-Heidelberg, 2004.

Part I

Background Notions

Chapter 1

Triangular Norms

In this chapter we provide the basic general background notions concerning triangular norms. We begin by discussing the properties of negation functions. Then we focus on t-norms (and t-conorms). Finally we review some general construction methods which allow the generation of new left-continuous t-norms.

1.1 Negations

We begin by introducing some basic concepts concerning negation functions, since they will turn out to be useful afterwards.

A non-increasing function $n : [0,1] \to [0,1]$ is called a *negation* if

$$n(0) = 1 \text{ and } n(1) = 0.$$

Non-increasingness means that n is order-reversing, i.e., for all $x, y \in [0,1]$, if $x \leq y$, then $n(y) \leq n(x)$.

A negation n is called *weak* if for all $x \in [0,1]$ $x \leq n(n(x))$.

A negation n is called a *strict negation* if n is continuous and strictly decreasing. A strict negation n is called a *strong negation* (also called *involutive negation*) if it enjoys the involutive property, i.e.:

$$n(n(x)) = x, \text{ for all } x \in [0,1].$$

A prototypical example of strong negation is the *standard negation* $n_s(x) = 1-x$. As proved by Trillas in [140], a mapping $n : [0,1] \to [0,1]$ is a strong negation if and only if there is a monotone bijection $\varsigma : [0,1] \to [0,1]$ such that for all $x \in [0,1]$

$$n(x) = \varsigma^{-1}(n_s(\varsigma(x))).$$

In other words, each strong negation is isomorphic to the standard involutive negation.

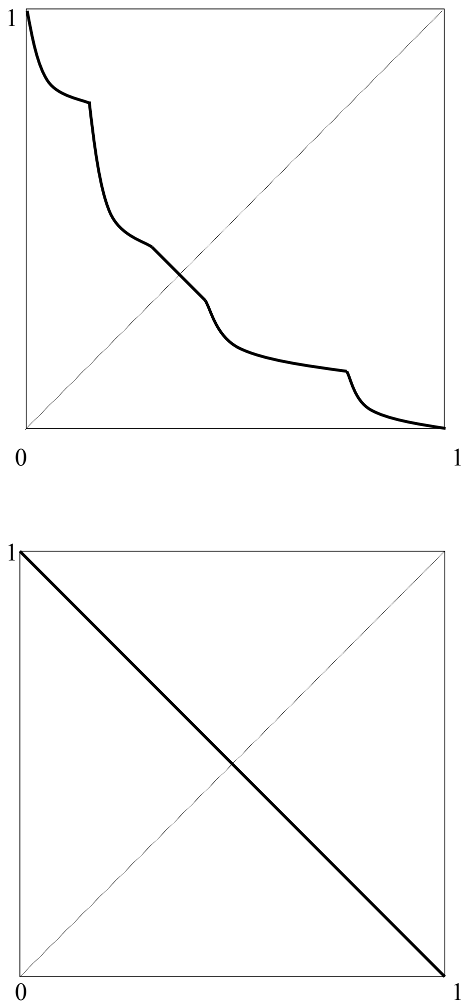

Figure 1.1: An involutive negation and the standard involutive negation.

From a geometric point of view every strong negation has a fixed point (i.e. a point x such that $n(x) = x$) which lies on the diagonal of the unit square, and it is symmetric w.r.t. $y = x$ (see Figure 1.1).

Weak negations are especially important in the field of t-norm based logics, since, as we will see later, they are the negations induced by left-continuous t-norms. This type of negations has been particularly studied by Esteva and Domingo in [46].

Weak negations are left-continuous and are symmetric w.r.t. $y = x$ (see Figure 1.2), i.e.:

i. For every $x \in [0, 1]$ being a discontinuity point, n is constant in the interval $(n(x^+), n(x^-))$, and equals x in that interval.

ii. For each maximal open interval (a, b) where n is constant and $n(x) = c$, n is discontinuous in c, so that $n(c^-) = b$, and $n(c^+) = a$.

Notice that a continuous weak negation necessarily is a strong negation.

Here, we particularly focus on weak negations with finitely many discontinuity points. Then, let n be a weak negation with k discontinuities $a_1, \ldots, a_k$. By symmetry n is constant in $(n(a_i^+), n(a_i^-))$. For each a_i we can take the points $\{a_i, n(a_i^+), n(a_i^-)\}$, and order them: $s_1, \ldots, s_r$. Hence we obtain a partition of the unit interval in $t + 1$ subintervals $I_1 = [0, s_1]$, $I_i = (s_{i-1}, s_i]$, $I_{t+1} = (s_t, 1]$, where n is either continuous and strictly decreasing or constant (notice that n cannot be constant in I_1).

Let K be the set of indices i of the subintervals I_i in which n is constant, and let K' be the set of indices i, j such that $I_i = (s_{i-1}, s_i]$, $I_j = (s_{j-1}, s_j]$, and $n(s_i) = s_{j-1}$. Then we have the following representation result.

Theorem 1.1.1 ([46]) *Given a weak negation n with a finite number of discontinuity points, there exist strong negations n_i on $[0, 1]$ such that for all $x \in [0, 1]$*

$$n(x) = \sum_{(i,j)\in K'} n_i(x) \cdot 1_{I_i \cup I_j}(x) + \sum_{i \in K} n_i(s_i) \cdot 1_{I_i}(x),$$

being 1_{I_i} the characteristic function of I_i.

Let n and n' be weak negations with a finite number of discontinuity points and let $I_1, \ldots, I_t$, and $I'_1, \ldots, I'_r$ be the associated intervals. We say that n and n' have an *analogous factorization* if

i. $t = r$;

ii. for each i, n is constant (strictly decreasing, resp.) in I_i, iff n' is constant (strictly decreasing, resp.) in I'_i;

iii. for each i there exists an increasing bijection from I_i into I'_i.

Theorem 1.1.2 ([46]) *Let n and n' be weak negations over $[0, 1]$ with a finite number of discontinuity points. Then n and n' are isomorphic iff they have an analogous factorization.*

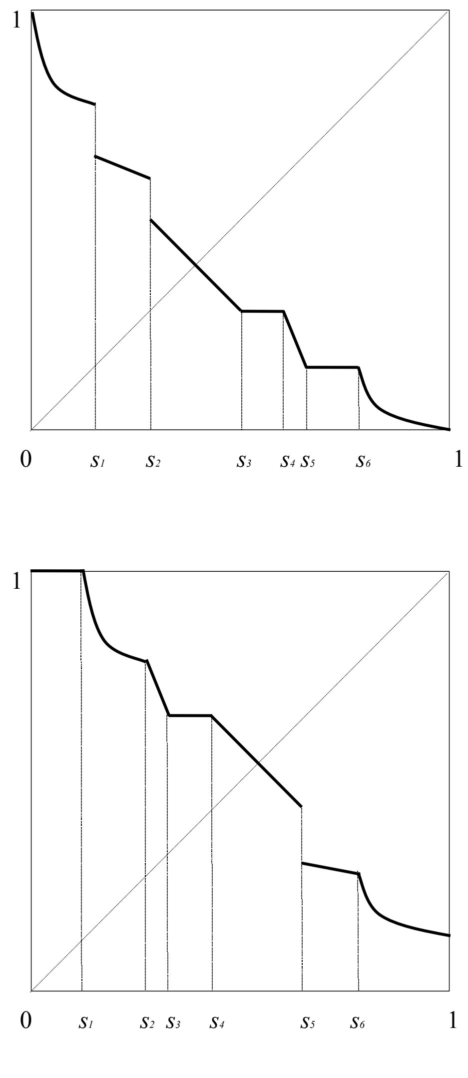

Figure 1.2: A weak negation and a quasi-weak negation.

As a consequence of Theorem 1.1.1 and Theorem 1.1.2 we obtain that each weak negation with a finite number of discontinuity points determines an equivalence class having a *canonical representative.* The canonical representative is the weak negation n obtained by taking the intervals I_i of equal length with n parallel to $1 - x$ on the intervals in which the negation is strictly decreasing.

Another interesting type of unary operators is that of *quasi-weak negations*, i.e. non-increasing functions $g : [0,1] \to [0,1]$ such that $g(0) = 1$ and $x \leq g(g(x))$, for all $x \in [0,1]$. Quasi-weak negations have been studied by De Baets in [36] and used in the characterization of idempotent uninorms [1] (see Appendix A). For a quasi-weak negation g, the region below its graph, i.e. the set $\{(x,y) \in [0,1]^2 : y \leq g(x)\}$ is symmetric w.r.t. $y = x$ (see Figure 1.2). It is easy to see that quasi-weak negations are left-continuous functions. Moreover we can prove the following:

Proposition 1.1.3 *A non-increasing function* $g : [0,1] \to [0,1]$ *is a quasi-weak negation iff it is either constantly equal to* 1*, or isomorphic to a weak negation on the subinterval* $[c,1]$*, where* $c = \sup\{x : g(x) = 1\}$*.*

Proof. Take a quasi-weak negation g, and let $c = \sup\{x : g(x) = 1\}$. The existence of c is guaranteed by left-continuity. If $0 < c < 1$, being g symmetric w.r.t. $y = x$, it follows that n(1)=c. Clearly, for all $x \in [c,1]$, $x \leq g(g(x))$, and since $g(1) = c$ and $g(c) = 1$, it immediately follows that for all $x \in [c,1]$, $g(x)$ is a weak negation. If $c = 0$ we can similarly see that g is indeed a weak negation. Finally, if $c = 1$, then $g(x) = 1$ for all $x \in [0,1]$.

To prove the converse just note that if g is non-increasing and it is isomorphic to a weak negation on $[c,1]$, then it immediately follows that $g(x) = 1$ for all $x \in [0,c]$. ■

An obvious consequence of Proposition 1.1.3 is that the above results concerning weak negations with finitely many discontinuity points can be easily adapted to the case of quasi-weak negations. Indeed, it is easy to see that every quasi-weak negation g with a finite number of discontinuity points determines a partition of the unit interval in finitely many subintervals $I_1, \ldots, I_r$ so that $g = 1$ on I_1, and it factorizes as a weak negation on the remaining subintervals. The concept of factorization and isomorphism between quasi-weak negations is then easily defined, along with the notion of canonical representative.

1.2 Triangular norms and conorms

A *triangular norm* $*$ (see the monograph [97] by Klement, Mesiar and Pap) is a binary commutative associative and monotone operation having 1 as a neutral element, i.e. for all $x, y, z \in [0,1]$:

i. $x * y = x * y$ [Commutativity],

[1] Actually,in [36] the term "quasi-weak negation" does not appear. However, we use here this term to make explicit the relationship with weak negations.

ii. $x * (y * z) = (x * y) * z$ [Associativity],

iii. $x * y \leq x * z$ whenever $y \leq z$ [Monotonicity],

iv. $x * 1 = x$ [Identity element].

Algebraically speaking, each t-norm makes $[0, 1]$ into a totally ordered integral commutative monoid. The three prototypical t-norms $*_g, *_\pi, *_l$ are given by, respectively (see Figure 1.3 and Figure 1.4):

- $x *_g y = \min(x, y)$, [minimum (Gödel) t-norm]
- $x *_\pi y = x \cdot y$, [Product t-norm]
- $x *_l y = \max(x + y - 1, 0)$, [Łukasiewicz t-norm].

A t-norm is continuous if for all convergent sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} \in [0, 1]^{\mathbb{N}}$,

$$\left(\lim_{n\to\infty} x_n\right) * \left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} (x_n * y_n).$$

Similarly, a t-norm is left-continuous iff for all convergent sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} \in [0, 1]^{\mathbb{N}}$,

$$\sup\{x_n\} * \sup\{y_n\} = \sup\{x_n * y_n\}.$$

Left-continuity of the t-norm is fundamental in order to guarantee the existence of a residual implication. Indeed, recall that a commutative, integral, lattice-ordered monoid $\langle A, *, \leq, 0, 1\rangle$ is residuated whenever there is a binary operation $\Rightarrow$, satisfying the adjointness property

$$x * y \leq z \text{ iff } x \leq y \Rightarrow z$$

(see [143], and Appendix B). Given a t-norm $*$, $\langle [0, 1], *, \Rightarrow_*, \leq, 0, 1\rangle$ is a commutative integral residuated lattice-ordered monoid (where $\Rightarrow_*$ denotes the residual implication) iff $*$ is left-continuous (see [97]). In this case the residuum $\Rightarrow$ is given by

$$x \Rightarrow_* y = \max\{z \in [0, 1] \mid x * z \leq y\}.$$

The residua of the three basic t-norms are given by

- $x \Rightarrow_l y = \min(1 - x + y, 1)$,
- $x \Rightarrow_g y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$,
- $x \Rightarrow_\pi y = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{otherwise} \end{cases}$.

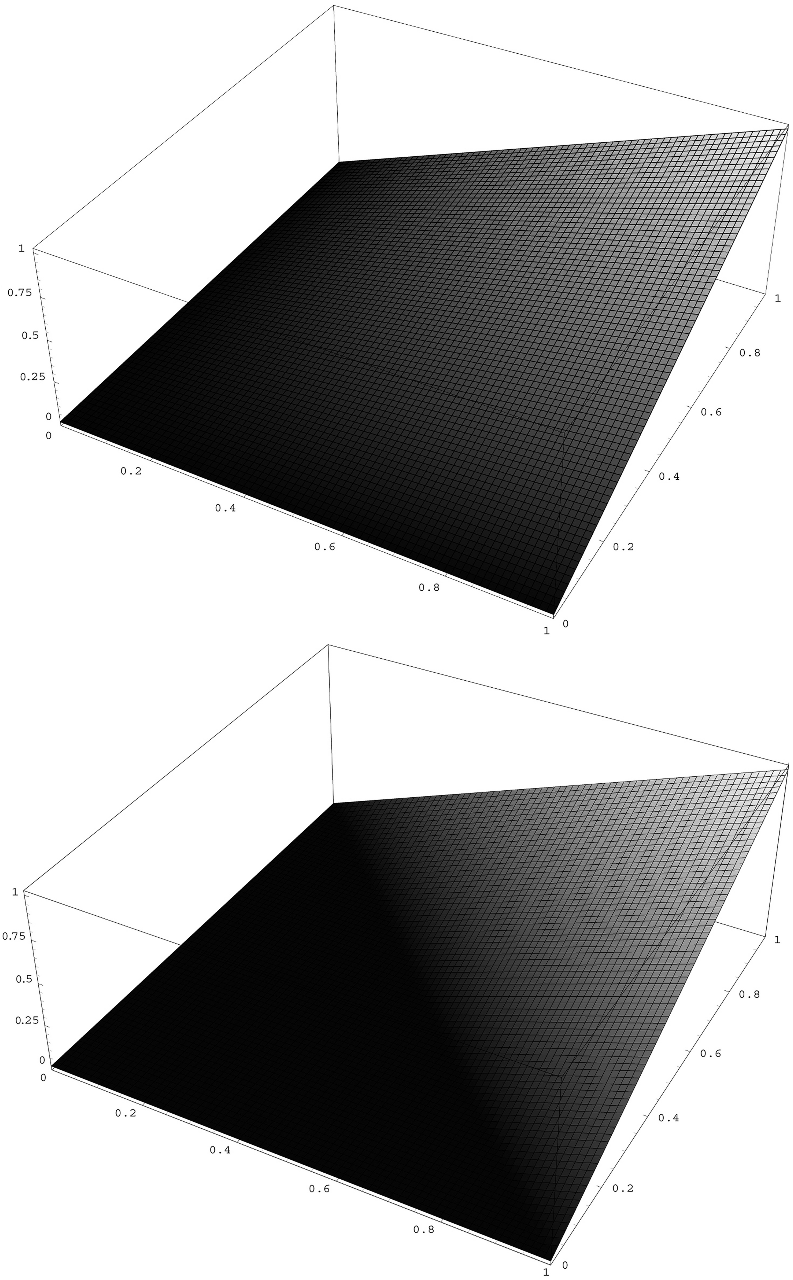

Figure 1.3: Product t-norm and Łukasiewicz t-norm.

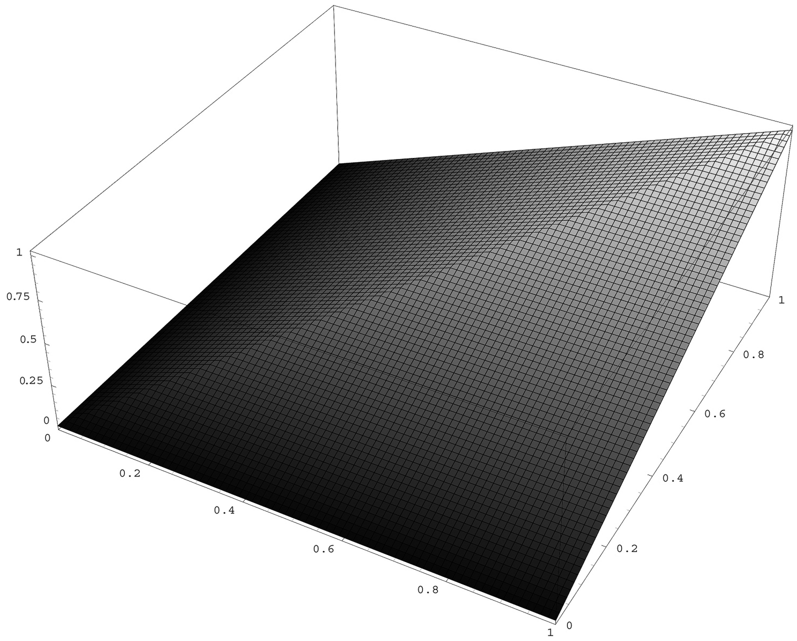

Figure 1.4: Gödel t-norm.

A t-subnorm (see [97]) is a function $* : [0,1] \times [0,1] \to [0,1]$ that is commutative, associative, monotone, and bounded by its arguments, i.e. $x * y \leq x$ for all $x, y \in [0,1]$. It is clear that each t-norm is a subnorm. Moreover, given a t-norm $*$ and $c \in]0,1[$, the operation

$$x *_c y = \frac{cx * cy}{c}$$

is a t-subnorm. T-subnorms play an important role in the construction of t-norms. Indeed, as shown in the following theorem, they can be taken as summands in the ordinal sum construction which allows the generation of new t-norms.

Theorem 1.2.1 ([92]) *Let $\mathcal{K} = \{[a_i, b_i]\}$ be a countable family of non-overlapping, closed, proper subintervals of $[0,1]$. With each $[a_i, b_i] \in \mathcal{K}$ associate a t-subnorm $*_i$ where for each $[a_i, b_i], [a_j, b_j] \in \mathcal{K}$ with $b_i = a_j$ and with zero-divisors*[2] *in $*_j$ we have that $*_i$ is a t-norm and for $[a_i, 1] \in \mathcal{K}$ we have that $*_i$ is a t-norm. Let $*$ be the function defined on $[0,1]^2$ by*

$$x * y = \begin{cases} a_i + (b_i - a_i) \cdot \left(\frac{x - a_i}{b_i - a_i} *_i \frac{y - a_i}{b_i - a_i} \right) & x, y \in]a_i, b_i]^2 \\ \min(x, y) & \text{otherwise} \end{cases} .$$

[2]Recall that, given a t-norm $*$, an element $a \in]0,1[$ is called a *zero-divisor* if there is some $b \in]0,1[$ such that $a * b = 0$.

Then $*$ *(also denoted by* $* = (\langle a_i, b_i, *_i \rangle)_{i \in \mathcal{K}}$*) is a t-norm.*

Notice that ordinal sums preserve left-continuity and continuity, i.e.: an ordinal sum is continuous (left-continuous) iff all of its summands are continuous (left-continuous).

Theorem 1.2.2 ([108]) *Let* $*$ *be a left-continuous t-norm obtained by ordinal sum, so that* $* = (\langle a_i, b_i, *_i \rangle)_{i \in \mathcal{K}}$*. The corresponding residual implication* $\Rightarrow_*$ *is given by*

$$x \Rightarrow_* y = \begin{cases} 1 & \text{if } x \leq y \\ a_i + (b_i - a_i) \cdot \left(\frac{x - a_i}{b_i - a_i} \Rightarrow_{*_i} \frac{y - a_i}{b_i - a_i} \right) & \text{if } a_i < y < x \leq b_i \\ y, & \text{otherwise} \end{cases} .$$

Recall that a t-norm $*$ is *strictly monotone* if, whenever $x > 0$ and $y < z$, $x * y < x * z$. A t-norm satisfies the *cancellation law* if $x * y = x * z$ implies $x = 0$ or $y = z$. A t-norm $*$ satisfies the *conditional cancellation law* if $x * y = x * z > 0$ implies $y = z$. A t-norm $*$ is *Archimedean* if for each $x, y \in]0, 1[^2$ there is an $n \in \mathbb{N}$ such that $x^n < y$, with

$$x^n = \underbrace{x * \cdots * x}_{n}.$$

A t-norm is strictly monotone if and only if it satisfies the cancellation law. Moreover, being strictly monotone implies the absence of zero-divisors and the absence of idempotent elements (except the trivial idempotents 0 and 1). The minimum t-norm has none of the above properties (due to the fact that each $x \in [0, 1]$ is an idempotent element for $*_g$), while the Product t-norm satisfies all of them. The Łukasiewicz t-norm is Archimedean and satisfies the conditional cancellation law, but not the cancellation law (in fact it is not strictly monotone). A t-norm is called *strict* if it is continuous and strictly monotone, and it is called *nilpotent* if it is continuous and for any $x \in]0, 1[$ there is an $n \in \mathbb{N}$ such that $x^n = 0$. Every continuous Archimedean t-norm is either strict or nilpotent (see [97]), and all strict t-norms are isomorphic to $*_\pi$, while all nilpotent t-norms are isomorphic to $*_l$.

Proposition 1.2.3 ([97]) *Let* $*_1$ *and* $*_2$ *be two continuous Archimedean t-norms. The following are equivalent:*

i. $*_1$ *and* $*_2$ *are isomorphic.*

ii. Either both $*_1$ *and* $*_2$ *are strict or both* $*_1$ *and* $*_2$ *are nilpotent.*

While for left-continuous t-norms a representation theorem is still lacking, we have a beautiful characterization of continuous t-norm encoded in the well-known Mostert-Shields theorem [118]. Indeed, every continuous t-norm can be represented as an ordinal sum of continuous Archimedean t-norms, i.e. of copies of the Łukasiewicz and the Product t-norm.

Theorem 1.2.4 (Mostert-Shields Theorem, [118, 97]) *For a function* $* : [0,1]^2 \to [0,1]$ *the following are equivalent:*

i. $*$ *is a continuous t-norm.*

ii. $*$ *is uniquely representable as an ordinal sum of continuous Archimedean t-norms.*

Triangular conorms (t-conorms for short) are functions $\diamond : [0,1] \times [0,1] \to [0,1]$ that generalize classical disjunction. They have the same properties as t-norms, but the neutral element is 0. For all $x, y, z \in [0,1]$:

i. $x \diamond y = y \diamond x$,

ii. $x \diamond (y \diamond z) = (x \diamond y) \diamond z$,

iii. $x \diamond y \leq x \diamond z$ whenever $y \leq z$,

iv. $x \diamond 0 = x$.

There are three basic t-conorms $\diamond_g, \diamond_\pi, \diamond_l$, given by, respectively:

- $x \diamond_g y = \max(x, y)$, [maximum]
- $x \diamond_\pi y = x + y - x \cdot y$, [probabilistic sum]
- $x \diamond_l y = \min(x + y, 1)$, [bounded sum].

T-conorms do not play a special role in logics based on t-norms. Indeed, as we will see below, those logics are usually defined from a conjunction operator, and disjunctions might be obtained as derived operators, but their interpretation often corresponds only to a very reduced class of t-conorms.

Given a t-norm $*$ we can define by means of a strict negation n a t-conorm $\diamond$ that is called n-dual of $*$ as follows:

$$x \diamond y = n^{-1}(n(x) * n(y)).$$

Notice that if the negation is strong, applying the above construction to $\diamond$, we obtain the t-norm $*$ we started with. In such cases we call structures $\langle *, \diamond, n \rangle$, where $*$ is a t-norm, $\diamond$ a t-conorm, and n an involutive negation, *De Morgan triples* [97].

Given a De Morgan triple, we can define a special type of implications called S-implications [121], which are operators of the form

$$x \Rightarrow_s y = n(x) \diamond y,$$

or equivalently,

$$x \Rightarrow_s y = n(x * n(y)),$$

where $*$ is the t-norm n-dual of $\diamond$. It is easy to see that S-implications generalize Boolean implication. Note that the only continuous t-norm whose residual implication corresponds to its related S-implication obtained by the De Morgan triple based on the standard negation n_s is the Łukasiewicz implication.

1.3 Annihilation, Rotation and Rotation-Annihilation

Annihilation, rotation and rotation-annihilation are construction methods first studied by Jenei in [89, 90, 91], in order to obtain new left-continuous t-norms whose associated negation is involutive. Those constructions have been generalized for associative aggregation operators by Jenei and De Baets in [94], and for commutative partially ordered semigroups in [93], again by Jenei. In particular, they can be applied to binary, commutative, associative and monotone left-continuous operations in order to obtain functions satisfying the following property,

$$x * y \leq z \text{ iff } y * n(z) \leq n(x),$$

w.r.t. a strong negation n (see [93]). Functions enjoying the above property are called *rotation-invariant*. Notice that the negation associated to rotation-invariant left-continuous t-norms is an involutive negation (see [93]).

The annihilation method arose from the study of the conditions under which, given a continuous t-norm $*$ and a strong negation n, the following operation is a t-norm:

$$x *' y = \begin{cases} 0 & x \leq n(y) \\ x * y & \text{otherwise} \end{cases} .$$

It is easy to se that $*'$ turns out to be monotone, commutative and with 1 as identity element, but, in general, it is not associative. The answer to the problem of finding a characterization for the annihilation of continuous t-norms was given in [89].

Theorem 1.3.1 ([89]) *A continuous t-norm $*$ can be n-annihilated iff the annihilated t-norm $*_a$ is an element of the Nilpotent Ordinal Sum Family:*

$$x *_a y = \begin{cases} 0 & x \leq n(y) \\ a + (n(a) - a) \cdot \left(\frac{x-a}{n(a)-a} *_1 \frac{y-a}{n(a)-a} \right) & a \leq x, y \leq n(a),\ x > n(y) \\ \min(x, y) & \text{otherwise} \end{cases} ,$$

where the unique fixed point of n is t, $a \in [0, t]$, and $*_1$ is the linear transformation of a nilpotent t-norm whose induced negation is n.

A typical example is given by the annihilation of the minimum t-norm, which yields a t-norm called the *Nilpotent Minimum* :

$$x *_{nm} y = \begin{cases} 0 & x \leq n(y) \\ \min(x, y) & \text{otherwise} \end{cases} .$$

The Łukasiewicz t-norm and the nilpotent minimum t-norm are limit cases of the family of Nilpotent Ordinal Sums. If $a \in]0, t[$, any two members of this family are isomorphic, and, in particular, they are isomorphic to the following t-norm

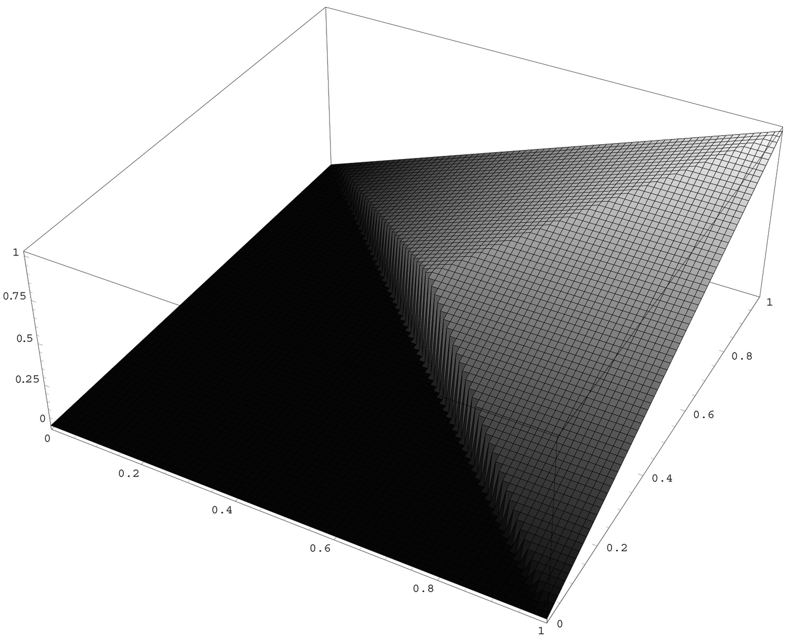

Figure 1.5: Nilpotent Minimum t-norm.

$$x *_n y = \begin{cases} 0 & x \leq 1 - y \\ \frac{1}{3} + x + y - 1 & \frac{1}{3} \leq x, y \leq \frac{2}{3} \text{ and } x > 1 - y \\ \min(x, y) & \text{otherwise} \end{cases} .$$

The Annihilation construction was generalized by Cignoli, Esteva, Godo and Montagna in [23], where the authors tried to characterize the conditions under which (similarly to the above case), given a weak negation n_w and a left-continuous t-norm $*$,

$$x *' y = \begin{cases} 0 & x \leq n_w(y) \\ x * y & \text{otherwise} \end{cases}$$

is a left-continuous t-norm.

Theorem 1.3.2 ([23]) *Given a weak negation n_w and a left-continuous t-norm $*$, the operation*

$$x *' y = \begin{cases} 0 & x \leq n_w(y) \\ x * y & \text{otherwise} \end{cases}$$

is a left-continuous t-norm, such that $n_w(x) = x \Rightarrow_{*'} 0$ iff, for all $x, y, z \in [0, 1]$

i. $x \Rightarrow_{*'} 0 \leq n_w(x)$, *and*

ii. *if* $y > n_w(z)$, *then* $n_w(y * z) = \max(n_w(y), y \Rightarrow_* n_w(z))$.

A weak negation satisfying conditions i. and ii. of the above theorem w.r.t. to a left-continuous t-norm $*$ is called *compatible* with $*$.

In [23], the authors also characterized which negations are compatible with the three fundamental t-norms:

Theorem 1.3.3 ([23]) *The following three statements hold:*

i. Every weak negation is compatible with the minimum t-norm.

ii. The only weak negation compatible with the Łukasiewicz t-norm is its own negation.

iii. The only weak negation compatible with the Product t-norm is its own negation.

Given a weak negation n_w and the minimum t-norm $*_g$, the t-norm generated by the above method is called *weak nilpotent minimum* t-norm

$$x *_w y = \begin{cases} 0 & x \leq n_w(y) \\ x * y & \text{otherwise} \end{cases}$$

whose residuum is given by

$$x \Rightarrow_{*_w} y = \begin{cases} 1 & x \leq y \\ \max(n_w(x), x \Rightarrow y) & \text{otherwise} \end{cases}.$$

Now, we focus on the Rotation and the Rotation-Annihilation.

The Rotation construction is characterized by the following theorem:

Theorem 1.3.4 (Rotation,[94]) *Let n be a strong negation, let t be its unique fixed point and let $*$ be a left-continuous t-norm. Let $*_1$ be the linear transformation of $*$ into $[t, 1]$, $I^+ =]t, 1]$, $I^- = [0, t]$, and define $*_r : [0, 1] \times [0, 1] \to [0, 1]$ by*

$$x *_r y = \begin{cases} x *_1 y & \text{if } x, y \in I^+ \\ n(x \Rightarrow_{*_1} n(y)) & \text{if } x \in I^+ \text{ and } y \in I^- \\ n(y \Rightarrow_{*_1} n(x)) & \text{if } x \in I^- \text{ and } y \in I^+ \\ 0 & \text{if } x, y \in I^- \end{cases}.$$

Then, $*_r$ is a left-continuous t-norm iff, either

C1) *$*$ has no zero-divisors, or*

C2) *there exists $c \in]0, 1]$ such that for any zero-divisor x of $*$ we have $x \Rightarrow_* 0 = c$.*

The rotation-annihilation is a combination of the above rotation method with the annihilation construction.

Theorem 1.3.5 (Rotation-Annihilation,[93]) *Let n be a strong negation, t be its unique fixed point, $d \in]t, 1[$ and define the following transformation of n[3]:*

[3] This transformation of a strong negation is called the *d-zoomed negation*.

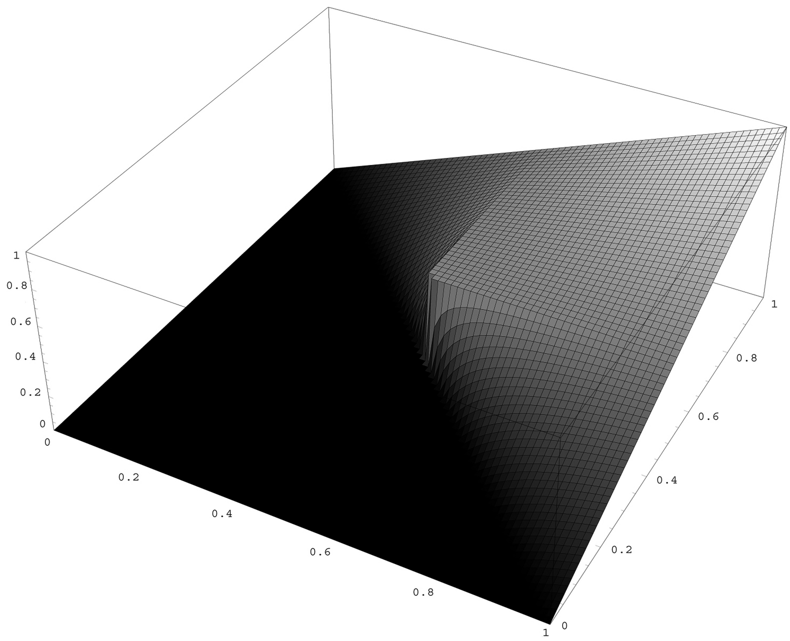

Figure 1.6: Rotation of the Product t-norm.

$$n_d(x) = \frac{n(x\cdot(d-n(d))+n(d))-n(d)}{d-n(d)}.$$

Let $*_1$ be a left-continuous t-subnorm.

1. *If $*_1$ has no zero-divisors, then let $*_2$ be a left-continuous t-subnorm which admits the rotation invariance property w.r.t. n_d. Further, let $I^- = [0, n(d)[$, $I^0 = [n(d), d]$ and $I^+ =]d, 1]$.*

2. *If $*_1$ has zero-divisors, then let $*_2$ be a left-continuous t-norm which admits the rotation invariance property w.r.t. n_d. Further, let $I^- = [0, n(d)]$, $I^0 =]n(d), d[$ and $I^+ = [d, 1]$.*

Let $*_3$ be the linear transformation of $*_1$ into $[d, 1]$, $*_4$ be the linear transformation of $*_2$ into $[n(d), d]$, and $*_5$ be the annihilation of $*_4$ given by

$$x *_5 y = \begin{cases} 0 & \text{if } x, y \in [n(d), d] \text{ and } x \leq n(y) \\ x *_4 y & \text{if } x, y \in [n(d), d] \text{ and } x > n(y) \end{cases}.$$

The binary operation $*_{ra} : [0,1]^2 \to [0,1]$ defined by

$$x *_{ra} y = \begin{cases} x *_3 y & \text{if } x, y \in I^+ \\ n(x \Rightarrow_{*_3} n(y)) & \text{if } x \in I^+ \text{ and } y \in I^- \\ n(y \Rightarrow_{*_3} n(x)) & \text{if } x \in I^- \text{ and } y \in I^+ \\ 0 & \text{if } x, y \in I^- \\ x *_5 y & \text{if } x, y \in I^0 \\ y & \text{if } x \in I^+ \text{ and } y \in I^0 \\ x & \text{if } x \in I^0 \text{ and } y \in I^+ \\ 0 & \text{if } x \in I^- \text{ and } y \in I^0 \\ 0 & \text{if } x \in I^0 \text{ and } y \in I^- \end{cases}$$

is a left-continuous rotation-invariant t-norm, also called the rotation-annihilation *of* $*_1$ *and* $*_2$.

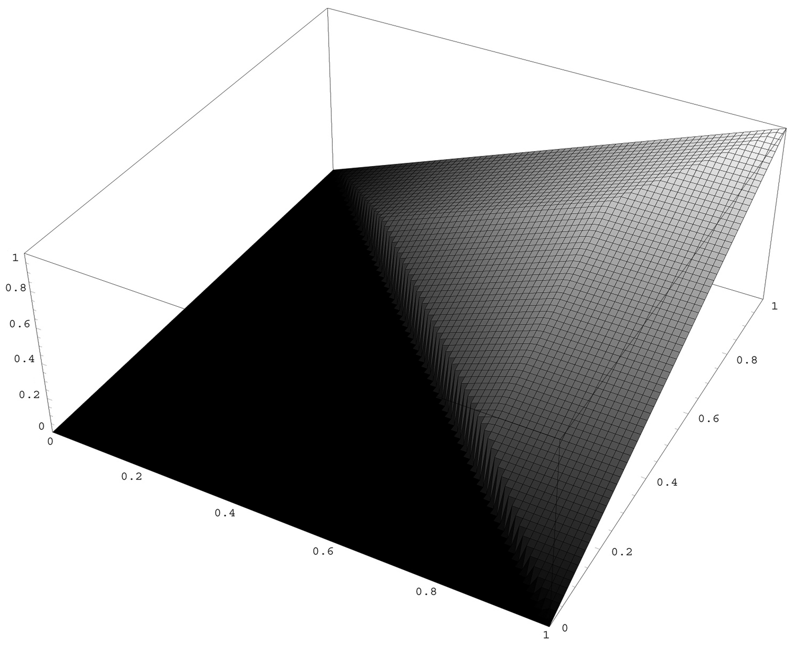

Figure 1.7: Rotation-annihilation of the minimum t-norm and the Łukasiewicz t-norm w.r.t. the standard negation.

Chapter 2

T-norm Based Logics

In this chapter we focus on logics based on left-continuous t-norms and review their basic properties.

In the next section, we survey the fundamental logical and algebraic results for MTL and its main schematic extensions, providing basic concepts needed in the next chapters.

In Section 2.2, we briefly recall some notions concerning expansions of t-norm based logics. We deal with expansions by means of the Baaz projector Δ [6], expansions obtained by adding truth-constants [55], and expansions built up from the combination of different t-norms.

Finally, in Section 2.3, we will be concerned with the first-order version of logics based on left-continuous t-norms. We will review the basic notions and provide the fundamental completeness results.

2.1 Logics based on t-norms

2.1.1 The Monoidal T-norm based Logic MTL

The language of MTL includes the binary connectives $\&, \to, \wedge$ and the truth-constant $\overline{0}$. The axiomatic system for MTL is given by the Hilbert-style calculus defined by the following axiom schemata:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $(\varphi \& \psi) \to \varphi$
(A3) $(\varphi \& \psi) \to (\psi \& \varphi)$[1]
(A4) $(\varphi \& (\varphi \to \psi)) \to (\varphi \wedge \psi)$
(A5) $(\varphi \wedge \psi) \to \varphi$
(A6) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
(A7) $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A8) $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

[1] Axiom (A3) was actually shown to be redundant by Cintula in [30].

(A9) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A10) $\overline{0} \to \varphi$.

The only inference rule of MTL is Modus Ponens: from φ and $\varphi \to \psi$ derive ψ.

Further connectives are defined as follows:

$$\begin{array}{rcl@{\qquad}rcl} \varphi \vee \psi & \text{is} & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), & \neg\varphi & \text{is} & \varphi \to \bar{0}, \\ \varphi \leftrightarrow \psi & \text{is} & (\varphi \to \psi) \& (\psi \to \varphi), & \bar{1} & \text{is} & \neg\bar{0}. \end{array}$$

Moreover, we use φ^n to denote $\underbrace{\varphi \& \ldots \& \varphi}_{n}$.

A *proof* in MTL is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that each φ_i either is an axiom of MTL or follows from some preceding φ_j, φ_k $(j, k < i)$ by modus ponens. As usual, a set of formulas is called a *theory*. We say that a formula φ can be derived from a theory Γ, denoted as $\Gamma \vdash \varphi$, if there is a proof of φ from a set $\Gamma' \subseteq \Gamma$. A theory Γ is said to be *consistent* if $\Gamma \nvdash \overline{0}$. MTL enjoys the following form of deduction theorem.

Theorem 2.1.1 ([50, 75]) *Let Γ be a theory over* MTL*, and let φ, ψ be formulas. Then, there is an $n \in \mathbb{N}$ such that*

$$\Gamma \cup \{\varphi\} \vdash \psi \textit{ iff } \Gamma \vdash \varphi^n \to \psi.$$

The algebraic semantics for MTL is given by MTL-algebras.

Definition 2.1.2 An MTL-algebra is a structure

$$\mathcal{A} = \langle A, \sqcup, \sqcap, *, \Rightarrow, 0, 1 \rangle$$

with four binary operations and two constants such that

- $\langle A, \sqcup, \sqcap, 0, 1 \rangle$ is a lattice with largest element 1 and least element 0 (w.r.t. the lattice ordering),
- $\langle A, *, 1 \rangle$ is a commutative semigroup with unit element 1,
- the operations $*$ and $\Rightarrow$ form an *adjoint pair*:

$$x * y \leq z \text{ iff } x \leq y \Rightarrow z,$$

- $\mathcal{A}$ satisfies the *prelinearity* equation

$$(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1.$$

Notice that a negation operator, called *pseudocomplement*, is defined from $\Rightarrow$ as $\neg x = x \Rightarrow 0$. An example of MTL-algebra is the structure

$$\langle [0, 1], \max, \min, *, \Rightarrow_*, 0, 1 \rangle,$$

where $*$ is a left-continuous t-norm and $\Rightarrow_*$ is its related residuum. Such a structure is called *standard* MTL-algebra, and will be denoted by $[0,1]_{\mathrm{MTL}_*}$. A linearly ordered MTL-algebra is also called an MTL-chain.

It is easy to see, by Birkhoff's theorem [15], that the class of MTL-algebras forms a variety (denoted by $\mathbb{MTL}$), since it is defined by a finite set of equations (see Appendix B).

Given an MTL-algebra $\mathcal{A}$, an $\mathcal{A}$-evaluation of propositional formulas is a mapping e assigning to each propositional variable p an element from A. Such an evaluation can be extended to all formulas as follows:

$$\begin{aligned} e(\overline{0}) &= 0, \\ e(\varphi \rightarrow \psi) &= e(\varphi) \Rightarrow e(\psi), \\ e(\varphi \& \psi) &= e(\varphi) * e(\psi), \\ e(\varphi \wedge \psi) &= e(\varphi) \sqcap e(\psi), \\ e(\varphi \vee \psi) &= e(\varphi) \sqcup e(\psi). \end{aligned}$$

An $\mathcal{A}$-evaluation e is an $\mathcal{A}$-*model* for a theory Γ, if, for all $\gamma \in \Gamma$, $e(\gamma) = 1$. A formula φ is an $\mathcal{A}$-tautology if $e(\varphi) = 1$ under all evaluations e. A formula φ is a 1-tautology if $e(\varphi) = 1$ under all evaluations e of all standard MTL-algebras.

MTL-algebras are the *equivalent algebraic semantics* for MTL, and so MTL is *algebraizable* in the sense of Blok and Pigozzi (see [10]). This property can be explained as follows. For every formula φ, let t^φ denote the term obtained from φ by replacing $\overline{0}$ by 0, any propositional variable p_i by the individual variable x_i, & by $*$, and $\rightarrow$ by $\Rightarrow$. Conversely, given a term t, let φ^t be the formula obtained from t by the inverse substitution, that is, by replacing x_i by p_i, 0 by $\overline{0}$, $*$ by &, and $\Rightarrow$ by $\rightarrow$. Let $\models_{\mathbb{MTL}}$ denote the consequence relation in the equational logic of MTL-algebras. Then we have:

- Let Γ be a set of MTL-formulas, and let φ be an MTL-formula. Let $\Gamma^t = \{t^\psi = 1 : \psi \in \Gamma\}$. Then, $\Gamma \vdash_{\mathrm{MTL}} \varphi$ iff $\Gamma^t \models_{\mathbb{MTL}} t^\varphi = 1$.
- Let Σ be a set of equations, and let ϵ be any equation in the equational logic of MTL-algebras. For every equation $\sigma : t = s$, let σ^φ denote the MTL-formula $\varphi^t \leftrightarrow \varphi^s$. Moreover, let $\Sigma^\varphi = \{\sigma^\varphi : \sigma \in \Sigma\}$. Then $\Sigma \models_{\mathbb{MTL}} \epsilon$ iff $\Sigma^\varphi \vdash_{\mathrm{MTL}} \epsilon^\varphi$.

Given that the equational logic of any variety is strongly complete w.r.t. the variety itself, we obtain:

Theorem 2.1.3 ([50]) *Let* Γ *be a theory over* MTL*, and* φ *be a formula. Then the following are equivalent:*

1. $\Gamma \vdash_{\mathrm{MTL}} \varphi$,
2. *for each* MTL*-algebra* $\mathcal{A}$ *and each* $\mathcal{A}$*-model* e *of* Γ, $e(\varphi) = 1$.

Given a class of algebras $\mathbb{K}$ of the same type, it is interesting to know if there is some subclass $\mathbb{K}'$ of $\mathbb{K}$ generating the whole variety, i.e. if $\mathcal{V}(\mathbb{K}) = \mathcal{V}(\mathbb{K}')$. In the case of t-norm based logics, for instance, it is interesting to know if the variety of certain algebras is generated by the class of linearly-ordered algebras, and by the class of standard algebras. The variety of MTL-algebras is generated by the class of MTL-chains. Indeed, the following theorem holds:

Theorem 2.1.4 ([50]) *Every* MTL*-algebra is isomorphic to a subdirect product of a family of linearly ordered* MTL*-algebras.*

Consequently:

Theorem 2.1.5 ([50]) *Let* Γ *be a theory over* MTL*, and* φ *be a formula. Then the following are equivalent:*

1. $\Gamma \vdash_{\mathrm{MTL}} \varphi$,
2. *for each* MTL*-chain* $\mathcal{A}$ *and each* $\mathcal{A}$*-model* e *of* Γ*,* $e(\varphi) = 1$*.*

The variety of MTL-algebras $\mathbb{MTL}$ was also shown, by Ciabattoni, Metcalfe and Montagna in [21], to be generated by the class of finite MTL-algebras, since $\mathbb{MTL}$ enjoys the finite embeddability property.

Definition 2.1.6 Let $\mathcal{A} = \langle A, \{h_i^{\mathcal{A}}\}_{i\in I}\rangle$ be an algebra, where the $h_i^{\mathcal{A}}$ denote its functions, and let $B \subseteq A$. Then a *partial sub-algebra* $\mathcal{B} = \langle B, \{h_i^{\mathcal{B}}\}_{i\in I}\rangle$ of $\mathcal{A}$ is an algebra of the same type as $\mathcal{A}$ and with functions defined as

$$h_i^{\mathcal{B}}(b_1, \ldots, b_n) = \begin{cases} h_i^{\mathcal{A}}(b_1, \ldots, b_n) & \text{if } h_i^{\mathcal{A}}(b_1, \ldots, b_n) \in B \\ \text{undefined} & \text{otherwise.} \end{cases}$$

A partial sub-algebra $\mathcal{B}$ of $\mathcal{A}$ is embedded into $\mathcal{A}$ if there is a injective function f from B to A such that, if $h_i^{\mathcal{B}}(b_i, \ldots, b_n)$ is defined, then

$$f(h_i^{\mathcal{B}}(b_1, \ldots, b_n)) = h_i^{\mathcal{A}}(f(b_1), \ldots, f(b_n)).$$

Given two algebras $\mathcal{A}$ and $\mathcal{B}$ of the same language, we say that $\mathcal{A}$ is *partially embeddable* into $\mathcal{B}$ if every finite partial sub-algebra of $\mathcal{A}$ is embeddable into $\mathcal{B}$.

Definition 2.1.7 A class of algebras $\mathbb{K}$ of the same type has the *finite embeddability property (FEP)* iff every finite partial subalgebra $\mathcal{A}'$ of any algebra $\mathcal{A} \in \mathbb{K}$ can be partially embedded into a finite algebra $\mathcal{B} \in \mathbb{K}$.

Theorem 2.1.8 ([21]) *The variety of* MTL*-algebras has the finite embeddability property.*

Hence the following result is immediate:

Theorem 2.1.9 *Let* Γ *be a theory over* MTL*, and* φ *be a formula. Then the following are equivalent:*

1. $\Gamma \vdash_{\mathrm{MTL}} \varphi$,
2. *for each finite* MTL*-algebra* $\mathcal{A}$ *and each* $\mathcal{A}$*-model* e *of* Γ*,* $e(\varphi) = 1$*.*

The most interesting issue is whether a t-norm based logic is complete w.r.t. the related class of standard algebras. Let $[0,1]_{\mathbb{MTL}}$ be the class of standard MTL-algebras. We talk about *strong standard completeness* whenever, given a theory Γ and a formula φ, $\Gamma \vdash_{\mathrm{MTL}} \varphi$ iff for all $\mathcal{A} \in [0,1]_{\mathbb{MTL}}$ and every $\mathcal{A}$-evaluation e, if $e(\gamma) = 1$ for all $\gamma \in \Gamma$, then $e(\varphi) = 1$. We talk about *finite strong standard completeness* when the above equivalence holds for finite theories only. We talk about *standard completeness* whenever Γ is empty. Clearly strong standard completeness implies finite strong standard completeness, which, in turn, implies standard completeness.

MTL was shown to be strongly standard complete by Jenei and Montagna in [95]. Given any countable MTL-chain $\mathcal{A} = \langle A, *, \Rightarrow, \leq, 0, 1\rangle$, the authors define a structure $\langle X, \star, \preceq, m, M\rangle$ which is shown to be a commutative dense linearly ordered integral monoid, where $\star$ is a left-continuous operation w.r.t. the order topology. $\langle X, \star, \preceq, m, M\rangle$ also is order-isomorphic to the structure $\langle \mathbb{Q} \cap [0,1], \star', \leq, 0, 1\rangle$. $\langle \mathbb{Q} \cap [0,1], \star', \leq, 0, 1\rangle$ can be embedded into an analogous structure $\langle [0,1], \hat{\star}, \leq, 0, 1\rangle$ over the real unit interval, which is the reduct of a linearly ordered MTL-algebra $\langle [0,1], \hat{\star}, \hat{\Rightarrow}, \leq, 0, 1\rangle$ where the initial MTL-chain $\mathcal{A}$ can be embedded. This means that every countable MTL-chain can be embedded into a standard MTL-algebra. Consequently:

Theorem 2.1.10 ([95]) *Let Γ be a theory over* MTL*, and φ be a formula. Then the following are equivalent:*

1. $\Gamma \vdash_{\mathrm{MTL}} \varphi$*,*
2. *for each* MTL*-algebra $\mathcal{A} \in [0,1]_{\mathbb{MTL}}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$.*

The obvious consequence of the above theorem is that the variety generated by $[0,1]_{\mathbb{MTL}}$ coincides with the whole variety of MTL-algebras. Hence, MTL is the logic of left-continuous t-norms and their residua.

2.1.2 Schematic extensions of MTL

We focus, now, on the main schematic extensions of MTL.

Definition 2.1.11 (1) An *axiom schema* given by a formula $\Phi(p_1, \ldots, p_n)$ is the set of all formulas $\Phi(\varphi_1, \ldots, \varphi_n)$ resulting by the substitution, for each $i = 1, \ldots, n$, of φ_i for p_i in $\Phi(p_1, \ldots, p_n)$.

(2) A logical calculus $\mathcal{L}$ is a *schematic extension* of MTL if it results from MTL by adding (finitely or infinitely many) axiom schemata to its axioms. The only deduction rule is modus ponens.

(3) Let $\mathcal{L}$ be a schematic extension of MTL and let $\mathcal{A}$ be an MTL-algebra. Then $\mathcal{A}$ is an $\mathcal{L}$*-algebra* if all the axioms of $\mathcal{L}$ are $\mathcal{A}$-tautologies.

The logic SMTL [76] is obtained from MTL by adding the axiom schema of *pseudocomplementation*

(PC) $\varphi \wedge \neg\varphi \to \overline{0}$.

The equivalent algebraic semantics for SMTL is given bt SMTL-algebras, i.e. MTL-algebras satisfying

$$\neg(x \sqcap \neg x) = 1, \tag{2.1}$$

that expresses the absence of zero-divisors. In a standard SMTL-algebra $*$ corresponds to a left-continuous t-norm without zero-divisors.

The logic ΠMTL [76] is obtained from MTL by adding the axiom schema of *cancellation* [115]

(C) $\neg\varphi \vee ((\varphi \to \varphi\&\psi) \to \psi)$.

A ΠMTL-algebra is an MTL-algebra satisfying

$$\neg x \sqcup ((x \Rightarrow (x * y)) \Rightarrow y) = 1. \tag{2.2}$$

In a standard ΠMTL-algebra $*$ corresponds to a left-continuous t-norm satisfying the cancellative property.

The logic IMTL is obtained from MTL by adding the axiom schema of *involution*

(Inv) $\neg\neg\varphi \to \varphi$.

The equivalent algebraic semantics for IMTL is given by IMTL-*algebras*, i.e. MTL-algebras satisfying

$$\neg\neg x = x. \tag{2.3}$$

In a standard IMTL-algebra $*$ corresponds to a left-continuous t-norm whose associated negation is involutive.

The logic WNM is obtained from MTL by adding the axiom schema of *weak nilpotent minimum*

(WNM) $(\varphi\&\psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi\&\psi)$.

A WNM-*algebra* is an MTL-algebra satisfying the following equation

$$(x * y \Rightarrow 0) \sqcup (x \sqcap y \Rightarrow x * y) = 1. \tag{2.4}$$

Clearly a WNM-standard algebra is based on a weak nilpotent minimum t-norm.

The logic NM is obtaind from WNM by adding the axiom schema (Inv). An NM-*algebra* is a WNM-algebra satisfying (2.3), and, clearly, the standard NM-algebra is based on the Nilpotent Minimum t-norm.

The logics BL, Ł, SBL, and Π (see [75]) are obtained from MTL, IMTL, SMTL, and ΠMTL, respectively, by adding the axiom schema of divisibility

(Div) $\varphi \wedge \psi \to \varphi\&(\varphi \to \psi)$.

Similarly, BL, MV, SBL, and Π-algebras are obtained from MTL, IMTL, SMTL, and ΠMTL-algebras, respectively, by adding the equation

$$x \sqcap y = x * (x \Rightarrow y). \tag{2.5}$$

Notice that for logics based on continuous t-norms $\varphi \wedge \psi$ is definable as $\varphi \& (\varphi \to \psi)$. This corresponds to the *divisibility property*, i.e. to the fact that for all continuous t-norms

$$x * (x \Rightarrow_* y) = \min(x, y).$$

In a BL-standard algebra, $*$ corresponds to a continuous t-norm; in a SBL-standard algebra it corresponds to a continuous t-norm without zero-divisors. In Ł the monoidal operation over $[0, 1]$ is the Łukasiewicz t-norm, while in Π it is the Product t-norm.

MV-algebras can be also seen as structures $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$ satisfying the following equations (see [22]):

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(MV2) $x \oplus y = y \oplus x$,

(MV3) $x \oplus 0 = x$,

(MV4) $\neg\neg x = x$,

(MV5) $x \oplus \neg 0 = 0$,

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

It is easy to see that in this form MV-algebras are termwise equivalent to commutative integral bounded residuated lattices satisfying prelinearity, divisibility and the involutive property. Indeed, the monoidal operation $*$ with neutral element 1 can be defined as

$$x * y = \neg(\neg x \oplus \neg y),$$

while the implication connective is definable as

$$x \Rightarrow y = \neg x \oplus y.$$

The order relation is obtained by

$$x \leq y \text{ iff } x \Rightarrow y = 1,$$

while lattice operations are given by

$$x \sqcap y = x * (\neg x \oplus y) \text{ and } x \sqcup y = (x * \neg y) \oplus y.$$

The Gödel Logic G [75] is obtained by adding to BL the axiom schema of Contraction

(Con) $\varphi \to \varphi \& \varphi$.

A G-*algebra* (Gödel algebra) is a BL-algebra satisfying the idempotence law:

$$x * x = x. \tag{2.6}$$

A Gödel algebra can be also seen as a Heyting algebra (see Appendix B) satisfying the prelinearity condition.

Finally, notice that Classical Logic can be obtained from MTL by adding the law of excluded middle :

(EM) $\varphi \vee \neg\varphi$.

The notions of proof, evaluation, model and tautology given for MTL are obviously extended to all the above logics. All the foregoing algebras form classes that are varieties, and constitute the equivalent variety semantics of the related logics. Furthermore, the following theorem holds:

Theorem 2.1.12 ([75]) *Let $\mathcal{L}$ be a schematic extension of* MTL*. Let Γ be a theory over $\mathcal{L}$, and φ be a formula. Then the following are equivalent:*

1. $\Gamma \vdash_{\mathcal{L}} \varphi$,
2. *for each $\mathcal{L}$-chain $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$,*
3. *for each $\mathcal{L}$-algebra $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$.*

IMTL, SMTL (see [21]), BL, and SBL (see [53, 2]) enjoy the finite embeddability property, hence:

Theorem 2.1.13 *Let $\mathcal{L}$ be any among* IMTL*,* SMTL*,* BL*, and* SBL*. Let Γ be a theory over $\mathcal{L}$, and φ be a formula. Then the following are equivalent:*

1. $\Gamma \vdash_{\mathcal{L}} \varphi$,
2. *for each finite $\mathcal{L}$-algebra $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$.*

Logics based on continuous t-norms have a nice characterization in terms of hoops, that are algebraic structures first studied by Büchi and Owens in [14], and recently deeply investigated by Blok and Ferreirim in [9].

Definition 2.1.14 A *hoop* is a structure $\mathcal{H} = \langle H, *, \Rightarrow, 1\rangle$ such that

i. $\langle H, *, \Rightarrow, 1\rangle$ is a commutative monoid,

ii. $\Rightarrow$ is a binary operation satisfying the following properties:

- $x \Rightarrow x = 1$,
- $x * (x \Rightarrow y) = y * (y \Rightarrow x)$,
- $x \Rightarrow (y \Rightarrow z) = (x * y) \Rightarrow y$.

Hoops can be shown to be partially ordered residuated integral commutative monoids, where the order is defined as

$$x \leq y \text{ iff } x \Rightarrow y = 1.$$

A *bounded hoop* is an algebra $\mathcal{H} = \langle H, *, \Rightarrow, 0, 1\rangle$, such that $\langle H, *, \Rightarrow, 1\rangle$ is a hoop and $0 \leq x$ for all $x \in H$.

Basic hoops are subdirect products of totally ordered hoops, and they form a variety axiomatized by the the equation:

$$(x \Rightarrow y) \Rightarrow z \leq ((y \Rightarrow x) \Rightarrow z) \Rightarrow z.$$

A BL-algebra is a bounded basic hoop (see [1]).

A *Wajsberg hoop* is a hoop $\mathcal{H}$ satisfying

$$(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x.$$

Wajsberg algebras are bounded Wajsberg hoops, and are termwise equivalent to MV-algebras (see [62]).

A *cancellative hoop* is a hoop satisfying

$$x \Rightarrow (x * y) = y.$$

Note that a cancellative hoop is a Wajsberg hoop.

Hoops can be obtained by means of an ordinal sum construction. Let $\langle I, \leq\rangle$ be a linearly ordered set with minimum i_0. For all $i \in I$, let $\mathcal{H}_i$ be a hoop such that for $i \neq j$, $H_i \cap H_j = \{1\}$. Then $\bigoplus_{i\in I} \mathcal{H}_i$ is called the *ordinal sum* of the family $(\mathcal{H}_i)_{i\in I}$, whose universe is given by $\bigcup_{i\in I} H_i$, and whose operations are given by

$$x \Rightarrow y = \begin{cases} x \Rightarrow_{\mathcal{H}_i} y & x, y \in H_i \\ y & j < i,\ x \in H_i,\ y \in H_j \\ 1 & i < j,\ x \in H_i\backslash\{1\},\ y \in H_j \end{cases},$$

and

$$x * y = \begin{cases} x *^{\mathcal{H}_i} y & x, y \in H_i \\ y & j < i,\ x \in H_i,\ y \in H_j\backslash\{1\} \\ x & i < j,\ x \in H_i\backslash\{1\},\ y \in H_j \end{cases}.$$

Notice that if $\mathcal{H}_{i_0}$ is a BL-chain and for all $i \in I\backslash\{i_0\}$, $\mathcal{H}_i$ is a linearly ordered hoop, then $\bigoplus_{i\in I} \mathcal{H}_i$ is a BL-algebra (see [2]). Moreover, the converse holds as well.

Theorem 2.1.15 ([2]) *Every linearly ordered* BL-*algebra is the ordinal sum of a family of Wajsberg hoops whose first element is bounded, i.e. is a Wajsberg algebra.*

The above theorem implies:

Theorem 2.1.16 ([2]) *Every continuous t-norm on* $[0, 1]$ *is the ordinal sum of a family of Wajsberg t-norms.*

As for SBL-chains, we have a similar characterization:

Theorem 2.1.17 ([114]) *Every linearly ordered* BL*-algebra* $\mathcal{A}$ *is an* SBL*-algebra iff* $\mathcal{A}$ *has the form* $\mathcal{A} = \mathbf{2} \oplus \mathcal{B}$*, where* $\mathbf{2}$ *is the two-element Wajsberg chain, and* $\mathcal{B}$ *is a (possibly trivial) linearly ordered* BL*-algebra.*

The most interesting issue is whether a t-norm based logic $\mathcal{L}$ is complete w.r.t. the related class of standard algebras $[0,1]_{\mathcal{L}}$. The following theorem characterizes the necessary and sufficient conditions for a logic $\mathcal{L}$ to enjoy some kind of standard completeness.

Theorem 2.1.18 ([55]) *Let* $\mathcal{L}$ *be a schematic extension on* MTL*, and let* $\mathbb{L}$ *be its equivalent variety semantics. Let* $\mathcal{QV}([0,1]_{\mathcal{L}})$ *be the quasivariety generated by the class of* $\mathcal{L}$*-standard algebras, and let* $\mathbf{ISP}([0,1]_{\mathcal{L}})$ *be the class of standard algebras closed under isomorphism, subalgebras and direct product (see Appendix B). Then:*

i. $\mathcal{L}$ *is standard complete iff* $\mathbb{L} = \mathcal{V}([0,1]_{\mathcal{L}})$

ii. $\mathcal{L}$ *is finitely strongly standard complete iff* $\mathbb{L} = \mathcal{QV}([0,1]_{\mathcal{L}})$ *iff every* $\mathcal{L}$*-chain is partially embeddable into a standard* $\mathcal{L}$*-algebra.*

iii. $\mathcal{L}$ *is strongly standard complete iff every countable chain of* $\mathbb{L}$ *belongs to* $\mathbf{ISP}([0,1]_{\mathcal{L}})$ *iff every countable subdirectly irreducible chain of* $\mathbb{L}$ *is embeddable into a standard* $\mathcal{L}$*-algebra.*

Only a few of the above logics enjoy strong standard completeness. Those logics were shown to have the following strong property.

Definition 2.1.19 [47] Let $\mathcal{L}$ be any schematic extension of MTL, and let $\mathbb{L}$ be its equivalent variety semantics. $\mathcal{L}$ has the *real embedding property* ($\mathbb{R}$-E), if any linearly ordered finite or countable structure of $\mathbb{L}$ can be embedded into a structure in $\mathbb{L}$ whose lattice reduct is the real unit interval $[0,1]$.

MTL [95], IMTL, SMTL [47], WNM, NM [50], and G [75] were shown to enjoy the real embedding property (see Table 2.1). Therefore they satisfy condition (iii) of the above theorem, and, consequently, they are strongly standard complete (hence finitely strongly standard complete and standard complete).

	MTL	IMTL	SMTL	WNM	NM	ΠMTL	BL	SBL	Π	G	Ł
$\mathbb{R}$-E	Yes	Yes	Yes	Yes	Yes	No	No	No	No	Yes	No

Table 2.1: *Real embedding property for the main* MTL *extensions.*

ΠMTL, BL, SBL, Π, and Ł are not strongly standard complete, and, in fact, they do not enjoy the real embedding property. However, they are standard complete and finitely strongly standard complete. Indeed, they satisfy condition (ii) of the above theorem. Consequently they also satisfy condition (i), and their related standard algebras generate the whole variety. ΠMTL was proved to be finitely strongly standard complete by Horčík in [84]. The same result was shown to hold for BL and SBL by Hájek in [74] and by Cignoli, Esteva, Godo and Torrens in [24]. Finite strong standard completeness for Product logic was proved by Hájek, Godo, and Esteva in [78, 75]. Finally completeness for Łukasiewicz logic was first proved by Rose and Rosser in [128] and by Chang in [18, 19].

We now recall some special features of the completeness property of some of the above logics.

Both, Łukasiewicz and Product logics are strongly connected to Abelian ℓ-groups [65] (see Appendix B). Given an ℓ-group G with a strong unit u, $\Gamma(G, u)$ denotes the MV-algebra $\langle [0, u], \oplus, \neg, 0\rangle$, where $x \oplus y = (x+y) \sqcap u$, and $\neg x = u - x$. Mundici showed in [119] that Γ is an equivalence functor from the category of Abelian ℓ-groups with strong unit to the category of MV-algebras (see also [22]).

Cignoli and Torrens proved that also Product algebras have a strong connection to Abelian ℓ-groups [25]. Indeed, let G be an Abelian ℓ-group and $\bot$ and element not included in G. Let G^- denote the negative cone of G, i.e. $\{x \in G : x \sqcap 0 = x\}$. We can define on the set $G^- \cup \{\bot\}$ the following operations: $x \odot y = x + y$ if $x, y \in G^-$, $x \odot y = \bot$ otherwise; $x \to y = 0 \sqcap (y - x)$ if $x, y \in G^-$, $x \to y = 0$ if $x = \bot$, and $x \to y = \bot$ if $x \in G^-$ and $y = \bot$. The algebra $\langle G^- \cup \{\bot\}, \odot, \to, \bot, 0\rangle$ is a Product algebra denoted by $\mathfrak{P}(G)$. In particular for any non-trivial Product algebra $\mathcal{A}$ there exists a unique (up to isomorphism) Abelian ℓ-group G such that $\mathcal{A} \cong \mathfrak{P}(G)$ (iff for all $z \in \mathcal{A}$, with $z > 0$, $z \to 0 = 0$). In this case the group is denoted by $\mathfrak{G}(\mathcal{A})$. Cignoli and Torrens showed that $\mathfrak{P}$ and $\mathfrak{G}$ are functors that define an equivalence between the category of Product algebras and the category of Abelian ℓ-groups.

Now, by a famous result of Gurevich and Kokorin [72] any linearly ordered Abelian group $\mathcal{G}$ is partially embeddable into the Abelian group of reals, i.e. for each finite $X \subseteq G$ there is a finite $Y \subseteq \mathbb{R}$ and an injective mapping f of X onto Y that is a partial isomorphism. This clearly implies that both Łukasiewicz and Product logics satisfy condition (ii) of Theorem 2.1.18.

As mentioned above BL-algebras and SBL-algebras are directly connected to the ordinal sum of hoops. Recall that they both enjoy the finite embeddability property, hence the variety of BL-algebras (SBL-algebras) is generated as a quasivariety by the finite linearly ordered members of the variety. Since each finite BL-chain (SBL-chain) has a finite number of components (as a consequence of Theorem 2.1.15, and Theorem 2.1.17), and each of them is partially embeddable into its related standard algebra, we obtain the following characterization.

Theorem 2.1.20 ([2, 114]) *The variety of* BL*-algebras is generated as a quasivariety by the class of all algebras which are ordinal sums of finitely many copies of the standard* MV*-algebras.*

Theorem 2.1.21 ([2, 114]) *The variety of* SBL*-algebras is generated as a quasivariety by both the class of all algebras of the form* $[0,1]_\Pi \oplus [0,1]_Ł \oplus \cdots \oplus [0,1]_Ł$*, or of the form* $[0,1]_G \oplus [0,1]_Ł \oplus \cdots \oplus [0,1]_Ł$*.*

Hence, both BL and SBL enjoy the partial embeddability property.

	MTL	IMTL	SMTL	WNM	NM	ΠMTL	BL	SBL	Π	G	Ł
SC	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
FSSC	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
SSC	Yes	Yes	Yes	Yes	Yes	No	No	No	No	Yes	No

Table 2.2: *Standard* (SC)*, finite strong standard* (FSSC) *and strong standard completeness* (SSC) *for the main* MTL *extensions.*

A general overview about standard, finite strong standard and strong standard completeness for the main extensions of MTL can be found in Table 2.2.

In Chapter 5 we will provide other completeness results for some of the above logics. Indeed, we will show that they are finitely strongly standard complete w.r.t. algebras based on t-norms definable as piecewise rational functions, i.e. t-norms definable in the ŁΠ$\frac{1}{2}$-logic.

2.2 Expansions of t-norm based logics

The above logics can be expanded by introducing new connectives in the language along with axioms defining their behavior. Given a logic $\mathcal{L}$, an expansion $\mathcal{L}'$ is called a *conservative expansion* of $\mathcal{L}$, if for all theories Γ and formula φ in the language of $\mathcal{L}$, we have: $\Gamma \vdash_{\mathcal{L}'} \varphi$ entails $\Gamma \vdash_{\mathcal{L}} \varphi$.

Most of the expansions of MTL have an equivalent algebraic semantics. Given an extension $\mathcal{L}$ of MTL, let $\mathcal{L}'$ be an expansion of $\mathcal{L}$ obtained adding the n-ary connectives λ_i to the language of $\mathcal{L}$, plus a set of formulas Γ in the language of $\mathcal{L}'$. If

$$\{\varphi_1 \leftrightarrow \psi_1, \ldots, \varphi_m \leftrightarrow \psi_m\} \vdash_{\mathcal{L}'} \lambda_i(\varphi_1, \ldots, \varphi_m) \Leftrightarrow \lambda_i(\psi_1, \ldots, \psi_m)$$

holds for each λ_i, then the equivalent algebraic semantics for $\mathcal{L}'$ is the variety of algebras in the language of $\mathcal{L}'$, axiomatized by the axioms of $\mathbb{L}$ (i.e. the equivalent algebraic semantics of $\mathcal{L}$), plus the equations $\{\varphi = 1 \mid \varphi \in \Gamma\}$.

Proposition 2.2.1 ([54, 10]) *Under the previous hypothesis, $\mathcal{L}'$ is a conservative expansion of $\mathcal{L}$ iff every $\mathcal{L}$-algebra is a subreduct of some $\mathcal{L}'$-algebra.*

From results in [31], it follows that every (finitary) expansion $\mathcal{L}$ of MTL satisfying the following prelinearity property, for each theory Γ,

$$(\star) \qquad \Gamma \cup \{\varphi \to \psi\} \vdash_{\mathcal{L}} \chi \text{ and } \Gamma \cup \{\psi \to \varphi\} \vdash_{\mathcal{L}} \chi \text{ imply } \Gamma \vdash_{\mathcal{L}} \chi,$$

inherits completeness w.r.t. the class of linearly ordered algebras. Indeed, it follows:

Proposition 2.2.2 ([32]) *Let $\mathcal{L}$ be an algebraizable expansion of* MTL *satisfying $(\star)$. Then, for every set of formulas $\Gamma \cup \{\varphi\}$ in the language of $\mathcal{L}$, it holds that*

$$\Gamma \vdash_{\mathcal{L}} \varphi \textit{ iff } \{\psi = \overline{1} \mid \psi \in \Gamma\} \models_{\mathcal{L}_{lo}} \varphi = \overline{1},$$

where $\mathcal{L}_{lo}$ is the class of linearly ordered $\mathcal{L}$-algebras.

Interesting expansions are obtained by introducing the Δ operator, rational truth constants, and an additional strong negation. Here, we briefly review the first two kinds of expansions. Expansions with an additional strong negation will be investigated in the next chapter.

Finally, we will also recall some results concerning logics obtained by joining different t-norms.

2.2.1 Delta expansions

The Delta projector Δ was first introduced by Baaz in [6]. Let $\mathcal{L}$ be any schematic extension of MTL, and expand its language by means of a unary operator Δ. In that language we can define the logic $\mathcal{L}_\Delta$ whose axioms are those of $\mathcal{L}$ plus the following:

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$,
(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$,
(Δ3) $\Delta\varphi \to \varphi$,
(Δ4) $\Delta\varphi \to \Delta\Delta\varphi$,
(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$.

The inference rule are modus ponens and *Necessitation*: from φ derive $\Delta\varphi$. The notion of proof is the usual one.

The algebraic semantics of $\mathcal{L}_\Delta$ is given by $\mathcal{L}_\Delta$-algebras, i.e. $\mathcal{L}$-algebras with a unary operator δ, satisfying the following conditions for all x, y:

(δ1) $\delta(x) \sqcup \neg\delta(x) = 1$
(δ2) $\delta(x \sqcup y) \leq (\delta(x) \sqcup \delta(y))$
(δ3) $\delta(x) \leq x$
(δ4) $\delta(x) \leq \delta(\delta(x))$
(δ5) $\delta(x \Rightarrow y) \leq (\delta(x) \Rightarrow \delta(y))$
(δ6) $\delta(1) = 1$

Notice that in a linearly ordered $\mathcal{L}_\Delta$-algebra $\delta(x) = 1$ if $x = 1$, and $\delta(x) = 0$ otherwise. The notions of evaluation, model and tautology are obviously adapted from the above case.

Notice that in general the following form of deduction theorem fails:

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ iff there is an } n \text{ such that } \Gamma \vdash \varphi^n \to \psi.$$

Indeed, $\varphi \vdash \Delta\varphi$, but for each n, $\not\vdash \varphi^n \to \Delta\varphi$. Take, for example, a strict continuous t-norm $*$, hence isomorphic to the Product. Then for all $0 < x < 1$, $x^n > 0$. However, every $\mathcal{L}$ satisfies a special form of deduction theorem:

Theorem 2.2.3 (Delta Deduction Theorem, [75]) *Let Γ be a theory over $\mathcal{L}_\Delta$, and let φ, ψ be formulas. Then,*

$$\Gamma \cup \{\varphi\} \vdash \psi \textit{ iff } \Gamma \vdash \Delta\varphi \to \psi.$$

Being $\mathcal{L}_\Delta$-chains simple algebras, the following holds:

Lemma 2.2.4 ([51]) *A $\mathcal{L}_\Delta$-algebra is subdirectly irreducible iff it is linearly ordered.*

Consequently, we have a subdirect decomposition theorem:

Theorem 2.2.5 ([51]) *Every $\mathcal{L}_\Delta$-algebra is isomorphic to a subdirect product of linearly ordered $\mathcal{L}_\Delta$-algebras.*

Theorem 2.2.6 ([31]) *Let Γ be a theory over $\mathcal{L}_\Delta$, and φ be a formula. Then the following are equivalent:*

1. $\Gamma \vdash_{\mathcal{L}_\Delta} \varphi$,
2. *for each $\mathcal{L}_\Delta$-chain $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$,*
3. *for each $\mathcal{L}_\Delta$-algebra $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$.*

It is easy to see that the following theorem holds (see [48]):

Theorem 2.2.7 *Let $\mathcal{L}$ be a schematic extension of* MTL. *Then:*

1. *if $\mathcal{L}_\Delta$ is standard complete so is $\mathcal{L}$;*
2. *if $\mathcal{L}_\Delta$ is strongly standard complete so is $\mathcal{L}$;*
3. *if $\mathcal{L}_\Delta$ is finitely strongly standard complete so is $\mathcal{L}$.*

Finally, we mention a series of results that will be particularly useful in the next chapter[2]:

Lemma 2.2.8 *Let $\mathcal{L}'_\Delta$ be an algebraizable conservative expansion of $\mathcal{L}_\Delta$. Let $\mathcal{A}$ be an $\mathcal{L}'_\Delta$-algebra and $\mathcal{A}^-$ its underlying $\mathcal{L}_\Delta$-algebra. Suppose that for every n-ary operation λ_i and for every $x_1, \ldots, x_m, y_1, \ldots, y_m \in A$,*

$$(+) \qquad \prod_{i=1}^{n} \delta(x_i \Leftrightarrow y_i) \leq (\lambda_i(x_1, \ldots, x_m) \Leftrightarrow \lambda_i(y_1, \ldots, y_m)).$$

Then $\mathcal{A}$ and $\mathcal{A}^-$ have the same congruences. In particular, $\mathcal{A}$ is subdirectly irreducible iff so is $\mathcal{A}^-$.

Consequently:

Lemma 2.2.9 *Every $\mathcal{L}'_\Delta$-algebra is subdirectly irreducible iff it is linearly ordered.*

And finally:

Theorem 2.2.10 *Suppose that an algebraizable conservative expansion $\mathcal{L}'_\Delta$ of $\mathcal{L}_\Delta$, satisfies the conditions of Lemma 2.2.8. Then, every $\mathcal{L}'_\Delta$-algebra is isomorphic to a subdirect product of linearly ordered $\mathcal{L}'_\Delta$-algebras.*

[2] Actually, this is proved for $\text{Ł}\Pi\frac{1}{2}$ and in general for expansions of MV_Δ-algebras in [52], but a very similar proof works for algebraizable conservative expansions of $\mathcal{L}_\Delta$.

2.2.2 Rational expansions

Expansions by means of truth-constants are particularly interesting. Indeed, they allow reasoning with partial degrees of truth by interpreting such constants as rationals from the unit interval. The expressive power of the language then is significantly improved. For example, we can express explicitly at the syntactic level that the truth-value of a formula φ equals a certain degree r, i.e. $\varphi \leftrightarrow \overline{r}$, is at most r, i.e. $\varphi \rightarrow \overline{r}$, or is at least r, i.e. $\overline{r} \rightarrow \varphi$. General treatments of rational expansions of logics based on t-norms have been given in [48, 55].

Let $\mathcal{L}$ be an extension of MTL being the logic of a given left-continuous t-norm $*$. Let $[0,1]_* = \langle [0,1], *, \Rightarrow, \min, \max, 0, 1\rangle$ be the standard algebra based on $*$. Let $\mathcal{C} = \langle C, *, \Rightarrow, \min, \max, 0, 1\rangle$ be a countable subalgebra of $[0,1]_*$. Expand the language of $\mathcal{L}$ by the set of constants $\{\overline{r} : r \in C\backslash\{0,1\}\}$. $\mathcal{RL}$ is the expansion of $\mathcal{L}$ in the expanded language obtained by adding the book-keeping axioms for every $r, s \in C$:

$$\overline{r}\&\overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \rightarrow \overline{s} \leftrightarrow \overline{r \Rightarrow s}$$

Let $*$ be a left-continuous t-norm and $\mathcal{C}$ a countable subalgebra of $[0,1]_*$. The algebraic counterparts are $\mathcal{RL}$-algebras, i.e. structures

$$\mathcal{A} = \langle A, *', \Rightarrow_{*'}, \sqcap, \sqcup, \{\overline{r} : r \in C\}\rangle$$

such that

(*i*) $\mathcal{A} = \langle A, *', \Rightarrow_{*'}, \sqcap, \sqcup, 0, 1\rangle$ is an $\mathcal{L}$-algebra, and

(*ii*) for every $r, s \in C$ the following indentities hold:

$$r *' s = r * s$$
$$r \Rightarrow_{*'} s = r \Rightarrow s.$$

The standard $\mathcal{RL}$-chain is the algebra

$$[0,1]_{\mathcal{RL}} = \langle [0,1], *, \Rightarrow, \min, \max, \{r : r \in C\}\rangle,$$

i.e., the expansion of $[0,1]_{\mathcal{L}}$ where the truth-constants are interpreted as themselves.

Each class of $\mathcal{RL}$-algebras forms a variety. Moreover, each $\mathcal{RL}$ is complete w.r.t. to its equivalent algebraic semantics (see [55]). Furthermore, all $\mathcal{RL}$-algebras are representable as a subdirect product of $\mathcal{RL}$-chains. Hence, each $\mathcal{RL}$ is also complete w.r.t. to the related class of linearly ordered algebras [55].

A very well-known example of rational expansion is the one obtained from Łukasiewicz logic by introducing rational truth constants for each rational $\overline{r} \in [0,1]$. Any $\overline{r}$, then, is a formula whose intended evaluation e is $e(\overline{r}) = r$. The logic so obtained is called Rational Pavelka Logic (RPL), and it allows to prove partially true conclusions from partially true premises. RPL was first studied by Pavelka in [125] and then simplified by Hájek in [75]. RPL-axioms are those of Łukasiewicz logic, plus the following bookkeeping axioms:

$$\begin{array}{rcl} \neg\overline{r} & \leftrightarrow & \overline{1-r} \\ \overline{r}\to\overline{s} & \leftrightarrow & \overline{\min(1,1-r+s)}. \end{array}$$

Given a theory Γ and a formula φ over RPL, we can define

- the *truth-degree* of φ over Γ, $\|\varphi\|_\Gamma = \inf\{e(\varphi) \mid e \text{ is a model of } \Gamma\}$;
- the *provability degree* of φ over Γ, $|\varphi|_\Gamma = \sup\{r \mid \Gamma \vdash \overline{r} \to \varphi\}$.

In [75], RPL is shown to be strongly complete w.r.t. finite theories, moreover it enjoys a special type of completeness called *Pavelka-style* completeness, i.e.:

$$\|\varphi\|_\Gamma = |\varphi|_\Gamma,$$

for each theory Γ and each formula φ.

Pavelka-style completeness for RPL strongly relies on the fact that all connectives in Łukasiewicz logic are interpreted by continuous functions over $[0,1]$ (see [75]). This type of completeness was studied for the rational expansion of Product logic RΠ (see [51]), obtained by adding the bookkeeping axioms

$$\begin{array}{rcl} \overline{r}\&\overline{s} & \leftrightarrow & \overline{r\cdot s} \\ \overline{r}\to\overline{s} & \leftrightarrow & \overline{r\Rightarrow_\pi s}, \end{array}$$

plus the infinitary rule:

$$\text{from } \varphi\to\overline{r}, \text{ for each } r>0, \text{ derive } \varphi\to\overline{0},$$

which was introduced in order to overcome the problem that the residuum of the Product t-norm is not continuous in $(0,0)$. RΠ was shown to enjoy Pavelka-style completeness [51].

Both Product and Gödel logics can be expanded with a countable set of truth constants, obtaining the logics RG and $\Pi(C)$ (see [54, 131]). They were proved to be standard complete, but not finitely strongly standard complete (and, consequently not strongly standard complete).

A peculiar kind of expansion which allows the representation of rational truth-constants is given by the Rational Łukasiewicz logic RŁ introduced by Gerla in [64]. RŁ is obtained by expanding Łukasiewicz logic by the unary connectives δ_n, for each $n \in \mathbb{N}$, plus the following axioms:

(D1) $\underbrace{\delta_n\varphi \oplus \cdots \oplus \delta_n}_{n}\varphi \leftrightarrow \varphi$

(D2) $\neg\delta_n\varphi \oplus \neg(\underbrace{\delta_n\varphi \oplus \cdots \oplus \delta_n\varphi}_{n-1})$.

The algebraic semantics for RŁ is given by DMV-algebras (divisible MV-algebras), i.e. structures $\mathcal{A} = \langle A, \oplus, \neg, \{\delta_n\}_{n\in\mathbb{N}}, 0, 1\rangle$ such that $\langle A, \oplus, \neg, 0, 1\rangle$ is an MV-algebra and the following equations hold for all $x \in A$ and $n \in \mathbb{N}$:

(δ_n1) $n.\delta_n x = x$;

(δ_n2) $\delta_n x * (n-1).\delta_n x = 0$;

where by $n.x$ we denote the element of A inductively defined by $0.x = 0$, $(n-1).x = n.x \oplus x$. Notice that an evaluation into the real unit interval is extended for the connectives δ_n as follows:

$$e(\delta_n \varphi) = \frac{e(\varphi)}{n}.$$

In this way we can define in RŁ all rationals in $[0, 1]$. Indeed,

- $\frac{1}{n}$ is given by $\delta_n \overline{1}$, i.e. $e(\delta_n \overline{1}) = \frac{1}{n} \cdot 1$,
- $\frac{m}{n}$ is given by $m.\delta_n \overline{1}$, i.e. $e(m.\delta_n \overline{1}) = \underbrace{\frac{1}{n} \oplus \cdots \oplus \frac{1}{n}}_{m}$.

RŁ was shown to enjoy both finite strong standard completeness and Pavelka-style completeness (see [64] for all details). In particular, notice that RPL can be faithfully interpreted in RŁ.

2.2.3 Combinations of t-norms

Other interesting kinds of expansions are those obtained by joining the logics of different t-norms or by adding specific t-norms to certain logics. An example of the former case is given by the logics ŁΠ and $\text{Ł}\Pi\frac{1}{2}$, which will be extensively treated in Chapter 4 and Chapter 5. An example of the latter case is given by the Product Łukasiewicz logic PŁ and its expansions.

PŁ was introduced by Horčík and Cintula in [85]. Basically, PŁ is an expansion of Łukasiewicz logic by means of the Product conjunction, and its language is built up from a countable set of propositional variables, three binary connectives $\&_l$ (Łukasiewicz conjunction), $\to_l$ (Łukasiewicz implication), $\&_\pi$ (Product conjunction), and the truth constant $\overline{0}$. The axioms of PŁ are those of Łukasiewicz logic, plus the following additional axioms:

(PŁ1) $\varphi \&_\pi (\psi \&_l (\chi \to_l \overline{0})) \leftrightarrow (\varphi \&_\pi \psi) \&_l ((\varphi \&_\pi \chi) \to_l \overline{0})$,

(PŁ2) $\varphi \&_\pi (\chi \&_\pi \psi) \leftrightarrow (\varphi \&_\pi \psi) \&_\pi \chi$,

(PŁ3) $\varphi \to_l \varphi \&_\pi \overline{1}$,

(PŁ4) $\varphi \&_\pi \psi \to_l \varphi$,

(PŁ5) $\varphi \&_\pi \psi \to_l \psi \&_\pi \varphi$.

PŁ$'$ extends PŁ by the deduction rule (ZD):

- if $\neg(\varphi \&_\pi \varphi)$, then $\neg \varphi$.

The algebraic semantics for PŁ is given by PŁ-algebras, i.e. structures $\mathcal{A} = \langle A, \oplus, \neg, *, 0, 1 \rangle$, where $\langle A, \oplus, \neg, 0, 1 \rangle$ is an MV-algebra and the following identities hold:

($\pi l1$) $(x * y) \ominus (x * z) = x * (y \ominus z)$,

($\pi l2$) $x * (y * z) = (x * y) * z$,

($\pi l3$) $x * 1 = x$,

($\pi l4$) $x * y = y * x$,

where $x \ominus y = \neg(\neg x \oplus y)$. Moreover, a PŁ$'$-algebra is a PŁ-algebra satisfying the following quasi-identity:

- if $x * x = 0$, then $x = 0$.

PŁ-algebras form a variety, while PŁ$'$-algebras form a quasi-variety. An example of standard PŁ-algebra (and standard PŁ$'$-algebra) is the structure

$$[0,1]_{\text{PŁ}} = \langle [0,1], \diamond_l, n_s, \cdot, \min, \max, 0, 1\rangle,$$

where $\diamond_l$ is the Łukasiewicz t-conorm, n_s is the standard involutive negation and $\cdot$ is the product of reals.

The following result was proved by Montagna in [113].

Theorem 2.2.11 ([113]) *The quasi-variety of* PŁ$'$*-algebras is generated by the standard algebra* $[0,1]_{\text{PŁ}}$.

Furthermore, Horčík and Cintula proved the following:

Theorem 2.2.12 ([85]) *The logic* PŁ$'$ *is finitely strongly standard complete, while the logic* PŁ *is not even standard complete.*

In [85], the authors also studied expansions of such logics by means of Baaz's Δ (plus axioms (Δ1)-(Δ5)), yielding the logics PŁ$_\Delta$ and PŁ$'_\Delta$. PŁ$'_\Delta$ can be equivalently defined as an extension of PŁ$_\Delta$ by the following axiom:

(ΔPŁ) $\Delta\neg(\varphi \&_\pi \varphi) \to \neg\varphi$.

PŁ$'_\Delta$ was shown to be finitely strongly standard complete.

The above logics have been also expanded by means of rational truth constants plus the related bookkeeping axioms, obtaining the logics RPPŁ, RPPŁ$'$ and the logics RPPŁ$_\Delta$ and RPPŁ$'_\Delta$ which also include the infinitary inference rule (IR):

- from $\overline{r} \to \varphi$, for all $r < 1$, derive φ,

(see [85] for all details). All those logics were shown to enjoy Pavelka-style completeness. Finite strong standard completeness was proved for RPPŁ$_\Delta$ and RPPŁ$'_\Delta$, that were also shown to be equivalent.

2.3 First-order logics

In this section we review the basic notions concerning the predicate calculi for MTL and for its schematic extensions. $\mathcal{L}\forall$ will stand for the predicate calculus of the logic $\mathcal{L}$.

We begin by expanding the propositional language with a set of predicates $Pred$, a set of object variables Var and a set of object constants $Const$, together with the two classical quantifiers $\forall$ and $\exists$. The notion of formula is easily generalized by saying that, if φ is a formula and $x \in Var$, then both $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulas.

Definition 2.3.1 Let $\mathcal{A}$ be a linearly ordered $\mathcal{L}$-algebra. An $\mathcal{A}$-*interpretation* for the predicate language of $\mathcal{L}\forall$ is a structure $\mathcal{M} = \langle M, (r_P)_{P\in Pred}, (m_c)_{c\in Const}\rangle$, where:

- M is a non-empty set,
- $r_P : M^{ar(P)} \to A$ for any $P \in Pred$, where $ar(P)$ stands for the ariety of the predicate P,
- $m_c \in M$ for each $c \in Const$.

For every evaluation of variables $v : Var \to M$, the truth value of a formula φ ($\|\varphi\|^{\mathcal{A}}_{M,v}$) is inductively defined as follows:

- $\|P(x, \ldots, c, \ldots)\|^{\mathcal{A}}_{M,v} = r_P(v(x), \ldots, m_c, \ldots)$, where $v(x) \in M$ for each variable x,
- The truth value commutes with the connectives of $\mathcal{L}$,
- $\|(\forall x)\varphi\|^{\mathcal{A}}_{M,v} = \inf\{\|\varphi\|^{\mathcal{A}}_{M,v'} : v(y) = v'(y)$ for all variables, except for $x\}$ and
  $\|(\exists x)\varphi\|^{\mathcal{A}}_{M,v} = \sup\{\|\varphi\|^{\mathcal{A}}_{M,v'} : v(y) = v'(y)$ for all variables, except for $x\}$,

if the infimum and supremum exist in $\mathcal{A}$, otherwise the truth value(s) remain(s) undefined.

A structure $\mathcal{M}$ is called $\mathcal{A}$-safe if all infima and suprema needed for the definition of the truth value of any formula exist in $\mathcal{A}$. In that case the truth value of a formula φ in a $\mathcal{A}$-safe structure $\mathcal{M}$ is just

$$\|\varphi\|^{\mathcal{A}}_{M} = \inf\{\|\varphi\|^{\mathcal{A}}_{M,v} : v : Var \to M\}.$$

Definition 2.3.2 The axioms for $\mathcal{L}\forall$ are those of $\mathcal{L}$ plus the following axioms for quantified formulas:

(∀1) $(\forall x)\varphi(x) \to \varphi(x/t)$ (t substitutable for x in φ),
(∀2) $(\forall x)(\psi \to \varphi) \to (\psi \to (\forall x)\varphi)$, ($x$ not free in ψ),
(∀3) $(\forall x)(\varphi \vee \psi) \to ((\forall x)\varphi \vee \psi)$, ($x$ not free in ψ),
(∃1) $\varphi(x/t) \to (\exists x)\varphi(x)$($t$ substitutable for x in φ),
(∃2) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to \psi)$, ($x$ not free in ψ).

Rules for $\mathcal{L}\forall$ are Modus Ponens and Generalization: from φ derive $(\forall x)\varphi$.

Definition 2.3.3 Let Γ be a theory over $\mathcal{L}\forall$.

(i) Γ is *consistent* if there is a formula φ unprovable in Γ.

(ii) Γ is *complete* if for each pair φ, ψ of closed formulas,

$$\Gamma \vdash \varphi \to \psi \text{ or } \Gamma \vdash \psi \to \varphi.$$

(iii) Γ is *Henkin* if for every closed formula of the form $(\forall x)\varphi(x)$ unprovable in Γ, there exists a constant c in the language of Γ such that $\varphi(c)$ is unprovable in Γ.

The notion of proof is an obvious adaptation of the one for $\mathcal{L}$. Moreover, the same deduction theorem holds.

The technique to prove completeness is the one carried out in [75] for BL$\forall$, that is:

(1) Given a theory Γ and a closed formula α such that $\Gamma \not\vdash \alpha$, there exists a complete Henkin supertheory $\hat{\Gamma}$ of Γ such that $\hat{\Gamma} \not\vdash \alpha$.

(2) For each complete Henkin theory Γ and every closed formula α such that $\Gamma \not\vdash \alpha$, there exist a linearly ordered $\mathcal{L}$-algebra $\mathcal{A}$ and an safe $\mathcal{A}$-model $\mathcal{M}$ of Γ such that $\|\alpha\|^{\mathcal{A}}_{M} < 1$.

Consequently:

Theorem 2.3.4 *Let Γ and φ be a theory and formula over $\mathcal{L}\forall$, respectively. Then $\Gamma \vdash \varphi$ iff for each linearly ordered $\mathcal{L}$-algebra $\mathcal{A}$ and each safe $\mathcal{A}$-model $\mathcal{M}$ of Γ,*

$$\|\varphi\|^{\mathcal{A}}_{M} = 1.$$

Standard completeness for first-order t-norm based logics does not hold for those logics which are not strongly standard complete. Then Ł$\forall$, ΠMTL$\forall$, BL$\forall$, SBL$\forall$, and $\Pi\forall$ are not complete w.r.t. to standard algebras. However, for the logics enjoying the real embedding property, it is easy to see that, under certain restrictions, standard completeness holds also for the predicate calculi.

Theorem 2.3.5 (Strong Standard Completeness) *Let $\mathcal{L}$ be a schematic extension of* MTL *satisfying the following properties:*

(a) *$\mathcal{L}$ enjoys the real embedding property,*

(b) *the real embedding preserves infima and suprema.*

Then for every $\mathcal{L}\forall$-theory Γ and for all $\mathcal{L}\forall$-formula φ, the following are equivalent:

i. $\Gamma \vdash_{\mathcal{L}\forall} \varphi$,

ii. for every safe-evaluation e in every standard $\mathcal{L}$-algebra such that $e(\gamma) = 1$ for all $\gamma \in \Gamma$, $e(\varphi) = 1$.

MTL∀, IMTL∀, and SMTL∀ are complete w.r.t. to standard algebras. Indeed, MTL, IMTL, and SMTL do enjoy the real embedding property, and, moreover, as shown in [116], the real embedding preserves infima and suprema. Strong standard completeness for WNM∀, NM∀, and G∀ was shown in [50, 75].

Part II

Functional Definability

Chapter 3

T-norm Based Logics with an Independent Involutive Negation

In the Monoidal T-norm based Logic MTL (as well as in the other t-norm based logics), the negation is definable from the implication and the truth constant $\overline{0}$, so that $\neg\varphi$ stands for $\varphi \to \overline{0}$. This negation behaves quite differently depending on the chosen left-continuous t-norm and, in general, is not a strong negation. This operator can be forced to be involutive by adding the axiom $\neg\neg\varphi \to \varphi$ to MTL. As shown in the previous chapter, the system so obtained was called IMTL (Involutive Monoidal T-norm based Logic) in [50]. However, in IMTL the negation does depend on the t-norm, so that IMTL singles out only those left-continuous t-norms which yield an involutive negation (hence having zero-divisors), and therefore operators like Gödel and Product t-norms are ruled out.

Esteva, Godo Hájek and Navara studied in [51] the logic SBL, i.e. the logic of continuous t-norms without zero-divisors, and the systems $SBL_{\sim}$, $G_{\sim}$ and $\Pi_{\sim}$ which extend SBL, Gödel and the Product logic by an independent involutive negation. Haniková successively improved in [73] Esteva, Godo Hájek and Navara's treatment by proving a general completeness theorem for schematic extensions of $SBL_{\sim}$.

In this chapter we aim at further generalizing that approach and at studying expansions of t-norm based logics by means of an independent strong negation. Our first step will consist in introducing the logic $MTL_{\sim}$ obtained by adding to MTL_{Δ} a unary connective $\sim$ and axioms which capture the behavior of involutive negations. Then, this approach will be extended to schematic extensions of MTL. We will show that, given any t-norm based logic satisfying some basic properties, its expansion by means of an involutive negation preserves algebraic and (finite) strong standard completeness. We will deal with both propositional and predicate logics.

The presence of an involutive negation is important if, for instance, we aim at defining connectives whose standard interpretation corresponds to t-conorms. Indeed, in any schematic extension $\mathcal{L}_\sim$ of $\mathrm{MTL}_\sim$ we can define a strong disjunction connective $\underline{\vee}$ as follows:

$$\varphi\underline{\vee}\psi \leftrightarrow \sim(\sim\varphi\&\sim\psi),$$

whose interpretation corresponds to the t-conorm dual of $*$ w.r.t. the strong negation. While in IMTL, the involutive negation $\neg$ would allow us to represent only a subclass of t-conorms, $\mathrm{MTL}_\sim$ provides the most general framework for representing (right-continuous) t-conorms. Then, $\mathrm{MTL}_\sim$ can be considered as the logic of left-continuous t-norms and right-continuous t-conorms.

Moreover, by means of the strong disjunction connective $\underline{\vee}$, we can also represent different types of implications. Indeed we can represent *S-implications* by means of the connective $\rightsquigarrow_s$ defined as

$$\varphi \rightsquigarrow_s \psi \text{ is } \sim\varphi\underline{\vee}\psi.$$

Furthermore, we can also represent an implication operator typical of Quantum Logic. Let, $\langle *, \diamond, n\rangle$ be a De Morgan triple. Then the operator

$$x \Rightarrow_q y = (n(x) \diamond (x * y))$$

is called a *Q-implication* [121]. In a logical framework $\Rightarrow_q$ can be represented by means of the connective $\rightsquigarrow_q$ defined as

$$\varphi \rightsquigarrow_q \psi \text{ is } \sim\varphi\underline{\vee}(\varphi\&\psi).$$

This chapter is structured as follows. In the next section we study $\mathrm{MTL}_\sim$ and its related algebraic structures. In Section 3, we investigate several schematic extensions of $\mathrm{MTL}_\sim$ and their respective algebras. In Section 4, we state and prove fundamental results which establish the basic requirements for a t-norm based logic to preserve (finite) strong standard completeness whenever an involutive negation is introduced. Finally, in the fifth section, we study first-order logics and provide another general result concerning standard completeness.

3.1 The $\mathrm{MTL}_\sim$ logic and $\mathrm{MTL}_\sim$-algebras

In this section we deal with the logic $\mathrm{MTL}_\sim$ (built up over MTL_Δ) and its related algebras ($\mathrm{MTL}_\sim$-algebras). We introduce the connective $\sim$ that is interpreted as an arbitrary involutive order-reversing mapping, not dependent on any other operator. We show the basic properties of $\mathrm{MTL}_\sim$ and provide algebraic completeness with respect to the class of linearly ordered $\mathrm{MTL}_\sim$-algebras.

Definition 3.1.1 The axioms of $\mathrm{MTL}_\sim$ are:

- The axioms of MTL.

- The axioms for Baaz's Delta.

- Axioms for the negation $\sim$:

  ($\sim$1) $\sim\sim\varphi \leftrightarrow \varphi$,

  ($\sim$2) $\Delta(\varphi \to \psi) \to (\sim\psi \to \sim\varphi)$.

The deduction rules of $\mathrm{MTL}_\sim$ are *Modus ponens* and *Necessitation*.

The notion of proof is easily adapted from the one for MTL. Moreover, $\mathrm{MTL}_\sim$ enjoys, as logics containing Δ do, the Delta Deduction Theorem (see the previous chapter).

The next proposition shows some peculiar properties of $\mathrm{MTL}_\sim$. In particular we are going to show that both De Morgan laws with respect to $\sim$ hold in $\mathrm{MTL}_\sim$.

Proposition 3.1.2 *The following inference rule and formulas are derivable in* $\mathrm{MTL}_\sim$*:*

i. From $\varphi \to \psi$ *derive* $\sim\psi \to \sim\varphi$ [Order Reversing (OR)],

ii. $\sim\overline{0}$,

iii. $\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$ [De Morgan Law],

iv. $\sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi)$ [De Morgan Law].

Proof.

i. Suppose $\varphi \to \psi$, then by necessitation $\Delta(\varphi \to \psi)$. By ($\sim$2), $\Delta(\varphi \to \psi) \to (\sim\psi \to \sim\varphi)$, and so by modus ponens $\sim\psi \to \sim\varphi$.

ii. $\overline{0} \to \sim\overline{1}$ is a theorem, so by (OR) and ($\sim$1), $\overline{1} \to \sim\overline{0}$. Finally, by modus ponens and $\overline{1}$ we get $\sim\overline{0}$.

iii. We begin with the left-right direction. $\varphi \to \varphi \vee \psi$ and $\psi \to \varphi \vee \psi$ are theorems and so are $\sim(\varphi \vee \psi) \to \sim\varphi$ and $\sim(\varphi \vee \psi) \to \sim\psi$ and (OR). $(\Phi_1 \to \Phi_2) \wedge (\Phi_1 \to \Phi_3) \to (\Phi_1 \to \Phi_2 \wedge \Phi_3)$ is an MTL-theorem, hence by modus ponens we easily obtain $\sim(\varphi \vee \psi) \to \sim\varphi \wedge \sim\psi$.

   To conclude, we prove the right-left direction. $\sim\varphi \wedge \sim\psi \to \sim\varphi$ and $\sim\varphi \wedge \sim\psi \to \sim\psi$ are theorems, and so are, by (OR) and ($\sim$2), $\varphi \to \sim(\sim\varphi \wedge \sim\psi)$ and $\psi \to \sim(\sim\varphi \wedge \sim\psi)$. Recall that $(\Phi_1 \to \Phi_3) \wedge (\Phi_2 \to \Phi_3) \to (\Phi_1 \vee \Phi_2 \to \Phi_3)$ is an MTL-theorem, hence $(\varphi \vee \psi) \to \sim(\sim\varphi \wedge \sim\psi)$ by modus ponens, and, by (OR) and ($\sim$2), $\sim\varphi \wedge \sim\psi \to \sim(\varphi \vee \psi)$.

iv. Notice that the proof is very similar to the one for (iii), hence it is omitted.

■

The algebraic counterpart of $\mathrm{MTL}_\sim$ is a class of algebras called $\mathrm{MTL}_\sim$-algebras.

Definition 3.1.3 An MTL$_\sim$-algebra is a structure $\mathcal{A} = \langle A, \sqcap, \sqcup, *, \Rightarrow, n, \delta, 0, 1\rangle$ such that $\langle A, \sqcap, \sqcup, *, \Rightarrow, \delta, 0, 1\rangle$ is an MTL$_\Delta$-algebra and, for every $x, y \in A$, the following properites are satisfied:

($n1$) $n(n(x)) = x$,
($n2$) $\delta(x \Rightarrow y) \leq (n(y) \Rightarrow n(x))$.

An example of MTL$_\sim$-algebra is given by the algebra $[0,1]_{\mathrm{MTL}^*_\sim}$ whose lattice reduct is the real unit interval, $*$ is a left-continuous t-norm, $\Rightarrow$ its related residuum, the lattice operations are given by the minimum and the maximum, and the negation n is any strong negation:

$$[0,1]_{\mathrm{MTL}^*_\sim} = \langle [0,1], *, \Rightarrow, \min, \max, \delta, n\rangle.$$

Let $\mathcal{A}$ be an MTL$_\sim$-algebra. Then the notions of $\mathcal{A}$-evaluation e and $\mathcal{A}$ model are easily adapted from those given for MTL$_\Delta$ by requiring that

$e(\sim\varphi) = n(e(\varphi))$.

Proposition 3.1.4 *The following rule and equations hold in every* MTL$_\sim$-*algebra:*

i. *If* $x \leq y$, *then* $n(y) \leq n(x)$,

ii. $n(0) = 1$,

iii. $n(x \sqcup y) = n(y) \sqcap n(x)$,

iv. $n(x \sqcap y) = n(y) \sqcup n(x)$.

Moreover, (i) *is equivalent to both* (iii) *and* (iv).

Proof.

i. Suppose that $x \leq y$. Then by ($\delta 5$) and ($\delta 6$) (see the previous chapter), $\delta(x \Rightarrow y) = (x \Rightarrow y) = 1$. Thus by ($n2$), $n(y) \leq n(x)$.

ii. Clearly $0 \leq n(1)$. Then, by (i) and ($n1$), $1 \leq n(0)$.

iii. To prove (iii), recall that $x, y \leq x \sqcup y$, then by (i), $n(x \sqcup y) \leq n(x), n(y)$. Therefore $n(x \sqcup y) \leq n(x) \sqcap n(y)$. Given that $n(x) \sqcap n(y) \leq n(x), n(y)$, by ($i$) and ($n1$), $x, y \leq n(n(x) \sqcap n(y))$. Hence, $x \sqcup y \leq n(n(x) \sqcap n(y))$, and, again by ($i$), $n(x) \sqcap n(y) \leq n(x \sqcup y)$.

iv. The proof for (iv) is similar to that for (iii), and so is omitted.

To prove that (i) and (iii) are equivalent we just have to prove that (i) can be derived by using (iii). Suppose that (iii) holds and $x \leq y$. Then $x \sqcup y = y$ and $n(x \sqcup y) = n(y)$. By (iii), $n(y) = n(x) \sqcap n(y)$. Hence $n(y) \leq n(x)$. ■

Proposition 3.1.5 *The class of* MTL$_\sim$-*algebras constitutes a variety.*

Proof. All the conditions of $\text{MTL}_\sim$-algebras can be expressed in an equational way. Thus, by Birkhoff Theorem (see Appendix B), it is clear that the class of all $\text{MTL}_\sim$-algebras constitutes a variety. ∎

Moreover, notice that it is obvious that the variety of $\text{MTL}_\sim$-algebras is the equivalent algebraic semantics for $\text{MTL}_\sim$ (see the previous chapter).

Now we are going to prove a completeness theorem for $\text{MTL}_\sim$. First, notice that:

Lemma 3.1.6 $\text{MTL}_\sim$ *is a conservative expansion of* MTL.

Proof. It is obvious that, for all Γ and φ in the language of MTL, $\Gamma \vdash_{\text{MTL}} \varphi$ entails $\Gamma \vdash_{\text{MTL}_\sim} \varphi$. ∎

Now, by Lemma 2.2.1 and Theorem 2.2.10 it immediately follows:

Lemma 3.1.7 *Let $\mathcal{A}$ be any* $\text{MTL}_\sim$*-algebra and let $\mathcal{A}^-$ be its underlying* MTL_Δ*-algebra. Then $\mathcal{A}$ and $\mathcal{A}^-$ have the same congruences.*

Theorem 3.1.8 *Every* $\text{MTL}_\sim$*-algebra is isomorphic to a subdirect product of a family of linearly ordered* $\text{MTL}_\sim$*-algebras.*

Consequently:

Theorem 3.1.9 (Completeness) *Let Γ be a theory over* $\text{MTL}_\sim$*, and φ be a formula. Then the following are equivalent:*

i. $\Gamma \vdash_{\text{MTL}_\sim} \varphi$,

ii. for each $\text{MTL}_\sim$*-algebra $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ,*

iii. for each $\text{MTL}_\sim$*-chain $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$.*

Proof. The equivalence between (i) and (ii) is immediate given that the variety of $\text{MTL}_\sim$-algebras is the equivalent algebraic semantics for $\text{MTL}_\sim$. To prove the equivalence with (iii) suppose that $\Gamma \nvdash \varphi$. Then there are an $\text{MTL}_\sim$-algebra $\mathcal{A}$ and an $\mathcal{A}$-evaluation v such that $v(\gamma) = 1$ for each $\gamma \in \Gamma$, but $v(\varphi) < 1$. By the above theorem, every $\text{MTL}_\sim$-algebra is isomorphic to a subdirect product of a family of linearly ordered $\text{MTL}_\sim$-algebras. Hence, the claim immediately follows. ∎

3.2 Schematic extensions of $\text{MTL}_\sim$

In this section we deal with some schematic extensions of $\text{MTL}_\sim$ and their related algebras. First, note that the notions of axiom schema and schematic extension are easily adapted from those given for MTL in the previous chapter. Then, particular schematic extensions of $\text{MTL}_\sim$ are obtained by adding the axioms characterizing the extensions of MTL, as shown in Chapter 2.

The logic $\mathrm{SMTL}_\sim$ is obtained from $\mathrm{MTL}_\sim$ by adding the axiom schema of pseudocomplementation (PC). $\Pi\mathrm{MTL}_\sim$ is obtained from $\mathrm{MTL}_\sim$ by adding the axiom schema of cancellation (C). $\mathrm{IMTL}_\sim$ is obtained from $\mathrm{MTL}_\sim$ by adding the axiom schema of involution (Inv). $\mathrm{WNM}_\sim$ is obtained from $\mathrm{MTL}_\sim$ by adding the axiom schema of weak nilpotent minimum (WNM). $\mathrm{NM}_\sim$ is obtained from $\mathrm{WNM}_\sim$ by adding the axiom schema of involution (Inv). The logics $\mathrm{BL}_\sim$, $\mathrm{SBL}_\sim$, Ł$_\sim$ and $\Pi_\sim$ are obtained from $\mathrm{MTL}_\sim$, $\mathrm{SMTL}_\sim$, $\mathrm{IMTL}_\sim$, and $\Pi\mathrm{MTL}_\sim$, respectively, by adding the axiom schema of divisibility (Div). Finally, the logic $\mathrm{G}_\sim$ is obtained by adding to $\mathrm{BL}_\sim$ the axiom schema of Contraction (Con).

The algebraic semantics of the above logics are similarly defined by adding the equations introduced in Chapter 2. The notions of proof, evaluation, model and tautology given for MTL are obviously extended to all the above logics. Notice that the Delta Deduction Theorem obviously holds for every $\mathcal{L}_\sim$. The following result immediately follows:

Proposition 3.2.1 *The class of $\mathcal{L}_\sim$-algebras forms a variety.*

Notice that for all the above logics, the related classes of algebras constitute their equivalent algebraic semantics.

Now, to prove completeness for $\mathcal{L}_\sim$, we proceed exactly as in the case of $\mathrm{MTL}_\sim$. First, notice that:

Lemma 3.2.2 *$\mathcal{L}_\sim$ is a conservative expansion of $\mathcal{L}$.*

Now, by Lemma 2.2.1 and Theorem 2.2.10 it immediately follows:

Lemma 3.2.3 *Let $\mathcal{A}$ be any $\mathcal{L}_\sim$-algebra and let $\mathcal{A}^-$ be its underlying $\mathcal{L}$-algebra. Then $\mathcal{A}$ and $\mathcal{A}^-$ have the same congruences.*

Theorem 3.2.4 *Every $\mathcal{L}_\sim$-algebra is isomorphic to a subdirect product of a family of linearly ordered $\mathcal{L}_\sim$-algebras.*

Consequently:

Theorem 3.2.5 (Completeness) *Let Γ be a theory over $\mathcal{L}_\sim$, and φ be a formula. Then the following are equivalent:*

i. $\Gamma \vdash_{\mathcal{L}_\sim} \varphi$,

ii. for each $\mathcal{L}_\sim$-algebra $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ,

iii. for each $\mathcal{L}_\sim$-chain $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$.

Let us now take a closer look at the logics resulting as schematic extensions of $\mathrm{SMTL}_\sim$, i.e.: $\mathrm{SMTL}_\sim$, $\Pi\mathrm{MTL}_\sim$, $\mathrm{SBL}_\sim$, $\Pi_\sim$ and $\mathrm{G}_\sim$ (see [26, 51] for more details). For the rest of this section, we will use $\mathcal{S}$ to denote either SMTL, ΠMTL, SBL, Π, or G. We are going to prove that for $\mathcal{S}_\sim$ it is possible to give an equivalent axiomatization obtained by expanding the language of $\mathcal{S}$ by means of the connective $\sim$ and introducing the axioms capturing the behavior of the involutive negation, plus a further axiom which makes the relationship between the two negations $\sim$ and $\neg$ clear. Baaz's operator can be then defined as $\neg{\sim}\varphi$.

Definition 3.2.6 The logic $\mathcal{S}'_{\sim}$ is obtained by adding the connective $\sim$ to the $\mathcal{S}$-language, along with the axiom ($\sim$1) and the following two further axioms:

($\sim$2)$'$ $\Delta'(\varphi \to \psi) \to \Delta'(\sim\psi \to \sim\varphi)$,

($A11$) $\neg\varphi \to \sim\varphi$,

where $\Delta'\varphi$ stands for $\neg\sim\varphi$.
The inference rules of $\mathcal{S}'_{\sim}$ are modus ponens and necessitation for Δ'.

Our aim is to prove that the new connective Δ' behaves exactly as Baaz's projector Δ.

As mentionted above, we are interested in the study of involutive negations that are totally independent from the t-norm, but it might seem that axiom ($A11$) binds, somehow, the choice of $\sim$ to the t-norm. Still, it is easy to see that in each linearly ordered $\mathcal{S}$-algebra the negation $\neg$ behaves like Gödel negation and thus ($A11$) is trivially satisfied, since any involutive negation is greater than Gödel negation.

Proposition 3.2.7 *The following formulas and the following rule are provable in* $\mathcal{S}'_{\sim}$*:*

i. $(\Delta'\varphi) \vee \neg(\Delta'\varphi)$,

ii. $\Delta'\varphi \to \varphi$,

iii. *if* $\varphi \to \psi$, *then* $\sim\psi \to \sim\varphi$,

iv. $\Delta'(\varphi \vee \psi) \to (\Delta'\varphi \vee \Delta'\psi)$,

v. $\Delta'\varphi \to \Delta'\Delta'\varphi$,

vi. $\Delta'(\varphi \to \psi) \to (\Delta'\varphi \to \Delta'\psi)$.

Thus the connective Δ' *does behave as Baaz's* Δ.

Proof.

i. It immediately follows by the fact that $\mathcal{S}$ proves $\neg\neg\psi \vee \neg\psi$, and by substituting ψ with $\sim\varphi$.

ii. It directly follows from ($A11$) and the involutive property of $\sim$.

iii. Suppose that $\varphi \to \psi$, then, by the rule of Δ'-necessitation, $\Delta'(\varphi \to \psi)$ and therefore, by $(\sim 2)'$ and modus ponens, $\Delta'(\sim\psi \to \sim\varphi)$. Now by ($ii$) and applying modus ponens once again, we get $\sim\psi \to \sim\varphi$. Clearly this means that the De Morgan laws (w.r.t. $\sim$) are derivable in $\mathcal{S}'_{\sim}$ (see Proposition 3.1.2).

iv. By the De Morgan law we have $\sim\varphi \wedge \sim\psi \to \sim(\varphi \vee \psi)$. Apply the order reversing rule w.r.t. $\neg$ (see [50]) to obtain $\neg\sim(\varphi \vee \psi) \to \neg(\sim\varphi \wedge \sim\psi)$. By the De Morgan law w.r.t. $\neg$ we have $\neg(\sim\varphi \wedge \sim\psi) \to (\neg\sim\varphi \vee \neg\sim\psi)$. Hence, by transitivity and definition of Δ' we obtain $\Delta'(\varphi \vee \psi) \to (\Delta'\varphi \vee \Delta'\psi)$.

v. $\Delta'(\overline{1} \to \varphi) \to \Delta'(\sim\varphi \to \sim\overline{1})$ is an instance of $(\sim 2)'$, thus we have $\Delta'\varphi \to \Delta'(\sim\varphi \to \sim\overline{1})$. By (ii) it is easy to prove $\Delta'(\sim\varphi \to \sim\overline{1}) \to (\neg\sim\overline{1} \to \neg\sim\varphi)$, being $(\sim\varphi \to \sim\overline{1}) \to (\neg\sim\overline{1} \to \neg\sim\varphi)$ derivable in $\mathcal{S}'_\sim$. Now, exploiting the fact that $\sim\overline{1} \leftrightarrow \overline{0}$ and $\sim\overline{0} \leftrightarrow \overline{1}$ (easy to prove), and applying the order reversing rule both w.r.t. $\neg$ and w.r.t. $\sim$ we obtain $(\neg\sim\overline{1} \to \neg\sim\varphi) \to (\overline{1} \to \neg\sim\neg\sim\varphi)$. Hence, we can easily derive, by transitivity and definition of Δ', $\Delta'\varphi \to \Delta'\Delta'\varphi$.

vi. From $(\sim 2)'$ and (ii) we can prove that $\Delta'(\varphi \to \psi) \to (\sim\psi \to \sim\varphi)$. It is easy to see that $(\sim\psi \to \sim\varphi) \to (\neg\sim\varphi \to \neg\sim\psi)$ is derivable in $\mathcal{S}'_\sim$. Now, by the definition of Δ', $(\sim\psi \to \sim\varphi) \to (\Delta'\varphi \to \Delta'\psi)$ and thus, by transitivity, $\Delta'(\varphi \to \psi) \to (\Delta'\varphi \to \Delta'\psi)$.

■

Proposition 3.2.7 shows that Δ' behaves exactly as the Baaz projector Δ. Therefore $\mathcal{S}'_\sim$ is equivalent to $\mathcal{S}_\sim$ (see also [27]). Indeed, as mentioned above, $(A11)$ holds in every $\mathcal{S}_\sim$-chain, and, furthermore, axiom $(\sim 2)'$ is derivable in $\mathcal{S}_\sim$ by means of the Delta Deduction Theorem. This implies that the direct introduction of the connective Δ becomes redundant for $\mathcal{S}'_\sim$.

3.3 Standard completeness

In this section we provide a general study of completeness for the family of schematic extensions of $\mathrm{MTL}_\sim$. The results we provide are general in the sense that we state the basic requirements to guarantee that the obtained extensions are complete w.r.t. algebras over the real unit interval.

In order to obtain these results, we first have to clarify the use we are going to make of some terms. To prove standard completeness for a logic $\mathcal{L}_\sim$ endowed with an involutive negation means, in our work, that a formula φ is provable in $\mathcal{L}_\sim$ if and only if it is a 1-tautology common to all the $\mathcal{L}_\sim$-algebras whose lattice reduct is the real unit interval. Unlike the approach of Esteva, Godo, Hájek and Navara in [51], we call *standard* an algebra even if its strong negation n does not behave like the standard negation $n_s(x) = 1 - x$. In fact it is easy to show that the following lemma holds.

Lemma 3.3.1 *Let $\mathcal{L}_\sim$ be any schematic extension of* $\mathrm{MTL}_\sim$*. For every standard $\mathcal{L}_\sim$-algebra*

$$[0,1]_n = \langle [0,1], \min, \max, *, \Rightarrow, \delta, n, 0, 1\rangle$$

where n is an arbitrary involutive negation, there exists a standard $\mathcal{L}_\sim$-algebra $[0,1]_{n_s}$, whose negation is the standard one, which is isomorphic to $[0,1]_n$.

Proof. Recall that every involutive negation n on $[0,1]$ is isomorphic to the standard negation n_s (see Chapter 1 and [140]). This means that the structures $\langle [0,1], \min, \max, n, 0, 1\rangle$ and $\langle [0,1], \min, \max, n_s, 0, 1\rangle$ are isomorphic. Now, let

$$h : \langle [0,1], \min, \max, n, 0, 1\rangle \to \langle [0,1], \min, \max, n_s, 0, 1\rangle$$

be such an isomorphism. Then it is easy to check that the function $*' : [0,1] \times [0,1] \to [0,1]$ defined as

$$x *' y = h(h^{-1}(x) * h^{-1}(y))$$

is a t-norm which is left-continuous iff so is $*$. Furthermore, h defines an isomorphism between the structures

$$[0,1]_n = \langle [0,1], \min, \max, *, \Rightarrow, \delta, n, 0, 1\rangle$$

and

$$[0,1]_{n_s} = \langle [0,1], \min, \max, *', \Rightarrow', \delta, n_s, 0, 1\rangle,$$

where $\Rightarrow'$ stands for the residuum of $*'$. ■

Note that, in [51], a $\Pi_\sim$-algebra over $[0,1]$ is called *semi-standard* whenever its involutive negation n is different from the standard one, and is called *standard* only if the strong negation is n_s. On the contrary, here, we do not distinguish between standard and semi-standard $\mathcal{L}_\sim$-algebras. The reason behind this choice comes from the fact that the two approaches (the one presented here and the one given in [51]) are different. In fact in [51] the authors fix the t-norm and change the strong negation, obtaining then non-isomorphic algebras. For instance, in the case of Product algebras, the structures

$$\langle [0,1], \min, \max, \cdot, \Rightarrow, \delta, n, 0, 1\rangle \text{ and } \langle [0,1], \min, \max, \cdot, \Rightarrow, \delta, n_s, 0, 1\rangle,$$

where $\cdot$ and $\Rightarrow$ correspond to the Product t-norm and its residuum respectively, are non-isomorphic. In our case, as shown in Lemma 3.3.1, we change the whole structure by transforming the arbitrary strong negation into the standard one. Each transformation defines an isomorphic algebra, where the t-norm obtained belongs to a certain class. Hence, when we deal with a Product algebra, in general the isomorphism will yield a strict t-norm isomorphic to the Product t-norm. Similarly, in the case of an MV-algebra, the transformation will give a nilpotent t-norm isomorphic to the Łukasiewicz t-norm. In proving completeness, we will then obtain that $\Pi_\sim$ and Ł$_\sim$ are the logics of strict t-norms and nilpotent t-norms with the additional standard negation, respectively[1].

[1] In a very recent work by Cintula, Klement, Mesiar and Navara (see [33]), the logic $\Pi_\sim$ is called CBL$_\sim$, since it is the logic of t-norms satisfying the cancellative property with an additional involutive negation.

3.3.1 Finite strong standard completeness

We begin by introducing a special property for the schematic extensions of MTL which we will need in order to prove general standard completeness.

Definition 3.3.2 Let $\mathcal{L}$ be any schematic extension of MTL. We say that $\mathcal{L}$ satisfies the *standard partial embeddability property* (SPEP), if and only if any linearly ordered $\mathcal{L}$-algebra $\mathcal{A}$ is partially embeddable into a standard $\mathcal{L}$-algebra $[0,1]_{\mathcal{L}}$.

We can now state a theorem which generalizes the result given by Haniková in [73], where only logics extending $\text{SBL}_{\sim}$ were considered.

Theorem 3.3.3 (Finite Strong Standard Completeness) *Let $\mathcal{L}$ be a schematic extension of* MTL *satisfying the standard partial embeddability property. Then, $\mathcal{L}_{\sim}$ is finitely strongly standard complete.*

Proof. Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be a $\mathcal{L}_{\sim}$-theory, let φ be a $\mathcal{L}_{\sim}$-formula, and assume that $\Gamma \nvdash_{\mathcal{L}_{\sim}} \varphi$. From the algebraic completeness of $\mathcal{L}_{\sim}$ w.r.t. the class of linearly ordered $\mathcal{L}_{\sim}$-algebras (see Theorem 3.1.9), it follows that there are a linearly ordered $\mathcal{L}_{\sim}$-algebra $\mathcal{A}$ and an $\mathcal{A}$-evaluation v, such that $v(\gamma_i) = 1$, for all γ_i, and $v(\varphi) < 1$. Take now the finite subset X of $\mathcal{A}$ containing the evaluations under v of all sub-formulas ϕ of φ and of each γ_i, plus 0 and 1. Namely

$$X = \{v(\phi) : \phi \sqsubseteq \{\varphi\} \cup \{\gamma_1, \ldots, \gamma_m\}\}^2 \cup \{0, 1\}.$$

Clearly X also is a finite subset of the $\{\delta, n\}$-free reduct $\mathcal{A}^-$ of $\mathcal{A}$. Moreover, by hypothesis, $\mathcal{L}$ has the standard partial embeddability property and $\mathcal{A}^-$ is a linearly ordered $\mathcal{L}$-algebra. Therefore there exist a standard $\mathcal{L}$-algebra $[0,1]_{\mathcal{L}} = \langle [0,1], \min, \max, *, \Rightarrow, 0, 1\rangle$ and an embedding f of X into $[0,1]_{\mathcal{L}}$ preserving the $\mathcal{A}^-$ operations. Now:

(i) Define a partial mapping $\rho : [0,1] \to [0,1]$ as $\rho(a) = 0$ if $a \neq 1$ and $\rho(1) = 1$.

(ii) For each pair of the form $(\phi, \sim\phi)$ with $\phi \sqsubseteq \{\varphi\} \cup \{\gamma_1, \ldots, \gamma_m\}$, define another partial mapping $\eta : [0,1] \to [0,1]$ as

$$\eta(f(v(\phi))) = f(v(\sim\phi)) = f(n(v(\phi))).$$

Moreover, $\eta(0) = 1$ and $\eta(1) = 0$.

It is easy to see that ρ exactly behaves as the δ operator. On the other hand, the partial mapping η induces a total involutive non-decreasing order-reversing mapping $\eta' : [0,1] \to [0,1]$ defined as the piecewise-linear function connecting all the pairs $(0,1)$, $(f(\phi), f(\sim\phi))$, $(f(\sim\phi), f(\phi))$ and $(1,0)$ of the unit square $[0,1]^2$. In general η' does not behave as the standard negation n_s. Hence we obtain a standard $\mathcal{L}_{\sim}$-algebra

$$[0,1]_\eta = \langle [0,1], \min, \max, *, \Rightarrow, \rho, \eta, 0, 1\rangle$$

[2] $\phi \sqsubseteq \{\varphi\} \cup \{\gamma_1, \ldots, \gamma_m\}$ means that ϕ is sub-formula of φ or of one of the γ_i's with $i = 1, \ldots, m$.

in which $\mathcal{A}$ can be embedded. As shown in Lemma 3.3.1, there is a standard $\mathcal{L}_\sim$-algebra $[0,1]_{n_s}$ equipped with n_s which isomorphic to $[0,1]_\eta$ via a bijection h. It is trivial to check that the composition $\nu = h \circ (f \circ e)$ is a $[0,1]_{n_s}$-evaluation such that $\nu(\gamma_i) = 1$ for each $\gamma_i \in \Gamma$, but $\nu(\varphi) < 1$. This completes the proof of the theorem. ■

We now want to check, given any logic among MTL, SMTL, IMTL, ΠMTL, WNM, NM, BL, SBL, Π, Ł, and G, whether it satisfies the requirements of the above theorem or not. This will allow us to determine which among the main schematic extensions of $\mathrm{MTL}_\sim$ are finitely strongly standard complete.

Lemma 3.3.4 *Let $\mathcal{L}$ be any logic among* MTL*,* SMTL*,* IMTL*,* ΠMTL*,* WNM*,* NM*,* BL*,* SBL*,* Π*,* Ł*, and* G*. Then, $\mathcal{L}$ enjoys the standard partial embeddability property.*

The proof of the above lemma just consists in checking the proof of completeness of the mentioned logics. However, this deserves some remarks.

Recall that given any extension $\mathcal{L}$ of MTL, $\mathcal{L}$ has the *real embedding property* ($\mathbb{R}$-E), if any linearly ordered finite or countable structure of $\mathbb{V}(\mathcal{L})$ (i.e the variety generated by the Lindenbaum sentence algebra over $\mathcal{L}$) can be embedded into a structure in $\mathbb{V}(\mathcal{L})$ whose lattice reduct is the real interval $[0,1]$ (see Definition 2.1.19).

As mentioned in the previous chapter (see also [47, 84]), MTL, IMTL, SMTL, WNM, NM, and Gödel logics do enjoy the real embedding property. Clearly this means that such logics also enjoy the standard partial embeddability property. In fact the following holds:

Lemma 3.3.5 *Let $\mathcal{L}$ be a schematic extension of* MTL *satisfying the real embedding property. Then $\mathcal{L}$ also satisfies the standard partial embeddability property.*

Proof. Let $\mathcal{A}$ be a countable linearly ordered $\mathcal{L}$-algebra, and let X be a finite subset of $\mathcal{A}$. Let now $\mathcal{A}_X$ be the countable $\mathcal{L}$-algebra generated by X. Clearly X can be embedded into $\mathcal{A}_X$, which, in turn, by the real embedding property, can be embedded into a standard $\mathcal{L}$-algebra. ■

By checking Theorem 2.1.18 we can see that a schematic extension $\mathcal{L}$ of MTL is finitely strongly standard complete if and only if every $\mathcal{L}$-chain is partially embeddable into the standard $\mathcal{L}$-chain. Hence, given that MTL, IMTL, SMTL, ΠMTL, WNM, NM, BL, Π, Ł, and G are finitely strongly standard complete, they all enjoy the standard partial embeddability property.

Consequently:

Corollary 3.3.6 *Let $\mathcal{L}_\sim$ be any of the above schematic extensions of* $\mathrm{MTL}_\sim$*. Then $\mathcal{L}_\sim$ is finitely strongly standard complete.*

3.3.2 Strong standard completeness

In the proof of Theorem 3.3.3, it has been crucial to suppose that $\mathcal{L}_\sim$ enjoys the standard partial embeddability property. As shown in Lemma 3.3.5 the real embedding property turns out to be stronger than the SPEP. Now, we will show that if we assume that a logic enjoys the real embedding property, we get a stronger form of completeness, i.e. standard completeness w.r.t. countable theories.

In [47] (Theorem 6, pagg. 213–214), the $\mathbb{R}$-E is shown to yield (finite) strong standard completeness for some logics extending MTL.

Theorem 3.3.7 ([47]) *Let $\mathcal{L}$ be any schematic extension of* MTL. *If $\mathcal{L}$ has the real embedding property, then $\mathcal{L}$ is (finitely) strongly standard complete.*

It is easy to see that the above theorem can be adapted to logics with an additional involutive negation:

Definition 3.3.8 Let $\mathcal{L}_\sim$ be any schematic extension of $\text{MTL}_\sim$, and let $\mathbb{L}_\sim$ be its equivalent variety semantics. $\mathcal{L}_\sim$ has the *real embedding property* ($\mathbb{R}$-E), if any linearly ordered finite or countable structure of $\mathbb{L}_\sim$ can be embedded into a structure in $\mathbb{L}_\sim$ whose lattice reduct is the real unit interval $[0,1]$ and whose additional negation n corresponds to any strong negation.

Theorem 3.3.9 *Let $\mathcal{L}_\sim$ be any schematic extension of* $\text{MTL}_\sim$. *If $\mathcal{L}_\sim$ has the real embedding property, then $\mathcal{L}_\sim$ is strongly standard complete.*

Proof. $\mathcal{L}_\sim$ is strongly complete w.r.t. linearly ordered $\mathcal{L}_\sim$-algebras. Then suppose that there are a countable theory Γ and a formula φ such that $\Gamma \not\vdash \varphi$. Then there are a linearly ordered $\mathcal{L}_\sim$-algebra $\mathcal{A}$ and an $\mathcal{A}$-evaluation e such that $e(\gamma) = 1$ for all $\gamma \in \Gamma$, but $e(\varphi) < 1$. Take the values under e of all subformulas appearing in $\Gamma \cup \{\varphi\}$ and take the $\mathcal{L}_\sim$-algebra $\mathcal{A}'$ generated by those values. $\mathcal{A}'$ clearly is a countable subalgebra of $\mathcal{A}$. Now, $\mathcal{L}_\sim$ enjoys the real embedding property, hence there is an embedding h of the algebra $\mathcal{A}'$ into the standard algebra over $[0,1]$. This means that the evaluation e can be extended to an evaluation (applying Lemma 3.3.1) $v = h \circ e$ from formulas to the real unit interval, so that $v(\gamma) = 1$ for all $\gamma \in \Gamma$, but $v(\varphi) < 1$. ■

In what follows, $\mathcal{L}$ will denote any logic among MTL, IMTL, SMTL, WNM, NM, and G. Recall that $\mathcal{L}$ enjoys the real embedding property. In this part we prove that also $\mathcal{L}_\sim$ enjoys the real embedding property. From this fact and the previous theorem, the strong standard completeness of $\mathcal{L}_\sim$ will easily follow. First of all we need the following:

Lemma 3.3.10 *For every countable linearly ordered $\mathcal{L}_\sim$-algebra*

$$\mathcal{A} = \langle A, *, \Rightarrow, \delta, n, \leq_A, 0_A, 1_A \rangle,$$

there are an ordered countable set $\langle X, \preceq \rangle$, a binary operation $\odot$, two monadic operations η and ρ, and a mapping $\Phi : A \to X$ such that the following conditions hold:

(1) *X is densely ordered, and has a maximum M and a minimum m.*

(2) $\langle X, \odot, \preceq, M\rangle$ *is a commutative linearly ordered integral monoid.*

(3) $\odot$ *is left-continuous w.r.t. the order topology on* $\langle X, \preceq\rangle$.

(4) η *is an order reversing involutive mapping.*

(5) *For any* $x \in X$, $\rho(x) = \begin{cases} M & \text{if } x = M \\ m & \text{otherwise} \end{cases}$.

(6) Φ *is an embedding of* $\langle A, *, \delta, n, \leq_A, 0_A, 1_A\rangle$ *into* $\langle X, \odot, \rho, \eta, \preceq, m, M\rangle$, *and for all* $a, b \in A$, $\Phi(a \Rightarrow b)$ *is the residuum of* $\Phi(a)$ *and* $\Phi(b)$ *in* $\langle X, \odot, \rho, \eta, \preceq, m, M\rangle$.

Proof. (1), (2) and (3) have been proved in [95, 116][3]. Notice, moreover, that $\mathcal{L}$ enjoys the real embedding property. Then we just have to show (4) and (5), and extend the proof of (6) given in [95, 116] so as to cope with the operations η and ρ.

For any $a \in A$, let $succ(a)$ be the successor of a if it exists, and take $succ(a) = a$ otherwise. Let

$$X = \{(s,1) \mid s \in A\} \cup \{(s,r) \mid \exists s', s = succ(s') >_A s', r \in \mathbb{Q} \cap (0,1)\}.$$

For any $(s,q), (t,r) \in X$, let

$$(s,q) \preceq (t,r) \text{ iff either } s <_A t, \text{ or } s = t \text{ and } q \leq r.$$

Clearly, $\preceq$ is a linear lexicographic order with a maximum $(1_A, 1)$ and a minimum $(0_A, 1)$.

To prove (4), define for any $(s,q) \in X$:

$$\eta(s,q) = \begin{cases} (n(s), 1) & \text{if } q = 1; \\ (succ(n(s)), 1-q) & \text{otherwise.} \end{cases}$$

We show that η is indeed an order reversing and involutive mapping.

(i) Clearly, $\eta(0_S, 1) = (1_S, 1)$, and $\eta(1_S, 1) = (0_S, 1)$

(ii) Suppose that $(s,q) \preceq (t,r)$, then we have two cases:

(a) $s <_S t$:

- Suppose that $q \neq 1$ and $r \neq 1$. Then, $\eta(t,r) \preceq \eta(s,q)$, since $succ(n(t)) <_S succ(n(s))$.

[3]Notice that in this proof the ordered set X we define does not correspond to the ordered set provided in [95], i.e.: $X' = \{(s,r) \mid s \in A \backslash \{0_A\}, r \in \mathbb{Q} \cap [0,1]\}$. Indeed X corresponds to the ordered set given in [116] which is an ordered submonoid of X'. This is due to the fact that the embedding of a countable $\mathcal{L}$-chain into X is a complete embedding, while the embedding into X' is not (see [116]). This will be useful for the first-order case (see next section). Moreover, the existence of the successor for an element s is not guaranteed in X', and so the definition of the involutive negation η would not work in that case.

- Suppose that $q \neq 1$ and $r = 1$. Then $n(t) <_S succ(n(s))$, hence $\eta(t,r) \preceq \eta(s,q)$.
- Suppose that $q = 1$ and $r \neq 1$. Then $succ(n(t)) \leq_S (n(s))$. Thus $\eta(t,r) \preceq \eta(s,q)$, since $succ(n(t))$ can be at most s, but in that case, $(1-r) \leq 1$.
- Suppose that $q = r = 1$. Then, $\eta(t,r) \preceq \eta(s,q)$, since $(n(t)) <_S (n(s))$.

(b) $s = t$ and $q \leq r$:

- Suppose $r \neq 1$. Then, given that $(1-r) \leq (1-q)$, $\eta(t,r) \preceq \eta(s,q)$.
- Suppose now that $r = 1$. Then $n(t) <_S succ(n(s))$, so, again $\eta(t,r) \preceq \eta(s,q)$.

(iii) Finally we have that $\eta(\eta(s,q)) = (succ(n(succ(n(s)))), 1-(1-q)) = (s,q)$, if $q \neq 1$; and $\eta(\eta(s,1)) = (n(n(s)), 1) = (s,1)$, otherwise.

Hence we have proved (4).

To prove (5), let, for any $(s,q) \in X$:

$$\rho(s,q) = \begin{cases} (\delta(s), 1) & \text{if } q = 1; \\ (0_S, 1) & \text{otherwise.} \end{cases}$$

Obviously $\rho(s,q) = (1_S, 1)$ iff $s = 1$ and $q = 1$, otherwise we have the minimum $(0_S, 1)$.

To prove (6), let for every $s \in A$, $\Phi(s) = (s,1)$. Clearly $\Phi(0_A) = (0_A, 1)$ and $\Phi(1_A) = (1_A, 1)$. Moreover,

$$\Phi(s) \odot \Phi(t) = (s,1) \odot (t,1) = (s * t, 1) = \Phi(s * t).$$

Finally, let $\Phi(n(s)) = (n(s), 1)$ and $\Phi(\delta(s)) = (\delta(s), 1)$, hence

$$\Phi(n(s)) = \eta(s,1), \text{ and } \Phi(\delta(s)) = \rho(s,1).$$

Thus, Φ is an embedding of partially ordered monoids equipped with an order-reversing involutive mapping. To conclude, notice that the fact that for all $s,t \in A$, $\Phi(s \Rightarrow t)$ is the residuum of $\Phi(s)$ and $\Phi(t)$ was shown in [95]. ∎

We now prove that $\mathcal{L}_\sim$ has the real embedding property.

Theorem 3.3.11 *$\mathcal{L}_\sim$ enjoys the real embedding property.*

Proof. As shown in the above lemma $\langle X, \preceq \rangle$ is a countable, dense, linearly-ordered set with maximum and minimum. Then $\langle X, \preceq \rangle$ is order-isomorphic to the rationals in $[0,1]$ with the natural order $\langle \mathbb{Q} \cap [0,1], \leq \rangle$. Let Ψ be such an isomorphism. Suppose that (1-6) hold, and let for $\alpha, \beta \in [0,1]$,

- $\alpha \odot' \beta = \Psi(\Psi^{-1}(\alpha) \odot \Psi^{-1}(\beta))$,
- $\eta'(\alpha) = \Psi(\eta'(\Psi^{-1}(\alpha))$,

- $\rho'(\alpha) = \Psi(\rho(\Psi^{-1}(\alpha)))$.

Moreover, let for all $s \in A$, $\Phi'(s) = \Psi(\Phi(s))$. Hence, we have a structure $\langle \mathbb{Q} \cap [0,1], \odot', \rho', \eta', \leq, 0, 1\rangle$, that, along with Φ', satisfies (1-6).

Now, we can assume, without loss of generality, that $X = \mathbb{Q} \cap [0,1]$ and that $\preceq$ is $\leq$. It is shown in [95] that such a structure is embeddable into an analogous structure $\langle [0,1], \hat{\odot}, \leq\rangle$ over the real unit interval, and that such an embedding preserves infima and suprema.

Now, define for all $\alpha \in [0,1]$

$$\hat{\eta}(\alpha) = \inf_{x \in X : x \leq \alpha} \eta(x).$$

We show that $\hat{\eta}$ is an order-reversing involutive mapping which extends η. First let

$$\xi(\alpha) = \sup_{y \in X : \alpha \leq y} \eta(y).$$

We prove that $\hat{\eta}(\alpha) = \xi(\alpha)$, which means that the negation defined is continuous. In general we have that $\xi(\alpha) \leq \hat{\eta}(\alpha)$. Suppose the inequality is strict: i.e. $\xi(\alpha) < \hat{\eta}(\alpha)$. This means that there is some $z \in \mathbb{Q}$ such that $\xi(\alpha) < \eta(z) < \hat{\eta}(\alpha)$. Therefore, for any $x \leq \alpha$, $\eta(z) < \eta(x)$ and, for any $y \geq \alpha$, $\eta(y) < \eta(z)$. Hence we have that, for any $x \leq \alpha$, $x < z$ and, for any $y \geq \alpha$, $z < y$. Then z must equal α, but $\alpha \in [0,1] \backslash \mathbb{Q} \cap [0,1]$, so we obtain a contradiction. Notice that if $\alpha \in \mathbb{Q}$, then the above equivalence clearly holds. Thus we have proved $\hat{\eta}(\alpha) = \xi(\alpha)$.

It is easy to see that $\hat{\eta}(0) = 1$, $\hat{\eta}(1) = 0$ and that $\hat{\eta}$ is order-reversing.

It remains to prove that $\hat{\eta}(\hat{\eta}(\alpha)) = \alpha$. Notice that

$$\hat{\eta}(\hat{\eta}(\alpha)) = \hat{\eta}(\inf_{x \leq \alpha} \eta(x)) = \sup_{x \leq \alpha} \hat{\eta}(\eta(x)) = \sup_{x \leq \alpha} \eta(\eta(x)) = \sup_{x \leq \alpha} x = \alpha.$$

Now, define for any $\alpha \in [0,1]$

$$\hat{\rho}(\alpha) = 0 \text{ if } \alpha \neq 1 \text{ and } \hat{\rho}(1) = 1.$$

Clearly $\hat{\rho}$ behaves like Baaz's Delta.

Then $\langle \mathbb{Q} \cap [0,1], \odot', \rho', \eta', \leq, 0, 1\rangle$ embeds into $\langle [0,1], \hat{\odot}, \hat{\rho}, \hat{\eta}, \leq, 0, 1\rangle$. Given left-continuity of $\hat{\odot}$ over $[0,1]$, $\langle [0,1], \hat{\odot}, \hat{\Rightarrow}, \hat{\rho}, \hat{\eta}, \leq, 0, 1\rangle$ is a linearly ordered $\mathcal{L}_\sim$-algebra, where the residuum $\hat{\Rightarrow}$ always exists. Hence the initial $\mathcal{L}_\sim$-chain $\mathcal{A}$ can be embedded into the standard algebra $\langle [0,1], \hat{\odot}, \hat{\Rightarrow}, \hat{\rho}, \hat{\eta}, \leq, 0, 1\rangle$. ■

Now, we immediately obtain the following strong standard completeness theorem.

Theorem 3.3.12 (Strong Standard Completeness) *The logic $\mathcal{L}_\sim$ is strongly standard complete.*

Proof. From Theorem 3.3.11 we know that $\mathcal{L}_\sim$ enjoys the real embedding property, and so by Theorem 3.3.9 it is strongly standard complete. ■

A general overview about finite strong standard and strong standard completeness for the main extensions of MTL$_\sim$ can be found in Table 3.1.

	SC	FSSC	SSC
$\mathrm{MTL}_\sim$	Yes	Yes	Yes
$\mathrm{IMTL}_\sim$	Yes	Yes	Yes
$\mathrm{SMTL}_\sim$	Yes	Yes	Yes
$\mathrm{WNM}_\sim$	Yes	Yes	Yes
$\mathrm{NM}_\sim$	Yes	Yes	Yes
$\Pi\mathrm{MTL}_\sim$	Yes	Yes	No
$\mathrm{BL}_\sim$	Yes	Yes	No
$\mathrm{SBL}_\sim$	Yes	Yes	No
$\Pi_\sim$	Yes	Yes	No
$\mathrm{G}_\sim$	Yes	Yes	Yes
Ł$_\sim$	Yes	Yes	No

Table 3.1: *Standard* (SC)*, finite strong standard* (FSSC) *and strong standard completeness* (SSC) *for the main* $\mathrm{MTL}_\sim$ *extensions.*

3.4 Predicate calculi

In this section we are going to study predicate calculi for $\mathrm{MTL}_\sim$ and for its schematic extensions. As before, $\mathcal{L}_\sim$ will stand, for the rest of this section, for any extension of $\mathrm{MTL}_\sim$. Clearly $\mathcal{L}\forall$ will stand for the predicate calculus of the $\{\Delta, \sim\}$-free fragment of $\mathcal{L}_\sim$, and $\mathcal{L}\forall_\sim$ for the predicate calculus related to $\mathcal{L}_\sim$.

As usual (see [50, 75] and the previous chapter) we begin by expanding the propositional language with a set of predicates $Pred$, a set of object variables Var and a set of object constants $Const$, together with the two classical quantifiers $\forall$ and $\exists$. The notion of formula is easily generalized by saying that, if φ is a formula and $x \in Var$, then both $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulas.

Definition 3.4.1 Let $\mathcal{A}$ be a linearly ordered $\mathcal{L}_\sim$-algebra. An $\mathcal{A}$-*interpretation* for a predicate language $\mathcal{L}_\sim$ is a structure $\mathcal{M} = \langle M, (r_P)_{P\in Pred}, (m_c)_{c\in Const}\rangle$, where:

- M is a non-empty set,
- $r_P : M^{ar(P)} \to A$ for any $P \in Pred$, where $ar(P)$ stands for the ariety of the predicate P,
- $m_c \in M$ for each $c \in Const$.

For every evaluation of variables $v : Var \to M$, the truth value of a formula φ ($\|\varphi\|^{\mathcal{A}}_{M,v}$) is inductively defined as follows:

- $\|P(x, \ldots, c, \ldots)\|^{\mathcal{A}}_{M,v} = r_P(v(x), \ldots, m_c, \ldots)$, where $v(x) \in M$ for each variable x,
- The truth value commutes with connectives of $\mathcal{L}_\sim$,

- $\|(\forall x)\varphi\|^{\mathcal{A}}_{M,v} = \inf\{\|\varphi\|^{\mathcal{A}}_{M,v'} : v(y) = v'(y)$ for all variables, except for $x\}$ and
  $\|(\exists x)\varphi\|^{\mathcal{A}}_{M,v} = \sup\{\|\varphi\|^{\mathcal{A}}_{M,v'} : v(y) = v'(y)$ for all variables, except for $x\}$,

if the infimum and supremum exist in $\mathcal{A}$, otherwise the truth value(s) remain(s) undefined.

A structure $\mathcal{M}$ is called $\mathcal{A}$-safe if all infima and suprema needed for the definition of the truth value of any formula exist in $\mathcal{A}$. In that case the truth value of a formula φ in an $\mathcal{A}$-safe structure $\mathcal{M}$ is just

$$\|\varphi\|^{\mathcal{A}}_{M} = \inf\{\|\varphi\|^{\mathcal{A}}_{M,v} : v : Var \to M\}.$$

Definition 3.4.2 The axioms for $\mathcal{L}\forall_{\sim}$ are those of $\mathcal{L}_{\sim}$ plus the following axioms for quantified formulas:
(∀1) $(\forall x)\varphi(x) \to \varphi(x/t)$ (t substitutable for x in φ),
(∀2) $(\forall x)(\psi \to \varphi) \to (\psi \to (\forall x)\varphi)$, ($x$ not free in ψ),
(∀3) $(\forall x)(\varphi \vee \psi) \to ((\forall x)\varphi \vee \psi)$, ($x$ not free in ψ),
(∃1) $\varphi(x/t) \to (\exists x)\varphi(x)$ (t substitutable for x in φ),
(∃2) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to \psi)$, ($x$ not free in ψ).
The rules of inference of $\mathcal{L}\forall_{\sim}$ are Modus Ponens, Necessitation and Generalization.

The introduction of an involutive negation improves the expressive power of the logics defined. Indeed, we are now going to show that in the first-order case we can use the involutive negation and the universal quantifier to define an existential quantifier. In fact, if we consider a language for the predicate calculus only containing the universal quantifier $\forall$ and we define a new quantifier $\exists'$ as follows,

$$(\exists' x)\varphi \text{ stands for } {\sim}(\forall x){\sim}\varphi,$$

then it is possible to prove that $\exists'$ satisfies the axioms (∃1) and (∃2). Therefore it behaves exactly as the usual quantifier $\exists$. The following proposition is a proof of this fact.

Proposition 3.4.3 *The following formulas*

(∃′1) $\varphi(x/t) \to {\sim}(\forall x){\sim}\varphi(x)$ *(for t substitutable for x in φ),*

(∃′2) $(\forall x)(\varphi \to \psi) \to ({\sim}(\forall x){\sim}\varphi \to \psi)$,

are provable from (∀1), (∀2) *and* (∀3).

Proof. We begin with (∃′1). $\forall x{\sim}\varphi(x) \to {\sim}\varphi(x/t)$ is an instance of (∀1) (where t is substitutable for x in φ). By (OR) (see Proposition 3.1.2) we obtain, $\varphi(x/t) \to {\sim}\forall x{\sim}\varphi(x)$.

To prove (∃′2) notice that $(\forall x)({\sim}\psi \to {\sim}\varphi) \to ({\sim}\psi \to (\forall x){\sim}\varphi)$ is an instance of (∀2). Now, by (OR) and the involutive property of $\sim$ it easily follows that $({\sim}\psi \to {\sim}\varphi) \leftrightarrow (\varphi \to \psi)$, and $({\sim}\psi \to (\forall x){\sim}\varphi) \leftrightarrow ({\sim}(\forall x){\sim}\varphi \to \psi)$. Hence, $\forall x(\varphi \to \psi) \to ({\sim}(\forall x){\sim}\varphi \to \psi)$ holds. ■

Remark. Notice that the above proposition does not imply that the usual axiomatization given in Definition 3.4.2 can somehow be simplified by replacing the axioms ($\exists$1) and ($\exists$2) by the formula

$$(\forall 4):\ (\exists x)\varphi \leftrightarrow {\sim}(\forall x){\sim}\varphi.$$

Indeed, it is not clear whether the usual axiomatization given in Definition 3.4.2 and the one obtained by replacing the axioms ($\exists$1) and ($\exists$2) by ($\forall$4) are equivalent in the following sense: call $\mathcal{L}\forall_{\sim}^{-}$ the predicate calculus obtained by replacing the axioms ($\exists$1) and ($\exists$2), by ($\forall$4), then

$$\mathcal{L}\forall_{\sim}^{-} \vdash (\exists 1) \wedge (\exists 2) \tag{3.1}$$

and

$$\mathcal{L}\forall_{\sim} \vdash (\forall 4). \tag{3.2}$$

The above proposition shows that (3.1) holds, but, on the other hand, to prove (3.2) seems a quite hard task from a purely syntactical point of view.

In [75], Hájek showed that Ł$\forall$ (the first-order Łukasiewicz logic) proves $(\exists x)\varphi \leftrightarrow \neg(\forall x)\neg\varphi$, where $\neg$ is the involutive Łukasiewicz negation (see [75], Lemma 5.4.1). In order to show that, Hájek could exploit the involutive property of Łukasiewicz negation, but also the fact that, in Ł the negation $\neg\varphi$ of a formula φ corresponds by definition to $\varphi \to \overline{0}$. Adding an independent involutive negation to a t-norm based logic $\mathcal{L}$ exactly means that this negation does not satisfy the above property of being definable from the $\mathcal{L}$-implication and the truth constant $\overline{0}$. Therefore the technique used by Hájek cannot be exploited in our case. However, we will see that $\mathcal{L}\forall_{\sim}^{-}$ and $\mathcal{L}\forall_{\sim}$ are equivalent as a byproduct of the completeness result.

Now we are going to prove general completeness for any $\mathcal{L}\forall_{\sim}$ and $\mathcal{L}\forall_{\sim}^{-}$ with respect to safe models built up over linearly ordered $\mathcal{L}_{\sim}$-algebras. First recall some definitions from the previous chapter:

Definition 3.4.4 [75] Let Γ be a theroy over $\mathcal{L}\forall_{\sim}$ ($\mathcal{L}\forall_{\sim}^{-}$).

i. Γ is *consistent* if there is a formula φ unprovable in Γ.

ii. Γ is *complete* if for each pair φ, ψ of closed formulas, $\Gamma \vdash \varphi \to \psi$ or $\Gamma \vdash \psi \to \varphi$.

iii. Γ is *Henkin* if for every closed formula of the form $(\forall x)\varphi(x)$ unprovable in Γ, there exists a constant c in the language of Γ such that $\varphi(c)$ is unprovable in Γ.

The idea to prove completeness is exactly the one used in [75] for BL$\forall$ (and mentioned in the previous chapter), that is:

(1) Given a theory Γ and a closed formula α such that $\Gamma \not\vdash \alpha$, there exists a complete Henkin supertheory $\hat{\Gamma}$ of Γ such that $\hat{\Gamma} \not\vdash \alpha$.

(2) For each complete Henkin theory Γ and every closed formula α such that $\Gamma \not\vdash \alpha$, there exist a linearly ordered $\mathcal{L}_\sim$-algebra $\mathcal{A}$ and a safe $\mathcal{A}$-model $\mathcal{M}$ of Γ such that $\|\alpha\|^{\mathcal{A}}_{M} < 1$.

Indeed, following exactly step by step the proof in [75] we immediately have the following result.

Theorem 3.4.5 *Let* Γ *and* φ *be a theory and formula over* $\mathcal{L}\forall_\sim$ $(\mathcal{L}\forall^-_\sim)$*, respectively. Then* $\Gamma \vdash \varphi$ *iff for each linearly ordered* $\mathcal{L}_\sim$*-algebra* $\mathcal{A}$ *and each safe* $\mathcal{A}$*-model* $\mathcal{M}$ *of* Γ*,*

$$\|\varphi\|^{\mathcal{A}}_{M} = 1.$$

We can now prove completeness w.r.t. real evaluations:

Theorem 3.4.6 (Strong Standard Completeness) *Let* $\mathcal{L}_\sim$ *be a schematic extension of* MTL$_\sim$ *satisfying the following properties:*

(a) $\mathcal{L}_\sim$ *enjoys the real embedding property,*

(b) *the real embedding preserves infima and suprema.*

Then for every $\mathcal{L}\forall_\sim$*- theory* ($\mathcal{L}\forall^-_\sim$*-theory*) Γ *and for all* $\mathcal{L}\forall_\sim$*-formula* ($\mathcal{L}\forall^-_\sim$*-formula*) φ*, the following are equivalent:*

i. $\Gamma \vdash_{\mathcal{L}\forall_\sim} \varphi \quad \left[\Gamma \vdash_{\mathcal{L}\forall^-_\sim} \varphi\right]$,

ii. For every safe evaluation e *in every standard* $\mathcal{L}_\sim$*-algebra*

$$[0,1]_{n_s} = \langle [0,1], \min \max, *, \Rightarrow, \delta, n_s, 0, 1\rangle$$

such that $e(\gamma) = 1$ *for all* $\gamma \in \Gamma$*,* $e(\varphi) = 1$*.*

Proof. We give a proof for $\mathcal{L}\forall_\sim$. The proof for $\mathcal{L}\forall^-_\sim$ is the same.
$(i) \Rightarrow (ii)$ Trivial.
$(ii) \Rightarrow (i)$ Suppose $\mathcal{L}\forall_\sim \not\vdash \varphi$. Then from the algebraic completeness of $\mathcal{L}\forall_\sim$, there exist a linearly ordered $\mathcal{L}_\sim$-algebra $\mathcal{A}$ and a safe evaluation v of $\mathcal{L}\forall_\sim$ into $\mathcal{A}$ such that $v(\varphi) < 1$. Exactly as in the proof of Theorem 3.3.12, let X be the countable $\mathcal{L}$-subset of the values, under v, of all the sub-formulas in $\Gamma \cup \{\varphi\}$, and let $\mathcal{A}_X$ be the countable $\mathcal{L}_\sim$-chain generated by X. Given that $\mathcal{L}_\sim$ enjoys the real embedding property, by hypothesis we can find a map g which embeds $\mathcal{A}_X$ into a standard $\mathcal{L}_\sim$ algebra $[0,1]_{n_s}$ and which preserves all the existing infima and suprema. Clearly this means that the composition $e = g \circ v$ is a safe-evaluation such that $e(\gamma) = 1$ for each $\gamma \in \Gamma$ and $e(\varphi) < 1$. Hence $\mathcal{L}\forall_\sim$ is strongly standard complete. ■

We can now prove the following

Corollary 3.4.7 *The systems* $\mathcal{L}\forall^-_\sim$ *and* $\mathcal{L}\forall_\sim$ *are equivalent.*

Proof. Following the above Remark, it is sufficient to show that $\mathcal{L}\forall_{\sim} \vdash (\forall 4)$. Given the standard completeness of $\mathcal{L}\forall_{\sim}$,

$$\mathcal{L}\forall_{\sim} \vdash (\exists x)\varphi \leftrightarrow {\sim}(\forall x){\sim}\varphi$$

iff for each standard $\mathcal{L}_{\sim}$-algebra $[0,1]_{n_s}$ and each $[0,1]_{n_s}$-model $\mathcal{M}$,

$$\|(\exists x)\varphi \leftrightarrow {\sim}(\forall x){\sim}\varphi\|_{M}^{[0,1]_{n_s}} = 1$$

iff for each standard $\mathcal{L}_{\sim}$-algebra $[0,1]_{n_s}$ and each $[0,1]_{n_s}$-model $\mathcal{M}$,

$$\sup\{\|\varphi\|_{M,v}^{[0,1]_{n_s}} \mid v : Var \to M\} = 1 - \inf\{1 - \|\varphi\|_{M,v}^{[0,1]_{n_s}} \mid v : Var \to M\}.$$

This concludes the proof, given that the last equality is trivially true. ∎

Finally, we obtain the following:

Theorem 3.4.8 *Let* $\mathcal{L}\forall_{\sim}$ *be any logic among* MTL$\forall_{\sim}$, IMTL$\forall_{\sim}$, SMTL$\forall_{\sim}$, WNM$\forall_{\sim}$, NM$\forall_{\sim}$, *and* G$\forall_{\sim}$. *Then for every theory* Γ *and for all formula* φ, *the following are equivalent:*

i. $\Gamma \vdash_{\mathcal{L}\forall_{\sim}} \varphi$,

ii. For every safe evaluation e *in every standard* $\mathcal{L}_{\sim}$*-algebra*

$$[0,1]_{n_s} = \langle [0,1], \min \max, *, \Rightarrow, \delta, n_s, 0, 1\rangle$$

such that $e(\gamma) = 1$ *for all* $\gamma \in \Gamma$, $e(\varphi) = 1$.

Proof. Soundness is obvious. To prove the converse just notice that by Theorem 3.3.11 $\mathcal{L}_{\sim}$ enjoys the real embedding property. The fact that the embedding preserves infima and suprema easily follows from results in [116, 75, 50]. Then, conditions (a) and (b) of Theorem 3.4.6 are satisfied, thus completeness immediately follows. ∎

Chapter 4

Ordered Fields and $Ł\Pi\frac{1}{2}$

The logic $Ł\Pi\frac{1}{2}$ certainly is the most powerful and most expressive t-norm based logic and is obtained by combining both Łukasiewicz and Product Logics. $Ł\Pi\frac{1}{2}$ was introduced by Esteva, Godo, and Montagna in [52], where the authors proved completeness with respect to the class of linearly ordered algebras and to the standard algebra over $[0, 1]$. The work by Cintula [26, 28] specially focused on providing a different axiomatization of $Ł\Pi\frac{1}{2}$ and investigating its expressive power by showing that logics associated to continuous t-norms representable as a finite ordinal sum can be framed in $Ł\Pi\frac{1}{2}$ (this result will be improved in the next chapter). Montagna provided a deep algebraic investigation of $Ł\Pi\frac{1}{2}$-algebras in [111], and a categorical analysis in [112]. Indeed, Montagna showed that $Ł\Pi\frac{1}{2}$-algebras are substructures of (ordered) fields extending the field of rational numbers. A functional representation theorem was given by Montagna and Panti in [117], where the authors proved that the set of functions definable in $Ł\Pi\frac{1}{2}$ exactly coincides with the set of piecewise rational functions, i.e. the suprema of fractions of polynomials with rational coefficients.

The connection with ordered fields and the definability of piecewise rational functions will allow us to provide in this chapter new results concerning $Ł\Pi\frac{1}{2}$ and $Ł\Pi\frac{1}{2}$-algebras. Indeed, we show that ordered fields can be framed in $Ł\Pi\frac{1}{2}$-algebras, so that Boolean combinations of polynomial equations and inequalities with rational coefficients can be translated into $Ł\Pi\frac{1}{2}$-equations (Section 4.2). Furthermore, we show that there is a strong connection between the theory of real closed fields and the equational theory of $Ł\Pi\frac{1}{2}$ (Section 4.3). We prove that the universal theory of real closed fields is faithfully interpretable in $Ł\Pi\frac{1}{2}$, and consequently functions definable over the field of real numbers (with rational coefficients) are definable in $Ł\Pi\frac{1}{2}$. We will also prove that there is a polynomial-time translation between $Ł\Pi\frac{1}{2}$-terms and quantifier-free formulas of the field of reals, and so the universal theory of real closed fields and the equational theory of $Ł\Pi\frac{1}{2}$ both belong to the same complexity class.

Finally, the correspondence between ordered fields and $Ł\Pi\frac{1}{2}$-algebras will be exploited in order to study the lattice of subvarieties of $Ł\Pi\frac{1}{2}$. This was an open problem proposed by Montagna in [111]. We prove here that the lattice of subvarieties has the cardinality of the continuum (Section 4.4).

We begin by recalling the basic properties of the logic $Ł\Pi\frac{1}{2}$, and $Ł\Pi\frac{1}{2}$-algebras.

4.1 The logic $Ł\Pi\frac{1}{2}$ and $Ł\Pi\frac{1}{2}$-algebras

The language of $Ł\Pi\frac{1}{2}$ is built from a countable set of propositional variables, three binary connectives $\rightarrow_{l}$ (Łukasiewicz implication), $\&_{\pi}$ (Product conjunction), $\rightarrow_{\pi}$ (Product implication), and the truth constants $\bar{0}$ and $\overline{\frac{1}{2}}$. The truth constant $\bar{1}$ is defined as $\varphi \rightarrow_{l} \varphi$. Moreover, many other connectives can be defined from those introduced above:

$$
\begin{array}{rll@{\qquad}rll}
\neg_{l}\varphi & \text{is} & \varphi \rightarrow_{l} \bar{0}, & \neg_{\pi}\varphi & \text{is} & \varphi \rightarrow_{\pi} \bar{0}, \\
\varphi \wedge \psi & \text{is} & \varphi\&(\varphi \rightarrow_{l} \psi), & \varphi \vee \psi & \text{is} & \neg_{l}(\neg_{l}\varphi \wedge \neg_{l}\psi), \\
\varphi \oplus \psi & \text{is} & \neg_{l}\varphi \rightarrow_{l} \psi, & \varphi\&\psi & \text{is} & \neg_{l}(\neg_{l}\varphi \oplus \neg_{l}\psi), \\
\varphi \ominus \psi & \text{is} & \varphi\&\neg_{l}\psi, & \varphi \leftrightarrow \psi & \text{is} & (\varphi \rightarrow_{l} \psi)\&(\psi \rightarrow_{l} \varphi), \\
\Delta\varphi & \text{is} & \neg_{\pi}\neg_{l}\varphi, & \nabla\varphi & \text{is} & \neg_{\pi}\neg_{\pi}\varphi.
\end{array}
$$

The logic $Ł\Pi\frac{1}{2}$ is defined Hilbert-style as the logical system whose axioms and rules are the following (see [52, 28]):

(i) Axioms of Łukasiewicz Logic.

(ii) Axioms of Product Logic.

(iii) The following additional axioms:

(ŁΠ1) $\varphi\&_{\pi}(\psi \ominus \chi)) \leftrightarrow (\varphi\&_{\pi}\psi) \ominus (\varphi\&_{\pi}\chi)$
(ŁΠ2) $\Delta(\varphi \rightarrow_{l} \psi) \rightarrow_{l} (\varphi \rightarrow_{\pi} \psi)$
(ŁΠ3) $\Delta(\varphi \rightarrow_{\pi} \psi) \rightarrow_{l} (\varphi \rightarrow_{l} \psi)$
(ŁΠ4) $\neg_{l}\overline{\frac{1}{2}} \leftrightarrow \overline{\frac{1}{2}}$

(iv) Deduction rules of $Ł\Pi\frac{1}{2}$ are modus ponens for $\rightarrow_{l}$ (modus ponens for $\rightarrow_{\pi}$ is derivable), and necessitation for Δ: from φ derive $\Delta\varphi$.

The notion of proof from a theory is the usual one (see Chapter 2). Notice that the logic ŁΠ (see [49, 52]) can be obtained from $Ł\Pi\frac{1}{2}$ just by excluding the constant $\overline{\frac{1}{2}}$ from the language and by omitting axiom (ŁΠ4).

The algebraic semantics for $Ł\Pi\frac{1}{2}$ is given by the class of $Ł\Pi\frac{1}{2}$-algebras, which forms a variety.

Definition 4.1.1 An $Ł\Pi\frac{1}{2}$-*algebra* (see [52, 29]) is a structure

$$\langle L, \oplus, \neg_{l}, *_{\pi}, \rightarrow_{\pi}, \sqcap, \sqcup, 0, 1, \tfrac{1}{2}\rangle$$

such that

- $\langle L, \oplus, \neg_{l}, 0, 1\rangle$ is an MV-algebra,

- $\langle L, *_\pi, \rightarrow_\pi, \sqcap, \sqcup, 0, 1\rangle$ is a Product algebra,

(such that the lattice-orders coincide), and the equations

$$\neg_l \tfrac{1}{2} = \tfrac{1}{2} \qquad \text{and} \qquad x *_\pi \neg_l(\neg_l y \oplus z) = \neg_l(\neg_l(x *_\pi y) \oplus (x *_\pi z))$$

hold.

In particular, other definable operations are the following:

$$\begin{array}{ccc} x \rightarrow_l y = (\neg_l x \oplus y); & \neg_\pi = x \rightarrow_\pi 0; & x \ominus y = \neg_l(\neg_l x \oplus y); \\ x \sqcup y = x \oplus (y \ominus x); & x \sqcap y = x \ominus (x \ominus y); & \delta(x) = \neg_\pi \neg_l x; \\ |x - y| = (x \ominus y) \oplus (y \ominus x); & & x *_l y = \neg_l(\neg_l x \oplus \neg_l y). \end{array}$$

In the Ł$\Pi\frac{1}{2}$-algebra $\mathbb{R}$Ł$\Pi\frac{1}{2}$ over $[0,1]$ we have the following interpretations:

$$\begin{array}{ccc} x *_l y = \max(x + y - 1, 0) & x \oplus y = \min(x + y, 1) & \neg_l x = 1 - x \\ x \rightarrow_l y = \min(1 - x + y, 1) & x \rightarrow_\pi y = \begin{cases} 1 & x \leq y \\ \frac{y}{x} & x > y \end{cases} & x *_\pi y = xy \\ x \ominus y = \max(x - y, 0) & x \sqcap y = \min(x, y) & x \sqcup y = \max(x, y) \\ |x - y| = \max(x - y, y - x) & \neg_\pi x = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases} & \delta(x) = \begin{cases} 1 & x = 1 \\ 0 & x < 1 \end{cases} \end{array}.$$

The notions of evaluation and model are the usual ones (see Chapter 2).

Theorem 4.1.2 ([52]) *Let Γ be a theory over Ł$\Pi\frac{1}{2}$, and φ be a formula. Then the following are equivalent:*

1. $\Gamma \vdash_{\text{Ł}\Pi\frac{1}{2}} \varphi$,
2. *for each Ł$\Pi\frac{1}{2}$-chain $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$,*
3. *for each Ł$\Pi\frac{1}{2}$-algebra $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$,*
4. *for the standard Ł$\Pi\frac{1}{2}$-algebra $\mathbb{R}$Ł$\Pi\frac{1}{2}$ and each $\mathbb{R}$Ł$\Pi\frac{1}{2}$-model e of Γ, $e(\varphi) = 1$.*

Thus Ł$\Pi\frac{1}{2}$ is complete w.r.t. the variety of Ł$\Pi\frac{1}{2}$-algebras, the class of linearly ordered Ł$\Pi\frac{1}{2}$-algebras and the Ł$\Pi\frac{1}{2}$ standard algebra, which hence generates the whole variety. We will see later that the variety of Ł$\Pi\frac{1}{2}$-algebras is also generated by the Ł$\Pi\frac{1}{2}$-algebra whose lattice reduct is the unit interval of any real closed field.

It is clear that Ł$\Pi\frac{1}{2}$ is an expansion of both Łukasiewicz and Product logics, but also Gödel logic can be defined in it. In fact, Gödel conjunction corresponds to the minimum operation $\sqcap$, and Gödel implication $\Rightarrow_g$ can be easily defined by the term $\delta(x \Rightarrow_l y) \sqcup y$. Hence the logics of the three fundamental t-norms can find a faithful interpretation in Ł$\Pi\frac{1}{2}$. In the next chapter we will show that there are many other logics definable in Ł$\Pi\frac{1}{2}$.

For each rational $r \in [0,1]$, a formula $\overline{r}$ is definable in Ł$\Pi\frac{1}{2}$ from the truth constant $\overline{\frac{1}{2}}$ and the connectives, so that $e(\overline{r}) = r$ for each evaluation e. Therefore, in the Ł$\Pi\frac{1}{2}$-language, we have a truth constant for each rational in $[0,1]$, as follows:

$$\overline{\tfrac{1}{2^n}} \text{ is given by } \underbrace{\overline{\tfrac{1}{2}} *_\pi \cdots *_\pi \overline{\tfrac{1}{2}}}_{n}; \qquad \overline{\tfrac{m}{2}} \text{ is given by } \underbrace{\overline{\tfrac{1}{2}} \oplus \cdots \oplus \overline{\tfrac{1}{2}}}_{m};$$

$$\overline{\tfrac{1}{n}} \text{ is given by } \overline{\tfrac{n}{2}} \Rightarrow_\pi \overline{\tfrac{1}{2}}; \qquad \overline{\tfrac{m}{n}} \text{ is given by } \overline{\tfrac{1}{m}} \Rightarrow_\pi \overline{\tfrac{1}{n}}.$$

Due to ŁΠ$\frac{1}{2}$-completeness, the following book-keeping axioms for rational truth constants are provable:

(RLΠ1)	$\neg_l \overline{r}$	$\leftrightarrow$	$\overline{1-r}$
(RLΠ2)	$\overline{r} \rightarrow_l \overline{s}$	$\leftrightarrow$	$\overline{\min(1, 1-r+s)}$
(RLΠ3)	$\overline{r} \&_\pi \overline{s}$	$\leftrightarrow$	$\overline{r \cdot s}$
(RLΠ4)	$\overline{r} \rightarrow_\pi \overline{s}$	$\leftrightarrow$	$\overline{r \Rightarrow_\pi s}$

where $r \Rightarrow_\pi s = 1$ if $r \leq s$, $r \Rightarrow_\pi s = \frac{s}{r}$ otherwise. This clearly means that Rational Pavelka Logic can be defined in ŁΠ$\frac{1}{2}$.

4.2 Ordered fields and ŁΠ$\frac{1}{2}$

In this section we show that ŁΠ$\frac{1}{2}$-algebras are strongly connected to ordered fields. Indeed, as shown in [111, 52], the operations of ŁΠ$\frac{1}{2}$-algebras can be defined over ordered fields, and consequently they can find an interpretation in those structures. Here we will show that also ordered fields can find a faithful interpretation in ŁΠ$\frac{1}{2}$-algebras.

Recall that a *field* is a a structure $\mathcal{F} = \langle F, +, \cdot, -, 0, 1\rangle$ where

- $+$ (addition) and $\cdot$ (multiplication) are commutative and associative operations having 0 and 1 as identity element, respectively.
- There exist multiplicative inverses, i.e. for every x there exists an element x^{-1} such that $x \cdot x^{-1} = 1$.
- There exist additive inverses, i.e. for every x there exists an element $-x$ such that $x + (-x) = 0$.
- Multiplication distributes over addition, i.e. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

An *ordered field* $\mathcal{F} = \langle F, +, \cdot, -, \leq, 0, 1\rangle$ is a field equipped with a total ordering $\leq$. Classical examples of ordered fields are the field $\mathcal{R}$ of real numbers $\mathbb{R}$, and the field $\mathcal{Q}$ of rational numbers $\mathbb{Q}$, that also is the smallest ordered field.

Definition 4.2.1 Let $\mathcal{F} = \langle F, +, \cdot, -, \leq, 0, 1\rangle$ be an ordered field. Let x^{-1} denote the multiplicative inverse of x, and $0^{-1} = 0$. Let $A = \{x \in F : 0 \leq x \leq 1\}$. Define for all $x, y \in A$, $x \oplus y = \min(x + y, 1)$ and $\neg_l x = 1 - x$. Denote by $\cdot$ the restriction of the product operation to A, and let

$$x \Rightarrow_\pi y = \begin{cases} 1 & x \leq y \\ y \cdot x^{-1} & \text{otherwise} \end{cases},$$

and $\frac{1}{2} = 2^{-1}$. The algebra $\langle A, \oplus, \neg_l, \cdot, \Rightarrow_\pi, 0, 1, \frac{1}{2}\rangle$ is called the *interval* ŁΠ$\frac{1}{2}$-*algebra* of $\mathcal{F}$.

It is easy to see that the interval $Ł\Pi\frac{1}{2}$-algebra of an ordered field is a linearly ordered $Ł\Pi\frac{1}{2}$-algebra (see [52, 111]). Furthermore, Montagna proved in [111] the following:

Theorem 4.2.2 ([111]) *Up to isomorphism, every linearly ordered $Ł\Pi\frac{1}{2}$-algebra is (isomorphic to) the interval algebra of exactly one ordered field. Moreover, the interval algebra of $\mathcal{Q}$ can be embedded in any $Ł\Pi\frac{1}{2}$-algebra.*

In the following, we denote by $\mathcal{R}_{\mathrm{alg}}$ the field of real algebraic numbers $\mathbb{R}_{\mathrm{alg}}$, and by $\mathcal{F}$ any ordered field such that $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{R}_{\mathrm{alg}}$. We denote by $\mathbb{Q}Ł\Pi\frac{1}{2}$ the $Ł\Pi\frac{1}{2}$-algebra over $\mathbb{Q} \cap [0,1]$, by $\mathbb{A}Ł\Pi\frac{1}{2}$ the $Ł\Pi\frac{1}{2}$-algebra over $\mathbb{R}_{\mathrm{alg}} \cap [0,1]$, and finally by $\mathbb{F}Ł\Pi\frac{1}{2}$ the $Ł\Pi\frac{1}{2}$-algebra over $\mathbb{F} \cap [0,1]$.

Now, our aim is to show that Boolean combinations of polynomial equations and inequalities in $\mathcal{F}$, in the language $\langle +, \cdot, -, \leq, 0, 1\rangle$, are interpretable in $\mathbb{F}Ł\Pi\frac{1}{2}$. In order to simplify the notation, we write $\Rightarrow$ for $\Rightarrow_l$, $\neg$ for $\neg_l$ and we omit the symbol of the product of two elements: i.e., whenever a term like $x *_\pi y$ appears, we simply write xy.

We start from the following functions:

$$f_1(x) = \tfrac{4x}{2x-1} \qquad f_2(x) = \tfrac{4-4x}{2x-1} \qquad f_1^{-1}(y) = \tfrac{y}{2y-4} \qquad f_2^{-1}(y) = \tfrac{y+4}{2y+4}.$$

Note that f_1 is a decreasing bijection from $(0, \frac{1}{2})$ onto $(-\infty, 0)$ and f_2 is a decreasing bijection from $(\frac{1}{2}, 1)$ onto $(0, +\infty)$. It follows that the function f defined by

$$f(x) = \begin{cases} f_1(x) & \text{if} \quad 0 < x < \frac{1}{2} \\ 0 & \text{if} \quad x = \frac{1}{2} \\ f_2(x) & \text{if} \quad \frac{1}{2} < x < 1 \end{cases}$$

is a bijection from $(0, 1)$ onto $\mathbb{R}$ whose inverse is

$$f^{-1}(x) = \begin{cases} f_1^{-1}(x) & \text{if} \quad x < 0 \\ \frac{1}{2} & \text{if} \quad x = 0 \\ f_2^{-1}(x) & \text{if} \quad x > 0 \end{cases} .$$

We will use the functions f, f^{-1} in order to define an isomorphic copy of $\mathcal{F}$ whose domain is $\mathbb{F} \cap (0, 1)$. More precisely, we define:

- $x +_0 y = f^{-1}(f(x) + f(y))$;
- $x \cdot_0 y = f^{-1}(f(x) \cdot f(y))$;
- $-_0 x = f^{-1}(-f(x))$;
- $0_0 = f^{-1}(0) = \frac{1}{2}$;
- $1_0 = f^{-1}(1) = \frac{5}{6}$;

- $x \leq_0 y$ iff whenever $0 < x, y < \frac{1}{2}$ or $\frac{1}{2} < x, y < 1$, then $f(y) \leq f(x)$; or whenever $0 < x < \frac{1}{2}$ and $\frac{1}{2} < y < 1$.

Then f is an isomorphism from $\mathcal{F}_0 = \langle \mathbb{F} \cap (0,1), +_0, \cdot_0, -_0, \leq_0, 0_0, 1_0 \rangle$ onto $\mathcal{F} = \langle \mathbb{F}, +, \cdot, -, \leq, 0, 1 \rangle$. We are going to prove that the structure $\mathcal{F}_0$ is definable in $\text{Ł}\Pi\frac{1}{2}$ by showing that, under f and f^{-1}, the operations of $\mathcal{F}$ can be translated into operations over $\text{Ł}\Pi\frac{1}{2}$.

$[-_0]$: We define $-_0 x$ by distinguishing three cases:

1) $0 < x < \frac{1}{2}$. Then $f(x) = f_1(x) = \frac{4x}{2x-1}$, $-f(x) = -\frac{4x}{2x-1} = \frac{4x}{1-2x}$. Note that $f(x) < 0$ and $-f(x) > 0$, therefore $-_0 x = f^{-1}(\frac{4x}{1-2x}) = f_2^{-1}(\frac{4x}{1-2x})$. Letting $y = \frac{4x}{1-2x}$, we have

$$f_2^{-1}(y) = \frac{y+4}{2y+4} = \frac{\frac{4x}{1-2x}+4}{\frac{8x}{1-2x}+4} = \frac{\frac{4-4x}{1-2x}}{\frac{4}{1-2x}} = \frac{4-4x}{4} = 1 - x = \neg x.$$

2) $x = \frac{1}{2}$. Then $-_0 x = f^{-1}(f(\frac{1}{2})) = f^{-1}(0) = \frac{1}{2} = \neg x$.

3) $\frac{1}{2} < x < 1$. Then $f(x) = f_2(x) = \frac{4-4x}{2x-1}$, $-f(x) = \frac{4-4x}{1-2x}$. Note that $f(x) > 0$ and $-f(x) < 0$, therefore $-_0 x = f^{-1}(\frac{4-4x}{1-2x}) = f_1^{-1}(\frac{4-4x}{1-2x})$. Letting $y = \frac{4-4x}{1-2x}$, we have

$$-_0 x = f_2^{-1}(y) = \frac{y}{2y-4} = \frac{\frac{4-4x}{1-2x}}{\frac{8-8x}{1-2x}-4} = \frac{\frac{4-4x}{1-2x}}{\frac{4}{1-2x}} = \frac{4-4x}{4} = 1 - x = \neg x.$$

In any case,

$$-_0 x = \neg x.$$

$[+_0]$: As for $+_0$, we distinguish the following cases:

a) $0 < x < \frac{1}{2}$ and $0 < y < \frac{1}{2}$. Then $f(x) = f_1(x) < 0$, $f(y) = f_1(y) < 0$, therefore $f(x) + f(y) < 0$, and $x +_0 y = f_1^{-1}(f_1(x) + f_1(y))$. Let $z = f_1(x) + f_1(y)$. We have:

$$z = \frac{4x}{2x-1} + \frac{4y}{2y-1} = \frac{16xy-4x-4y}{(2x-1)(2y-1)}.$$

Moreover,

$$2z - 4 = \frac{32xy-8x-8y-16xy+8x+8y-4}{(2x-1)(2y-1)} = \frac{16xy-4}{(2x-1)(2y-1)}.$$

Hence

$$x +_0 y = \frac{z}{2z-4} = \frac{\frac{16xy-4x-4y}{(2x-1)(2y-1)}}{\frac{16xy-4}{(2x-1)(2y-1)}} = \frac{16xy-4x-4y}{16xy-4} = \frac{\frac{1}{4}x+\frac{1}{4}y-xy}{\frac{1}{4}-xy}.$$

Now recalling that $x, y < \frac{1}{2}$ we have that $\frac{1}{4} - xy > \frac{1}{4}x + \frac{1}{4}y - xy > 0$. Therefore, letting $x \ominus y = x *_l (\neg_l y)$ and $x \oplus y = \neg x \Rightarrow y$, we have that $x +_0 y$ is represented by the term

$$S_a(x, y) = \left(\tfrac{1}{4} \ominus xy\right) \Rightarrow_\pi \left(\left(\tfrac{1}{4}x \oplus \tfrac{1}{4}y\right) \ominus xy\right).$$

Case (a) has a *characteristic term*, namely a term $a(x, y)$ such that for all $x, y \in (0, 1)$, $a(x, y) = 1$ if x, y satisfy case (a) and $a(x, y) = 0$ otherwise. Such a term is defined by

$$a(x, y) = \neg\delta(\neg x \sqcup \neg y) \sqcap \neg\delta\left(\frac{1}{2} \Rightarrow (x \sqcup y)\right).$$

b) $x = \frac{1}{2}$. Then $x +_0 y = y$. Thus we set $S_b(x, y) = y$. The characteristic term of case b) is

$$b(x, y) = \delta\left(x \Leftrightarrow \frac{1}{2}\right) \sqcap \neg\delta(y \sqcup \neg y).$$

c) $y = \frac{1}{2}$. Then $x +_0 y = x$. Thus we set $S_c(x, y) = x$. The characteristic term of case c) is

$$c(x, y) = \delta\left(y \Leftrightarrow \frac{1}{2}\right) \sqcap \neg\delta(x \sqcup \neg x).$$

d) $0 < x < \frac{1}{2}$, $\frac{1}{2} < y < 1$, and $\neg x < y$. Note that in this case $x + y > 1$, $-_0 x >_0 y$ and $x +_0 y <_0 0$. Indeed, $\frac{1}{2} < \neg x = -_0 x < y$, and since f_2 is decreasing, $f_2(-_0 x) > f_2(y)$, and finally $-_0 x >_0 y$, which in turn implies that $x +_0 y <_0 0$. It follows that if case d) occurs, then $f(x) + f(y) = f_1(x) + f_2(y) < 0$, and that $x +_0 y = f_1^{-1}(f_1(x) + f_2(y))$. We start with a computation of $z = f_1(x) + f_2(y)$. We have:

$$z = \tfrac{4x}{2x-1} + \tfrac{4-4y}{2y-1} = \tfrac{8xy-4x+8x-4-8xy+4y}{(2x-1)(2y-1)} = \tfrac{4x+4y-4}{(2x-1)(2y-1)}.$$

Thus we have:

$$2z - 4 = \tfrac{8x+8y-8-16xy+8x+8y-4}{(2x-1)(2y-1)} = \tfrac{-16xy+16x+16y-12}{(2x-1)(2y-1)},$$

and recalling that $f_1^{-1}(z) = \frac{z}{2z-4}$, we obtain:

$$x +_0 y = f^{-1}(z) = \tfrac{4x+4y-4}{-16xy+16x+16y-12} = \tfrac{1-x-y}{3-4x-4y+4xy} = \tfrac{\frac{1}{8}x+\frac{1}{8}y-\frac{1}{8}}{\frac{x}{2}+\frac{y}{2}-\frac{3}{8}-\frac{xy}{2}}.$$

Note that $x+y>1$, therefore $\frac{1}{8}x+\frac{1}{8}y-\frac{1}{8}>0$. Also, $x+y-\frac{3}{4}-xy>0$. Indeed, since $x>1-y$, we have $x=1-y+c$ with $c>0$. Thus $x+y-\frac{3}{4}-xy=\frac{1}{4}+c-y\cdot(1-y+c)>\frac{1}{4}-y\cdot(1-y)$. Now the maximum of the function $y\cdot(1-y)$ in $[0,1]$ is $\frac{1}{4}$, therefore $x+y-\frac{3}{4}-xy>0$, and $\frac{x}{2}+\frac{y}{2}-\frac{3}{8}-\frac{xy}{2}>0$
It follows that in case d) $x+_0 y$ is definable by the term

$$S_d(x,y)=\left(\left(\tfrac{1}{2}x\oplus\tfrac{1}{2}y\right)\ominus\left(\tfrac{3}{8}\oplus\tfrac{1}{2}xy\right)\right)\Rightarrow_\pi\left(\left(\tfrac{1}{8}x\oplus\tfrac{1}{8}y\right)\ominus\tfrac{1}{8}\right).$$

Note also that the characteristic term of case d) is

$$d(x,y)=a(x,\neg y)\sqcap\neg\delta(y\Rightarrow\neg x).$$

e) $\neg x=y$. Then we have $x+_0 y=\frac{1}{2}$. Thus let $S_e(x,y)=\frac{1}{2}$.
The characteristic term of case e) is

$$e(x,y)=\delta(\neg x\Leftrightarrow y)\sqcap\neg\delta(\neg x\sqcup x\sqcup\neg y\sqcup y).$$

f) $0<x<\frac{1}{2}$, $\frac{1}{2}<y<1$ and $\neg x>y$. Then let $x'=-_0x=\neg x$ and $y'=-_0y=\neg y$. Clearly y' and x' satisfy the assumptions of case d), therefore $y'+_0x'=S_d(y',x')$. Moreover $y+_0x=-_0(-_0y+_0(-_0x))=\neg S_d(\neg y,\neg x)$. Thus let $S_f(x,y)=\neg S_d(\neg y,\neg x)$.
Finally the characteristic term for case f) is $f(x,y)=d(\neg y,\neg x)$.

g) $0<y<\frac{1}{2}$, $\frac{1}{2}<x<1$ and $\neg x<y$. This case is symmetric to case d), therefore $x+_0 y=S_d(y,x)$. Thus let $S_g(x,y)=S_d(y,x)$.
The characteristic term is $g(x,y)=d(y,x)$.

h) $0<y<\frac{1}{2}$, $\frac{1}{2}<x<1$ and $\neg x>y$. This case is symmetric to case f), therefore $x+_0 y=\neg S_d(\neg x,\neg y)$. Thus let $S_h(x,y)=\neg S_d(\neg x,\neg y)$.
The characteristic term is $h(x,y)=d(\neg x,\neg y)$.

i) $\frac{1}{2}<x<1$ and $\frac{1}{2}<y<1$. Then $\neg x$ and $\neg y$ satisfy case a), and $x+_0y=-_0(-_0x+_0(-_0y))=\neg S_a(\neg x,\neg y)$. Thus let $S_i(x,y)=\neg S_a(\neg x,\neg y)$.
Moreover the charcateristic term is $i(x,y)=a(\neg x,\neg y)$.

This concludes the definition of $+_0$ in $\text{Ł}\Pi\frac{1}{2}$. Notice that we can find a uniform term which includes all cases, namely:

$$S(x,y)=\sqcup_{\alpha\in\{a,b,c,d,e,f,g,h,i\}}\alpha(x,y)\sqcap S_\alpha(x,y).$$

$[\cdot_0]$: Also for $\cdot_0$ the definition is given by cases.

l) $x=\frac{1}{2}$ or $y=\frac{1}{2}$. Then clearly $x\cdot_0 y=\frac{1}{2}$. Thus we define $P_l(x,y)=\frac{1}{2}$.
The characteristic term is

$$l(x,y)=\left(\delta\left(x\Leftrightarrow\frac{1}{2}\right)\sqcup\delta\left(y\Leftrightarrow\frac{1}{2}\right)\right)\sqcap\neg\delta(\neg x\sqcup x\sqcup\neg y\sqcup y).$$

i) $\frac{1}{2} < x < 1$ and $\frac{1}{2} < y < 1$. Then $x \cdot_0 y = f_2^{-1}(f_2(x) \cdot f_2(y))$. We compute $z = f_2(x) \cdot f_2(y)$. We have:

$$z = \frac{4-4x}{2x-1} \cdot \frac{4-4y}{2y-1} = \frac{16+16xy-16x-16y}{(2x-1)(2y-1)}.$$

Now $x \cdot_0 y = f_2^{-1}(z) = \frac{z+4}{2z+4}$. Thus we compute:

$$z+4 = \frac{16+16xy-16x-16y}{(2x-1)(2y-1)} + 4 = \frac{16+16xy-16x-16y+16xy-8x-8y+4}{(2x-1)(2y-1)} = \frac{20+32xy-24x-24y}{(2x-1)(2y-1)}.$$

$$2z+4 = \frac{32+32xy-32x-32y}{(2x-1)(2y-1)} + 4 = \frac{32+32xy-32x-32y+16xy-8x-8y+4}{(2x-1)(2y-1)} = \frac{36+48xy-40x-40y}{(2x-1)(2y-1)}.$$

Hence

$$x \cdot_0 y = \frac{20+32xy-24x-24y}{36+48xy-40x-40y} = \frac{5+8xy-6x-6y}{9+12xy-10x-10y} = \frac{\frac{1}{6}+\frac{4}{15}xy-\frac{1}{5}x-\frac{1}{5}y}{\frac{3}{10}+\frac{3}{5}xy-\frac{1}{3}x-\frac{1}{3}y}.$$

Now $z = f_2(x) + f_2(y) > 0$, therefore $z+4 > 0$ and $2z+4 > 0$. Since $x > \frac{1}{2}$ and $y > \frac{1}{2}$, we also have $(2x-1)(2y-1) > 0$. So, $\frac{1}{6} + \frac{4}{15}xy - \frac{1}{5}x - \frac{1}{5}y > 0$ and $\frac{3}{10} + \frac{3}{5}xy - \frac{1}{3}x - \frac{1}{3}y > 0$. Thus let

$$P_i(x,y) = \left(\left(\tfrac{3}{10} \oplus \tfrac{3}{5}xy\right) \ominus \tfrac{1}{3}\,(x \oplus y)\right) \Rightarrow_\pi \left(\left(\tfrac{1}{6} \oplus \tfrac{4}{15}xy\right) \ominus \tfrac{1}{5}\,(x \oplus y)\right).$$

Finally, the characteritic term is $i(x,y)$.

a) $0 < x < \frac{1}{2}$ and $0 < y < \frac{1}{2}$. Then $\neg x$ and $\neg y$ satisfy case i), and $x \cdot_0 y = (-_0 x) \cdot_0 (-_0 y) = \neg x \cdot_0 \neg y = P_i(\neg x, \neg y)$. Thus let $P_a(x,y) = P_i(\neg x, \neg y)$.
Clearly the characteristic term is $a(x,y)$.

m) $0 < x < \frac{1}{2}$ and $\frac{1}{2} < y < 1$. Then $\neg x$ and y satisfy case i), therefore $\neg x \cdot_0 y = P_i(\neg x, y)$, and $x \cdot_0 y = -_0(\neg x \cdot_0 y) = \neg P_i(\neg x, y)$. Thus let $P_m(x,y) = \neg P_i(\neg x, y)$. The characteristic term of case m) is $m(x,y) = i(\neg x, y)$.

n) $0 < y < \frac{1}{2}$ and $\frac{1}{2} < x < 1$. This case is symmetric to case m), therefore $P_n(x,y) = P_m(y,x)$ and $n(x,y) = m(y,x)$.

Hence we have a uniform term representing $x \cdot_0 y$, namely

$$P(x,y) = \sqcup_{\alpha \in \{l,i,a,m,n\}} \alpha(x,y) \sqcap P_\alpha(x,y).$$

Now for every term t in the language of ordered fields we define a term t_0 of Ł$\Pi\frac{1}{2}$-algebras in the following inductive way:

- If t is a variable, then $t_0 = t$.
- If $t = 0$, then $t_0 = \frac{1}{2}$.
- If $t = 1$, then $t_0 = \frac{5}{6}$.
- If $t = -s$, then $t_0 = \neg s_0$.
- If $t = s + u$, then $t_0 = S(s_0, u_0)$.
- If $t = s \cdot u$, then $t_0 = P(s_0, u_0)$.

Next we observe that, since f is decreasing in $(0, \frac{1}{2})$ and in $(\frac{1}{2}, 1)$, we have that $x \leq_0 y$ iff either $x \leq \frac{1}{2} \leq y$ or $x \geq y$ and either $x, y < \frac{1}{2}$ or $x, y > \frac{1}{2}$. The characteristic term of this relation is:

$$\delta\left((x \sqcup \neg y) \Rightarrow \frac{1}{2}\right) \sqcup \left(\delta(y \Rightarrow x) \sqcap \left(\neg\delta\left(\frac{1}{2} \Rightarrow (x \sqcup y)\right) \sqcup \neg\delta\left((x \sqcap y) \Rightarrow \frac{1}{2}\right)\right)\right).$$

This term will be denoted by $t^{\leq}(x, y)$.

The formula $x = y$ is translated by $\delta(x \Leftrightarrow y)$ (this term will be denoted by $t^{=}(x, y)$. Of course, the characteristic term of $x < y$ is $t^{<}(x, y) = t^{\leq}(x, y) \sqcap \neg t^{=}(x, y)$.

Next we define for every quantifier-free formula Φ in the language of ordered fields, a term t^{Φ} in the following inductive way:

- If Φ is $s = u$, then $t^{\Phi} = t^{=}(s_0, u_0)$.
- If Φ is $s \leq u$, then $t^{\Phi} = t^{\leq}(s_0, u_0)$.
- If Φ is $\Gamma \sqcup \Sigma$ ($\Gamma \sqcap \Sigma$, $\neg\Gamma$ respectively), then $t^{\Phi} = t^{\Gamma} \sqcup t^{\Sigma}$ ($t^{\Gamma} \sqcap t^{\Sigma}$, $\neg t^{\Gamma}$ respectively).

Hence we obtain the following theorem.

Theorem 4.2.3 *Let $\mathcal{F}$ be an ordered subfield of the field of real algebraic numbers, and let $\Phi(x_1, ..., x_n)$ be a quantifier-free formula in the language of ordered fields with coefficients in $\mathbb{Q}$. Then, for all $a_1, ..., a_n \in \mathbb{F}$, the following are equivalent:*

(i) $\mathcal{F} \models \Phi(a_1, ..., a_n)$.

(ii) $\mathbb{F}Ł\Pi\frac{1}{2} \models t^{\Phi}(f^{-1}(a_1), ..., f^{-1}(a_n)) = 1$.

Proof. Take a term $t(x_1, ..., x_n)$ in $\mathcal{F}$. Notice that given that f is an isomorphism between $\mathcal{F}$ and $\mathcal{F}_0$, then for every $a_1, ..., a_n \in \mathbb{F}$, one has that $f^{-1}(t(a_1, ..., a_n)) = t^{\mathcal{F}_0}(f^{-1}(a_1), ..., f^{-1}(a_n))$, where $t^{\mathcal{F}_0}(f^{-1}(a_1), ..., f^{-1}(a_n))$ is the translation in $\mathcal{F}_0$ of $t(a_1, ..., a_n)$, under f.

Moreover, for every quantifier-free formula $\Psi(x_1, ..., x_n)$ in the language of ordered fields and for every $b_1, ..., b_n \in (0, 1) \cap \mathbb{F}$, we can easily prove by induction on the complexity of $\Psi(x_1, ..., x_n)$ that if $\mathcal{F}_0 \models \Psi(b_1, ..., b_n)$, then there is an equation in the language of $Ł\Pi\frac{1}{2}$-algebras such that $t^{\Psi}(b_1, ..., b_n) = 1$, and $t^{\Psi}(b_1, ..., b_n) = 0$ otherwise.

Consequently we have that $\mathcal{F} \models \Phi(a_1, ..., a_n)$ iff $\mathcal{F}_0 \models \Phi^{\mathcal{F}_0}(f^{-1}(a_1), ..., f^{-1}(a_n))$ iff $\mathbb{F}Ł\Pi\frac{1}{2} \models t^{\Phi}(f^{-1}(a_1), ..., f^{-1}(a_n)) = 1$, where $\Phi^{\mathcal{F}_0}(f^{-1}(a_1), ..., f^{-1}(a_n))$ is the translation of $\Phi(a_1, ..., a_n)$ under the isomorphism f. ■

In the last section of this chapter we will rely on the above theorem to study the lattice of subvarieties of $Ł\Pi\frac{1}{2}$.

4.3 Real closed fields and $Ł\Pi\frac{1}{2}$

In this section we investigate the connections between $Ł\Pi\frac{1}{2}$ and real closed fields. First we show that the universal theory of real closed fields can be faithfully interpreted in $Ł\Pi\frac{1}{2}$. This means that functions (with rational coefficients) definable over real closed fields will also be definable over $Ł\Pi\frac{1}{2}$. This will lead us to the study of functions and sets definable in $Ł\Pi\frac{1}{2}$. Finally we will show that the universal theory of real closed fields and $Ł\Pi\frac{1}{2}$ both share the same computational complexity class, since their mutual translatability can be carried out in polynomial time.

A *real closed field* $\mathcal{RF} = \langle R, +, \cdot, -, \leq, 0, 1\rangle$ (see [11]) is a field with a unique ordering whose positive cone is the set of squares of R, and every polynomial of $R[X]$, of odd degree, has a root in R. Given an ordered field $\mathcal{F}$, the *real closure* of $\mathcal{F}$ is an algebraic extension $\mathcal{G}$ which is a real closed field and with a unique ordering extending the ordering of $\mathcal{F}$. The field $\mathcal{R}$ of real numbers $\mathbb{R}$ is a real closed field, while the field $\mathcal{Q}$ of rational numbers $\mathbb{Q}$ is not real closed. The real closure of $\mathcal{Q}$ is the field $\mathcal{R}_{\text{alg}}$ of real algebraic numbers $\mathbb{R}_{\text{alg}}$, i.e. the real roots of polynomials with integer coefficients

$$a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0.$$

$\mathcal{R}_{\text{alg}}$ also is the smallest real closed field. Real closed fields form an elementary class of structures since their fundamental properties are axiomatizable in first-order logic (see [20]). A very important and well-known result, proved by Tarski [139], is that the first-order theory of real closed fields admits quantifier elimination in the language $\langle +, \cdot, -, \leq, 0, 1\rangle$.

Theorem 4.3.1 ([20, 11]) *Let $\Phi(x_1, \ldots, x_n)$ be a formula in the language of ordered fields $\langle +, \cdot, -, \leq, 0, 1\rangle$ with coefficients in an ordered ring $\mathcal{D}$ contained in the real closed field $\mathcal{RF}$. Then there is a quantifier-free formula $\Psi(x_1, \ldots, x_n)$ with coefficients in D such that for every $\langle x_1, \ldots, x_n\rangle \in R^n$, the formula $\Phi(x_1, \ldots, x_n)$ is true iff so is $\Psi(x_1, \ldots, x_n)$.*

A consequence of the above result is that the theory of real closed fields is complete and decidable (see [20, 139]). Moreover, being all real closed fields an elementary extension of $\mathcal{R}_{\text{alg}}$, they are all elementary equivalent to $\mathcal{R}_{\text{alg}}$, and, thus, to each other.

It is easy to show that the universal theory of real closed fields can be faithfully interpreted in the equational theory of $\text{Ł}\Pi\frac{1}{2}$-algebras. By applying exactly the same construction carried out in Section 4.2, we can show that any quantifier-free formula in the language of ordered fields with rational coefficients can be translated into an equation in the language of $\text{Ł}\Pi\frac{1}{2}$.

Let $x_1, ..., x_n$ be the variables in Φ; then let

$$t_0^\Phi = (\delta(x_1 \sqcup \neg x_1) \sqcap ... \sqcap \delta(x_n \sqcup \neg x_n)) \to t^\Phi.$$

Then the following theorem holds.

Theorem 4.3.2 *Let $\Phi(x_1, ..., x_n)$ be a quantifier-free formula in the language of ordered fields with coefficients in $\mathbb{Q}$. Then:*

1) *For all $a_1, ..., a_n \in \mathbb{R}$, the following are equivalent:*

   1.1) $\mathcal{R} \models \Phi(a_1, ..., a_n)$.

   1.2) $\mathbb{R}\text{Ł}\Pi\frac{1}{2} \models t^\Phi(f^{-1}(a_1), ..., f^{-1}(a_n)) = 1$.

2) *The following are equivalent:*

   2.1) $\mathcal{R} \models \forall x_1 ... \forall x_n \Phi(x_1, ..., x_n)$.

   2.2) $\mathbb{R}\text{Ł}\Pi\frac{1}{2} \models \forall x_1 ... \forall x_n (t_0^\Phi(x_1, ..., x_n) = 1)$.

Proof. The proof is an easy adaptation of the proof given for Theorem 4.2.3. ■

4.3.1 Definable sets and definable functions

In this section we investigate $\text{Ł}\Pi\frac{1}{2}$-definable functions and sets by exploiting the connection between the universal theory of real closed fields and $\text{Ł}\Pi\frac{1}{2}$.

Definition 4.3.3 [11] Given a real closed field $\mathcal{RF} = \langle R, +, \cdot, -, \leq, 0, 1 \rangle$, a *semialgebraic set* is a subset of R^n of the form

$$(\star) \qquad \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in R^n \mid f_{i,j}(x) *_{i,j} 0\},$$

where $f_{i,j}(x) \in R[X_1, \ldots, X_n]$ and $*_{i,j}$ is either $<$ or $=$, for $i = 1, \ldots, s$, and $j = 1, \ldots, r_i$.

It is easy to see that semialgebraic subsets of R are exactly finite unions of points and open intervals. In particular, every semialgebraic subset of R^n can be written as a finite union of semialgebraic sets of the form:

$$\{x \in R^n \mid f_1(x) = \cdots = f_l(x) = 0, g_1(x) > 0, \ldots, g_m(x) > 0\},$$

where $f_1, \ldots, f_l, g_1, \ldots, g_m \in R[X_1, \ldots X_n]$. In other words, semialgebraic sets are subsets of a real closed field defined by a finite Boolean combination of polynomial equations and inequalities.

Definition 4.3.4 A set $S \subseteq \mathbb{R}^n$ is $\mathbb{Q}$-*semialgebraic* if it has the form ($\star$), where each $f_{i,j}(x)$ is a polynomial with rational coefficients.

Definition 4.3.5 (a) A function g from $[0,1]^n$ into $[0,1]$ is said to be *term-definable* (without parameters) in $\text{Ł}\Pi\frac{1}{2}$ if there is a term $t(x_1, ..., x_n)$ of $\text{Ł}\Pi\frac{1}{2}$-algebras such that for all $a_1, ..., a_n \in [0,1]$ one has

$$t(a_1, ..., a_n) = g(a_1, ..., a_n).$$

(b) A set $X \subseteq [0,1]^n$ is said to be *definable* in $\text{Ł}\Pi\frac{1}{2}$ (without parameters) if its characteristic function is term-definable in $\text{Ł}\Pi\frac{1}{2}$.

(c) A function f is said to be *implicitly definable* (without parameters) in $\text{Ł}\Pi\frac{1}{2}$ if its graph is definable in $\text{Ł}\Pi\frac{1}{2}$.

(d) A set $S \subseteq \mathbb{R}^n$ is said to be *definable* in $\mathcal{R}$ (without parameters) if there is a first-order formula $\Phi(x_1, ..., x_n)$ such that

$$S = \{(a_1, ..., a_n) : \mathcal{R} \models \Phi(a_1, ..., a_n)\}.$$

(e) A function is said to be *definable* in $\mathcal{R}$ iff its graph is definable in $\mathcal{R}$.

Example 4.3.6 *Every term-definable function is implicitly definable, but the converse does not hold:* $\sqrt{x}$ *is implicitly definable in* $\text{Ł}\Pi\frac{1}{2}$*, because the characteristic function of its graph is* $\delta(x \Leftrightarrow y^2)$*. We will see below why* $\sqrt{x}$ *is not term-definable in* $\text{Ł}\Pi\frac{1}{2}$*.*

Definition 4.3.7 An $\text{Ł}\Pi\frac{1}{2}$-*hat* over $[0,1]^n$ is a function $h : [0,1]^n \to [0,1]$ such that there exist a $\mathbb{Q}$-semialgebraic set $S \subseteq [0,1]^n$ and polynomials $f(x_1, ..., x_n)$, $g(x_1, ..., x_n) \in \mathbb{Q}[X_1, ..., X_n]$ such that $g(x_1, ..., x_n)$ has no zeros on S, $h = \frac{f(x_1,...,x_n)}{g(x_1,...,x_n)}$ on S, and $h = 0$ on $[0,1]^n \backslash S$.

A function $h : [0,1]^n \to [0,1]$ is said to be *piecewise rational* if it is the supremum of finitely many $\text{Ł}\Pi\frac{1}{2}$-hats.

The next theorem, whose proof can be found in [117], characterizes term-definable functions.

Theorem 4.3.8 ([117]) *A function* $h : [0,1]^n \to [0,1]$ *is term-definable in* $\text{Ł}\Pi\frac{1}{2}$ *iff it is a piecewise rational function.*

Clearly, it follows that functions as $\sqrt{x}$ or $\sqrt{1-x^2}$ cannot be defined by terms in $\text{Ł}\Pi\frac{1}{2}$, not being piecewise rational.

The next theorem gives a characterization of definable sets and therefore of implicitly definable functions in $\text{Ł}\Pi\frac{1}{2}$.

Theorem 4.3.9 *A set* $S \subseteq [0,1]^n$ *is definable in* $\text{Ł}\Pi\frac{1}{2}$ *iff it is definable in* $\mathcal{R}$ *by a formula with rational coefficients iff it is* $\mathbb{Q}$*-semialgebraic. Thus a function* $h : [0,1]^n \to [0,1]$ *is implicitly definable in* $\text{Ł}\Pi\frac{1}{2}$ *iff its graph is* $\mathbb{Q}$*-semialgebraic.*

Proof. Suppose that $S \subseteq [0,1]^n$ is definable in $\text{Ł}\Pi\frac{1}{2}$. Then its characteristic function is term-definable in $\text{Ł}\Pi\frac{1}{2}$. It follows that there is a piecewise rational function h such that $S = \{(a_1, ..., a_n) : h(a_1, ..., a_n) = 1\}$. Now it is an easy exercise to prove that every piecewise rational function is definable in $\mathcal{R}$, therefore S is in turn definable in $\mathcal{R}$. For the other direction, suppose that $S \subseteq [0,1]^n$ is defined in $\mathcal{R}$ by a formula $\Phi(x_1, ..., x_n)$, with rational coefficients. Since the theory of $\mathcal{R}$ has quantifier elimination, we can assume without loss of generality that Φ is quantifier-free. Now, by Theorem 4.2.3, we have that for all $a_1, ..., a_n \in \mathbb{R}$ one has

$$\mathcal{R} \models \Phi(a_1, ..., a_n) \text{ iff } \mathbb{R}\text{L}\Pi\tfrac{1}{2} \models t^{\Phi}(f^{-1}(a_1), ..., f^{-1}(a_n)) = 1.$$

Now if $x \in [0,1]$ we have that $f^{-1}(x)$ is defined by the term

$$t^{f^{-1}}(x) = \left(\delta\,(\neg x) \sqcap \tfrac{1}{2}\right) \sqcup \left(\neg\delta\,(\neg x) \sqcap \left(\left(\tfrac{1}{4}x \oplus \tfrac{1}{2}\right) \Rightarrow_{\pi} \left(\tfrac{1}{8}x \oplus \tfrac{1}{2}\right)\right)\right).$$

Hence $S = \left\{(a_1, ..., a_n) : t^{\Phi}(t^{f^{-1}}(a_1), ..., t^{f^{-1}}(a_n)) = 1\right\}$, and the claim follows. ∎

4.3.2 Computational complexity

Hájek and Tulipani showed in [79] that the 1-satisfiability problem for the $\text{Ł}\Pi\frac{1}{2}$ logic is in PSPACE. This was proved by showing that $\text{Ł}\Pi\frac{1}{2}$ is polynomially reducible to the universal theory of $\mathcal{R}$, which was proved to be in PSPACE by Canny in [17]. The next result shows that also the converse is true, therefore $\text{Ł}\Pi\frac{1}{2}$ and the theory of $\mathcal{R}$ have exactly the same complexity.

Theorem 4.3.10 *There is a polynomial-time reduction of the universal theory of $\mathcal{R}$ to $\text{Ł}\Pi\frac{1}{2}$.*

Proof. Theorem 4.3.2 (2) says that for every quantifier-free formula in the language of $\mathcal{R}$ we have that

$$\mathcal{R} \models \forall x_1 ... \forall x_n \Phi(x_1, ..., x_n) \text{ iff } \mathbb{R}\text{L}\Pi\tfrac{1}{2} \models \forall x_1 ... \forall x_n (t_0^{\Phi}(x_1, ..., x_n) = 1).$$

Since $\text{Ł}\Pi\frac{1}{2}$ is complete with respect to $\mathbb{R}\text{Ł}\Pi\frac{1}{2}$, the map $\Phi \mapsto t_0^{\Phi}$ reduces the universal theory of $\mathcal{R}$ to $\text{Ł}\Pi\frac{1}{2}$. However this map is not P-time in general, because the presence of long terms occurring in Φ may force t_0^{Φ} to be exponentially longer than Φ. Thus we proceed as follows: say that a formula Φ in the language of $\mathcal{R}$ is *in normal form* iff every term t in it which is not a variable is either a constant or a term of the form $x + y$ or $x \cdot y$, where x, y are variables, and only occurs in atomic formulas of the form $v = t$, where v is a variable. The result is an obvious consequence of the following lemmas.

Lemma 4.3.11 *If $\forall x_1 ... \forall x_n \Phi(x_1, ..., x_n)$ is a universal formula in normal form in the language of $\mathcal{R}$, then t_0^{Φ} can be computed in polynomial time from Φ.*

Lemma 4.3.12 *There is a polynomial time algorithm that computes for every universal formula in the language of $\mathcal{R}$ a formula in normal from which is equivalent to it.*

Proof of Lemma 4.3.11. t_0^Φ is computed in linear time from t^Φ, therefore it is sufficient to prove the claim for t^Φ. Now t^Φ is obtained from Φ replacing all the atomic subformulas of the form $v = 0$ by $\delta(v \Leftrightarrow \frac{1}{2})$, every subformula of the form $v = 1$ by $\delta(v \Leftrightarrow \frac{5}{6})$, every formula of the form $v = x + y$ by $t^=(v, S(x, y))$, every subformula of the form $x = xy$ by $t^=(v, P(x, y))$, every subformula of the form $x = y$ by $t^=(x, y)$ and every subformula of the form $x \le y$ by $t^\le(x, y)$. Let k be the maximum length of $t^=(v, S(x, y))$, $t^=(v, P(x, y))$ and $t^\le(x, y)$. Let lth denote the length function. Then $lth(t^\Phi) \le k \cdot lth(\Phi)$, and Lemma 4.3.11 is proved.

Proof of Lemma 4.3.12. Any quantifier-free formula Φ can be written as $\Psi(t_1, ..., t_n)$ for some terms $t_1, ..., t_n$ and for some quantifier-free formula $\Psi(v_1, ..., v_n)$ such that every term occurring in it is a variable. Let $T = \{s_1, ..., s_k\}$ denote the set of all subterms of $t_1, ..., t_n$. Note that the total length of T is linear in $lth(\Phi)$. For all $s_i \in T$, introduce a new variable v_{s_i}. Now for every $s_i \in T$, define a formula χ_{s_i} as follows: if s_i is a constant c, then χ_{s_i} is $v_{s_i} = c$; if s_i is a variable v, then χ_{s_i} is $v_{s_i} = v$; if s_i is $s_j + s_h$ for some $s_j, s_h \in T$, then χ_{s_i} is $v_{s_i} = v_{s_j} + v_{s_h}$; if $s_i = s_j \cdot s_h$ for some $s_i, s_h \in T$, then χ_{s_i} is $v_{s_i} = v_{s_j} \cdot v_{s_h}$. Let χ denote the conjunction of all $\chi_{s_i} : s_i \in T$. Now let $x_1, ..., x_h$ be the variables in Φ. Then it is easily seen that $\forall x_1 ... \forall x_h \Phi(x_1, ..., x_k)$ is logically equivalent to the universal closure of $\chi \Rightarrow \Psi(x_{t_1}, ..., x_{t_n})$. ■

4.4 Subvarieties of $\text{Ł}\Pi\frac{1}{2}$

In this section we will study the lattice of subvarieties of $\text{Ł}\Pi\frac{1}{2}$-algebras.

4.4.1 The algebra of real algebraic numbers

The following theorem is an immediate generalization of a result proven by Montagna in [111]:

Theorem 4.4.1 ([111]) *The interval* $\text{Ł}\Pi\frac{1}{2}$*-algebra of any real closed field generates the whole variety of* $\text{Ł}\Pi\frac{1}{2}$*-algebras.*

Proof. Trivially, if an equation holds in every $\text{Ł}\Pi\frac{1}{2}$-algebra, then it also holds in the $\text{Ł}\Pi\frac{1}{2}$-algebra of any real closed field.

To prove the converse, recall that the variety of $\text{Ł}\Pi\frac{1}{2}$-algebras is generated by the class of linearly ordered $\text{Ł}\Pi\frac{1}{2}$-algebras, and thus suppose that there is an equation ϵ that does not hold in some $\text{Ł}\Pi\frac{1}{2}$-chain $\mathcal{A}$. $\mathcal{A}$ is isomorphic the interval algebra $\mathcal{B}$ of an ordered field $\mathcal{H}$. We can extend the field $\mathcal{H}$ to a real closed field $\mathcal{RF}$ (by taking its real closure) whose $\text{Ł}\Pi\frac{1}{2}$ interval algebra $\mathcal{C}$ is a superstructure of $\mathcal{B}$, in which then ϵ does not hold. Now, all real closed fields are elementarily equivalent, hence it follows that the interval algebra of any real closed field generates the whole variety. ■

Corollary 4.4.2 $\mathbb{A}Ł\Pi\frac{1}{2}$ *generates the whole variety of* $Ł\Pi\frac{1}{2}$*-algebras.*

We show now that any $Ł\Pi\frac{1}{2}$-algebra that is a strict subalgebra of $\mathbb{A}Ł\Pi\frac{1}{2}$ generates a different variety. The idea consists in showing that we can find an equation in the language of $Ł\Pi\frac{1}{2}$ that, in some sense, translates the fact that the root of a polynomial is not contained in the structure. That equation will not be satisfied only if that root can be mapped in the lattice reduct of $Ł\Pi\frac{1}{2}$.

Lemma 4.4.3 *Let* $\mathcal{F}$ *be an ordered field such that* $\mathbb{Q} \subseteq \mathbb{F} \subset \mathbb{R}_{\mathrm{alg}}$*. Then the algebra* $\mathbb{F}Ł\Pi\frac{1}{2}$ *does not generate the whole variety of* $Ł\Pi\frac{1}{2}$*-algebras.*

Proof. Take an ordered field $\mathcal{F}$ such that $\mathbb{Q} \subseteq \mathbb{F} \subset \mathbb{R}_{\mathrm{alg}}$ and take the related $Ł\Pi\frac{1}{2}$-algebra $\mathbb{F}Ł\Pi\frac{1}{2}$ (i.e. the algebra whose lattice reduct is $[0,1] \cap \mathbb{F}$). $\mathbb{F}Ł\Pi\frac{1}{2}$ clearly is a subalgebra of $\mathbb{A}Ł\Pi\frac{1}{2}$. Take now any real algebraic number α such that $\alpha \in \mathbb{R}_{\mathrm{alg}} \cap [0,1] \backslash \mathbb{F} \cap [0,1]$. We show that there is an equation that holds in $\mathbb{F}Ł\Pi\frac{1}{2}$, but not in $\mathbb{A}Ł\Pi\frac{1}{2}$, and so it does not hold in the whole variety. Given α, notice that $f(\alpha)$ (where f is the isomorphism defined above) still is a real algebraic number, since the function f maps real algebraic numbers into real algebraic numbers. By definition, the element $f(\alpha)$ is the root of a polynomial with integer coefficients

$$a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0.$$

Following the previous section, let $t^p(x)$ be the translation of such a polynomial into an $Ł\Pi\frac{1}{2}$-term, and let s and r be two rational numbers such that $f(\alpha) \in [f(r), f(s)]$, while all the other roots of the polynomial lie outside the interval. Note that $f(\alpha) \in [f(r), f(s)]$ iff $\alpha \in [s, r]$. Then we claim that the equation

$$\neg_\pi \left(\delta \left(t^p(x) \Leftrightarrow \tfrac{1}{2}\right) \sqcap \delta\left(x \Rightarrow r\right) \sqcap \delta\left(s \Rightarrow x\right)\right) = 1$$

holds in $\mathbb{F}Ł\Pi\frac{1}{2}$, but not in $\mathbb{A}Ł\Pi\frac{1}{2}$. To see that, notice first that for all $x \in [0,1] \backslash [s,r]$ the equation is satisfied. Suppose now that $x \in [s, r]$. The term $t^p(x)$ will equal $\frac{1}{2}$ if and only if x is α, in that case the equation does not hold (recall the the other roots lie outside the interval). Since α is not contained in $[0,1] \cap \mathbb{F}$, then the equation will always be satisfied.

Hence the claim is proven. ■

The proof of the following theorem is now obvious.

Theorem 4.4.4 $\mathbb{A}Ł\Pi\frac{1}{2}$ *is the smallest subalgebra of* $\mathbb{R}Ł\Pi\frac{1}{2}$ *generating the whole variety.*

Proof. Immediate from Corollary 4.4.2 and Lemma 4.4.3. ■

We then obtain the following:

Theorem 4.4.5 *The* $Ł\Pi\frac{1}{2}$ *logic is finitely strongly standard complete w.r.t. the interval algebra of any real closed field.*

Proof. Immediate from Theorem 4.4.1. ■

4.4.2 The lattice of subvarieties

Now, we want to study the lattice of subvarieties of $\text{Ł}\Pi\frac{1}{2}$ to show that such a lattice has the cardinality of the continuum. Recall that, given two ordered fields $\mathcal{F} = \langle F, +, \cdot, -, \leq, 0, 1\rangle$ and $\mathcal{G} = \langle G, +, \cdot, -, \leq, 0, 1\rangle$, $\mathcal{F}$ is an extension of $\mathcal{G}$ iff there is a monomorphism $i : \mathcal{G} \to \mathcal{F}$ (see [137]). This is denoted by $\mathcal{F}/\mathcal{G}$, and in this case $\mathcal{G}$ is a subfield of $\mathcal{F}$. The field of real algebraic numbers $\mathcal{R}_{\text{alg}}$, for instance, is an extension of the field of rational numbers $\mathcal{Q}$.

Given a field $\mathcal{G}$, an extension $\mathcal{F}/\mathcal{G}$, and any subset L of F, the subfield generated by $G \cup L$ is written $\mathcal{G}[L]$, and is obtained from G by adjoining all elements in L. Denote the subsets of $\mathbb{R}_{\text{alg}}$ by $\mathbf{A}, \mathbf{B}, \dots$: $\mathcal{Q}[\mathbf{A}]$ is the extension of $\mathcal{Q}$ obtained by adjoining all elements in $\mathbf{A}$. For instance, the field $\mathcal{Q}[\sqrt{3}]$ is defined as

$$\mathcal{Q}[\sqrt{3}] = \{a + \sqrt{3}b \mid a, b \in \mathbb{Q}\}.$$

Lemma 4.4.6 *The lattice of ordered subfields of $\mathcal{R}_{\text{alg}}$ has the cardinality of the continuum.*

Proof. First notice that, since real algebraic numbers form a countable set, there exist up to $2^{\aleph_0}$ extensions of $\mathcal{Q}$ being subfields of $\mathcal{R}_{\text{alg}}$. To see that the lattice of ordered subfields of $\mathcal{R}_{\text{alg}}$ has exactly the cardinality of the continuum we can just restrict ourselves to subfields generated by square roots of prime numbers. Indeed, take the subsets $\mathbf{P}_i$ of the set of square roots of prime numbers $\{\sqrt{p_1}, \sqrt{p_2}, \dots\}$, and generate, for each the extension field $\mathcal{Q}[\mathbf{P}_i]$. It is easy to see that there is a one-to-one correspondence between the set of extensions $\mathcal{Q}[\mathbf{P}_i]$ and the set of subsets of the set of square roots of prime numbers. This is clear considering that any two distinct $\mathbf{P}_i$ and $\mathbf{P}_j$ generate a different field. The cardinality of the set of subsets of the set of square roots of prime numbers is $2^{\aleph_0}$, hence the claim is proven. ■

We can now exploit the above results and see how the subvarieties of $\text{Ł}\Pi\frac{1}{2}$ inherit the above properties.

Lemma 4.4.7 *Any two different ordered subfields of $\mathcal{R}_{\text{alg}}$ have interval $\text{Ł}\Pi\frac{1}{2}$-algebras which generate two different subvarities.*

Proof. We know that every linearly ordered $\text{Ł}\Pi\frac{1}{2}$-algebra is the interval algebra of exactly one ordered field, up to isomorphism. Take then two different subfields of the real algebraic numbers, $\mathcal{F}_1$ and $\mathcal{F}_2$. Without any loss of generality we can suppose that there is some $\alpha \in \mathbb{F}_2$ not contained in $\mathbb{F}_1$. Being α a real algebraic number, it is the root of a polynomial $g(x)$ with integer coefficients. Proceeding as in Lemma 4.4.3, we can find an equation that holds in $\mathcal{F}_1$ but not in $\mathcal{F}_2$, being α not contained in $\mathbb{F}_1$. ■

We are now ready to prove the main theorem.

Theorem 4.4.8 *The lattice of subvarieties of $\text{Ł}\Pi\frac{1}{2}$-algebras has the cardinality of the continuum.*

Proof. By Lemma 4.4.6 the lattice of ordered subfields of $\mathcal{R}_{\mathrm{alg}}$ has the cardinality of the continuum. Up to isomorphism, every linearly ordered Ł$\Pi\frac{1}{2}$-algebra is the interval algebra of exactly one ordered field. By Lemma 4.4.7, each $\mathcal{F}$ such that $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{R}_{\mathrm{alg}}$ has an interval Ł$\Pi\frac{1}{2}$-algebra which generates a different variety. Hence, the claim follows. ■

Chapter 5

Triangular Norms Definable in $Ł\Pi\frac{1}{2}$

In the previous chapter we have investigated the strong relation between the equational theory of $Ł\Pi\frac{1}{2}$ and real closed fields. This has helped us analyze the definability of functions and sets. In particular, we have seen that the set of functions term-definable in $Ł\Pi\frac{1}{2}$ coincides with the set of piecewise rational functions, and definable sets precisely are $\mathbb{Q}$-semialgebraic sets, i.e. sets determined by polynomial equations and inequalities with rational coefficients. In this chapter we exploit those results in order to study the definability of t-norms in $Ł\Pi\frac{1}{2}$.

In the next section we give negative results concerning left-continuous t-norms. In fact, we show that left-continuous t-norms having a set of discontinuity points that is either dense or composed by infinitely many isolated points are not definable in $Ł\Pi\frac{1}{2}$.

On the other hand, we give a complete characterization of term-definable weak nilpotent minimum t-norms (Section 5.1) and of term-definable continuous t-norms (Section 5.2). Indeed, we show that a weak nilpotent minimum t-norm is term-definable iff its induced negation has a finite number of discontinuity points, and we prove that a continuous t-norm is term-definable iff it is representable as a finite ordinal sum.

Section 5.3 will be devoted to the study of definability of construction methods. We will show that the class of term-definable left-continuous t-norms is closed under the annihilation, the rotation and the rotation-annihilation constructions.

In Section 5.4 we give important completeness results. Indeed we show that several well-known t-norm based logics are complete w.r.t to the related class of standard algebras based on t-norms term-definable in $Ł\Pi\frac{1}{2}$. Such results will cover MTL, SMTL, IMTL, BL, SBL, and WNM.

Finally, we specifically exploit the connection with real closed fields in order to give decidability and complexity results. Indeed we show that every logic complete w.r.t to an implicitly definable t-norm is in PSPACE, and every logic complete w.r.t. a class of implicitly definable t-norms and having a finite axiomatization is decidable (Section 5.5). This is due to the fact that implicitly definable t-norms are, in turn, definable in the universal theory of real closed fields, which, as mentioned above, is decidable and in PSPACE.

5.1 Left-continuous t-norms

To begin our investigation of definability of t-norms, notice that we already have at our disposal the three fundamental t-norms, i.e. Łukasiewicz, Product and Gödel, since they correspond to operations of $Ł\Pi\frac{1}{2}$-algebras. Hence, we call them *trivially term-definable t-norms.* Clearly, these are not the only t-norms representable in $Ł\Pi\frac{1}{2}$, since every piecewise rational function can be defined in it. However, they can be regarded as a special kind of building blocks. Indeed they can be directly used to get new (left-continuous) t-norms.

Notice that given a term-definable left-continuous t-norm $*$, its residuum is not always term-definable. Take for instance the following t-norm, isomorphic to the nilpotent minimum, obtained by annihilation of the minimum t-norm by means of the strong negation $n(x) = \sqrt{1 - x^2}$:

$$x * y = \begin{cases} 0 & x^2 + y^2 > 1 \\ \min(x, y) & \text{otherwise} \end{cases} .$$

The above t-norm is clearly definable by the term $\delta(x^2 \Rightarrow \neg y^2) \sqcap (x \sqcap y)$, but its residuum, given by

$$x \Rightarrow y = \begin{cases} 1 & x \leq y \\ \max(\sqrt{1 - x^2}, y) & \text{otherwise} \end{cases} ,$$

is not term-definable, since the negation $n(x)$ is not piecewise rational.

Notice, however, that the residuum of an implicitly definable t-norm is implicitly definable. Indeed, if $*$ is term-definable in $Ł\Pi\frac{1}{2}$, then its graph is definable in $\mathcal{R}$ by a quantifier-free formula $\Phi(x, y, z)$, and so is the graph of its residuum $\rightarrow_*$ by means of the first-order formula

$$\forall u \forall v(\Phi(u, x, v) \rightharpoondown (u \leq z \leftrightharpoons v \leq y)),$$

where $\rightharpoondown$ denotes the classical implication and $\varphi \leftrightharpoons \psi$ denotes $(\varphi \rightharpoondown \psi) \sqcap (\psi \rightharpoondown \varphi)$. Hence, by Theorem 4.3.9 the residuum is implicitly definable in $Ł\Pi\frac{1}{2}$.

Now, we aim at studying the definability of left-continuous t-norms. We begin by recalling the following notions from topology.

Definition 5.1.1 The *closure* of a set X is the smallest closed set containing X. The *interior* of a set X is the union of all open sets contained in X.

A set X is *dense* in Y if the only closed subset of Y containing X is Y itself. A set X is *nowhere dense* if the interior of its closure is empty. A set X is *first category* if it is a countable union of nowhere dense sets.

The following result given by Jenei and Montagna provides a topological characterization of the set of continuity and discontinuity points of a left-continuous t-norm.

Theorem 5.1.2 ([96]) *Every left-continuous t-norm has a dense set of continuity points. Moreover, the set of discontinuity points is a first-category set, and its measure is zero.*

Now, we give a general result concerning some sets definable from left-continuous t-norms which are implicitly definable.

Theorem 5.1.3 *Suppose that a left-continuous t-norm $*$ is implicitly definable. Then every subset of $[0,1]^n$ that is first-order definable (without parameters) in the language $\{*, +, \cdot, \leq, 0, 1\}$ is $\mathbb{Q}$-semialgebraic. In particular:*

- (a) *The set of discontinuity points of $*$ is $\mathbb{Q}$-semialgebraic, and its closure has measure zero.*
- (b) *The set of idempotent elements of $*$ is $\mathbb{Q}$-semialgebraic. If $*$ is an ordinal sum of infinitely many t-norms then all of them but a finite number are isomorphic to the Gödel t-norm.*
- (c) $[0,1]$ *can be partitioned into a finite number of intervals $I_1, ..., I_{m+1}$ such that in each I_i the negation $\neg$ associated to $*$ is continuous and either constant or strictly increasing. In particular, $\neg$ has only finitely many discontinuity points.*

Proof. Notice first, that if $*$ is implicitly definable, then it is definable in the real field, and so is any set definable from $*$ in the reals, which then is $\mathbb{Q}$-semialgebraic.

- (a) The set of discontinuity points of an implicitly definable left-continuous t-norm is $\mathbb{Q}$-semialgebraic, being definable by the formula

  $$\{(x, y) : \exists a(a > 0) \wedge \forall b(b > 0 \rightarrow \exists c_1 \exists c_2 \exists z \exists u \Psi)\},$$

  where Ψ is the conjunction of the following formulas:

  $(x - c_1)^2 + (y - c_2)^2 < b$,

  $\Phi^*(x, y, z)$,

  $\Phi^*(c_1, c_2, u)$,

  $(z - u > a) \vee (u - z > a)$,

  and Φ^* is a formula which defines the graph of $*$ in the reals.

  By Theorem 5.1.2, the set of discontinuity points of a left-continuous t-norm is a first-category set, and its measure is zero. Since the boundary of a semialgebraic set obviously has measure zero, it is then clear that the closure of the set of discontinuity points also has measure zero.

(b) The set of idempotents of $*$ is definable as $\{x : x * x = x\}$, and therefore it is the union of finitely many (possible degenerate) intervals. Suppose now that $*$ is an ordinal sum of infinitely many t-norms. The minimum m of each component must be an idempotent (clearly $m * m \leq m$, but at the same time $m * m$ is in the component, therefore $m \leq m * m$). Thus if $*$ has infinitely many non-Gödel components, then there are infinitely many intervals (namely, the non-Gödel components) containing both an idempotent (the minimum) and a non-idempotent (since the component is not Gödel). Thus the set of idempotents can not be a union of finitely many (possibly degenerate) intervals. Hence, if $*$ is a definable t-norm obtained by an infinite ordinal sum the number of non-Gödel components must be finite.

(c) The residuum of a definable left-continuous t-norm is implicitly definable, and consequently so is its associated negation $\neg$. This means that the set of discontinuities of $\neg$ is definable in the reals and then, by Theorem 4.3.9 it must be a $\mathbb{Q}$-semialgebraic set, which means that it must be the union of finitely many intervals. Indeed, the set of discontinuities of a decreasing function is countable, hence it cannot be a whole non-degenerate interval. Therefore the set of discontinuities of $\neg$ must be finite.

Now, as shown in [46] (see Chapter 1), every weak negation $\neg$ with a finite number of discontinuity points determines a partition of the real interval in finitely many subintervals in which $\neg$ is either involutive or constant. Hence the claim follows.

■

In the following theorem we show that left-continuous t-norms with a dense set of discontinuity points or with infinitely many isolated discontinuity points are not definable.

Theorem 5.1.4 *If the set of discontinuity points of a left-continuous t-norm is dense or it is composed by infinitely many isolated points, then the t-norm is not definable.*

Proof. Suppose that the set of discontinuity points D of a t-norm is definable, and dense. Assuming definability, D must be a $\mathbb{Q}$-semialgebraic set. As noted in the proof of Theorem 5.1.3, D is first-category and being, $\mathbb{Q}$-semialgebraic, its closure must have an empty-interior. Therefore, D cannot be dense, otherwise the interior of its closure would be $[0,1]^2$ and consequently it would not be empty.

To conclude the proof, notice that being $\mathbb{Q}$-semialgebraic, D has finitely many components, therefore it cannot have infinitely many isolated points. ■

An example of a left-continuous t-norm that is not definable is given by the Smutná t-norm [136], whose set of discontinuities is dense in the unit square:

$$x * y = \begin{cases} 0 & \text{if } \min(x,y) = 0 \\ \sum\limits_{i=1}^{\infty} \frac{1}{2^{x_i + y_i - i}}, & \text{otherwise} \end{cases},$$

where, for $x, y \in]0, 1]$

$$x = \sum_{i=1}^{\infty} \frac{1}{2^{x_i}} \qquad y = \sum_{i=1}^{\infty} \frac{1}{2^{y_i}}$$

are the unique infinite dyadic expansions of x and y, respectively, and $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are strictly increasing sequences of natural numbers.

Now, recall that *weak nilpotent minimum* t-norms (Chapter 1) are left-continuous t-norms defined from a weak negation n_w as

$$x * y = \begin{cases} 0 & x \leq n_w(y) \\ \min(x, y) & \text{otherwise} \end{cases},$$

so that their induced negation corresponds to n_w.

Theorem 5.1.5 *Let $*$ be a weak nilpotent minimum t-norm. The following are equivalent:*

i. *Up to isomorphism, $*$ is implicitly definable in* $\text{Ł}\Pi\frac{1}{2}$.

ii. *Up to isomorphism, $*$ is term-definable in* $\text{Ł}\Pi\frac{1}{2}$.

iii. *The negation associated to $*$ has a finite number of discontinuity points.*

Proof. We prove (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (i) is trivial, while (i) $\Rightarrow$ (iii) follows from Theorem 5.1.3. Then we prove (iii) $\Rightarrow$ (ii).

As seen in [46] (see Chapter 1), if a weak negation $\neg$ has finitely many discontinuity points, then $[0, 1]$ can be divided into finitely many intervals $I_1 = [0 = a_0, a_1]$, $I_i = (a_{i=1}, a_i]$, $I_{r+1} = (a_r, a_{r+1} = 1]$, such that $\neg$ is either continuous and involutive or constant on each I_i. Up to isomorphism we can assume that the endpoints $a_1, \ldots, a_{r+1}$ as well as the corresponding values $\neg a_1, \ldots, \neg a_{r+1}$ and the right-limits $b_i = \lim_{x \to a_i^+} \neg x$ are rational numbers. Now, for $x \in I_i$, define

$$\neg' x = \begin{cases} b_{i-1} & \text{if } \neg \text{ is constant in } I_i \\ b_{i-1} + \frac{(\neg a_i - b_{i-1})(x - a_{i-1})}{a_i - a_{i-1}} & \text{if } \neg \text{ is involutive in } I_i \end{cases}.$$

We can easily see that $\neg'$ is isomorphic to $\neg$, and that the t-norm is definable in $\text{Ł}\Pi\frac{1}{2}$ by the term $\delta(x \to \neg' y) \sqcap (x \sqcap y)$.

Hence the theorem is proved. ■

5.2 Continuous t-norms

We now investigate definability of continuous t-norms. First of all we prove:

Theorem 5.2.1 *Any finite ordinal sum of implicitly definable t-norms is definable up to isomorphism.*

Proof. Up to isomorphism we can assume that the cut-points in the ordinal sum are rationals $0 = a_0 < \cdots < a_n = 1$. Now, in the case of finitely many components, the formula in Theorem 1.2.1 defines t-norms as ordinal sums which can be easily represented in the language of $\mathcal{R}$ by a first-order formula (long, but fairly easy to construct). Thus, up to isomorphism, such ordinal sums are definable in $\mathcal{R}$, and, from Theorem 4.3.9, it follows that they also are implicitly definable in $Ł\Pi\frac{1}{2}$. ■

We now show that if all the components are term-definable, so is their finite ordinal sum (up to isomorphism)[1]. Let $[a_i, b_i] \in [0, 1]$, with $1 \leq i \leq n$, $b_i \leq a_{i+1}$, and $a_i, b_i \in \mathbb{Q} \cap [0, 1]$. Let $*$ be a term-definable t-norm. We can define the linear transformation of $*$ into an interval $[a_i, b_i]$ by means of the following term:

$$\ell_i^*(x, y) = [((b_i \ominus a_i) \to_\pi (x \ominus a_i)) * ((b_i \ominus a_i) \to_\pi (y \ominus a_i))] \cdot (b_i \ominus a_i) \oplus a_i.$$

If $*$ is left-continuous then it has a residuum $\to_*$ whose linear transformation (assuming term-definability) is represented by

$$\overrightarrow{\ell_i}(x, y) = [((b_i \ominus a_i) \to_\pi (x \ominus a_i)) \to_* ((b_i \ominus a_i) \to_\pi (y \ominus a_i))] \cdot (b_i \ominus a_i) \oplus a_i.$$

Take now a finite number of non-overlapping intervals $[a_i, b_i]$ with rational cut-points, and a finite family of term-definable t-norms $*_i$. The ordinal sum of all $*_i$ over $[a_i, b_i]$ is defined by the following term:

$$\sigma^*(x, y) = \bigsqcap_{i=1}^{n} [(\ell_i^*(x, y) \sqcap \iota_i(x, y)) \sqcup ((x \sqcap y) \sqcap \neg\iota_i(x, y))],$$

where $\iota_i(x, y) = \delta[(a_i \to_\pi (x \sqcap y)) \sqcap ((x \sqcup y) \to_\pi b_i))]$. If each $*_i$ appearing as a summand in the ordinal sum is left-continuous, then it admits a residuum, which is represented by the term below (assuming term-definability):

$$\overrightarrow{\sigma}(x, y) = \bigsqcup_{i=1}^{n} \left[\left(\overrightarrow{\ell_i}(x, y) \sqcap \iota_i(x, y) \right) \sqcup (y \sqcap \neg\iota_i(x, y)) \right] \sqcup \delta(x \to y).$$

Given the previous construction, it is now easy to check that the following proposition holds.

Proposition 5.2.2 *Let $*_i$ be a finite family of term-definable t-norms and $[a_i, b_i]$ a finite family of subintervals of $[0, 1]$ having rational cut-points. Then, the ordinal sum σ^* of $*_i$ is term-definable.*

Moreover, if each $*_i$ is left-continuous and admits a term-definable residuum, the residuum $\overrightarrow{\sigma}$ is term-definable.

Proof. Suppose we have a finite family of non-overlapping intervals $[a_i, b_i]$ with rational cutpoints, and a family of definable t-norms $*_i$. Let $*$ be the t-norm obtained by ordinal sum out of the family $*_i$. We have to prove that there is a term $\sigma^*(x, y)$ defining exactly that t-norm.

Take any interval $[a_i, b_i]$ and suppose that $x, y \in [a_i, b_i]$. Hence

[1] Notice that this construction, along with Proposition 5.2.2, basically corresponds to the one given by Cintula in [28] concerning term-definability of finite ordinal sums of isomorphic copies of the Łukasiewicz and the Product t-norms.

$$[(\ell_i^*(x,y) \sqcap \iota_i(x,y)) \sqcup ((x \sqcap y) \sqcap \neg\iota_i(x,y))] = \ell_i^*(x,y),$$

while for all $j \neq i$,

$$[(\ell_i^*(x,y) \sqcap \iota_i(x,y)) \sqcup ((x \sqcap y) \sqcap \neg\iota_i(x,y))] = x \sqcap y.$$

Given that $*_i$ is term-definable and $x *_i y \leq x \sqcap y$, we have that for $x, y \in [a_i, b_i]$, $\sigma^*(x,y) = \ell_i^*(x,y)$, and so definability of $*$ directly follows.

Suppose now that x and y belong to different intervals. Then for all i

$$[(\ell_i^*(x,y) \sqcap \iota_i(x,y)) \sqcup ((x \sqcap y) \sqcap \neg\iota_i(x,y))] = x \sqcap y.$$

Hence in such cases $\sigma^*(x,y) = x \sqcap y$, which means that $*$ is term-definable being the Gödel t-norm trivially term-definable.

Thus we have proved that $*$ is term-definable. The proof for the residuum is similar and so is omitted. ■

We now give a characterization of all the continuous t-norms definable in $\text{Ł}\Pi\frac{1}{2}$. Indeed, the term-definable and implicitly definable t-norms are only those t-norms representable as finite ordinal sums in the sense of the Mostert-Shields theorem.

Theorem 5.2.3 *Let $*$ be a continuous t-norm. The following are equivalent:*

i. *Up to isomorphism, $*$ is implicitly definable in* $\text{Ł}\Pi\frac{1}{2}$.

ii. *Up to isomorphism, $*$ is term-definable in* $\text{Ł}\Pi\frac{1}{2}$.

iii. *$*$ is representable as a finite ordinal sum.*

Proof. We prove (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (i) is trivial; (i) $\Rightarrow$ (iii) follows by Theorem 5.2.1. Recall that by the Mostert-Shield Theorem, every continuous t-norm is representable as an ordinal sum of Łukasiewicz and Product t-norms, which are both trivially term-definable. If there are only finitely many components, then (iii) $\Rightarrow$ (ii) follows from Proposition 5.2.2. ■

An example of a continuous t-norm (see [97]) that is not definable is given by the following t-norm

$$x * y = \begin{cases} \frac{1}{2^n} + 2^n \left(x - \frac{1}{2^n}\right)\left(y - \frac{1}{2^n}\right) & x, y \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]^2 \\ \min(x,y) & \text{otherwise} \end{cases}.$$

5.3 Construction methods

We now focus on the definability of some constructions methods. Our aim consists in showing that:

Theorem 5.3.1 *The class of term-definable left-continuous t-norms is closed under finite ordinal sum, annihilation, rotation and rotation-annihilation.*

In order to prove the theorem we only have to show that the mentioned construction methods can be defined in $Ł\Pi\frac{1}{2}$.

Notice first, that definability of finite ordinal sum has been defined above. Then, recall that for all the methods below we need to use a strong negation $\sim$ in order to construct a new conjunctive operator. We will clearly suppose that the negation $\sim$ is term-definable, i.e. it is a piecewise rational function.

We begin with the annihilation construction. Recall that a left-continuous t-norm $*$ can be annihilated w.r.t. a strong negation $\sim$ if it admits the rotation-invariance property, i.e.:

$$\text{if } x * y \leq z \text{ then } y * {\sim}z \leq {\sim}x.$$

The annihilation is encoded in the following term:

$$\alpha^*(x, y) = (x * y) \sqcap \neg\delta(x \Rightarrow {\sim}y).$$

We now focus on left-continuous t-norms obtained by means of the rotation method. Given a left-continuous t-norm without zero-divisors, or whose zero-divisors are confined in a sub-square of $[0, 1]^2$, the rotation can be defined w.r.t. a strong negation. Clearly, we need the starting operation $*$, its residuum $\Rightarrow_*$ and the involutive negation $\sim$ to be term-definable. Let e be the unique rational fixed point of the negation, i.e. ${\sim}e = e$. The rotation is represented by the following term:

$$\varrho^*(x, y) = \left[\sim \overrightarrow{\ell}(y, {\sim}x) \sqcap [\delta(x \Rightarrow e) \sqcap \neg\delta(y \Rightarrow e)]\right] \sqcup$$
$$\left[\sim \overrightarrow{\ell}(x, {\sim}y) \sqcap [\neg\delta(x \Rightarrow e) \sqcap \delta(y \Rightarrow e)]\right] \sqcup [\ell^*(x, y) \sqcap [\neg\delta(x \Rightarrow e) \sqcap \neg\delta(y \Rightarrow e)]],$$

where,

$$\ell^*(x, y) = e \oplus (\neg e) \cdot [(\neg e \Rightarrow_\pi (x \ominus e)) * (\neg e \Rightarrow_\pi (y \ominus e))],$$

and

$$\overrightarrow{\ell}(x, y) = e \oplus (\neg e) \cdot [(\neg e \Rightarrow_\pi (x \ominus e)) \Rightarrow_* (\neg e \Rightarrow_\pi (y \ominus e))].$$

We now focus on the rotation-annihilation construction. Let $\sim$ be a term-definable strong negation, with e as a rational fixed point, and $d \in]e, 1] \cap \mathbb{Q}$. The d-zoomed negation is easily definable as follows:

$$\sim_d x = (d \ominus {\sim}d) \Rightarrow_\pi [\sim(x \cdot (d \ominus {\sim}d) \oplus {\sim}d) \ominus {\sim}d].$$

Let $*_1$ be a term-definable left-continuous t-norm, and let ℓ_1^* and $\overrightarrow{\ell_1}$ be the linear transformations into $[d, 1]$ of $*_1$ and its residuum, respectively, i.e.:

$$\ell_1^*(x, y) = d \oplus (\neg d) \cdot [(\neg d \Rightarrow_\pi (x \ominus d)) * (\neg d \Rightarrow_\pi (y \ominus d))],$$

and

$$\overrightarrow{\ell_1}(x, y) = d \oplus (\neg d) \cdot [(\neg d \Rightarrow_\pi (x \ominus d)) \Rightarrow_* (\neg d \Rightarrow_\pi (y \ominus d))].$$

Suppose that if $x, y > 0$ implies that $x *_1 y > 0$. Then let $*_2$ be a term-definable left-continuous t-subnorm that is rotation-invariant w.r.t. $\sim_d$. Let

$$\iota(x, y) = \delta[(\sim d \Rightarrow (x \sqcap y)) \sqcap ((x \sqcup y) \Rightarrow d)],$$

and let ℓ_2^* and $\overrightarrow{\ell_2}$ be the linear transformations into $[\sim d, d]$ of $*_2$ and its residuum, respectively, i.e.:

$$\ell_2^*(x, y) = [((d \ominus \sim d) \Rightarrow_\pi (x \ominus \sim d)) * ((d \ominus \sim d) \Rightarrow_\pi (y \ominus \sim d))] \cdot (d \ominus \sim d) \oplus \sim d,$$

and

$$\overrightarrow{\ell_2}(x, y) = [((d \ominus \sim d) \Rightarrow_\pi (x \ominus \sim d)) \Rightarrow_* ((d \ominus \sim d) \Rightarrow_\pi (y \ominus \sim d))] \cdot (d \ominus \sim d) \oplus \sim d.$$

We can define a term representing the $\sim_d$-rotation-annihilation of $*_1$ and $*_2$,

$$\begin{gathered}\rho_\alpha^*(x, y) = [\ell_1^*(x, y) \sqcap \neg\delta(x \Rightarrow d) \sqcap \neg\delta(y \Rightarrow d)] \sqcup \\ [\sim \overrightarrow{\ell_1}(x, \sim y) \sqcap \neg\delta(x \Rightarrow d) \sqcap \neg\delta(\sim d \Rightarrow y)] \sqcup \\ [\sim \overrightarrow{\ell_1}(y, \sim x) \sqcap \neg\delta(y \Rightarrow d) \sqcap \neg\delta(\sim d \Rightarrow x)] \sqcup [\ell_2^*(x, y) \sqcap \iota(x, y) \sqcap \neg\delta(x \Rightarrow \sim y)] \sqcup \\ [y \sqcap \neg\delta(x \Rightarrow d) \sqcap \iota(y, y)] \sqcup [x \sqcap \neg\delta(y \Rightarrow d) \sqcap \iota(x, x)].\end{gathered}$$

Suppose now that $*_1$ has zero divisors, i.e. there exist $x, y > 0$ such that $x *_1 y = 0$. Let $*_2$ be a term-definable left-continuous t-norm that is rotation-invariant w.r.t. $\sim_d$. Define the linear transformations exactly as above. In this case the $\sim_d$-rotation-annihilation of $*_1$ and $*_2$ is given by the following term

$$\begin{gathered}\rho_\alpha^*(x, y) = [\ell_1^*(x, y) \sqcap \delta(d \Rightarrow (x \sqcap y))] \sqcup \\ [\sim \overrightarrow{\ell_1}(x, \sim y) \sqcap \delta(y \Rightarrow \sim d)) \sqcap \delta(d \Rightarrow x)] \sqcup [\sim \overrightarrow{\ell_1}(y, \sim x) \sqcap \delta(x \Rightarrow \sim d) \sqcap \delta(d \Rightarrow y)] \sqcup \\ [\ell_2^*(x, y) \sqcap \neg\delta(d \Rightarrow x) \sqcap \neg\delta(x \Rightarrow \sim d) \sqcap \neg\delta(d \Rightarrow y) \sqcap \neg\delta(y \Rightarrow \sim d) \sqcap \neg\delta(x \Rightarrow \sim y)] \sqcup \\ [y \sqcap \delta(d \Rightarrow x) \sqcap \neg\delta(d \Rightarrow y) \sqcap \neg\delta(y \Rightarrow \sim d)] \sqcup \\ [x \sqcap \delta(d \Rightarrow y) \sqcap \neg\delta(d \Rightarrow x) \sqcap \neg\delta(x \Rightarrow \sim d)].\end{gathered}$$

5.4 Completeness

The aim of this section is to prove that BL, SBL, MTL, IMTL, SMTL, and WNM are all complete with respect to the classes of term-definable t-norms which are respectively, continuous, continuous and without zero-divisors, left-continuous, left-continuous with an involutive negation, left-continuous without zero-divisors, and weak nilpotent minimum.

Let $\mathcal{L}$ be a schematic extension of MTL, and denote by $\mathbb{R}\mathcal{L}_{\text{Ł}\Pi\frac{1}{2}}$ the $\mathcal{L}$-standard algebras based on definable left-continuous t-norms. We say that $\mathcal{L}$ is Ł$\Pi\frac{1}{2}$-complete if it is complete w.r.t. to the corresponding class of Ł$\Pi\frac{1}{2}$-definable left-continuous t-norms. Let $\mathcal{QV}(\mathbb{R}\mathcal{L}_{\text{Ł}\Pi\frac{1}{2}})$ be the quasivariety generated by the class of algebras $\mathbb{R}\mathcal{L}_{\text{Ł}\Pi\frac{1}{2}}$. We prove the following theorem that is similar to Theorem 2.1.18.

Theorem 5.4.1 *Let $\mathcal{L}$ be a schematic extension of* MTL*, and let $\mathbb{L}$ be its equivalent variety semantics. Then:*

(1) $\mathcal{L}$ *is* $\text{Ł}\Pi\frac{1}{2}$*-complete iff* $\mathbb{L} = \mathcal{V}(\mathbb{R}\mathcal{L}_{\text{Ł}\Pi\frac{1}{2}})$

(2) $\mathcal{L}$ *is finitely strongly* $\text{Ł}\Pi\frac{1}{2}$*-complete iff* $\mathbb{L} = \mathcal{QV}(\mathbb{R}\mathcal{L}_{\text{Ł}\Pi\frac{1}{2}})$.

Proof. We only prove (2), since (1) will immediately follow. Take an arbitrary $\mathcal{L}$-equation ϵ and a set of $\mathcal{L}$-equations Σ. Furthermore, consider the formula ϵ^{φ} associated to the equation ϵ, the formulas σ^{γ} associated to the equations σ in Σ, and denote the set of σ^{γ} by Σ^{γ} (see Chapter 2). Now,

- $\Sigma \models_{\mathbb{L}} \epsilon$ iff

- $\{\sigma_1^{\gamma} = \overline{1}, \ldots, \sigma_n^{\gamma} = \overline{1}\} \models_{\mathbb{L}} \epsilon^{\varphi} = \overline{1}$ iff

- $\Sigma^{\gamma} \vdash_{\mathcal{L}} \epsilon^{\varphi}$ iff

- $\{\sigma_1^{\gamma} = \overline{1}, \ldots, \sigma_n^{\gamma} = \overline{1}\} \models_{\mathbb{R}\mathcal{L}_{\text{Ł}\Pi\frac{1}{2}}} \epsilon^{\varphi} = \overline{1}$ iff

- $\Sigma \models_{\mathbb{R}\mathcal{L}_{\text{Ł}\Pi\frac{1}{2}}} \epsilon$ iff

- $\Sigma \models_{\mathcal{QV}(\mathbb{R}\mathcal{L}_{\text{Ł}\Pi\frac{1}{2}})} \epsilon$.

Then the quasivariety $\mathcal{QV}(\mathbb{R}\mathcal{L}_{\text{Ł}\Pi\frac{1}{2}})$ coincides with the variety $\mathbb{L}$.

Conversely, suppose that there are a theory Γ and a formula φ such that $\Gamma \nvdash \varphi$. Then, by completeness w.r.t. the equivalent variety semantics $\Gamma^t \not\models_{\mathbb{L}} \varphi^t = 1$, and consequently $\Gamma^t \not\models_{\mathcal{QV}(\mathbb{R}\mathcal{L}_{\text{Ł}\Pi\frac{1}{2}})} \varphi^t = 1$. ∎

5.4.1 Continuous t-norms

Recall that, as shown in Chapter 2, the varieties of BL-algebras and SBL-algebras both enjoy the finite embeddability property, and hence they are generated by their finite members. Moreover, the variety of BL-algebras is generated by the class of BL-algebras that are ordinal sums of finitely many copies of standard Wajsberg-algebras, and the the variety of SBL-algebras is generated by the class of SBL-algebras which are ordinal sums of the two-element Wajsberg algebra followed by finitely many copies of standard Wajsberg algebras.

Theorem 5.4.2 BL *is finitely strongly complete w.r.t. the class of continuous t-norms term-definable in* $\text{Ł}\Pi\frac{1}{2}$.

Proof. Suppose that for a finite set Γ of formulas, $\Gamma \nvdash_{\text{BL}} \varphi$. The variety of BL-algebras is generated as a quasivariety by the class of finite BL-chains (see [114]). Then, there are a finite BL-chain $\mathcal{A}$ and an $\mathcal{A}$-evaluation e such that $e(\gamma) = 1_{\mathcal{A}}$ for all $\gamma \in \Gamma$, while $e(\varphi) < 1_{\mathcal{A}}$. $\mathcal{A}$ is the ordinal sum of finitely many finite Wajsberg hoops, and each of them is the reduct of a Wajsberg algebra (see Chapter 2) which can be embedded in the standard Wajsberg algebra on $[0, 1]$. Thus $\mathcal{A}$ embeds into an ordinal sum of finitely many Łukasiewicz components, which by Theorem 5.2.3 is definable in $\text{Ł}\Pi\frac{1}{2}$. Now it is easy to extend the evaluation e to an evaluation v over $[0, 1]$ such that $v(\gamma) = 1$, for all γ, and $v(\varphi) < 1$. ∎

Theorem 5.4.3 SBL *is finitely strongly complete w.r.t. the class of continuous t-norms without zero-divisors term-definable in* $Ł\Pi\frac{1}{2}$.

Proof. Suppose that for a finite set Γ of formulas, $\Gamma \not\vdash_{\mathrm{SBL}} \varphi$. The variety of SBL-algebras is generated as a quasivariety by the class of finite BL-chains (see [114]). Then, there are a finite SBL-chain $\mathcal{A}$ and an $\mathcal{A}$-evaluation e such that $e(\gamma) = 1_{\mathcal{A}}$ for all $\gamma \in \Gamma$, while $e(\varphi) < 1_{\mathcal{A}}$. $\mathcal{A}$ is the ordinal sum of finitely many finite Wajsberg components $\mathcal{W}_k$, where the first component is the two-element Wajsberg algebra $\mathbf{2}$, i.e.: $\mathbf{2} \oplus \mathcal{W}_{k_1} \oplus \cdots \oplus \mathcal{W}_{k_n}$. Thus $\mathcal{A}$ embeds into a finite ordinal sum either of the form $[0,1]_\Pi \oplus [0,1]_Ł \oplus \cdots \oplus [0,1]_Ł$, or of the form $[0,1]_G \oplus [0,1]_Ł \oplus \cdots \oplus [0,1]_Ł$, which by Theorem 5.2.3 are both definable in $Ł\Pi\frac{1}{2}$. Now it is easy to extend the evaluation e to an evaluation v over $[0,1]$ such that $v(\gamma) = 1$, for all γ, and $v(\varphi) < 1$. ∎

Hence by Theorem 5.4.1 we immediately obtain.

Corollary 5.4.4 *The variety of* BL*-algebras and the variety of* SBL*-algebras are generated (as quasi-varieties) by the class of* BL*-algebras based on term-definable continuous t-norms, and the class of* SBL*-algebras based on term-definable continuous t-norms without zero-divisors, respectively.*

5.4.2 Left-continuous t-norms

We now consider logics based on left-continuous t-norms. Recall first that the varieties of MTL-algebras, IMTL-algebras and SMTL-algebras enjoy the finite embeddability property, and hence are complete w.r.t. their finite members (see Chapter 2). Now we prove the following:

Theorem 5.4.5 MTL *is finitely strongly complete with respect to the class of all algebras* $\mathbb{R}\mathrm{MTL}_{Ł\Pi\frac{1}{2}}$ *where the monoidal operation is a term-definable left-continuous t-norm.*

Proof. Suppose that $\Gamma \not\vdash_{\mathrm{MTL}} \varphi$. Then we know that there is a finite and totally ordered MTL-algebra $\mathcal{A}$ and a evaluation v_0 such that $v_0(\psi) = 1$ for all $\psi \in \Gamma$ and $v_0(\varphi) < 1$. Let $\mathcal{A}$ consist of $0 = a_0 < a_1 < ... < a_n = 1$. Let $*_A$ and $\Rightarrow_A$ denote the monoidal operation and the residuum in $\mathcal{A}$, respectively. Without any loss of generality we can assume that $a_i = \frac{i}{n}$ for $i = 0, ..., n$. Now we perform a construction which is quite similar to the one given in [95] (the only difference is that now we use reals in $(0,1]$ and not rationals in $(0,1]$). Let

$$S = \{(0,1)\} \cup \left\{(\tfrac{i}{n}, r) : i = 1, ..., n, \quad r \in (0,1]\right\}.$$

Define: $\left(\frac{i}{n}, r\right) \preceq \left(\frac{j}{n}, s\right)$ if either $i < j$ or $i = j$ and $r \leq s$. Note that $\preceq$ is a total lexicographic order on S. Let $\min'\left(\left(\frac{i}{n}, r\right), \left(\frac{j}{n}, s\right)\right)$ and $\max'\left(\left(\frac{i}{n}, r\right), \left(\frac{j}{n}, s\right)\right)$ denote the minimum and the maximum of $\left(\frac{i}{n}, r\right), \left(\frac{j}{n}, s\right)$ with respect to $\preceq$. Define:

$$\left(\frac{i}{n},\right) \odot \left(\frac{j}{n}, s\right) = \begin{cases} \left(\frac{i}{n} *_A \frac{j}{n}, 1\right) & \text{if} \quad \frac{i}{n} *_A \frac{i}{n} < \min\left(\frac{i}{n}, \frac{j}{n}\right) \\ \min'\left(\left(\frac{i}{n}, r\right), \left(\frac{j}{n}, s\right)\right) & \text{otherwise} \end{cases} .$$

As in [95] we can prove that $\langle S, \odot, (1,1), \preceq \rangle$ is a totally ordered commutative integral monoid. Moreover the order is dense and complete, and it has a maximum and a minimum. It is easy to see that $\odot$ is left-continuous with respect to the order topology (all discontinuities being to the right) and so has a residuum $\Rightarrow$. Thus, we obtain an MTL-algebra

$$\mathcal{S} = \left\langle S, \odot, \Rightarrow, \max', \min', (1,1), (0,1) \right\rangle.$$

Again, as in [95], we see that the map $\frac{i}{n} \mapsto \left(\frac{i}{n}, 1\right)$ is an embedding of $\mathcal{A}$ into $\mathcal{S}$. Therefore we obtain an evaluation in $\mathcal{A}$ such that $v(\psi) = (1,1)$ for all $\psi \in \Gamma$ and $v(\varphi) < (1,1)$. Now, the function $l\left(\frac{i}{n}, r\right) = \frac{i+r-1}{n}$ is an order isomorphism from $\mathcal{S}$ into $[0,1]$. Let $i(x)$ be the minimum integer i such that $x \leq \frac{i}{n}$ and $d(x) = nx - i(x) + 1$. Hence, we have that the inverse of l is $l^{-1}(x) = \left(\frac{i(x)}{n}, d(x)\right)$. Therefore letting $x * y = l(l^{-1}(x) \odot l^{-1}(y))$ we obtain a t-norm $*$ on $[0,1]$ such that $\langle S, \odot, \preceq \rangle$ and $\langle [0,1], *, \leq \rangle$ are isomorphic as lattice-ordered monoids. This immediately implies that they are isomorphic as residuated lattices, therefore they are also isomorphic as MTL-algebras.

It remains to prove that $*$ is term-definable. Note that:

- If $x = 0$ or $y = 0$, then $x * y = 0$.
- Suppose that $x, y > 0$. Let $k(x,y)$ be the unique natural number such that $\frac{i(x)}{n} *_A \frac{i(y)}{n} = \frac{k(x,y)}{n}$. Then it is easy to check that:

$$x * y = \begin{cases} \frac{k(x,y)}{n} & \text{if} \quad \frac{i(x)}{n} *_A \frac{i(y)}{n} < \min\left(\frac{i(x)}{n}, \frac{i(y)}{n}\right) \\ \min(x,y) & \text{otherwise} \end{cases} .$$

Clearly $*$ is piecewise rational, and therefore it is term-definable in $\text{Ł}\Pi\frac{1}{2}$. This concludes the proof. ■

Theorem 5.4.6 SMTL *is finitely strongly complete with respect to the class of all algebras* $\mathbb{R}\text{SMTL}_{\text{Ł}\Pi\frac{1}{2}}$ *where the monoidal operation is a term-definable left-continuous t-norm without zero-divisors.*

Proof. The proof for SMTL is the same as the proof for MTL. The only difference is that in this case the finite algebra $\mathcal{A}$ is a totally ordered SMTL-algebra and the construction of $\mathcal{S}$ and its isomorphic copy on $[0,1]$ preserves the absence of zero-divisors. Indeed, note that either $x * y = \min(x,y)$ or $x * y = \frac{k(x,y)}{n}$. Moreover $*_A$ has no zero-divisors, because $\mathcal{A}$ is an SMTL-algebra. Thus if $x, y \neq 0$, then $\min(x,y) \neq 0$ and $\frac{k(x,y)}{n} = \frac{i(x)}{n} *_A \frac{i(y)}{n} \neq 0$. So $x * y \neq 0$ and the claim is proved. ■

Theorem 5.4.7 IMTL *is finitely strongly complete with respect to the class of all algebras* $\mathbb{R}\text{IMTL}_{\text{Ł}\Pi\frac{1}{2}}$ *where the monoidal operation is a term-definable left-continuous t-norm with an involutive negation.*

Proof. In the case of IMTL we use a modified construction as in [47]. Once again we have a finite and totally ordered IMTL-algebra $\mathcal{A}$ and an evaluation v_0 such that $v_0(\psi) = 1$ for all $\psi \in \Gamma$ and $v_0(\varphi) < 1$. Without loss of generality we may assume $\mathcal{A} = \left\{\frac{i}{n} : i = 0, 1, ..., n\right\}$ for some natural number n. Note that the negation $\neg_A$ in $\mathcal{A}$ must satisfy the condition $\neg_A\left(\frac{i}{n}\right) = \frac{n-i}{n}$. Next let $\mathcal{S} = \left\langle S, \odot, \max', \min', (1,1), (0,1)\right\rangle$ as in Theorem 5.4.5, and define:

$$\left(\tfrac{i}{n}, r\right) \odot' \left(\tfrac{j}{n}, s\right) = \begin{cases} (0,1) & \text{if} \quad i + j = n + 1 \text{ and } r + s \le 1 \\ \left(\tfrac{i}{n}, r\right) \odot \left(\tfrac{j}{n}, s\right) & \text{otherwise} \end{cases}.$$

As in [47] we can see that $\mathcal{S}' = \left\langle S, \odot', \max', \min', (1,1), (0,1)\right\rangle$ is a totally ordered commutative and integral monoid. Moreover, the order is complete and $\odot'$ is left-continuous w.r.t. the order topology (this is easily proved using the left continuity of $\odot$). It follows that $\odot'$ has a residuum $\Rightarrow'$, so that $\mathcal{S}' = \left\langle S, \odot', \Rightarrow', \max', \min', (1,1), (0,1)\right\rangle$ is an MTL-algebra in which $\mathcal{A}$ can be embedded. Finally, the negation in $\mathcal{S}'$ is

$$\neg'\left(\tfrac{i}{n}, s\right) = \begin{cases} \left(\frac{n-i+1}{n}, 1 - s\right) & \text{if} \quad i > 0 \\ (1,1) & \quad \text{otherwise} \end{cases}.$$

Clearly, $\neg'$ is an involutive negation, therefore $\mathcal{S}'$ is an IMTL-algebra. Indeed, recall that the residuum can be defined in an IMTL-algebra from the negation and the monoidal operation as

$$\left(\tfrac{i}{n}, r\right) \Rightarrow_{\odot'} \left(\tfrac{j}{n}, s\right) = \neg'\left(\left(\tfrac{i}{n}, r\right) \odot' \neg'\left(\tfrac{j}{n}, s\right)\right).$$

Now define $x *' y = l(l^{-1}(x) \odot' l^{-1}(y))$, where l is the function defined in the proof of MTL. Then we obtain an IMTL-algebra $[0,1]_{*'}$ isomorphic to $\mathcal{S}'$, and it remains to prove that $*'$ is term-definable in $Ł\Pi\frac{1}{2}$. To prove this, just note that, with reference to the t-norm $*$ and to the notation introduced in case of MTL, we have

$$x *' y = \begin{cases} 0 & \text{if} \quad i(x) + i(y) = n + 1 \text{ and } d(x) + d(y) \le 1 \\ x * y & \text{otherwise} \end{cases}.$$

By an easy computation we see that

$$x *' y = \begin{cases} 0 & \text{if} \quad x + y \le 1 \\ x * y & \text{otherwise} \end{cases}$$

clearly is piecewise rational (since $*$ is piecewise rational), and so it is term-definable in $Ł\Pi\frac{1}{2}$.

This concludes the proof. ■

Finally, we prove that also WNM is complete w.r.t. to the class of term-definable weak nilpotent minimum t-norms, which are exactly those weak nilpotent minimum t-norms whose associated negation is a piecewise rational function.

Theorem 5.4.8 WNM *is finitely strongly complete with respect to the class of all algebras* $\mathbb{R}\mathrm{WNM}_{\mathrm{L\Pi}\frac{1}{2}}$ *where the monoidal operation is a term-definable weak nilpotent minimum t-norm.*

Proof. The proof is basically the same as the one given in [50]. Suppose that $\Gamma \not\vdash_{\mathrm{WNM}} \varphi$. Then we know that there are a totally ordered WNM-algebra $\mathcal{A}$ and an $\mathcal{A}$-evaluation v such that $v(\psi) = 1$ for all $\psi \in \Gamma$ and $v(\varphi) < 1$. Let X be the finite set of all values of all subformulas γ of $\Gamma \cup \{\varphi\}$, under v, plus $\neg\gamma$, $\neg\neg\gamma$, 0 and 1 (i.e. the top and the bottom of $\mathcal{A}$). Let

$$X \cap \neg(A) = \{0 = a_0 < \cdots < a_m = 1\}.$$

Now, let $h : X \to [0,1]$ be the order-preserving mapping such that $h(a_i) = \frac{i}{m}$. We define the following weak negation:

$$n_w(x) = \begin{cases} 1 - x & \text{if } x \in \{\frac{i}{m}\} \cup \left(\bigcup_{I \in \mathcal{I}} I\right) \\ \frac{m-i-1}{m} & \text{if } x \in \left(\frac{i}{m}, \frac{i+1}{m}\right) \text{ and } \left[\frac{i}{m}, \frac{i+1}{m}\right] \notin \mathcal{I} \end{cases},$$

where

$$\mathcal{I} = \left\{ \left[\tfrac{i}{m}, \tfrac{i+1}{m}\right] \mid \left(\left(\tfrac{i}{m}, \tfrac{i+1}{m}\right) \cup \left(\tfrac{m-i-1}{m}, \tfrac{m-1}{m}\right)\right) \cap h(X) = \emptyset \right\},$$

with $0 \leq i \leq m-1$.

Define now a weak nilpotent minimum t-norm $*$ from n_w. Clearly, h becomes a morphism from $\mathcal{A}$ into $\langle [0,1], *, \Rightarrow_*, \min, \max, 0, 1\rangle$. Thus, we can define an evaluation $e = h \circ v$, such that $e(\gamma) = 1$ for all $\gamma \in \Gamma$ and $e(\varphi) < 1$. It can be easily seen that $*$ is term-definable, hence the theorem is proven. ■

Then by Theorem 5.4.1 it directly follows:

Corollary 5.4.9 *The varieties of* MTL*,* IMTL*,* SMTL*, and* WNM*-algebras are generated by the class of standard algebras based on term-definable left-continuous t-norms, term-definable left-continuous t-norms with an involutive negation, term-definable left-continuous t-norms without zero-divisors and term-definable weak nilpotent minimum t-norms, respectively.*

5.5 Decidability and complexity

In this section we investigate the complexity of logics associated to left-continuous t-norms implicitly definable in $\mathrm{L\Pi}\frac{1}{2}$. Notice that many results concerning complexity for t-norm based logics are already well-known (for an overview see [3]). Here we show that a complexity bound can be obtained in a uniform way for logics based on t-norms implicitly definable in $\mathrm{L\Pi}\frac{1}{2}$.

We start from the following theorem.

Theorem 5.5.1 *If a left-continuous t-norm* $*$ *is implicitly definable in* $\mathrm{L\Pi}\frac{1}{2}$*, then* $\mathcal{L}_*$ *is in* PSPACE*.*

Proof. Recall that if $*$ is implicitly definable in $\text{Ł}\Pi\frac{1}{2}$, then its graph is definable in $\mathcal{R}$ by a quantifier-free formula $\Phi(x,y,z)$. Set $(\&(\Phi))(x,y,z) = \Phi(x,y,z)$. As pointed out above, also the residuum $\Rightarrow_*$ of $*$ is definable by means of the first-order formula $(\Rightarrow(\Phi))(x,y,z) : \forall u \forall v(\Phi(u,x,v) \rightharpoondown (u \leq z \rightleftharpoons v \leq y))$, where $\rightharpoondown$ denotes the classical implication and $\varphi \rightleftharpoons \psi$ denotes $(\varphi \rightharpoondown \psi) \wedge (\psi \rightharpoondown \varphi)$. Now $(\Rightarrow(\Phi))$ can be replaced by a quantifier-free equivalent formula, which we will still denote by $(\Rightarrow(\Phi))$. Clearly, lattice operations are implicitly definable by the quantifier-free formulas $(\sqcup(\Phi))(x,y,z) : (x \leq y \sqcap z = y) \sqcup (y < x \sqcap z = x)$ and $(\sqcap(\Phi))(x,y,z) : (x \leq y \sqcap z = x) \sqcup (y < x \sqcap z = y)$. Now let φ be any formula of $\mathcal{L}_*$, and let $S = \{\varphi_1, ..., \varphi_n\}$ be the set of its subformulas. Let $S_\& = \{(\chi,\gamma,\psi) \in S : \chi = \gamma\&\psi\}$, $S^\rightarrow = \{(\chi,\gamma,\psi) \in S : \chi = \gamma \rightarrow \psi\}$, $S_\vee = \{(\chi,\gamma,\psi) \in S : \chi = \gamma \vee \psi\}$ and $S_\wedge = \{(\chi,\gamma,\psi) \in S : \chi = \gamma \wedge \psi\}$. Let us associate to each φ_i a variable v_{φ_i} (different variables for different subformulas). For $\ominus \in \{\&, \rightarrow, \vee, \wedge\}$, let $K^\ominus$ denote the conjunction of all formulas of the form $(\ominus(\Phi))(v_\chi, v_\gamma, v_\psi)$ such that $(\chi,\gamma,\psi) \in S^\ominus$. Furthermore, let $K^\perp : v_\perp = 0$ and $K^\top : v_\top = 1$.

Then it is easy to prove that $\mathcal{L}_* \vdash \varphi$ iff the formula

$$\varphi(\Phi) : \forall v_{\varphi_1} ... \forall v_{\varphi_n}((K^\& \wedge K^\rightarrow \wedge K^\vee \wedge K^\wedge \wedge K^\perp \wedge K^\top) \Rightarrow v_\varphi = 1)$$

is true in $\mathcal{R}$. Note that $\varphi(\Phi)$ is a universal formula that can be computed from φ in polynomial time. Since the universal theory of $\mathcal{R}$ is in PSPACE (see [17]), the theorem is proved. ■

Theorem 5.5.2 *Let $\mathcal{K}$ be a class of left-continuous t-norm implicitly definable in* $\text{Ł}\Pi\frac{1}{2}$ *and let $\mathcal{L}_\mathcal{K}$ be its associated logic. If $\mathcal{L}_\mathcal{K}$ is finitely axiomatizable, then it is decidable.*

Proof. Under the above assumption, $\mathcal{L}_*$ is recursively enumerable, hence it is sufficient to show that the complement of $\mathcal{L}_*$ is recursively enumerable. For every formula $\Phi(x,y,z)$ in the language of $\mathcal{R}$, let $T(\Phi)$ denote the conjunction of the following formulas (expressing that the set defined by Φ is the graph of a left-continuous t-norm; being left-continuous and conjunctive is equivalent to the existence of a residuum, which is expressed by (T_5); the fact that 1 is the neutral element is encoded in (T_3)):

(T_1) $\forall x \forall y \forall z \forall u \forall v \forall w(\Phi(x,y,z) \wedge \Phi(z,u,v) \wedge \Phi(y,u,w) \rightharpoondown \Phi(x,w,v))$;

(T_2) $\forall x \forall y \forall z(\Phi(x,y,z) \rightharpoondown \Phi(y,x,z))$;

(T_3) $\forall x(\Phi(x,1,x))$;

(T_4) $\forall x \forall y \forall z((\Phi(x,y,z) \wedge \Phi(x,u,w) \wedge y \leq u) \rightharpoondown z \leq w)$;

(T_5) $\forall x \forall y \exists z \forall u \forall v(\Phi(x,u,v) \rightharpoondown (u \leq z \rightleftharpoons v \leq y))$.

Now, for every axiom ψ of $\mathcal{L}_*$ (finitely many!) consider the formula $\psi(\Phi)$ defined as in the proof of the previous theorem. We have that $\mathcal{L}_* \not\vdash \varphi$ iff there is a formula $\Phi(x,y,z)$ such that for every axiom ψ of $\mathcal{L}_*$, the formulas $\psi(\Phi)$ and $T(\Phi)$ are true in $\mathcal{R}$, but $\varphi(\Phi)$ is false in $\mathcal{R}$. Since truth in $\mathcal{R}$ is decidable, the claim is proved. ■

Let $\mathcal{L}_{*_1 ... *_n}$ denote the logic of finitely many left-continuous t-norms $*_1, ..., *_n$. Then from Theorem 5.5.1 it immediately follows:

Corollary 5.5.3 *Let $*_1, ..., *_n$ be finitely many left-continuous t-norms implicitly definable in* $\text{Ł}\Pi\frac{1}{2}$, *and let $\mathcal{L}_{*_1 \dots *_n}$ be their associated logic. Then $\mathcal{L}_{*_1 \dots *_n}$ is in* PSPACE.

Notice that the result of Hájek and Tulipani [79] that says that $\text{Ł}\Pi\frac{1}{2}$ is in PSPACE can be obtained from the previous corollary as a particular case. However, it is not fair to present our result as a real improvement of [79], because the strategies behind the proofs are quite similar.

Part III

Applications and Open Problems

Chapter 6

Logical Representation of Uncertainty Theories

In this chapter we deal with the representation of uncertainty measures in the framework of t-norm based logics.

Most representations of uncertainty are based on a set of *possible situations* (or worlds), sometimes called a *sample space* or a *frame of discernment*, which represents all the possible outcomes. A typical example is the toss of a die. In this case, the sample space is given by six different situations, each of them corresponding to a certain outcome. An event can be simply regarded as a subset of the sample space corresponding to the set of those situations in which the event is true. In the case of the toss of a die, for instance, the event "the outcome will be an even number" corresponds to the set given by $\{2, 4, 6\}$. Complex events can be seen as Boolean combinations of subsets of the sample space. For instance, the event "the outcome will be an even number and it will be strictly greater than 4" is nothing but the intersection of the sets $\{2, 4, 6\}$ and $\{5, 6\}$. Measures of uncertainty are normally defined over the Boolean algebra generated by subsets of a given sample space.

An event can be also identified with the proposition whose meaning is the set of situations that make it true. From a logical point of view, we can associate to a proposition the set of classical evaluations in which the proposition is true. Each of those evaluations, in fact, corresponds to a possible situation.

In the rest of this chapter we will use the words "event" and "proposition" with the same meaning, and they will refer to a set of situations, or equivalently to a set of classical evaluations. Given that measures are defined over the Boolean algebra of subsets of a sample space, we will then treat measures as defined over the Boolean algebra of provably equivalent classical propositions.

In general, measures of uncertainty aim at formalizing our degree of confidence in the occurrence of an event by assigning a value from a partially ordered bounded scale. This is encoded in the concept of plausibility measure introduced by Halpern (see [81]). Given a set W of possible situations, a plausibility measure is a mapping ρ from the Boolean algebra of subsets of W into a partially ordered bounded set $\langle A, \leq, 0, 1\rangle$ satisfying the following properties:

i. $\rho(\bot) = 0$,

ii. $\rho(\top) = 1$,

iii. if $\vdash \varphi \to \psi$ then $\rho(\varphi) \leq \rho(\psi)$.

The first two conditions mean that the certain event $\top$ and the impossible event $\bot$ have measure 1 and 0, respectively. Indeed, the certain event will be satisfied in every possible situation, while the impossible event will never occur. The third condition corresponds to monotonicity, i.e. if the situations in which an event can occur are included in those that support another event, then the degree of uncertainty of the former is smaller than the degree of uncertainty of the latter.

Many uncertainty measures are defined as real valued functions where the partially ordered scale is identified with the real unit interval $[0, 1]$. Measures of this kind are called *fuzzy measures* and were first introduced by Sugeno in [138]. Then, fuzzy measures are plausibility measures assigning values from $[0, 1]$ to elements of the Boolean algebra of events. We denote the class of fuzzy measures by $\mathcal{M}$ and a particular fuzzy measure by μ.

Besides such common properties, each kind of fuzzy measure differs from the others in how the measure associated to compound propositions is computed from the marginal ones. In other words, what specifies the behavior of a fuzzy measure is the way how from assessments of uncertainty concerning separated events we can determine the degree associated to their combination. In a certain sense we can say that classes of fuzzy measures are characterized by the satisfaction of some compositional properties. However, it is well known that fuzzy measures cannot be fully compositional. This means that the degree of confidence in any compound proposition φ cannot be always computed from the degree assigned to its subformulas.

Theorem 6.0.4 ([44]) *Let μ be any fuzzy measure. If μ is fully compositional then it collapses into a two-valued function.*

As mentioned above, classes of fuzzy measures differ from each other in the satisfaction of peculiar compositional properties. Typical examples of classes of fuzzy measures are probability measures, lower and upper probability measures, possibility measures and necessity measures[1].

Probability measures, first introduced from a measure-theoretic perspective by Kolmogorov in [99], are fuzzy measures defined over a σ*-algebra*. Recall that, given a set W, a σ-algebra is a collection of subsets of W closed under complementation and countable unions. Notice that a σ-algebra clearly is a Boolean algebra, more precisely it is a complete Boolean algebra (see Appendix

[1]Notice that we do not discuss here the appropriateness of a class of measures w.r.t. uncertainty phenomena and we do not compare them to each other. For an analysis of that kind see the papers by Smets [134, 135], Halpern's book [81] and the references therein.

B). Probabilities measures over σ-algebras are fuzzy measures which satisfy the law of countable additivity, i.e.: for any countable sequence of pairwise disjoint events $(\varphi_1, \varphi_2, \dots)$:

$$\mu\left(\bigvee_{i=1}\varphi_i\right) = \sum_{i=1}\varphi_i.$$

These measures are also called *countably additive probabilities*. If we do not require the algebra to be closed under countable unions, we define the class of probability measures, called *finitely additive probabilities*, as the class of all those fuzzy measures (over Boolean algebras) which satisfy the law of finite additivity

$$\text{if} \vdash \varphi \wedge \psi \leftrightarrow \bot \quad \text{then} \quad \mu(\varphi \vee \psi) = \mu(\varphi) + \mu(\psi).$$

In the rest of this chapter we deal with finitely additive probabilities only. We denote the class of probability measures by $\mathcal{P}$, and each measure in $\mathcal{P}$ by P.

Possibility measures (first introduced by Zadeh in [145], and deeply studied by Dubois and Prade [41, 43]) are a class of fuzzy measures satisfying the following law of composition w.r.t. the maximum t-conorm:

$$\mu(\varphi \vee \psi) = \max(\mu(\varphi), \mu(\psi)).$$

We denote the class of possibility measures by $\mathcal{P}i$, and each measure in $\mathcal{P}i$ by Π. Similarly, Necessity measures [41] are fuzzy measures satisfying the following law of composition w.r.t. the minimum t-norm:

$$\mu(\varphi \wedge \psi) = \min(\mu(\varphi), \mu(\psi)).$$

We denote the class of necessity measures by $\mathcal{N}$, and each measure in $\mathcal{N}$ by N. Possibility and Necessity measures are dual in the sense that, given a possibility measure Π (a necessity measure N), one can derive its dual necessity measure (possibility measure) from it by means of the standard involutive negation $n_s(x) = 1 - x$. Indeed

$$N(\varphi) = 1 - \Pi(\neg\varphi) \qquad [\Pi(\varphi) = 1 - N(\neg\varphi)]\,.$$

Probability measures are, on the contrary, self-dual, since the dual measure of a probability measure still is a probability measure:

$$P(\varphi) = 1 - P(\neg\varphi).$$

Given a set of probability measures P_i over the same Boolean algebra, the *upper probability* $P^{\vartriangle}(\varphi)$ is defined as $\sup\{P_i(\varphi)\}$ and the *lower probability* $P_{\triangledown}(\varphi)$ is defined as $\inf\{P_i(\varphi)\}$ (see [81]). Upper and lower probabilities are dual, since from an upper probability we can define a lower probability as

$$P_{\triangledown}(\varphi) = 1 - P^{\vartriangle}(\neg\varphi),$$

and viceversa.

Upper and lower probabilities can be also seen as classes of fuzzy measures. Indeed, as shown by Anger and Lembcke in [5], any upper probability is a fuzzy measure μ such that for all natural numbers m, n, k, and all $\varphi_1, \ldots, \varphi_m$, if $\{\{\varphi_1, \ldots, \varphi_m\}\}$ is an (n, k)-cover[2] of $(\varphi, \top)$, then

$$(\sharp) \qquad k + n\mu(\varphi) \leq \sum_{i=1}^{m} \mu(\varphi_i).$$

Notice that Halpern and Pucella proved in [82] that when the sample space is finite there are only finitely many instances of the the above property. Indeed, there exist constants $k_0, k_1, \ldots$ such that if W is a finite set, for all natural numbers $m, n, k \leq k_{|W|}$, and all $\varphi_1, \ldots, \varphi_m$, if $\{\{\varphi_1, \ldots, \varphi_m\}\}$ is an (n, k)-cover of $(\varphi, \top)$, then $(\sharp)$ holds.

Similarly we can see any lower probability as a fuzzy measure μ such that for all natural numbers m, n, k, and all $\varphi_1, \ldots, \varphi_m$, if $\{\{\varphi_1, \ldots, \varphi_m\}\}$ is an (n, k)-cover of $(\varphi, \top)$, then

$$(\sharp\sharp) \qquad k + n\mu(\varphi) \geq \sum_{i=1}^{m} \mu(\varphi_i).$$

Obviously, as mentioned above, by the standard strong negation we can define from a lower probability the related dual upper probability, and viceversa. The classes of upper probabilities and lower probabilities are denoted by $\mathcal{P}^{\vartriangle}$ and $\mathcal{P}_{\triangledown}$, respectively.

The degree of confidence in the occurrence of an event might have to be changed when new information comes at hand. This results in an update of the sample space which is commonly treated in theories of uncertainty by the concept of *conditioning.*

Historically the most famous example of conditioning is that of conditional probability. The updated probability measure $P(\cdot|\chi)$ (i.e. the probability of an event given the occurrence of χ) called *conditional probability*, is defined as

$$P(\varphi|\chi) = \tfrac{P(\varphi \wedge \chi)}{P(\chi)},$$

with $P(\chi) > 0$. If $P(\chi) = 0$ the conditional probability remains undefined. However, assessments of zero-probability cannot be avoided in practice. Then, this clearly yields both technical and conceptual problems[3].

---

[2] A proposition φ is said to be *covered* n times by a multiset $\{\{\varphi_1, \ldots, \varphi_m\}\}$ of propositions if every situation in which φ is true makes true at least n propositions from $\varphi_1, \ldots, \varphi_m$ as well. An (n, k)-*cover* of $(\varphi, \top)$ is a multiset $\{\{\varphi_1, \ldots, \varphi_m\}\}$ that covers $\top$ k times and covers φ $n + k$ times.

[3] Two well-known proposals which aim at solving this problem are either to adopt a non-standard probability approach (where events are measured on the hyper-real interval $[0, 1]$ rather than on the usual real interval), or to take conditional probability as a primitive notion. In the first case, the assignment of zero probability is only allowed to impossible events, while other events can take on an infinitesimal probability. This clearly permits to avoid situations in which the conditioning event has null probability.

The second approach (that goes back to de Finetti, Rényi and Popper among others) considers conditional probability and conditional events as basic notions, not derived from the notion of unconditional probability, and also provides an adequate axiomatization. We will focus here on this approach in a more general setting.

The notion of conditioning for possibility measures has received different treatments. The first were those proposed by Zadeh [145], Hisdal [83] and Nguyen [122]. In general, the conditional possibility $\Pi(\varphi|\chi)$ can be viewed as the solution to the equation

$$\Pi(\varphi \wedge \chi) = x * \Pi(\chi),$$

where $*$ is a continuous t-norm (continuity guarantees the existence of a solution), and $\Pi(\varphi|\chi)$ is defined as the greatest solution. This would then be equivalent to the following equation [38]:

$$\Pi(\varphi|\chi) = \Pi(\chi) \Rightarrow_* \Pi(\varphi \wedge \chi),$$

where $\Rightarrow_*$ is the residuum of the t-norm $*$. However, as shown by De Baets, Mesiar and Tsiporkova in [39], not any t-norm can be used if we want the mapping $\Pi(\cdot|\chi)$ to be a possibility measure. If we rely on an arbitrary space, $*$ must be a strict t-norm, i.e. continuous, Archimedean and without zero-divisors. If the universe is finite, $*$ needs not be Archimedean, and then we can rely on the minimum t-norm, following the classical treatment proposed by Dubois and Prade to obtain a qualitative definition [42].

Notice that conditioning for necessity measures is not in general defined from marginal necessities, but it is derived from conditional possibilities (see [43], for the details). Furthermore, there is no clear notion of conditional lower or upper probability as derived from a single measure. Conditional lower and upper probabilities are rather defined from a set of probabilities (see [81]).

A general solution for avoiding problems with conditioning is to define conditioning as a primitive notion: a measure is defined not over events, but directly over *conditional events*, i.e. objects of the form $\varphi|\chi$, where φ and ψ are classical propositions. Unconditional measures are then recovered by taking the certain event as the conditioning event . Such a treatment of conditional measures can be found specially in the work carried out by Halpern [81], and Coletti and Scozzafava [35].

In order to provide a general treatment, Halpern introduced the notion of *conditional plausibility measures* (see [81]). Recall that a *Popper algebra* is a structure $\mathcal{B} \times \mathcal{B}'$, where $\mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{B}'$ is closed under supersets, i.e.: if $\varphi \in \mathcal{B}'$, $\varphi \to \psi$, and $\psi \in \mathcal{B}$, then $\psi \in \mathcal{B}'$. A conditional plausibility measure is a mapping ϱ from a Popper algebra $\mathcal{B} \times \mathcal{B}'$ into a partially ordered bounded set $\langle A, \leq, 0, 1\rangle$ such that:

i. $\varrho(\bot|\chi) = 0$,

ii. $\varrho(\top|\chi) = 1$,

iii. if $\vdash \varphi \to \psi$ then $\varrho(\varphi|\chi) \leq \varrho(\psi|\chi)$,

iv. $\varrho(\varphi|\chi) = \varrho(\varphi \wedge \chi|\chi)$.

Given a sample set W, a Popper algebra $\mathcal{B} \times \mathcal{B}'$ of events over W, and a conditional plausibility measure ϱ, the structure $\langle W, \mathcal{B} \times \mathcal{B}', \varrho, \langle A, \leq, 0, 1\rangle\rangle$ is called a *conditional plausibility space*. A conditional plausibility space is called *acceptable* if $\chi \in \mathcal{B}'$ and $\varrho(\varphi|\chi) \neq 0$ implies that $\varphi \wedge \chi \in \mathcal{B}'$. A conditional plausibility space is called *algebraic* if it is acceptable and there are functions $\otimes$, $\circledcirc$ from $A \times A$ into A such that:

(i) $\varrho(\varphi \vee \psi|\chi) = \varrho(\varphi|\chi) \circledcirc \varrho(\psi|\chi)$, if $\varphi \wedge \psi \leftrightarrow \bot$ and $\chi \in \mathcal{B}'$,

(ii) $\varrho(\varphi \wedge \psi|\chi) = \varrho(\varphi|\psi \wedge \chi) \otimes \varrho(\psi|\chi)$, if $\psi \wedge \chi \in \mathcal{B}'$, and $\varphi, \psi, \chi \in \mathcal{B}$,

(iii) $\otimes$ distributes over $\circledcirc$.

Here, we are interested in measures assigning values from the real unit interval $[0, 1]$. Another general treatment, that resembles the above one provided by Halpern, was introduced by Coletti and Scozzafava [35]. In this case, measures are defined over a structure $\mathcal{E} = \mathcal{E}' \times \mathcal{H}^0$, where $\mathcal{E}'$ is a Boolean algebra, $\mathcal{H}^0 \subseteq \mathcal{E}$ is an additive set (i.e. closed under finite unions), and conditional events in $\mathcal{E}$ are assigned values from $[0, 1]$. Notice that in general in a Popper algebra $\mathcal{B} \times \mathcal{B}'$, $\mathcal{B}'$ is an additive set, but a structure like $\mathcal{E} = \mathcal{E}' \times \mathcal{H}^0$ needs not be a Popper algebra[4].

Definition 6.0.5 [35] A real function $\xi : \mathcal{E} \to [0, 1]$ defined on $\mathcal{E} = \mathcal{E}' \times \mathcal{H}^0$, is a conditional measure[5], if there exist two commutative, associative and increasing operations $\otimes, \circledcirc$ from $\xi(\mathcal{E}) \times \xi(\mathcal{E})$ to $[0, 1]$, having, respectively, 1 and 0 as neutral elements, and with $\otimes$ distributive over $\circledcirc$, such that:

(D1) $\xi(\varphi|\chi) = \xi(\varphi \wedge \chi|\chi)$, for every $\varphi \in \mathcal{E}'$ and $\chi \in \mathcal{H}^0$,

(D2) given $\chi \in \mathcal{H}^0$, for any $\varphi, \psi \in \mathcal{E}'$, with $\psi \wedge \varphi \wedge \chi \leftrightarrow \emptyset$, we have

$$\xi(\varphi \vee \psi|\chi) = \xi(\varphi|\chi) \circledcirc \xi(\psi|\chi),\ \xi(\top|\chi) = 1,\ \xi(\bot|\chi) = 0,$$

(D3) for every $\psi \in \mathcal{E}'$ and $\varphi, \chi, \varphi \wedge \psi \in \mathcal{H}^0$,

$$\xi(\varphi \wedge \psi|\chi) = \xi(\varphi|\chi) \otimes \xi(\psi|\varphi \wedge \chi).$$

Particular classes of conditional measures can be obtained by specifying the behavior of the operations $\otimes$ and $\circledcirc$.

[4]To see this, take a Popper algebra $\mathcal{B} \times \mathcal{B}'$, and suppose that $\chi, \chi' \in \mathcal{B}'$. We have that $\chi \vee \chi' \in \mathcal{B}$, and $\chi \to \chi \vee \chi'$. Then, clearly $\chi \vee \chi' \in \mathcal{B}'$. Hence, $\mathcal{B}'$ is an additive set. Conversely, let $\mathcal{E}'$ be the Boolean algebra of all subsets of $\{\chi_1, \ldots, \chi_6\}$, and let $\mathcal{H}^0 = \{\{\chi_1, \chi_2\}, \{\chi_3, \chi_4\}, \{\chi_1, \chi_2, \chi_3, \chi_4\}, \{\chi_1, \ldots, \chi_6\}\}$. Clearly, $\mathcal{H}^0$ is closed under finite unions, but it is not closed under supersets. Hence $\mathcal{E}' \times \mathcal{H}^0$ is not a Popper algebra.

[5]Note that in [35], conditional measures are called $(\otimes, \circledcirc)$-decomposable conditional measures. However, for the sake of simplicity here we will simply refer to them by the term "conditional measures".

- A *primitive conditional probability* is a conditional measure such that $\circledcirc$ and $\otimes$ correspond to the sum and product of real numbers, respectively (see [35]). A function being a primitive conditional probability is denoted by $P_{\mathcal{C}}$, while the whole class of primitive conditional probabilities is denoted by $\mathcal{CP}$.

- A *primitive conditional possibility* is a conditional measure such that $\circledcirc$ and $\otimes$ correspond to the maximum t-conorm and to the minimum t-norm, respectively (see [12]). A function being a primitive conditional possibility is denoted by $\Pi_{\mathcal{C}}$, while the whole class of primitive conditional possibilities is denoted by $\mathcal{CP}i$.

- A *generalized conditional possibility* is a conditional measure such that $\circledcirc$ and $\otimes$ correspond to the maximum t-conorm and to any t-norm, respectively (see [13]). A function being a generalized conditional possibility is denoted by $\Pi_{\mathcal{GC}}$, while the whole class of primitive conditional possibilities is denoted by $\mathcal{GCP}i$.

- Given a De Morgan triple $\langle *, \diamond, n \rangle$, it is possible to define the concept of generalized conditional necessity from that of generalized conditional possibility as a conditional function $N_{\mathcal{GC}}$ dual of $\Pi_{\mathcal{GC}}$, i.e.:

  $$N_{\mathcal{GC}}(\varphi|\chi) = n(\Pi_{\mathcal{GC}}(\neg\varphi|\chi)),$$

  for every $\chi \in \mathcal{H}$. Then, a real function $N_{\mathcal{GC}}$ defined on $\mathcal{E} = \mathcal{E}' \times \mathcal{H}$, where $\mathcal{E}$ is a Boolean algebra, $\mathcal{H}^0$ an additive set, with $\mathcal{H}^0 \subseteq \mathcal{E}'$ and $\mathcal{H}^0 = \mathcal{H} \backslash \{\emptyset\}$, is a *generalized conditional necessity* if the following conditions hold:

  (i) $N_{\mathcal{GC}}(\varphi|\chi) = N_{\mathcal{GC}}(\varphi \wedge \chi|\chi)$, for every $\varphi \in \mathcal{E}'$ and $\chi \in \mathcal{H}$

  (ii) $N_{\mathcal{GC}}(\cdot|\chi)$ is a necessity measure, for any given $\chi \in \mathcal{H}$

  (iii) $N_{\mathcal{GC}}(\varphi \vee \psi|\chi) = N_{\mathcal{GC}}(\varphi|\chi) \diamond N_{\mathcal{GC}}(\psi|\neg\varphi \wedge \chi)$, for every $\psi, \varphi \in \mathcal{E}'$ and $\neg\varphi, \chi \in \mathcal{H}$, $\neg\varphi \wedge \chi \in \mathcal{H}^0$ for a t-conorm $\diamond$ n-dual of $*$.

  The class of generalized conditional necessities is denoted by $\mathcal{GCN}$.

- A *primitive conditional necessity* is a particular generalized conditional necessity where $\diamond$ corresponds to the maximum t-conorm, and can be obtained as a dual measure from a primitive conditional possibility by means of the standard strong negation n_s (see [12]). A function being a primitive conditional necessity is denoted by $N_{\mathcal{C}}$, while the whole class of primitive conditional necessities is denoted by $\mathcal{CN}$.

Notice that in the following, in general, we will feel free not to use the word "primitive" whenever it is clear from the context whether a conditional measure is derived from marginal ones or not.

In the next sections we will show how the above measures can be represented in the framework of t-norm based logics. Indeed, the degree of uncertainty of any proposition φ can be interpreted as the degree of truth of the sentence "φ

is plausible (probable, believable)" (the same holds for conditional events and conditional measures). The higher our degree of confidence in φ is, the higher the degree of truth of the above sentence will be. In some sense, the predicate "is plausible (believable, probable)" can be regarded as a modal operator over the proposition φ. Given a class of measures of uncertainty $\mathcal{M}$, we can define many-valued modal formulas $\kappa(\varphi)$, whose interpretation is given by a real number corresponding to the degree of uncertainty assigned to φ under the measure $\mu \in \mathcal{M}$. Therefore, we can translate the peculiar axioms governing the behavior of an uncertainty measure into formulas of a certain t-norm based logic, depending on the operations we need to represent. An adequate analysis of functional definability will allow an adequate choice among several possible logics. In fact, not every t-norm based logic might be suitable for representing a specific class of measures $\mathcal{M}$, but only those in which the basic operations of $\mathcal{M}$ can be defined.

This chapter is structured as follows. In Section 6.1, we deal with the representation of fuzzy measures. We build up a logic $\mathcal{M}(\mathcal{L})$ for fuzzy measures based on an arbitrary t-norm based logic $\mathcal{L}$. We describe the properties of $\mathcal{M}(\mathcal{L})$ and provide a completeness result w.r.t. special Kripke models equipped with a measure $\mu \in \mathcal{M}$. Then we show how to extend this treatment to some particular classes of measures such as probability, possibility and necessity, and lower and upper probabilities.

In Section 6.2, we adopt the same strategy w.r.t. primitive conditional measures. In particular we deal with conditional probability and with (generalized) conditional possibility and necessity.

In Section 6.3, we will see that we can prove by purely logical means interesting properties of the measures represented. In fact, we will show that, by relying on t-norm based logics with rational truth constants, we can build up theories whose consistency is equivalent to the coherence of related assessments of uncertainty. Then, if the logic $\mathcal{L}$ satisfies certain properties, we will be able to prove the compactness of those assessments.

We end with some final remarks about related works.

6.1 Logics for fuzzy measures

6.1.1 The base logic

Let $\mathcal{L}$ be any t-norm based logic, or any of its expansions, and let $\mathcal{M}$ be any class of fuzzy measures. $\mathcal{M}(\mathcal{L})$ is built up over $\mathcal{L}$ extending its language by including modal formulas which represent the uncertainty given by a fuzzy measure $\mu \in \mathcal{M}$. We define the language in two steps. First, we have classical Boolean formulas φ, ψ, etc., defined in the usual way from the classical connectives $(\wedge, \neg)$ and from a countable set V of propositional variables $p, q, \ldots$, etc. The set of Boolean formulas is denoted by L. Moreover, given any set $D \subseteq L$, we denote by $Con(D)$ the set of sentences which logically follow from D in classical logic. Moreover, $Sat(L)$ and $Taut(L)$ will denote the set of classically satisfiable formulas and the set of classical tautologies, respectively.

Elementary modal sentences are formulas of the form $\kappa(\varphi)$, where κ is a unary operator taking Boolean sentences as arguments. Compound modal formulas are built by means of the $\mathcal{L}$-connectives. Nested modalities are not allowed.

Definition 6.1.1 The axioms of the logic $\mathcal{M}(\mathcal{L})$ are the following:

(i) The set $Taut(L)$ of classical Boolean tautologies

(ii) Axioms of $\mathcal{L}$ for modal formulas

(iii) The following axiom:
$(\mathcal{M}1) \qquad \neg\kappa(\bot)$

Deduction rules of $\mathcal{M}(\mathcal{L})$ are those of $\mathcal{L}$, plus:

(iv) *modalization*: from $\vdash \varphi$ (i.e. φ is derivable in Classical Logic) derive $\kappa(\varphi)$

(v) *monotonicity*: from $\vdash \varphi \rightarrow \psi$ derive $\kappa(\varphi) \rightarrow \kappa(\psi)$.

The language we have defined clearly is a hybrid language. Indeed, any theory (set of formulas) we will deal with will be of the form $\Gamma = D \cup T$, where D contains only non-modal formulas and T contains only modal formulas. Notice that there is no direct interaction between non-modal and modal formulas, with the exception of the application of the above rules of inference. The role of modalization and monotonicity only consists in generating new modal formulas which can then be used in the deduction. Therefore, we are led to define in $\mathcal{M}(\mathcal{L})$ the notion of proof from a theory, written $\vdash_{\mathcal{M}(\mathcal{L})}$, in a non-standard way, at least when the theory contains non-modal formulas.

Definition 6.1.2 The proof relation $\vdash_{\mathcal{M}(\mathcal{L})}$ between sets of formulas and formulas is defined by:

1. $D \cup T \vdash_{\mathcal{M}(\mathcal{L})} \varphi$ if $\varphi \in Con(D)$;

2. $T \vdash_{\mathcal{M}(\mathcal{L})} \Phi$ if Φ follows from T in the usual way from the above axioms and rules;

3. $D \cup T \vdash_{\mathcal{M}(\mathcal{L})} \Phi$ if $T \cup D^{\mathcal{M}} \vdash_{\mathcal{M}(\mathcal{L})} \Phi$;

where $D^{\mathcal{M}} = \{\kappa(\varphi) : \varphi \in Con(D)\} \cup \{\kappa(\varphi) \rightarrow \kappa(\psi) : \varphi \rightarrow \psi \in Con(D)\}$. We require the theory D to have a decidable set of consequences.

We now define the semantics for $\mathcal{M}(\mathcal{L})$ by introducing *$\mathcal{M}$-Kripke structures.*

Definition 6.1.3 A $\mathcal{M}$-Kripke model is a structure $K = \langle W, \mathcal{U}, e, \mu \rangle$, where:

- W is a non-empty set of possible worlds.
- $\mathcal{U}$ is a Boolean algebra of subsets of W.

- $e : V \times W \to \{0, 1\}$ is a *Boolean* evaluation of the propositional variables, that is, $e(p, w) \in \{0, 1\}$ for each propositional variable $p \in V$ and each world $w \in W$. Any given truth-evaluation $e(\cdot, w)$ is extended to Boolean propositions as usual. For a Boolean formula φ, we will denote by $[\varphi]_W$ the set of worlds in which φ is true, i.e. $[\varphi]_W = \{w \in W \mid e(\varphi, w) = 1\}$.
- $\mu : \mathcal{U} \to [0, 1]$ is a fuzzy measure over $\mathcal{U}$, such that $[\varphi]_W$ is μ-measurable for any non-modal φ.
- $e(\cdot, w)$ is extended to elementary modal formulas by defining

$$e(\kappa(\varphi), w) = \mu([\varphi]_W),$$

and to arbitrary modal formulas according to the $\mathcal{L}$-semantics.

A structure K is a model for Φ, written $K \models \Phi$, if $e^K(\Phi) = 1$. If T is a set of formulas, we say that K is a model of T if $K \models \Phi$ for all $\Phi \in T$. The notion of logical entailment relative to a class of structures $\mathcal{K}$, written $\models_{\mathcal{K}}$, is then defined as follows:

$$\Gamma \models_{\mathcal{K}} \Phi \text{ iff } K \models \Phi \text{ for each } K \in \mathcal{K} \text{ model of } \Gamma.$$

If $\mathcal{K}$ denotes the whole class of $\mathcal{M}$-Kripke structures we shall write $\Gamma \models_{\mathcal{M}(\mathcal{L})} \Phi$. When $\models_{\mathcal{K}} \Phi$ holds we will say that Φ is *valid* in $\mathcal{K}$, i.e. when Φ gets value 1 in all structures $K \in \mathcal{K}$.

Lemma 6.1.4 *Axiom* ($\mathcal{M}1$) *is valid in the class of $\mathcal{M}$-Kripke structures. Moreover the modalization rule and the monotonicity rule preserve validity in a model.*

Proof.

- $e^K(\mathcal{M}1) = 1$, given that $\mu(\perp) = 0$.
- As for the modalization rule, suppose that $K \models \varphi$, then $[\varphi]_W = W$. Hence $e(\kappa(\varphi), w) = 1$, that is $K \models \kappa(\varphi)$.
- As for the monotonicity rule, suppose that $K \models \varphi \to \psi$. Clearly we have that $e(\kappa(\varphi)) = \mu(\varphi) \leq \mu(\psi) = e(\kappa(\psi))$, hence $K \models \kappa(\varphi) \to \kappa(\psi)$.

■

Proposition 6.1.5 (Soundness) *The logic $\mathcal{M}(\mathcal{L})$ is sound with respect to the class of $\mathcal{M}$-Kripke structures.*

Proof. Suppose that $\Gamma \vdash_{\mathcal{M}(\mathcal{L})} \Phi$ and recall Definition 6.1.2. If Φ is non-modal then the result is obvious. Thus assume that Φ is modal, and suppose that Γ is modal as well. By the above lemma, we have $\Gamma \models_{\mathcal{M}(\mathcal{L})} \Phi$. Finally, let $\Gamma = D \cup T$ where D is non-modal and T modal. Let $K = (W, \mathcal{U}, e, \mu)$ be such that $K \models D \cup T$: we have to show that $K \models \Phi$. Since $K \models D$, then $[\psi]_W = W$ for every $\psi \in Con(D)$. Moreover, $[\varphi]_W \subseteq [\psi]_W$ for every $\varphi \to \psi \in Con(D)$, and so $\mu(\varphi) \leq \mu(\psi)$. This means that $K \models D^{\mathcal{M}}$, and consequently $K \models T \cup D^{\mathcal{M}}$. Now, $T \cup D^{\mathcal{M}}$ is a modal theory, hence $K \models \Phi$ holds. ■

Let $D \subset L$ be any given non-modal (propositional) theory (possibly empty). For any $\varphi, \psi \in L$, define $\varphi \sim_D \psi$ iff $\varphi \leftrightarrow \psi$ follows from D in classical propositional logic, i.e. if $\varphi \leftrightarrow \psi \in Con(D)$. The relation $\sim_D$ is an equivalence relation in L and $[\varphi]_D$ will denote the equivalence class of φ. Obviously, the quotient set $L/_{\sim_D}$ forms a Boolean algebra which is isomorphic to a subalgebra $\mathcal{B}(\Omega_D)$ of the power set of the set Ω_D of Boolean interpretations of the crisp language L which are model of D[6]. For each $\varphi \in L$, we shall identify the equivalence class $[\varphi]_D$ with the set $\{\omega \in \Omega_D \mid \omega(\varphi) = 1\} \in \mathcal{B}(\Omega_D)$ of models of D that make φ true. We shall denote by $\mathcal{M}(D)$ the set of fuzzy measures defined over $L/_{\sim_D}$ or, equivalently, over $\mathcal{B}(\Omega_D)$.

Notice that each fuzzy measure $\mu \in \mathcal{M}(D)$ induces an $\mathcal{M}$-Kripke structure $\langle \Omega_D, \mathcal{B}(\Omega_D), e_\mu, \mu\rangle$ where $e_\mu(p, \omega) = \omega(p) \in \{0, 1\}$ for each $\omega \in \Omega_D$ and each propositional variable p. We shall denote by $\mathcal{K}_D$ the class of $\mathcal{M}$-Kripke structures which are models of D, and by $\mathcal{MS}(D)$ the class of $\mathcal{M}$-Kripke models $\{(\Omega_D, \mathcal{B}(\Omega_D), e_\mu, \mu) \mid \mu \in \mathcal{M}(D)\}$. Obviously, $\mathcal{MS}(D) \subset \mathcal{K}_D$.

Abusing the language, we will say that a fuzzy measure $\mu \in \mathcal{M}(D)$ is a *model* of a modal theory T whenever the induced Kripke structure $\langle \Omega_D, \mathcal{B}(\Omega_D), e_\mu, \mu\rangle$ is a model of T (obviously $\langle \Omega_D, \mathcal{B}(\Omega_D), e_\mu, \mu\rangle$ is a model of D as well).

Given the above notions, we now prove a completeness result for $\mathcal{M}(\mathcal{L})$.

Theorem 6.1.6 ((Finite) Strong completeness) *Let $\mathcal{L}$ be any t-norm based logic (or any of its expansions). If $\mathcal{L}$ is finitely strongly standard complete, then let T be a finite modal theory over $\mathcal{M}(\mathcal{L})$, D a finite non-modal theory and Φ a modal formula. Then*

$$T \cup D \vdash_{\mathcal{M}(\mathcal{L})} \Phi \textit{ iff } e_\mu(\Phi) = 1$$

for each fuzzy measure $\mu \in \mathcal{M}(D)$ model of T.
Moreover, if $\mathcal{L}$ is strongly standard complete the same holds for infinite theories.

Proof. First, we translate theories over $\mathcal{M}(\mathcal{L})$ into theories over $\mathcal{L}$. We define a theory, called $\mathcal{F}$, as follows:
(i) take as propositional variables of the theory variables of the form f_φ, where φ is a classical proposition from L.
(ii) take as axioms of the theory the following ones, for each φ and ψ:

($\mathcal{F}1$) f_φ, for $\varphi \in Con(D)$,
($\mathcal{F}2$) $f_\varphi \to f_\psi$, whenever $\varphi \to \psi \in Con(D)$,
($\mathcal{F}3$) $\neg f_\perp$.

Then define the mapping $\star$ from modal formulas to $\mathcal{L}$-formulas as follows:

- $(\kappa(\varphi))^\star = f_\varphi$,
- $(\Phi \ominus \Psi)^\star = \Phi^\star \ominus \Psi^\star$, for $\ominus$ being a binary $\mathcal{L}$-connective,

[6] Actually, $\mathcal{B}(\Omega_D) = \{\{\omega \in \Omega_D \mid \omega(\varphi) = 1\} \mid \varphi \in L\}$. Needless to say, if the language has only finitely many propositional variables then the algebra $\mathcal{B}(\Omega_D)$ is just the whole power set of Ω_D, otherwise it is a strict subalgebra.

- $(\natural(\Phi))^\star = \natural(\Phi^\star)$, for $\natural$ being a unary $\mathcal{L}$-connective,

- $(s)^\star = s$, for each truth-constant s.

Let us denote by $T^\star$ and $(D^{\mathcal{M}})^\star$ the sets of all formulas translated from T and $D^{\mathcal{M}}$.

Then, by the construction of $\mathcal{F}$ and $(D^{\mathcal{M}})^\star$, one can easily check that for any Φ,

$$T \cup D \vdash_{\mathcal{M}(\mathcal{L})} \Phi \text{ iff } T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star \vdash_{\mathcal{L}} \Phi^\star. \tag{1}$$

Now we prove that the semantical analogue of (1) also holds, that is,

$$T \cup D \models_{\mathcal{M}(\mathcal{L})} \Phi \text{ iff } T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star \models_{\mathcal{L}} \Phi^\star. \tag{2}$$

Assume $T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star \not\models_{\mathcal{L}} \Phi^\star$. This means that there exists an $\mathcal{L}$-evaluation e which is model of $T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star$ such that $e(\Phi^\star) < 1$. We show that there is an $\mathcal{M}$-Kripke model K_e of $T \cup D$, such that $K_e \not\models \Phi$. Define a fuzzy measure μ_e on $\mathcal{B}(\Omega_D)$ as follows:

$$\mu_e([\varphi]_D) = e(f_\varphi).$$

Moreover, let

$$e'(p, w) = w(p)$$

for each propositional variable p. We prove that μ_e is a fuzzy measure by showing that, given e, the axioms of fuzzy measures do hold.

(i) By $\mathcal{F}1$ we have that for any $\varphi \in Con(D)$, $e(f_\varphi) = 1$. Then $\mu_e(\varphi) = 1$.

(ii) Given $\mathcal{F}3$, $e(\neg f_\perp) = 1$. But, $e(\neg f_\perp) = e(f_\perp) \Rightarrow 0$ where $\Rightarrow$ is the residuum of a left-continuous t-norm over $[0, 1]$. Therefore, $e(f_\perp) \Rightarrow 0 = 1$, which means $e(f_\perp) = 0$, and consequently $\mu_e(\perp) = 0$

(iii) By $\mathcal{F}2$ if $\varphi \to \chi$ belongs to $Con(D)$, then $f_\varphi \to f_\chi$ is an axiom. Consequently $\mu_e(\varphi) \leq \mu_e(\chi)$.

Therefore, we have proved that μ_e actually is a fuzzy measure. Then, it is clear that the model $K_e = \langle \Omega_D, \mathcal{B}(\Omega_D), \mu_e, e' \rangle$ is a model of D. Indeed, for any $w \in \Omega_D$, $e'(\varphi, w) = 1$ for any $\varphi \in D$, and the truth degree of modal formulas Ψ coincides with the truth evaluation $e(\Psi^\star)$ since it only depends on the values of μ_e and e over the elementary modal formulas $\mu(\varphi)$ and atoms f_φ respectively. Therefore $e'(\Psi, w) = e(\Psi^\star)$ for every modal formula Ψ, and in particular $e'(\Phi, w) = e(\Phi^\star) < 1$.

Conversely, suppose that $T \cup D \not\models_{\mathcal{M}(\mathcal{L})} \Phi$, i.e., assume that there is an $\mathcal{M}$-Kripke structure $K = (W, \mathcal{U}, e, \mu)$ which is a model of $T \cup D$, but $K \not\models \Phi$. Thus, K is also a model of $D^{\mathcal{M}}$. It is easy to see that there also exists an $\mathcal{L}$-evaluation e_K model of $T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star$ such that $e_K(\Psi^\star) = e(\Psi, w)$ for any modal formula Ψ and any $w \in W$. Indeed, take an arbitrary $w \in W$, and define:

$$e_K(f_\varphi) = e(\kappa(\varphi), w) = \mu([\varphi]_W).$$

Clearly, e_K is a model of axioms $\mathcal{F}1 - \mathcal{F}3$ since μ is a fuzzy measure. Therefore $e_K(\Psi^\star) = 1$ for every $\Psi^\star \in T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star$ but $e_K(\Phi^\star) < 1$, as desired. Hence we have proved the equivalence (2).

From (1) and (2), to prove the theorem it remains to show that

$$T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star \vdash_{\mathcal{L}} \Phi^\star \text{ iff } T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star \models_{\mathcal{L}} \Phi^\star.$$

The initial theories T and D are finite, and so is $T^\star$. However $\mathcal{F}$ contains infinitely many instances of axioms $\mathcal{F}1 - \mathcal{F}3$, and $(D^{\mathcal{M}})^\star$ is built up on all the consequences of D. If $\mathcal{L}$ is strongly standard complete, then the above equivalence is obvious. Then, we immediately obtain that $\mathcal{M}(\mathcal{L})$ is strongly standard complete, and the second part of the theorem is proved.

If $\mathcal{L}$ is only finitely strongly standard complete, then we have some additional work to do. Indeed, it is easy to prove that the infinitely many formulas in $T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star$ can be replaced by only finitely many instances by using propositional normal forms, following the lines of [75, 8.4.12]. Indeed, take n propositional variables $p_1, \ldots, p_n$ containing at least all variables in $T \cup D$. For any formula φ built from these propositional variables, take the corresponding disjunctive normal form $(\varphi)_{dnf}$. Notice that there are only finitely many different such formulas. Then, when translating a modal formula Φ into $\Phi^\star$, we replace each atom f_φ by $f_{(\varphi)_{dnf}}$ to obtain its normal translation $\Phi^\star_{dnf}$. The theory $T^\star_{dnf}$ is the (finite) set of all $\Psi^\star_{dnf}$, where $\Psi \in T$. The theory $(D^{\mathcal{M}})^\star_{dnf}$ is the finite set of formulas obtained from $(D^{\mathcal{M}})^\star$ by means of propositional normal forms. The theory $\mathcal{F}_{dnf}$ is the *finite* set of instances of axioms $\mathcal{F}1 - \mathcal{F}3$ for disjunctive normal forms of Boolean formulas built from propositional variables $p_1, \ldots, p_n$. Following [75, 8.4.13], it is now easy to prove the following:

(a) $T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star \vdash_{\mathcal{L}} \Phi^\star$ iff $T^\star_{dnf} \cup \mathcal{F}_{dnf} \cup (D^{\mathcal{M}})^\star \vdash_{\mathcal{L}} \Phi^\star_{dnf}$,

(b) $T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star \models_{\mathcal{L}} \Phi^\star$ iff $T^\star_{dnf} \cup \mathcal{F}_{dnf} \cup (D^{\mathcal{M}})^\star \models_{\mathcal{L}} \Phi^\star_{dnf}$.

Finally, we obtain the following chain of equivalences:

$$\begin{array}{lll} T \cup D \vdash_{\mathcal{M}(\mathcal{L})} \Phi & \text{iff} & T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star \vdash_{\mathcal{L}} \Phi^\star \\ & \text{iff} & T^\star_{dnf} \cup \mathcal{F}_{dnf} \cup (D^{\mathcal{M}})^\star \vdash_{\mathcal{L}} \Phi^\star_{dnf} \\ & \text{iff} & T^\star_{dnf} \cup \mathcal{F}_{dnf} \cup (D^{\mathcal{M}})^\star \models_{\mathcal{L}} \Phi^\star_{dnf} \\ & \text{iff} & T^\star \cup \mathcal{F} \cup (D^{\mathcal{M}})^\star \models_{\mathcal{L}} \Phi^\star \\ & \text{iff} & T \cup D \models_{\mathcal{M}(\mathcal{L})} \Phi. \end{array}$$

Then the proof is complete. ■

6.1.2 Classes of measures

We now see how we can easily extend the above logic in order to treat particular classes of measures of uncertainty. Let $\mathcal{L}$ be any t-norm based logic (or any of its expansions), and let $\mathcal{M}'$ be a class of fuzzy measures. We say that $\mathcal{L}$ is *compatible* with $\mathcal{M}'$ if the real valued operations needed to compute the values of compound propositions are definable in $\mathcal{L}$. It is immediately clear now, that a careful analysis of functional definability can help us know which logics are suitable for representing certain measures.

Probability. As for probabilities, we need the sum and the standard involutive negation, which are only available in expansions of the Łukasiewicz logic, that are then the only t-norm based logics compatible with probability measures.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{P}$. Then, the logic $\mathcal{P}(\mathcal{L})$ is obtained from $\mathcal{M}(\mathcal{L})$ by adding the following axioms (where $\rightarrow$ and $\neg$ correspond to Łukasiewicz implication and negation, respectively):

($\mathcal{M}$2) $\kappa(\varphi \rightarrow \psi) \leftrightarrow (\kappa(\varphi) \rightarrow \kappa(\varphi))$,

($\mathcal{M}$3) $\kappa(\varphi \vee \psi) \leftrightarrow (\kappa(\varphi) \rightarrow \kappa(\varphi \wedge \psi)) \rightarrow \kappa(\psi)$,

($\mathcal{M}$4) $\kappa(\neg\varphi) \leftrightarrow \neg\kappa(\varphi)$.

Notice that in presence of axiom ($\mathcal{M}$2) the monotonicity rule is derivable. $\mathcal{P}$-Kripke models are $\mathcal{M}$-Kripke models where μ is a finitely additive probability measure.

Possibility and Necessity. As for possibility measures we only need the minimum t-norm, hence every t-norm based logic is compatible with $\mathcal{P}i$. Then, the logic $\mathcal{P}i(\mathcal{L})$ is obtained from $\mathcal{M}(\mathcal{L})$ by adding the following axiom:

($\mathcal{M}$5) $\kappa(\varphi \vee \psi) \leftrightarrow \kappa(\varphi) \vee \kappa(\psi)$.

$\mathcal{P}i$-Kripke models are $\mathcal{M}$-Kripke models where μ is a possibility measure.

As for necessity measures we only need the maximum t-conorm, hence every t-norm based logic is compatible with $\mathcal{N}$. Then, the logic $\mathcal{N}(\mathcal{L})$ is obtained from $\mathcal{M}(\mathcal{L})$ by adding the following axiom:

($\mathcal{M}$6) $\kappa(\varphi \wedge \psi) \leftrightarrow \kappa(\varphi) \wedge \kappa(\psi)$.

$\mathcal{N}$-Kripke models are $\mathcal{M}$-Kripke models where μ is a necessity measure.

Lower and Upper Probability. As for upper probabilities, notice that the condition ($\sharp$) is equivalent to

$$\tfrac{k}{m} + \tfrac{n}{m}\mu(\varphi) \leq \sum_{i=1}^{m} \tfrac{\mu(\varphi_i)}{m},$$

given that $n, k \leq m$. It is then clear that $\sum_{i=1}^{m} \frac{\mu(\varphi_i)}{m} \leq 1$, and so it makes sense to rely on t-norm based logics. It is evident that a logic is compatible with the class $\mathcal{P}^{\vartriangle}$ only if it allows the representation rational numbers, the product of rationals and formulas, and the sum. Thus, any expansion of RŁ, or RPPŁ$'_{\Delta}$ (see Chapter 2) represents a suitable choice. Furthermore, the presence of the standard involutive negation makes possible to define also lower probabilities.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{P}^{\vartriangle}$. The logic $\mathcal{P}^{\vartriangle}(\mathcal{L})$ is obtained from $\mathcal{M}(\mathcal{L})$ by adding the rule (UP):

- if

$$\varphi \to \bigvee_{\left(\substack{J\subseteq\{1,\ldots,m\}\\ |J|=k+n}\right)} \bigwedge_{j\in J} \varphi_j,$$

and

$$\bigvee_{\left(\substack{J\subseteq\{1,\ldots,m\}\\ |J|=k}\right)} \bigwedge_{j\in J} \varphi_j$$

are propositional tautologies, then derive

$$k.\delta_m \oplus n.\delta_m \kappa(\varphi) \to \bigoplus_{j=1}^{m} \delta_m \kappa(\varphi_j),$$

if $\mathcal{L}$ is an expansion of RŁ, or derive

$$\overline{\tfrac{k}{m}} \oplus \left(\overline{\tfrac{n}{m}} *_\pi \kappa(\varphi)\right) \to \bigoplus_{j=1}^{m} \left(\overline{\tfrac{1}{m}} *_\pi \kappa(\varphi_j)\right),$$

if $\mathcal{L}$ is an expansion of RPPŁ$'_\Delta$.

The semantics for $\mathcal{P}^\Delta$ is given by $\mathcal{P}^\Delta$-Kripke models, i.e. $\mathcal{M}$-Kripke models where μ is an upper probability measure.

As for the class of lower probability measures, given that they are fuzzy measures characterized by (♯♯), it is obvious that the logics compatible with $\mathcal{P}^\Delta$ are the same that are compatible with $\mathcal{P}_\nabla$. Furthermore, notice that (♯♯) is equivalent to

$$\sum_{i=1}^{m} \tfrac{\mu(\varphi_i)}{m} \ominus \tfrac{k}{m} \leq \tfrac{n}{m}\mu(\varphi).$$

It is then clear that $\frac{n}{m}\mu(\varphi) \leq 1$, and so, again, t-norm based logics provide an adequate framework.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{P}_\nabla$. The logic $\mathcal{P}_\nabla(\mathcal{L})$ is obtained from $\mathcal{M}(\mathcal{L})$ by adding the rule (LP):

- if

$$\varphi \to \bigvee_{\left(\substack{J\subseteq\{1,\ldots,m\}\\ |J|=k+n}\right)} \bigwedge_{j\in J} \varphi_j,$$

and

$$\bigvee_{\left(\substack{J\subseteq\{1,\ldots,m\}\\ |J|=k}\right)} \bigwedge_{j\in J} \varphi_j$$

are propositional tautologies, then derive

$$\bigoplus_{j=1}^{m} \delta_m \kappa(\varphi_j) \ominus k.\delta_m \to n.\delta_m \kappa(\varphi),$$

if $\mathcal{L}$ is an expansion of RŁ, or derive

$$\bigoplus_{j=1}^{m}\left(\overline{\tfrac{1}{m}} *_\pi \kappa(\varphi_j)\right) \ominus \overline{\tfrac{k}{m}} \to \left(\overline{\tfrac{n}{m}} *_\pi \kappa(\varphi)\right),$$

if $\mathcal{L}$ is an expansion of $\mathrm{RPPŁ}'_\Delta$.

The semantics for $\mathcal{P}_\triangledown$ is given by $\mathcal{P}_\triangledown$-Kripke models, i.e. $\mathcal{M}$-Kripke models where μ is a lower probability measure.

We can now prove the following completeness theorem.

Theorem 6.1.7 *Let $\mathcal{L}$ be any t-norm based logic, and let $\mathcal{M}'$ be any class of measures among $\mathcal{P}$, $\mathcal{P}i$, $\mathcal{N}$, $\mathcal{P}^\Delta$, and $\mathcal{P}_\triangledown$. If the following conditions are satisfied:*

1. *$\mathcal{L}$ is compatible with $\mathcal{M}'$,*
2. *$\mathcal{L}$ is (finitely) strongly standard complete,*

then $\mathcal{M}'(\mathcal{L})$ is (finitely) strongly standard complete[7]*.*

Proof. The proof for $\mathcal{P}(\mathcal{L})$ and $\mathcal{N}(\mathcal{L})$ is an easy adaptation of the one given above for $\mathcal{M}(\mathcal{L})$, and is basically contained in [77, 75].

As for $\mathcal{P}i(\mathcal{L})$ the only difference w.r.t. $\mathcal{M}(\mathcal{L})$ is that the theory $\mathcal{F}$ has to be enlarged by adding the following axiom

($\mathcal{F}$4) $f_{\varphi\vee\psi} \leftrightarrow f_\varphi \vee f_\psi$.

Thus, we just have to proof the analogue of (2) of Theorem 6.1.6, which only consists in showing that μ_e is a possibility measure Π. This is easy, since from $\mathcal{F}4$, $\mu_e(\varphi\vee\psi) = \max(\mu_e(\varphi), \mu_e(\psi))$, which means that μ_e is in fact a possibility measure Π. The rest of the proof is an obvious adaptation of the one for $\mathcal{M}(\mathcal{L})$.

The proof for $\mathcal{P}^\Delta(\mathcal{L})$ and $\mathcal{P}_\triangledown(\mathcal{L})$ is almost the same. Then, we carry out the proof for $\mathcal{P}^\Delta(\mathcal{L})$ only. In this case the theory $\mathcal{F}$ has to be enlarged by adding either the axiom

($\mathcal{F}$5) $\quad k.\delta_m \oplus n.\delta_m f_\varphi \to \bigoplus_{j=1}^{m} \delta_m f_{\varphi_j}$, with

$$\varphi \to \bigvee\nolimits_{\binom{J\subseteq\{1,\dots,m\}}{|J|=k+n}} \bigwedge_{j\in J} \varphi_j,$$

and

$$\bigvee\nolimits_{\binom{J\subseteq\{1,\dots,m\}}{|J|=k}} \bigwedge_{j\in J} \varphi_j$$

belonging to $Con(D)$, if $\mathcal{L}$ is an expansion of RŁ, or the axiom

($\mathcal{F}$5') $\quad \overline{\tfrac{k}{m}} \oplus \left(\overline{\tfrac{n}{m}} *_\pi f_\varphi\right) \to \bigoplus_{j=1}^{m}\left(\overline{\tfrac{1}{m}} *_\pi f_{\varphi_j}\right)$, with

$$\varphi \to \bigvee\nolimits_{\binom{J\subseteq\{1,\dots,m\}}{|J|=k+n}} \bigwedge_{j\in J} \varphi_j,$$

[7]Here "(finite) strong standard completeness" has to be read in the sense of Theorem 6.1.6.

and

$$\bigvee_{\binom{J\subseteq\{1,\ldots,m\}}{|J|=k}} \bigwedge_{j\in J} \varphi_j$$

belonging to $Con(D)$, if $\mathcal{L}$ is an expansion of RPPŁ$'_{\Delta}$.

The analogue of (2) of Theorem 6.1.6 is immediate by construction. We have to show that

$$T^{\star} \cup \mathcal{F} \cup (D^{\mathcal{P}^{\Delta}})^{\star} \vdash_{\mathcal{L}} \Phi^{\star} \text{ iff } T^{\star} \cup \mathcal{F} \cup (D^{\mathcal{P}^{\Delta}})^{\star} \models_{\mathcal{L}} \Phi^{\star}.$$

If $\mathcal{L}$ was strongly standard complete, then the above equivalence would be obvious, we would immediately obtain that $\mathcal{M}(\mathcal{L})$ is strongly standard complete, and the second part of the theorem would be proved. However, notice that all the logics that here we regard as compatible with $\mathcal{P}^{\Delta}$ are expansions of the Łukasiewicz logic, which is not strongly standard complete, and so, neither is any of those logics.

If $\mathcal{L}$ is finitely strongly standard complete, the solution is to take disjunctive normal forms as in the proof of Theorem 6.1.6. Still, we have to be careful, since $\mathcal{F}5$ holds for all $n, m, k \in \mathbb{N}$. However, notice that there are finitely many propositional variables in T and D, and so the related Boolean algebra of provably equivalent propositions has finitely many atoms. Then, as proven by Halpern and Pucella (see above and [82]), there are only finitely many instances of ($\sharp$), and similarly there are only finitely many instances of $\mathcal{F}5$, in which we can substitute disjunctive normal forms. Then, the whole theory can be reduced to a finite set. The rest of the proof is exactly as above. ■

6.1.3 Expansions: truth-constants and definability of conditional measures

The expressive power of the above logics can be significantly increased if we aim at representing other features of fuzzy measures. First of all, notice that relying on a t-norm based logic including rational truth-constants would allow to represent several statements concerning assessment of rational values like

- "the degree of uncertainty of φ is 0.8" as $\kappa(\varphi) \leftrightarrow \overline{0.8}$,
- "the degree of uncertainty of φ is at least 0.8" as $\overline{0.8} \to \kappa(\varphi)$,
- "the degree of uncertainty of φ is at most 0.8" as $\kappa(\varphi) \to \overline{0.8}$.

Not having truth constants would yield a purely qualitative treatment in which only comparative statements can be expressed. The advantage of the presence of truth constants will be made even clearer in the third section of this chapter.

Another way to enhance the expressive power would be to allow the definition of conditional measures from the simple marginal measures represented in the logic.

Conditional probability. In order to define conditioning from marginal probabilities we need to be able to represent division. Clearly, the only possibility to express division in t-norm based logics is given by logics containing the Product implication. Hence, any expansion of Łukasiewicz logic having the Product implication, like ŁΠ and ŁΠ$\frac{1}{2}$, is a suitable choice.

Conditional possibility. As mentioned above, if the universe is finite, conditioning can be defined by means of the residuum of a continuous t-norm without zero-divisors. If the universe is infinite such a t-norm must be also Archimedean. Hence, two natural definitions for conditional possibility are the following:

- $\Pi_g(\varphi|\chi) = \Pi(\chi) \Rightarrow_g \Pi(\varphi \wedge \chi)$,
- $\Pi_\pi(\varphi|\chi) = \Pi(\chi) \Rightarrow_\pi \Pi(\varphi \wedge \chi)$.

Clearly, in order to model such operators in t-norm based logics we need Gödel implication and Product implication, respectively. Hence, such types of derived conditioning can be framed in any expansion of Gödel logic, the former, and in any expansion of Product logic, the latter.

Finally, recall, as mentioned above, that possibility and necessity measures are dual. This means that one measure can defined from the other by using the standard involutive negation n_s. Clearly if $\mathcal{L}$ is a t-norm based logic (or any expansion) in which n_s is definable, then necessity measures are definable in $\mathcal{P}i(\mathcal{L})$ and possibility measures are definable in $\mathcal{N}(\mathcal{L})$. Indeed:

Theorem 6.1.8 *Let $\mathcal{L}$ be a t-norm based logic (or any of its expansions) in which the standard involutive negation is definable, and let $\sim$ be the connective corresponding to such an operation. Define, both in $\mathcal{P}i(\mathcal{L})$ and $\mathcal{N}(\mathcal{L})$ a new operator*

$$\kappa'(\varphi) \quad \text{as} \quad {\sim}\kappa(\neg\varphi).$$

Then, $\mathcal{P}i(\mathcal{L})$ and $\mathcal{N}(\mathcal{L})$ are termwise equivalent.

Proof. We work out the proof for $\mathcal{P}i(\mathcal{L})$ only, since the proof for $\mathcal{N}(\mathcal{L})$ is very similar. Basically, we have to prove that the axioms of necessity measure are derivable in $\mathcal{P}i(\mathcal{L})$.

i. $\neg{\sim}\kappa(\neg\bot)$ is a theorem, and so is $\neg\kappa'(\bot)$.

ii. ${\sim}\kappa(\bot)$ is a theorem, and so is $\kappa'(\top)$.

iii. Suppose that $\varphi \to \psi$ is a theorem, then so is $\neg\psi \to \neg\varphi$. By the monotonicity rule $\kappa(\neg\psi) \to \kappa(\neg\varphi)$. Finally, by properties of the involutive negation $\sim$, we have ${\sim}\kappa(\neg\varphi) \to {\sim}\kappa(\neg\psi)$, and so $\kappa'(\varphi) \to \kappa'(\psi)$.

iv. $\kappa(\neg\varphi \vee \neg\psi) \leftrightarrow \kappa(\neg\varphi) \vee \kappa(\neg\varphi)$ is a theorem, and so is ${\sim}\kappa(\neg\varphi \vee \neg\psi) \leftrightarrow {\sim}(\kappa(\neg\varphi) \vee \kappa(\neg\varphi))$. Hence ${\sim}\kappa(\neg(\varphi \wedge \psi)) \leftrightarrow {\sim}(\kappa(\neg\varphi)) \wedge {\sim}(\kappa(\neg\varphi))$, and consequently $\kappa'(\varphi \wedge \psi) \leftrightarrow \kappa'(\varphi) \wedge \kappa'(\psi)$.

■

Finally, we prove a similar result for logics of lower and upper probabilities.

Theorem 6.1.9 *Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{P}^{\vartriangle}(\mathcal{L})$ and $\mathcal{P}_{\triangledown}(\mathcal{L})$ in which the standard involutive negation is definable, and let $\sim$ be the connective corresponding to such an operation. Define, both in $\mathcal{P}^{\vartriangle}(\mathcal{L})$ and $\mathcal{P}_{\triangledown}(\mathcal{L})$ a new operator*

$$\kappa'(\varphi) \quad \text{as} \quad {\sim}\kappa(\neg\varphi).$$

Then $\mathcal{P}^{\vartriangle}(\mathcal{L})$ and $\mathcal{P}_{\triangledown}(\mathcal{L})$ are termwise equivalent.

Proof. To see that $\mathcal{P}^{\vartriangle}(\mathcal{L})$ and $\mathcal{P}_{\triangledown}(\mathcal{L})$ are termwise equivalent, just take $\mathcal{P}^{\vartriangle}(\mathcal{L})$ and define the new connective $\kappa'(\varphi)$ as above. Then, it is obvious that the interpretation of κ' corresponds to a lower probability, and then by completeness the axioms of $\mathcal{P}_{\triangledown}(\mathcal{L})$ hold in $\mathcal{P}^{\vartriangle}(\mathcal{L})$. The other direction is obviously symmetric. ■

6.2 Logics for conditional measures

6.2.1 The base logic

Now we introduce the logic $\mathcal{CM}(\mathcal{L})$ for dealing with classes of conditional measures. Notice that in the following the conditional measures we consider are all primitive. Let $\mathcal{L}$ be any t-norm based logic (or any of its expansions) in which two commutative, associative and increasing operations $\circledcirc, \otimes$, having, respectively, 0 and 1 as neutral elements, and with $\otimes$ distributive over $\circledcirc$, are definable. Let us denote by $\hat{\circledcirc}$ and $\hat{\otimes}$ the connectives whose interpretation corresponds to $\circledcirc$ and $\otimes$, respectively. Let $\mathcal{CM}$ be a class of conditional measures. We expand the $\mathcal{L}$-language by including modal formulas which represent the uncertainty of conditional events. As in the case of simple measures, we define the language in two steps. First, we have classical Boolean formulas φ, ψ, etc., defined in the usual way from the classical connectives $(\wedge, \neg)$ and from a countable set V of propositional variables $p, q, \ldots$, etc. The set of Boolean formulas is denoted by L. Given any set $D \subset L$, we denote by $Con(D)$ the set of sentences which logically follow from D in classical logic, while we denote by $Sat(D) = \{\varphi \mid \neg\varphi \notin Con(D)\}$ the set of formulas not in contradiction with D. Moreover, $Sat(L)$ and $Taut(L)$ will denote the set of classically satisfiable formulas and the set of classical tautologies, respectively.

Elementary modal sentences are formulas of the form $\zeta(\varphi|\chi)$, where ζ is a unary operator taking as arguments conditional events $\varphi|\chi$, such that φ and χ are Boolean sentences, and $\chi \in Sat(L)$. Compound modal formulas are built by means of the $\mathcal{L}$-connectives. Nested modalities are not allowed.

Definition 6.2.1 The axioms of the logic $\mathcal{CM}(\mathcal{L})$ are the following:

- (i) The set $Taut(L)$ of classical Boolean tautologies
- (ii) Axioms of $\mathcal{L}$ for modal formulas

(*iii*) Conditional modal axioms:

($\mathcal{CM}$1)	$\zeta(\varphi\|\chi) \to \zeta(\varphi \wedge \chi\|\chi)$
($\mathcal{CM}$2)	$\zeta(\varphi \wedge \psi\|\chi) \leftrightarrow \zeta(\varphi\|\chi)\hat{\otimes}\zeta(\psi\|\varphi \wedge \chi)$
($\mathcal{CM}$3)	$\neg\zeta(\bot\|\chi)$

Deduction rules of $\mathcal{CM}(\mathcal{L})$ are those of $\mathcal{L}$, plus:

(*iv*) *modalization*: from $\vdash \varphi$ derive $\zeta(\varphi|\chi)$

(*v*) *substitution of equivalents* for the conditioning event: from $\vdash \chi \leftrightarrow \chi'$, derive $\zeta(\varphi|\chi) \leftrightarrow \zeta(\varphi|\chi')$

(*vi*) *monotonicity*: from $\vdash \varphi \to \psi$ derive $\zeta(\varphi|\chi) \to \zeta(\psi|\chi)$,

(*vii*) *join*: from $\vdash (\varphi \wedge (\psi \wedge \chi)) \leftrightarrow \bot$ derive $\zeta(\varphi \vee \psi|\chi) \leftrightarrow \zeta(\varphi|\chi)\hat{\oplus}\zeta(\psi|\chi)$.

Again, the language we have defined clearly is a hybrid language. Indeed, any theory (set of formulas) we will deal with will be of the form $\Gamma = D \cup T$, where D contains only non-modal formulas and T contains only modal formulas. Notice that there is no direct interaction between non-modal and modal formulas, with the exception of the application of the above rules of inference. The role of modalization, substitution of equivalents, join, and monotonicity only is to generate new modal formulas which can then be used in the deduction. On the other hand, in proofs from Γ, we want to avoid inconsistencies. In fact, the application of the above inference rules can yield modal formulas with conditioning events contradictory with D. As an example, if $D = \{\neg p\}$, where p is a propositional variable, then from D one could derive $\zeta(\neg p \mid p)$ by applying the modalization rule, obtaining something in clear contradiction with $\zeta(p \mid p)$. Therefore, we are led to define in $\mathcal{CM}(\mathcal{L})$ the notion of proof from a theory, written $\vdash_{\mathcal{CM}(\mathcal{L})}$, in a non-standard way, at least when the theory contains non-modal formulas.

In what follows let D denote a classical propositional theory with a decidable set of consequences, T a modal theory, φ a non-modal formula and Φ a modal formula. Moreover we will denote by $Ax(D)$ the set of instances of the axioms ($\mathcal{CM}$1)-($\mathcal{CM}$3) where the formulas appearing in the conditioning part of modal formulas are only from $Sat(D)$.

Definition 6.2.2 The proof relation $\vdash_{\mathcal{CM}(\mathcal{L})}$ between sets of formulas and formulas is defined by:

1. $D \cup T \vdash_{\mathcal{CM}(\mathcal{L})} \varphi$ if $\varphi \in Con(D)$;
2. $T \vdash_{\mathcal{CM}(\mathcal{L})} \Phi$ if Φ follows from $T \cup D^{\mathcal{CM}}$ in the usual way from the axioms $Ax(D)$ and the above inference rules restricted to formulas with conditioning part from $Sat(D)$;

where $D^{\mathcal{CM}}$ is the set of modal formulas obtained by applying the inference rules to formulas in $Con(D)$, with the restriction that every conditioning formula χ belongs to $Sat(D)$ and appears as conditioning in subformulas of Φ.

We now define the semantics for $\mathcal{CM}(\mathcal{L})$ by introducing $\mathcal{CM}$*-Kripke structures.*

Definition 6.2.3 A $\mathcal{CM}$-Kripke model is a structure $K = \langle W, \mathcal{U}, e, \xi \rangle$, where:

- W is a non-empty set of possible worlds.
- $\mathcal{U}$ is a Boolean algebra of subsets of W.
- $e : V \times W \to \{0, 1\}$ is a *Boolean* evaluation of the propositional variables, that is, $e(p, w) \in \{0, 1\}$ for each propositional variable $p \in V$ and each world $w \in W$. Any given truth-evaluation $e(\cdot, w)$ is extended to Boolean propositions as usual. For a Boolean formula φ, we will denote by $[\varphi]_W$ the set of worlds in which φ is true, i.e. $[\varphi]_W = \{w \in W \mid e(\varphi, w) = 1\}$.
- $\xi : \mathcal{U} \times \mathcal{U}^0 \to [0, 1]$ is a conditional measure over $\mathcal{U} \times \mathcal{U}^0$, where $\mathcal{U}^0 = \mathcal{U} \setminus \{\emptyset\}$, and such that $([\varphi]_W, [\chi]_W)$ is ξ-measurable for any non-modal φ and χ *(*with $[\chi]_W \neq \emptyset$*)*.[8]
- $e(\cdot, w)$ is extended to elementary modal formulas by defining

  $$e(\zeta(\varphi|\chi), w) = \xi([\varphi]_W \mid [\chi]_W),$$

  and to arbitrary modal formulas according to $\mathcal{L}$-semantics.

We call a Kripke structure $K = \langle W, \mathcal{U}, e, \xi \rangle$ *safe* for a formula Φ if $e(\Phi, w)$ is defined for every world w. Trivially, any Kripke structure is safe for all non-modal formulas. Notice that if K is a safe model for a modal formula Φ, then the truth-evaluation $e(\Phi, w)$ depends only on the conditional measure ξ and not on the particular world w. In this case, we will also write $e^K(\Phi)$ to denote $e(\Phi, w)$ for any $w \in W$. Furthermore, given a set of non-modal formulas D, we will say that K is D*-safe*, if K is safe for any formula $\zeta(\varphi|\chi)$ with $\chi \in Sat(D)$.

If K is safe for Φ, then we say that K is a model for Φ, written $K \models \Phi$, if $e^K(\Phi) = 1$. If T is a set of formulas, we say that K is a model of T if K is safe for all formulas in T and $K \models \Phi$ for all $\Phi \in T$.

Remark. $K = \langle W, \mathcal{U}, e, \xi \rangle$ is safe for $\zeta(\varphi \mid \chi)$ iff $[\chi]_W \neq \emptyset$ iff $K \not\models \neg\chi$.

$K = \langle W, \mathcal{U}, e, \xi \rangle$ is safe for a modal formula Φ iff K is so for every elementary modal subformula of Φ.

The notion of logical entailment relative to a class of structures $\mathcal{K}$, written $\models_{\mathcal{K}}$, is then defined as follows:

$T \models_{\mathcal{K}} \Phi$ iff $K \models \Phi$ for each $K \in \mathcal{K}$ model of T which is D-safe for Φ.

[8]Notice that in our definition the factors of the Cartesian product are the same Boolean algebra. This is clearly a special case of what stated in Definition 6.0.5.

If $\mathcal{K}$ denotes the whole class of $\mathcal{CM}$-Kripke structures we shall write $T \models_{\mathcal{CM}(\mathcal{L})} \Phi$. When $\models_{\mathcal{K}} \Phi$ holds we will say that Φ is *valid* in $\mathcal{K}$, i.e. when Φ gets value 1 in all structures $K \in \mathcal{K}$ safe for Φ.

Remark. $\models_{\mathcal{K}} \Phi$ does not mean $e^K(\Phi) = 1$ in each structure $K \in \mathcal{K}$, but only in those structures which are safe for Φ.

Lemma 6.2.4 *The axioms ($\mathcal{CM}1$)-($\mathcal{CM}3$) are valid in the class of $\mathcal{CM}$-Kripke structures. Moreover the modalization rule, the substitution of equivalents rule, the monotonicity rule and the join rule preserve validity in a model.*

Proof.

- To prove that for any model K, $e^K(\mathcal{CM}1) = 1$, just notice that by axiom (i) of conditional measures $\xi(\varphi|\chi) = \xi(\varphi \wedge \chi|\chi)$.
- We have that $e^K(\mathcal{CM}2) = 1$, since $\xi(\varphi \wedge \psi|\chi) = \xi(\varphi|\chi) \circledcirc \xi(\psi|\varphi \wedge \chi)$ corresponds to axiom (iii) of conditional measures.
- $e^K(\mathcal{CM}3) = 1$, given that $\xi(\perp|\chi) = 0$

We now check that the inference rules do preserve validity in a model.

- As for the modalization rule, suppose that $K \models \varphi$ and K is safe for $\zeta(\varphi \mid \chi)$, then $[\varphi]_W = W$ and $[\chi]_W \neq \emptyset$, hence $e(\zeta(\varphi \mid \chi), w) = 1$, that is $K \models \zeta(\varphi \mid \chi)$.
- As for substitution of equivalents, let $K = (W, \mathcal{U}, e, \xi)$ be such that $K \models \chi \leftrightarrow \chi'$ and K is safe for $\zeta(\varphi \mid \chi)$ and $\zeta(\varphi \mid \chi')$. Then, $[\chi]_W = [\chi']_W \neq \emptyset$ and hence obviously $e(\zeta(\varphi \mid \chi), w) = e(\zeta(\varphi \mid \chi'), w)$ for all $w \in W$, that is, $K \models \zeta(\varphi \mid \chi) \leftrightarrow \zeta(\varphi \mid \chi')$.
- As for the monotonicity rule, suppose that $K \models \varphi \to \psi$ and that K is safe for both $\zeta(\varphi|\chi)$ and $\zeta(\psi|\chi)$. Clearly we have that $e(\zeta(\varphi|\chi)) = \xi(\varphi|\chi) \leq \xi(\psi|\chi) = e(\zeta(\psi|\chi))$, hence $K \models \zeta(\varphi|\chi) \to \zeta(\psi|\chi)$.
- As for the join rule suppose that $K \models (\varphi \wedge (\psi \wedge \chi)) \leftrightarrow \perp$ and that K is safe for both $\zeta(\varphi \vee \psi|\chi)$, $\zeta(\varphi|\chi)$ and $\zeta(\psi|\chi)$. ξ is a conditional measure, then for any $\chi \in Sat(L)$, $\xi(\varphi \vee \psi|\chi) = \xi(\varphi|\chi) \circledcirc \xi(\psi|\chi)$.

■

Proposition 6.2.5 (Soundness) *The logic $\mathcal{CM}(\mathcal{L})$ is sound with respect to the class of $\mathcal{CM}$-Kripke structures.*

Proof. Assume $\Gamma \vdash_{\mathcal{CM}(\mathcal{L})} \Phi$ and recall Definition 6.2.2. If Φ is non-modal it is obvious, thus assume Φ is modal. Now, let us suppose that Γ is modal as well. Then, by the above lemma, we also have $\Gamma \models_{\mathcal{CM}(\mathcal{L})} \Phi$. Finally, let $\Gamma = D \cup T$ where D is non-modal and T modal. Let $K = (W, \mathcal{U}, e, \xi)$ be such that $K \models D \cup T$ and K is D-safe for Φ, we have to show that $K \models \Phi$. Since K is

D-safe for Φ, it means that $[\chi]_W \neq \emptyset$ for every χ in elementary modal formulas $\zeta(\cdot \mid \chi)$ appearing in Φ. On the other hand, it is easy to see that $K \models D^{\mathcal{CM}}$, by simply applying the inference rules and following the above lemma. Hence $K \models \Phi$ holds. ■

Similarly to the case of simple measures, for each $\varphi \in L$, we shall identify the equivalence class $[\varphi]_D$ with the set $\{\omega \in \Omega_D \mid \omega(\varphi) = 1\} \in \mathcal{B}(\Omega_D)$ of models of D that make φ true. We shall denote by $\mathcal{CM}(D)$ the set of conditional measures defined over $L/_{\sim_D} \times (L/_{\sim_D} \setminus [\bot])$ or, equivalently, on $\mathcal{B}(\Omega_D) \times \mathcal{B}(\Omega_D)^0$ (where $\mathcal{B}(\Omega_D)^0$ is $\mathcal{B}(\Omega_D)\setminus\{\emptyset\}$).

Notice that each conditional measure $\xi \in \mathcal{CM}(D)$ induces a $\mathcal{CM}$-Kripke structure $\langle \Omega_D, \mathcal{B}(\Omega_D), e_\xi, \xi\rangle$ where $e_\xi(p, \omega) = \omega(p) \in \{0, 1\}$ for each $\omega \in \Omega_D$ and each propositional variable p. We shall denote by $\mathcal{K}_D$ the class of $\mathcal{CM}$-Kripke structures which are models of D, and by $\mathcal{CMS}(D)$ the class of $\mathcal{CM}$-Kripke models $\{(\Omega_D, \mathcal{B}(\Omega_D), e_\xi, \xi) \mid \xi \in \mathcal{CM}(D)\}$. Obviously, each $K \in \mathcal{CMS}(D)$ is D-safe, and $\mathcal{CMS}(D) \subset \mathcal{K}_D$.

Abusing the language, we will say that a conditional measure $\xi \in \mathcal{CM}(D)$ is a *model* of a modal theory T whenever the induced Kripke structure $\langle \Omega_D, \mathcal{B}(\Omega_D), e_\xi, \xi\rangle$ is a model of T (obviously $\langle \Omega_D, \mathcal{B}(\Omega_D), e_\xi, \xi\rangle$ is a model of D as well).

Theorem 6.2.6 ((Finite) Strong completeness) *Let $\mathcal{L}$ be a t-norm based logic (or any expansion). If $\mathcal{L}$ is finitely strongly standard complete, then let T be a finite modal theory over $\mathcal{CM}(\mathcal{L})$, D a finite non-modal theory and Φ a modal formula with the following constraint: any modal formula $\zeta(\varphi|\chi)$ appearing (as subformula) in $T \cup \{\Phi\}$ is such that $\chi \in Sat(D)$. Then*

$$T \cup D \vdash_{\mathcal{CM}(\mathcal{L})} \Phi \text{ iff } e_\xi(\Phi) = 1$$

for each conditional measure $\xi \in \mathcal{CM}(D)$ model of T.

Moreover, if $\mathcal{L}$ is strongly standard complete the same holds for infinite theories.

Proof. The proof follows the lines of that given above for simple measures.

First, we have to translate theories over $\mathcal{CM}(\mathcal{L})$ into theories over $\mathcal{L}$. We define a theory, called $\mathcal{F}$, as follows:
(i) take as propositional variables of the theory variables of the form $f_{\varphi|\chi}$, where φ and χ are classical propositions from L, and $\chi \in Sat(D)$.
(ii) take as axioms of the theory the following ones, for each φ, ψ and χ:

($\mathcal{F}1$) $f_{\varphi|\chi}$, for $\varphi \in Con(D)$, and χ such that $\chi \in Sat(D)$,
($\mathcal{F}2$) $f_{\varphi|\chi} \leftrightarrow f_{\varphi|\chi'}$, for any χ, χ' such that $\chi \leftrightarrow \chi' \in Con(D)$ and $\chi, \chi' \in Sat(D)$,
($\mathcal{F}3$) $f_{\varphi|\chi} \to f_{\psi|\chi}$, whenever $\varphi \to \psi \in Con(D)$, with $\chi \in Sat(D)$,
($\mathcal{F}4$) $f_{\varphi|\chi} \to f_{\varphi\wedge\chi|\chi}$,
($\mathcal{F}5$) $f_{\varphi\vee\psi|\chi} \leftrightarrow f_{\varphi|\chi} \circledcirc f_{\psi|\chi}$, whenever $\varphi \wedge \psi \wedge \chi \leftrightarrow \bot \in Con(D)$, with $\chi \in Sat(D)$,

($\mathcal{F}6$) $f_{\varphi\wedge\psi|\chi} \leftrightarrow f_{\psi|\varphi\wedge\chi} \otimes f_{\varphi|\chi}$,
($\mathcal{F}7$) $\neg f_{\perp|\chi}$.

Then define the mapping $\star$ from modal formulas to $\mathcal{L}$-formulas as follows:

- $(\zeta(\varphi|\chi))^\star = f_{\varphi|\chi}$,
- $(\Phi \ominus \Psi)^\star = \Phi^\star \ominus \Psi^\star$, for $\ominus$ being a binary $\mathcal{L}$-connective,
- $(\natural(\Phi))^\star = \natural(\Phi^\star)$, for $\natural$ being a unary $\mathcal{L}$-connective,
- $(s)^\star = s$, for each truth-constant s.

Let us denote by $T^\star$ and $(D^{\mathcal{CM}})^\star$ the sets of all formulas translated from T and $D^{\mathcal{CM}}$.

Then, by the construction of $\mathcal{F}$ and $(D^{\mathcal{CM}})^\star$, one can easily check that for any Φ,

$$T \cup D \vdash_{\mathcal{CM}(\mathcal{L})} \Phi \text{ iff } T^\star \cup \mathcal{F} \cup (D^{\mathcal{CM}})^\star \vdash_{\mathcal{L}} \Phi^\star. \tag{1}$$

Now we prove that the semantical analogue of (1) also holds, that is,

$$T \cup D \models_{\mathcal{CM}(\mathcal{L})} \Phi \text{ iff } T^\star \cup \mathcal{F} \cup (D^{\mathcal{CM}})^\star \models_{\mathcal{L}} \Phi^\star. \tag{2}$$

Assume $T^\star \cup \mathcal{F} \cup (D^{\mathcal{CM}})^\star \not\models_{\mathcal{L}} \Phi^\star$. This means that there exists an $\mathcal{L}$-evaluation e which is model of $T^\star \cup \mathcal{F} \cup (D^{\mathcal{CM}})^\star$ such that $e(\Phi^\star) < 1$. We show that there is a Kripke model K_e of $T \cup D$, safe for Φ such that $K_e \not\models \Phi$. Define a conditional measure ξ_e on $\mathcal{B}(\Omega_D) \times \mathcal{B}(\Omega_D)^0$ as follows:

$$\xi_e([\varphi]_D \mid [\chi]_D) = e(f_{\varphi|\chi}),$$

with $\varphi, \chi \in L$ and $\chi \in Sat(D)$. Moreover, let

$$e'(p, w) = w(p)$$

for each propositional variable p. We prove that ξ_e is a conditional measure by showing that, given e, the axioms of conditional measures do hold.

(i) By $\mathcal{F}4$, $e(f_{\varphi|\chi} \to (f_{\varphi\wedge\chi|\chi})) = 1$, i.e. $e(f_{\varphi|\chi}) \leq e(f_{\varphi\wedge\chi|\chi})$. Hence, $\xi_e(\varphi|\chi) \leq \xi_e(\varphi \wedge \chi|\chi)$. Given that $\varphi \wedge \chi \to \varphi$ is a Boolean tautology, by $\mathcal{F}3$ we obtain the axiom $f_{\varphi\wedge\chi|\chi} \to f_{\varphi|\chi}$, and by an argument similar to the foregoing $\xi_e(\varphi \wedge \chi|\chi) \leq \xi_e(\varphi|\chi)$. Consequently we have $\xi_e(\varphi|\chi) = \xi_e(\varphi \wedge \chi|\chi)$.

(ii) We have to prove that $\xi_e(\cdot|\chi)$ is a fuzzy measure (compositional w.r.t. $\vee$):

- By $\mathcal{F}1$ we have that for every formula $\varphi \in Con(D)$, $e(f_{\varphi|\chi}) = 1$. Then $\xi_e(\varphi|\chi) = 1$.

- Given $\mathcal{F}7$, $e(\neg f_{\perp|\chi}) = 1$. But, $e(\neg f_{\perp|\chi}) = e(f_{\perp|\chi}) \Rightarrow 0 = 1$, which means that $e(f_{\perp|\chi}) = \xi_e(\perp|\chi) = 0$.

- By $\mathcal{F}3$ if $\varphi \to \psi \in Con(D)$, then $f_{\varphi|\chi} \to f_{\psi|\chi}$ is an axiom. Therefore $\xi_e(\varphi|\chi) \leq \xi_e(\psi|\chi)$.

- By $\mathcal{F}5$ if $\varphi \wedge \psi \wedge \chi \leftrightarrow \perp \in Con(D)$, we have that $\xi_e(\varphi \vee \psi|\chi) = \xi_e(\varphi|\chi) \circledcirc \xi_e(\psi|\chi)$.

(iii) Finally, given $\mathcal{F}6$, we have that $\xi_e(\varphi \wedge \psi|\chi) = \xi_e(\psi|\varphi \wedge \chi) \circledcirc \xi_e(\varphi|\chi)$.

Therefore, we have proved that ξ_e actually is a conditional measure. Then, it is clear that the model $K_e = \langle \Omega_D, \mathcal{B}(\Omega_D) \times \mathcal{B}(\Omega_D)^0, \xi_e, e' \rangle$ is a model of D. Indeed, for any $w \in \Omega_D$, $e'(\varphi, w) = 1$ for any $\varphi \in D$, and the truth degree of modal formulas Ψ coincides with the truth evaluation $e(\Psi^\star)$ since it only depends on the values of ξ_e and e over the elementary modal formulas $\zeta(\varphi|\chi)$ and atoms $f_{\varphi|\chi}$ respectively. Moreover, K_e clearly is a D-safe model. Therefore $e'(\Psi, w) = e(\Psi^\star)$ for every modal formula Ψ, and in particular $e'(\Phi, w) = e(\Phi^\star) < 1$.

Conversely, assume $T \cup D \not\models_{\mathcal{CM}(\mathcal{L})} \Phi$, that is, assume that there is a $\mathcal{CM}$-Kripke structure $K = (W, \mathcal{U}, e, \xi)$ which is a model of $T \cup D$ (hence D-safe for T), D-safe for Φ but $K \not\models \Phi$. Thus, it is easy to see that K also is a model of $D^{\mathcal{CM}}$, since for each elementary modal subformula $\zeta(\cdot \mid \chi)$ appearing in formulas in $D^{\mathcal{CM}}$, $[\chi]_W \neq \emptyset$, and every $\Phi \in D^{\mathcal{CM}}$ is obtained by applying the inference rules to $Con(D)$. Therefore, there also exists an $\mathcal{L}$-evaluation e_K model of $T^\star \cup \mathcal{F} \cup (D^{\mathcal{CM}})^\star$ such that $e_K(\Psi^\star) = e(\Psi, w)$ for any modal formula Ψ and any $w \in W$. Indeed, take an arbitrary $w \in W$, and define:

$$e_K(f_{\varphi|\chi}) = \begin{cases} e(\zeta(\varphi \mid \chi)) = \xi([\varphi]_W \mid [\chi]_W), & \text{if } [\chi]_W \neq \emptyset \\ \textit{arbitrary}, & \text{otherwise} \end{cases}.$$

Clearly, e_K is a model of axioms $\mathcal{F}1 - \mathcal{F}7$ since ξ is a conditional measure. For any modal formula $\Psi \in T \cup D^{\mathcal{CM}}$ we have $e_K(\Psi^\star) = e(\Psi, w)$ since this value is defined (K is D-safe for $T \cup \{\Phi\}$, hence also for $D^{\mathcal{CM}}$), and moreover it only depends on ξ. Therefore $e_K(\Psi^\star) = 1$ for every $\Psi^\star \in T^\star \cup \mathcal{F} \cup (D^{\mathcal{CM}})^\star$ but $e_K(\Phi^\star) < 1$, as desired. Hence we have proved the equivalence (2).

From (1) and (2), to prove the theorem it remains to show that

$$T^\star \cup \mathcal{F} \cup (D^{\mathcal{CM}})^\star \vdash_{\mathcal{L}} \Phi^\star \text{ iff } T^\star \cup \mathcal{F} \cup (D^{\mathcal{CM}})^\star \models_{\mathcal{L}} \Phi^\star.$$

The above equivalence can be proved exactly following the lines of Theorem 6.1.6. ■

6.2.2 Classes of conditional measures

We now proceed to the treatment of particular classes of conditional measures by extending and/or modifying the logic $\mathcal{CM}(\mathcal{L})$. Again, if we want to represent a specific class of measures we need the logic $\mathcal{L}$ to have enough expressive power to allow the definition of functions required by the axioms of the class we are interested in. As above, when that happens, we say that a logic $\mathcal{L}$ is compatible with the class of conditional measure $\mathcal{CM}$. We now focus on the specific cases of probability, possibility and necessity.

Conditional Probability. The functions required by the axioms of conditional probability are the standard involutive negation for the probability of the negation of a proposition, plus the sum and the product of real numbers. Thus, any finitely strongly standard complete expansion of the Product Łukasiewicz logic PŁ′ can be used, like, for instance ŁΠ and ŁΠ$\frac{1}{2}$.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{CP}$. Then, the logic $\mathcal{CP}(\mathcal{L})$ is obtained from $\mathcal{CM}(\mathcal{L})$ by omitting the join rule and adding the following axioms:

($\mathcal{CM}$4) $\zeta(\varphi \to \psi|\chi) \leftrightarrow (\zeta(\varphi|\chi) \to \zeta(\varphi|\chi))$,

($\mathcal{CM}$5) $\zeta(\varphi \vee \psi|\chi) \leftrightarrow (\zeta(\varphi|\chi) \to_l \zeta(\varphi \wedge \psi|\chi)) \to_l \zeta(\psi|\chi)$,

($\mathcal{CM}$6) $\zeta(\neg\varphi|\chi) \leftrightarrow \neg_l\zeta(\varphi|\chi)$,

where $\to_l$ and $\neg_l$ are Łukasiewicz implication and negation, respectively, and the connective $\hat{\otimes}$ in ($\mathcal{CM}$2) is interpreted as the Product t-norm. Notice that the monotonicity rule becomes redundant in presence of axiom ($\mathcal{CM}$4).

The semantics for $\mathcal{CP}(\mathcal{L})$ is given by $\mathcal{CP}$-Kripke models, which are $\mathcal{CM}$-Kripke models where ξ is a conditional probability measure $P_{\mathcal{C}}$.

Conditional Possibility and Necessity. Both for conditional possibility and necessity we just need the maximum t-norm and the minimum t-conorm. Hence, any t-norm based logic might represent a suitable choice. If from conditional possibility we want to define conditional necessity as a dual measure (or viceversa), we need the standard involutive negation. In that case IMTL, NM, and Łukasiewicz certainly are adequate choices. Furthermore, any t-norm based logic $\mathcal{L}_\sim$ with an independent involutive negation would work. Needless to say, the chosen logic must be (finitely) strongly standard complete.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{CP}i$. Then, the logic $\mathcal{CP}i(\mathcal{L})$ is obtained from $\mathcal{CM}(\mathcal{L})$ by omitting the join rule and adding the following axiom:

($\mathcal{CM}$7) $\zeta(\varphi \vee \psi|\chi) \leftrightarrow \zeta(\varphi|\chi) \vee \zeta(\psi|\chi)$.

Furthermore, the connective $\hat{\otimes}$ in ($\mathcal{CM}$2) is interpreted as the minimum t-norm. The semantics for $\mathcal{CP}i(\mathcal{L})$ is given by $\mathcal{CP}i$-Kripke models, which are $\mathcal{CM}$-Kripke models where ξ is a conditional possibility measure $\Pi_{\mathcal{C}}$.

Since conditional necessities are not exactly conditional measures but are defined from conditional possibilities, any conditional necessity can be defined from a conditional possibility in $\mathcal{CP}i(\mathcal{L})$ in presence of the standard strong negation $\sim$. Indeed, we can introduce a new connective η such that $\eta(\varphi|\chi)$ is $\sim\zeta(\neg\varphi|\chi)$ (see [105]). Alternatively, we can directly introduce a logic for conditional necessity, as follows. Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{CN}$. Then, the logic $\mathcal{CN}(\mathcal{L})$ is obtained from $\mathcal{CM}(\mathcal{L})$ by omitting the join rule and axiom ($\mathcal{CM}$2), and adding axiom ($\mathcal{CM}$4) plus the following axioms:

($\mathcal{CM}$8) $\zeta(\varphi \wedge \psi|\chi) \leftrightarrow \zeta(\varphi|\chi) \wedge \zeta(\psi|\chi)$,

($\mathcal{CM}$9) $\zeta(\varphi \vee \psi|\chi) \leftrightarrow \zeta(\varphi|\chi) \vee \zeta(\psi|\neg\varphi \wedge \chi)$.

The semantics for $\mathcal{CN}(\mathcal{L})$ is given by $\mathcal{CN}$-Kripke models, which are $\mathcal{CM}$-Kripke models where ξ is a conditional necessity measure $N_{\mathcal{C}}$.

Generalized Conditional Possibility and Necessity. As for generalized conditional possibility we need the maximum t-conorm and any t-norm, while for generalized conditional necessity we need the minimum t-norm and any t-conorm. A general (and qualitative) choice would be MTL$_{\sim}$, since, besides the maximum and the minimum, we can represent in general left-continuous t-norms and their dual t-conorms. Furthermore, the presence of an involutive negation allows the definition of possibilities from necessities and viceversa. If we look for a representation of generalized conditional possibility in which the t-norm is specified, then any t-norm based logic might be a suitable choice. If we look for a representation of generalized conditional necessity in which the t-conorm is specified, then we need to exploit an involutive negation in order to obtain the t-conorm from the t-norm of the chosen logic. In that case, any expansion of Łukasiewicz logic or any t-norm based logic (of a specific t-norm) $\mathcal{L}$ with an additional involutive negation might be adequate for the purpose. Furthermore, as the analysis of functional definability in Ł$\Pi\frac{1}{2}$ worked out in Chapter 5 has shown, several (left-continuous) t-norms are definable in Ł$\Pi\frac{1}{2}$, along with their dual t-conorms. Then, Ł$\Pi\frac{1}{2}$ provides a very powerful and expressive framework for the representation of both generalized conditional possibility and necessity.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{GC}\Pi$. Then, the logic $\mathcal{GCP}i(\mathcal{L})$ is obtained from $\mathcal{CM}(\mathcal{L})$ by omitting the join rule and adding axiom ($\mathcal{CM}$7) and the connective $\hat{\otimes}$ in ($\mathcal{CM}$2) is interpreted either as any member of a class of left-continuous t-norms, or as a particular t-norm definable in $\mathcal{L}$. The semantics for $\mathcal{GCP}i(\mathcal{L})$ is given by $\mathcal{GCP}i$-Kripke models, which are $\mathcal{CM}$-Kripke models where ξ is a generalized conditional possibility measure whose behavior is determined by the interpretation of $\hat{\otimes}$.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{GCN}$. Then, the logic $\mathcal{GCN}(\mathcal{L})$ is obtained from $\mathcal{CM}(\mathcal{L})$ by omitting the join rule and axiom ($\mathcal{CM}$2), and adding axioms ($\mathcal{CM}$4) and ($\mathcal{CM}$8), plus the following axiom:

($\mathcal{CM}$10) $\zeta(\varphi \vee \psi|\chi) \leftrightarrow \zeta(\varphi|\chi)\hat{\circledcirc}\zeta(\psi|\neg\varphi \wedge \chi)$,

where $\hat{\circledcirc}$ is interpreted either as any member of a class of t-conorms, or as a particular t-conorm definable in $\mathcal{L}$. The semantics for $\mathcal{GCN}(\mathcal{L})$ is given by $\mathcal{GCN}$-Kripke models, which are $\mathcal{CM}$-Kripke models where ξ is a generalized conditional necessity measure whose behavior is determined by the interpretation of $\hat{\circledcirc}$.

Theorem 6.2.7 *Let $\mathcal{L}$ be any t-norm based logic, and let $\mathcal{CM}'$ be any class among $\mathcal{CP}$, $\mathcal{CP}i$, $\mathcal{GCP}i$, $\mathcal{CN}$ and $\mathcal{GCN}$. If the following conditions are satisfied:*

1. *$\mathcal{L}$ is compatible with $\mathcal{CM}'$,*
2. *$\mathcal{L}$ is (finitely) strongly standard complete,*

then $\mathcal{CM}'(\mathcal{L})$ is (finitely) strongly standard complete.

Proof. The proof is an easy adaptation of the one given above for conditional measures in general and is given for probabilities in [70], and for possibilities and necessities in [105]. ■

6.3 Consistency, coherence and compactness

In this section we lay out a link between the consistency of modal theories and the coherence of rational assessments of fuzzy measures and conditional measures. In order to do so, we need some previous notions and results concerning satisfiability, compactness and consistency.

A detailed investigation of compactness of many logics based on continuous t-norms was presented in [34]. The notion of *satisfiability* proposed there generalizes the classical one by admitting various degrees of simultaneous satisfiability.

Definition 6.3.1 [34] For a set Γ of formulas in a t-norm based logic and $K \subseteq [0,1]$, we say that Γ is *K-satisfiable* if there exists an evaluation e such that $e(\varphi) \in K$ for all $\varphi \in \Gamma$. The set Γ is said to be *finitely K-satisfiable* if each finite subset of Γ is K-satisfiable. We say that a logic is *K-compact* if K-satisfiability is equivalent to finite K-satisfiability. A logic satisfies the *compactness property* if it is K-compact for each closed subset of $[0,1]$.

In particular t-norm based logics only having continuous truth-functions, like Łukasiewicz Logic, do enjoy the compactness property.

Theorem 6.3.2 ([16, 34]) *Let $\mathcal{L}$ be a given t-norm based logic whose connectives only have continuous truth-functions. Then $\mathcal{L}$ has the compactness property.*

The above result clearly still holds when we deal with theories in which the interpretations of all connectives correspond to continuous truth functions.

Proposition 6.3.3 *Let $\mathcal{L}$ be a t-norm based logic, and let Γ be any theory in $\mathcal{L}$ in which the connectives only have continuous truth-functions. Then $\mathcal{L}$ has the compactness property restricted to each such Γ.*

Among the t-norm based logics investigated in [34], besides Łukasiewicz logic, the only logics enjoying the compactness property are Gödel and its expansions G_Δ and $G_\sim$, but only with a finite number of propositional variables (see [34] for all the details).

For every t-norm based logic $\mathcal{L}$ having a model is tantamount to $\{1\}$-satisfiability. Now, given Theorem 6.3.2 we can show that $\{1\}$-satisfiability is equivalent to consistency for t-norm based logics having a finitary notion of derivability (i.e. having inference rules which refer to only finitely many premises).

Theorem 6.3.4 *Let $\mathcal{L}$ be a given t-norm based logic whose connectives are interpreted as continuous truth-functions, and having a finitary notion of derivability. Then any countable $\mathcal{L}$-theory Γ is consistent iff it has a model.*

Proof. Suppose that Γ has a model. Then Γ is $\{1\}$-satisfiable, and so is each finite subtheory $\Gamma_i \subseteq \Gamma$, by Theorem 6.3.2. Hence every Γ_i is consistent, which implies that $\Gamma \not\vdash \overline{0}$.

Conversely, suppose that Γ is consistent. This yields that there is no $\Gamma_i \subseteq \Gamma$ such that $\Gamma_i \vdash \overline{0}$. Thus, all Γ_i are $\{1\}$-satisfiable, and by Theorem 6.3.2 so is Γ. ∎

We now apply the above results to the notion of coherence for fuzzy measures (coherence for conditional measures will be an easy adaptation of the following results, as shown below).

In many real-life situations assessments of uncertainty are not precisely made over a set of events with a specific algebraic structure. Still, such assessments must be required to be coherent, that is: they must satisfy the axioms of a fuzzy measure whenever they are extended over the whole Boolean algebra generated by those events.

Definition 6.3.5 Let $\mathcal{M}$ be a class of fuzzy measures, $\mathcal{C}$ be a countable set of events, and μ be a real-valued assessment defined on $\mathcal{C}$. We call μ a *$\mathcal{M}$-coherent fuzzy measure* if there is a fuzzy measure $\mu' \in \mathcal{M}$ over the Boolean algebra generated by $\mathcal{C}$ such that $\mu(\varphi) = \mu'(\varphi)$ for all $\varphi \in \mathcal{C}$.

It is clear that by relying on a t-norm based logic $\mathcal{L}$ in which rational truth constants are definable we can represent rational assessments w.r.t. to a fuzzy measure. This will allow us to show that checking the coherence of a rational assessment over a countable set of events is tantamount to checking consistency of a suitably defined theory in $\mathcal{M}(\mathcal{L})$.

First of all, a clarification has to be made. Here, given a class of fuzzy measures $\mathcal{M}'$, $\mathcal{M}'(\mathcal{L})$ will denote an extension of $\mathcal{M}(\mathcal{L})$ over a t-norm based logic with rational truth constants compatible with $\mathcal{M}'$ being complete w.r.t. to $\mathcal{M}'$-Kripke models. For instance, $\mathcal{M}'(\mathcal{L})$ might correspond to either $\mathcal{P}(\mathcal{L})$, $\mathcal{P}i(\mathcal{L})$, or $\mathcal{N}(\mathcal{L})$. Now, we need theories of the form $\Gamma = \{\kappa(\varphi_i) \leftrightarrow \overline{\alpha_i}\}$ in order to have models in which assessments of fuzzy measures are not only 1-valued. Of course, we cannot take into account real-valued assessments, since we only have rational truth-constants in our language. Then we obtain that for any rational assessment its coherence is equivalent to the consistency of its related theory in $\mathcal{M}'(\mathcal{L})$, given that its extension induces a $\mathcal{M}'$-Kripke structure which is a model of such a theory. However, there is an important restriction. Indeed, to obtain the mentioned result we need the logic $\mathcal{L}$ to have the compactness property, or the connective $\leftrightarrow$ to be continuous, since we need to exploit the above compactness results. Since $\varphi \leftrightarrow \psi$ is defined as $(\varphi \leftrightarrow \psi)\&(\psi \leftrightarrow \varphi)$, it is obvious that both $\&$ and $\rightarrow$ must have continuous truth functions. Up to isomorphism, the only continuous t-norm having a continuous residuum is the

Łukasiewicz t-norm. This implies, in this case, that $\mathcal{L}$ must be an expansion of RPL.

Theorem 6.3.6 *Let $\mathcal{M}'$ be a class of fuzzy measures and let $\theta = \{\mu^*(\varphi_i) = \alpha_i\}$ be a rational assessment. Suppose that the following conditions are satisfied:*

i. $\mathcal{L}$ is a t-norm based logic with rational truth constants

ii. $\mathcal{L}$ either has the compactness property or is an expansion of RPL

iii. $\mathcal{L}$ has a finitary notion of derivability

iv. $\mathcal{L}$ is compatible with $\mathcal{M}'$

v. $\mathcal{M}'(\mathcal{L})$ is (finitely) strongly complete w.r.t. $\mathcal{M}'$-Kripke structures.

Then θ is $\mathcal{M}'$-coherent iff the theory $\Gamma_\theta = \{\kappa(\varphi_i) \leftrightarrow \overline{\alpha}_i\}$ is consistent in $\mathcal{M}'(\mathcal{L})$, i.e. $\Gamma_\theta \not\vdash_{\mathcal{M}'(\mathcal{L})} \overline{0}$.

Proof. Suppose that Γ_θ is consistent. As shown in the completeness proof (equivalence (1)), also the translated $\mathcal{L}$-theory $\Gamma^\star_\theta$ is consistent. By assumption either $\mathcal{L}$ has the compactness property or the connective $\leftrightarrow$ has an interpretation that corresponds to a continuous truth-function and has a finitary notion of derivability. Then, by Thorem 6.3.4, the theory $\Gamma^\star_\theta$ has a model e. From the completeness proof we know that e induces a $\mathcal{M}'$-Kripke structure which is a model K_e of Γ_θ. K_e is equipped with a fuzzy measure hence, the assessment $\theta = \{\mu^*(\varphi_i) = \alpha_i\}$ is $\mathcal{M}'$-coherent.

The converse is similar and so left to the reader. ■

Given the above theorems, it is now easy to prove a compactness result for coherent assessments. This means that when we have a rational assessment to a countable set of events, such an assessment is coherent if and only if its restriction to each finite subset of that set also is coherent. Indeed, since any of such coherent restrictions can be translated into a theory which is consistent by Theorem 6.3.6, the whole corresponding theory is consistent, and consequently, again by Theorem 6.3.6, the corresponding assessment is coherent. Notice that this result concerns rational assessments of fuzzy measures only, and it is proved by purely logical means.

Theorem 6.3.7 *Let $\mathcal{C} = \{\varphi_i\}$ be a countable family of events, let $\theta = \{\mu^*(\varphi_i) = \alpha_i\}$ be a rational assessment over $\mathcal{C}$, and let $\mathcal{M}'$ be a class of fuzzy measures. Suppose that the following conditions are satisfied:*

i. $\mathcal{L}$ is a t-norm based logic with rational truth constants

ii. $\mathcal{L}$ either has the compactness property or is an expansion of RPL

iii. $\mathcal{L}$ has a finitary notion of derivability

iv. $\mathcal{L}$ is compatible with $\mathcal{M}'$

v. $\mathcal{M}'(\mathcal{L})$ *is (finitely) strongly complete w.r.t.* $\mathcal{M}'$*-Kripke structures.*

Let $\theta_{\downarrow_\mathcal{I}}$ *be the restriction of* θ *to each finite* $\mathcal{I}$*, such that* $\mathcal{I} \subset \mathcal{C}$*. Then:*

$$\theta \text{ is } \mathcal{M}'\text{-coherent iff } \theta_{\downarrow_\mathcal{I}} \text{ is } \mathcal{M}'\text{-coherent for every } \mathcal{I}.$$

Proof. Obviously, if θ is $\mathcal{M}'$-coherent, then for any finite subset $\mathcal{I} \subset \mathcal{C}$ also $\theta_{\downarrow_\mathcal{I}}$ is $\mathcal{M}'$-coherent. Conversely, suppose that $\theta_{\downarrow_\mathcal{I}}$ is $\mathcal{M}'$-coherent for every $\mathcal{I}$. Hence, by Theorem 6.3.6, each theory $\Gamma_{\theta_{\downarrow_\mathcal{I}}} = \{\kappa(\varphi_i) \leftrightarrow \overline{\alpha}_i\}$ is consistent in $\mathcal{M}'(\mathcal{L})$. Then, $\Gamma_\theta = \{\kappa(\varphi_i) \leftrightarrow \overline{\alpha}_i\}$ consistent in $\mathcal{M}'(\mathcal{L})$, so by Theorem 6.3.6 θ is $\mathcal{M}'$-coherent. ■

Now, we can easily extend to above results to conditional measures.

Definition 6.3.8 Let $\mathcal{C}$ be a countable family of conditional events, ξ a real-valued function defined on $\mathcal{C}$, and $\mathcal{CM}$ a class of conditional measures. We call ξ a $\mathcal{CM}$*-coherent conditional measure* if for every $\mathcal{C}' \supseteq \mathcal{C}$, where $\mathcal{C}' = \mathcal{E} \times \mathcal{H}$, with $\mathcal{E}$ a Boolean algebra, $\mathcal{H}$ an additive set, $\mathcal{H} \subseteq \mathcal{E}$ and $\emptyset \notin \mathcal{H}$, there exists a conditional measure $\xi' \in \mathcal{CM}$ defined on $\mathcal{C}'$ extending ξ.

Theorem 6.3.9 *Let* $\theta = \{\xi^*(\varphi_i|\chi_i) = \alpha_i\}$ *be a rational conditional assessment, and let* $\mathcal{CM}'$ *be a class of conditional measures. Suppose that the following conditions are satisfied:*

i. $\mathcal{L}$ *is a t-norm based logic with rational truth constants*

ii. $\mathcal{L}$ *either has the compactness property or is an expansion of* RPL

iii. $\mathcal{L}$ *has a finitary notion of derivability*

iv. $\mathcal{L}$ *is compatible with* $\mathcal{CM}'$

v. $\mathcal{CM}'(\mathcal{L})$ *is (finitely) strongly complete w.r.t.* $\mathcal{CM}'$*-Kripke structures.*

Then θ *is* $\mathcal{CM}'$*-coherent iff the theory* $\Gamma_\theta = \{\zeta(\varphi_i|\chi_i) \leftrightarrow \overline{\alpha}_i\}$ *is consistent in* $\mathcal{CM}'(\mathcal{L})$*, i.e.* $\Gamma_\theta \not\vdash_{\mathcal{CM}'(\mathcal{L})} \bar{0}$.

Proof. The proof is an obvious adaptation of the above one for unconditional measures. ■

Again, it is now easy to prove a compactness result for coherent conditional assessments.

Theorem 6.3.10 *Let* $\mathcal{C} = \{\varphi_i|\chi_i\}$ *be a countable family of conditional events, let* $\theta = \{\xi^*(\varphi_i|\chi_i) = \alpha_i\}$ *be a rational conditional assessment over* $\mathcal{C}$*, and let* $\mathcal{CM}'$ *be a class of conditional measures. Suppose that the following conditions are satisfied:*

i. $\mathcal{L}$ *is a t-norm based logic with rational truth constants* *ii.* $\mathcal{L}$ *either has the compactness property or is an expansion of* RPL

iii. $\mathcal{L}$ *has a finitary notion of derivability*

iv. $\mathcal{L}$ *is compatible with* $\mathcal{CM}'$

v. $\mathcal{CM}'(\mathcal{L})$ *is (finitely) strongly complete w.r.t.* $\mathcal{CM}'$*-Kripke structures.*

Let $\theta_{\downarrow_\mathcal{I}}$ *be the restriction of* θ *to each finite* $\mathcal{I}$*, such that* $\mathcal{I} \subset \mathcal{C}$*. Then:*

$$\theta \text{ is } \mathcal{CM}'\text{-coherent iff } \theta_{\downarrow_\mathcal{I}} \text{ is } \mathcal{CM}'\text{-coherent for every } \mathcal{I}.$$

Proof. Easy from the unconditional case. ■

6.4 Comparisons with other treatments

As mentioned in the introduction, previous particular results concerning the representation of measures of uncertainty by relying on t-norm based logics were presented in several works. However, in all those cases uncertainty measures were represented by choosing a specific t-norm based logic, and furthermore only in a few cases reasoning with classical formulas was allowed.

The treatment of probability measures and necessity measures was proposed by Esteva, Hájek, and Godo in [77, 75] where the authors introduced the logics FP and FPS by relying on RPL. The same authors studied in [67] a more powerful logic for probabilities by relying on $Ł\Pi\frac{1}{2}$. This logic, called FP(ŁΠ), allowed the representation of conditional probability from the marginal ones by means of the Product implication.

The treatment of primitive conditional probability was carried out by the present author and Godo in [106, 69, 70], where the logic FCP(ŁΠ) was introduced by relying, again, on $Ł\Pi\frac{1}{2}$. While in [106] FCP(ŁΠ) did not allow reasoning with Boolean formulas, this possibility was fully developed in [70].

The treatment of (generalized) conditional possibility and necessity was studied by the present author in [104] for modal theories only and in [105] for theories with both modal and classical formulas (actually, (generalized) conditional necessity was derived from (generalized) conditional possibility and not treated separately). While in [104] the chosen logic was $Ł\Pi\frac{1}{2}$ for all those classes of measures, in [105] conditional possibilities and necessities were defined over RPL by introducing the logic FCΠ. The classes of generalized conditional possibilities and necessities were defined by relying on $Ł\Pi\frac{1}{2}$ or, alternatively, on $\mathrm{MTL}_\sim$. The logics defined were called $\mathrm{FC\Pi}(\mathcal{T}^{\mathrm{R}})$ (where $\mathcal{T}^R$ refers to a finitely constructed continuous t-norm) and GFCΠ.

Flaminio and Montagna introduced in [59] the logic FP(SŁΠ) for reasoning about simple and conditional non-standard probabilities. FP(SŁΠ) is based on the logic SŁΠ which expands $Ł\Pi\frac{1}{2}$ by means of a unary connective S. SŁΠ-algebras are $Ł\Pi\frac{1}{2}$algebras equipped with an idempotent endomorphism σ over the reduct $\langle \oplus, \neg, \cdot, 0, 1\rangle$. SŁΠ was shown to be finitely strongly complete w.r.t. evaluations into SŁΠ-algebras whose lattice reduct is an ultrapower of $\mathbb{R}Ł\Pi\frac{1}{2}$.

Basically, SŁΠ-connectives are interpreted by functions over hyper-real numbers (for basic notions of non-standard analysis see Robinson's [127]), and the connective S is introduced to recover the standard part of the evaluation (see [59] for all details). FP(SŁΠ) is then constructed following the strategy carried out in this chapter, but it allows to represent non-standard probabilities.

Another treatment that exploits t-norm based logics was given by Godo, Hájek and Esteva in [68] in order to represent belief functions. Belief functions [132] are fuzzy measures satisfying the following law

$$\mu\left(\bigcup_{i=1}^{n}\varphi_i\right) \geq \sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}} (-1)^{|I|+1}\,\mu\left(\bigcap_{i\in I}\varphi_i\right),$$

and form a subclass of lower probabilities. In order to define a logic for belief functions, Godo, Hájek and Esteva relied on $\text{Ł}\Pi\frac{1}{2}$, and introduced a modal operator not over classical Boolean formulas, but over modal formulas of the modal logic S5 [87]. Indeed, they take as a base logic S5, and introduce a many-valued modal operator P over formulas of S5. The logic obtained is called FB($\text{Ł}\Pi\frac{1}{2}$), and its axioms are exactly the axioms of the logic FP(ŁΠ) (and of $\mathcal{P}(\mathcal{L})$), while its semantics is given by Kripke models equipped with a probability measure. Belief functions are then representable, since the evaluation of $P(\Box\varphi)$ corresponds to a belief function. FB($\text{Ł}\Pi\frac{1}{2}$) was shown to be finitely strongly complete.

Neither FB($\text{Ł}\Pi\frac{1}{2}$) nor FP(SŁΠ) is directly covered by the approach carried out in this chapter. However, it is easy to see that small adjustments, like allowing the definition of measures over two-valued modal formulas and including t-norm based logics with non-standard semantics, enable to recover both FB($\text{Ł}\Pi\frac{1}{2}$) and FP(SŁΠ) under our framework.

The above mentioned works present quite a novel approach in the logical representation of uncertainty which has been developed in the last ten years. In fact, in the literature we can find many logical treatments of uncertainty, specially for probability theory, but such logics are basically two-valued expansions of the classical Boolean logic which are in general equipped with a modal operator representing a class of measures.

As for probability measures[9] we might mention the papers by Nilsson [120], by Halpern [80], by Fagin, Halpern and Megiddo [56], by Fattorosi-Barnaba and Amati [57], by Ognjanović and Rašković [123] by van der Hoek [142] and the book by Bacchus [7]. A specific treatment of conditional probabilities (derived from marginal ones) was given by in Rašković, Ognjanović and Marković in [126]. A logic for upper (and lower) probabilities was proposed by Halpern and Pucella in [82].

As for possibility measures, a remarkable and very well-known example is given by the *Possibilistic Logic*, particularly studied by Dubois, Lang and Prade [40]. In Possibilistic Logic formulas are pairs $(\varphi\ \alpha)$, where φ is a first-order

[9]We do not aim at giving an exhaustive list of works on probabilistic logics, but we only mention some which we believe to be of particular interest. The reader can find further references in the cited papers and books.

Boolean formula and α is a real number which represents the weight attached to φ. The interpretation of such formulas is that $N(\varphi) \geq \alpha$, i.e. the necessity of φ is at least α. A similar approach was given by Saffiotti in [130] for belief functions. In this case, first-order formulas are weighted by intervals, so that $\varphi[\alpha, \beta]$ represents that the belief in the truth of φ is α, while the belief in the falsity of φ is β.

Bendová and Hájek introduced in [8] the logic CPMPL for comparative possibilities. CPMPL is an expansion of classical logic by means of a binary modality $\triangleleft$, so that $\varphi \triangleleft \psi$ means that the possibility of φ is less than or equal to the possibility of ψ. Similarly, in [75], Hájek adapted this approach to belief functions introducing the logic CBMPL (having the same language as CPMPL) for representing comparisons between beliefs assigned to propositions. Both CPMPL and CBMPL are two-valued logics and allow a purely qualitative representation.

All the above treatments focus on simple measures. The only treatment for primitive conditional measures is the one given by Ikodinović and Ognjanović in [88], where the authors study a logic to represent conditional probability in the sense of Coletti and Scozzafava [35].

As far as we know, the only comprehensive logical treatment of uncertainty measures is the one proposed by Halpern [81]. This approach basically is an adaptation of the logical analysis of probabilities in [80, 56], and has been extended by Halpern to other kinds of measures. In such a work, a modal operator ℓ, standing for likelihood, is applied over Boolean formulas, so that $\ell(\varphi)$ is a likelihood term interpreted as "the uncertainty of φ". A basic likelihood formula is an expression of the form

$$a_1\ell(\varphi_1) + \cdots + a_k\ell(\varphi_k) > b,$$

where $a_1, \ldots, a_k, b$ are real numbers and $k \geq 1$. Likelihood formulas are Boolean combinations of basic likelihood formulas. The language resulting from the foregoing description is called $\mathcal{L}^{QU}$, where QU stands for *quantitative uncertainty*[10].

From $\mathcal{L}^{QU}$ we can then build up a logic for a class of measures, by introducing the adequate axioms. Given that likelihood formulas are linear inequalities, we also have to introduce all substitution instances of valid linear inequality formulas as axioms. The semantics is given by Kripke models $\langle W, \mathcal{U}, e, \mu \rangle$ where W is a set of possible worlds, $\mathcal{U}$ is a Booean algebra of subsets of W, e is a classical evaluation, and μ is a measure belonging to the chosen class. Halpern showed how to treat probabilities, possibilities, belief functions and upper probabilities obtaining sound and complete axiomatizations.

Fagin, Halpern and Megiddo also studied in [56] a logic for probabilities including the axioms of real closed fields. This allows to have not only linear

[10] Actually Halpern defines $\mathcal{L}^{QU}$ including the possibility of having indexed likelihood terms ℓ_i, so that each ℓ_i corresponds to the uncertainty framework of a (possibly) different agent. In other words, $\mathcal{L}^{QU}$ can include many likelihood terms, all corresponding to the same class of measures, aiming at formalizing the uncertainty assigned by different agents. However, for our purposes and for making the comparison simpler, we can simply restrict ourselves to the case where there is only one agent.

inequalities but also polynomial inequalities, and hence the possibility to represent multiplication of terms. This approach has not been extended to other measures, but it should not be difficult to work it out. In the case of probabilities, the presence of product allows the representation of independence which can be expressed by the formula $\ell(\varphi \wedge \psi) = \ell(\varphi) \cdot \ell(\psi)$. In our approach probabilistic independence can be represented by relying on any t-norm based logic expanding PŁ$'$, by means of the formula

$$\kappa(\varphi \wedge \psi) \leftrightarrow \kappa(\varphi) *_{\pi} \kappa(\psi).$$

Halpern's approach, however, is not extended to primitive conditional measures, and, as mentioned above, besides [88] there is no treatment of primitive conditional measures. We have shown here that we can find an adequate approach for representing those measures over t-norm based logics. This allows, for instance, to represent statements about independence such as

$$\zeta(\varphi|\chi \wedge \psi) \leftrightarrow \zeta(\varphi|\chi),$$

which can be read as "φ and ψ are independent given χ".

Halpern's approach is strongly based on the presence of axioms of linear inequalities (and in some cases polynomial inequalities) which allow to represent basic operations between formulas. Our approach exploits the advantage given by the fact that in t-norm based logics the operations associated to the evaluation of the connectives are functions defined over the real unit interval $[0,1]$, which correspond, directly or up to some combinations, to operations used to compute degrees of uncertainty. Then such algebraic operations can be embedded in the connectives of the many-valued logical framework, resulting in clear and elegant formalizations. Given that there is a whole family of t-norm based logics, the choice of the logic to exploit to represent a specific class of measures will clearly depend on the operations we need to represent. This permits to avoid the introduction of instances of linear inequalities, since they are directly given by the functions associated to the connectives of some logics. For instance, as mentioned above, Łukasiewicz logic and its expansions allow the representation of piecewise linear functions, and hence are the most suitable choice for the representation of linear equalities and inequalities. Furhermore, if we need to have polynomial inequalities we do not need to rely on real closed fields as done in [56]. As shown in Chapter 4, polynomial inequalities definable in real closed fields can be translated into Ł$\Pi\frac{1}{2}$-formulas. Moreover, in the case of possibility and necessity measures, for instance, we might not even need to use linear inequalities. What we need are just the minimum and the maximum operators plus the possibility of expressing comparative statements which is immediately given by the implication connective.

Therefore, in our treatment we do not need to add axioms for having peculiar operations, since the possible presence of those operations just relies on an adequate choice of the base logic. The representation of uncertainty then clearly result in elegant and simple formalizations.

Having functions embedded in our logics also implies that some properties of the chosen logic might be inherited by the kind of measures we define in it. Indeed, once proven the connection between the consistency of a suitably defined theory in our logic and the coherence of the related assessment, properties like compactness for those assessments can be easily studied by purely logical means.

To conclude, we would like to point out that we do not deem that the t-norm based approach is better than the others. The study carried out in this chapter might be just an overt example of the advantages t-norm based logics can provide.

Chapter 7

Open Problems

Throughout this work we have given many new results concerning functional definability and some applications. Still, many interesting problems deserve further investigation. We mention here some issues which are of particular interest.

Involutive Negations

An open problem concerns the definition of a logic expanding MTL by means of an independent involutive negation without Baaz's Delta. Such a logic would expand MTL just by axiom ($\sim$1) and the order-reversing rule (OR)(see Proposition 3.1.2). Notice that we cannot add the stronger axiom

$$(\varphi \to \psi) \to (\sim\psi \to \sim\varphi),$$

since this is not valid w.r.t. to some t-norms. Indeed, Product and Gödel implication, for instance, do not satisfy the above formula. Its algebraic counterpart would be similarly defined, and the class of algebras would be a variety[1]. However, a deduction theorem (in the usual form for t-norm based logics without Delta) seems to fail, as well as the fact that an algebra $\mathcal{A}$ and its n-free reduct $\mathcal{A}^-$ have the same congruences (see Lemma 3.1.7). On the other hand the presence of the connective Δ seems quite a strong requirement for the logics introduced in Chapter 3.

Problem. Find a uniform way to introduce an independent involutive negation in t-norm based logics without using Δ.

Ordered Fields and $\text{Ł}\Pi\frac{1}{2}$-Algebras

In [111] Montagna proposed some problems about $\text{Ł}\Pi\frac{1}{2}$ and $\text{Ł}\Pi\frac{1}{2}$-algebras. All those problems have been solved (one of them in this dissertation) with the exception of two.

[1]This is an easy consequence of the equivalences between (i) and (iii), and (i) and (iv) of Proposition 3.1.4.

The first problem concerns the decidability of the $\text{Ł}\Pi\frac{1}{2}$-chain $\mathbb{Q}\text{Ł}\Pi\frac{1}{2}$ over the rationals:

Problem. Is $\mathbb{Q}\text{Ł}\Pi\frac{1}{2}$ decidable? And if so, what is its computational complexity?

It is evident that this problem is linked to the decidability of the universal theory of $\mathcal{Q}$.

Theorem 7.0.1 *The universal theory of $\mathcal{Q}$ is decidable iff so is $\mathbb{Q}\text{Ł}\Pi\frac{1}{2}$.*

Decidability of the universal theory of the rational field was a problem proposed by Kokorin in [98], and it is still open. Furthermore, similarly to the case of real fields, we can translate formulas of $\mathcal{Q}$ into equations over $\mathbb{Q}\text{Ł}\Pi\frac{1}{2}$ and viceversa in polynomial time.

Theorem 7.0.2 *There is a polynomial-time reduction of the universal theory of $\mathcal{Q}$ to $\mathbb{Q}\text{Ł}\Pi\frac{1}{2}$.*

If decidability held, then both $\mathcal{Q}$ and $\mathbb{Q}\text{Ł}\Pi\frac{1}{2}$ would belong to the same complexity class.

Problem. Is the universal theory of $\mathcal{Q}$ decidable? And if so, what is its computational complexity?

The second still unsolved problem among those raised by Montagna concerns the possibility of finding a normal form theorem for $\text{Ł}\Pi\frac{1}{2}$.

Problem. Can we find a normal form theorem of $\text{Ł}\Pi\frac{1}{2}$?

Theorem 4.3.10 is of special importance since it clearly states that the universal theory of the field of real numbers and $\text{Ł}\Pi\frac{1}{2}$ share the same complexity class, and they are linked by a polynomial-time translation. It is well-known that the universal theory of the field of reals is in PSPACE, but it is unknown whether it also is PSPACE-complete.

Problem. Is the universal theory of real closed fields PSPACE-complete?

A solution to this long-standing open problem would immediately yield that also $\text{Ł}\Pi\frac{1}{2}$ is PSPACE-complete.

Completeness w.r.t. Term-definable T-norms

In Section 5.4 we have given completeness results for several logics based on t-norms. Still, the question regarding completeness for ΠMTL remains unanswered.

Problem. Is the logic ΠMTL finitely strongly standard complete w.r.t. the class of left-continuous cancellative t-norms term-definable in $\text{Ł}\Pi\frac{1}{2}$?

The treatment provided in Chapter 6 focused on fuzzy measures and conditional measures whose events are Boolean. It would be then interesting to find a general treatment for measures of uncertainty of many-valued propositions. A first step in that direction has been recently made by Flaminio and Godo in [58], where the authors extend and improve the initial attempt of Hájek in [75] to define a logic for the probability of many-valued events.

Problem. Find a general logical treatment to represent the uncertainty of many-valued events.

In a very recent paper, Hájek gave a general study of complexity for probabilistic logics based on continuous t-norms. Hájek's treatment covers both the probability of Boolean events and the probability of fuzzy events. That might be a starting point for the study of computational complexity for the logics representing uncertainty introduced above.

Problem. Find a general treatment to characterize the computational complexity of logics of uncertainty.

Appendix A

Further Results on Uninorms

In this appendix we focus on operators called uninorms. Such functions are interesting since they generalize both t-norms and t-conorms. In particular, they have been recently used to define logics more general than logics based on t-norms (see the work by Metcalfe in [109], by Metcalfe and Montagna in [110], and by Gabbay and Metcalfe in [63]). Therefore, we naturally aim at extending the previous analysis of functional definability, carried out in Chapter 5, to uninorms as well.

This appendix is structured as follows. In the first and in the second section we review the basic notions concerning uninorms and uninorm based logics, respectively. In Section A.3, we study definability of some classes of uninorms, while in Section A.4 we show that the complexity and decidability results proved in Chapter 5 can be easily extended to logics based on uninorms. In Section A.5 we prove that the Uninorm Mingle Logic UML and the Basic Uninorm Logic BUL are finitely strongly standard complete w.r.t. to the class of definable left-continuous conjunctive idempotent uninorms and w.r.t. to the class of definable uninorms continuous on $[0, 1)$, respectively. We end with some open problems.

A.1 Uninorms

A *uninorm* $*_u$ is a binary commutative and associative aggregation operation having a neutral element $e \in [0, 1]$. As a class of operators, uninorms were introduced by Yager and Rybalov in [144], but examples of uninorms were known even before, like the *Cross Ratio Uninorm* introduced by Silvert in [133]:

$$x *_u y = \begin{cases} \frac{xy}{xy+(1-x)(1-y)} & (x, y) \in [0, 1]^2 \backslash \{(0, 1), (1, 0)\} \\ 0 & \text{otherwise} \end{cases} .$$

Uninorms clearly generalize both t-norms and t-conorms, that can be seen as extremal examples of uninorms where the identity element e coincides with

1 and 0, respectively. Indeed, each uninorm $*_u$ behaves like a t-norm over $[0, e]$, like a t-conorm over $[e, 1]$, and $\min(x, y) \leq x *_u y \leq \max(x, y)$ if $x \leq e \leq y$ or $y \leq e \leq x$ (see [61]). Notice that a uninorm which is continuous necessarily is either a t-norm or a t-conorm, hence, in a certain sense, there is no continuous uninorm.

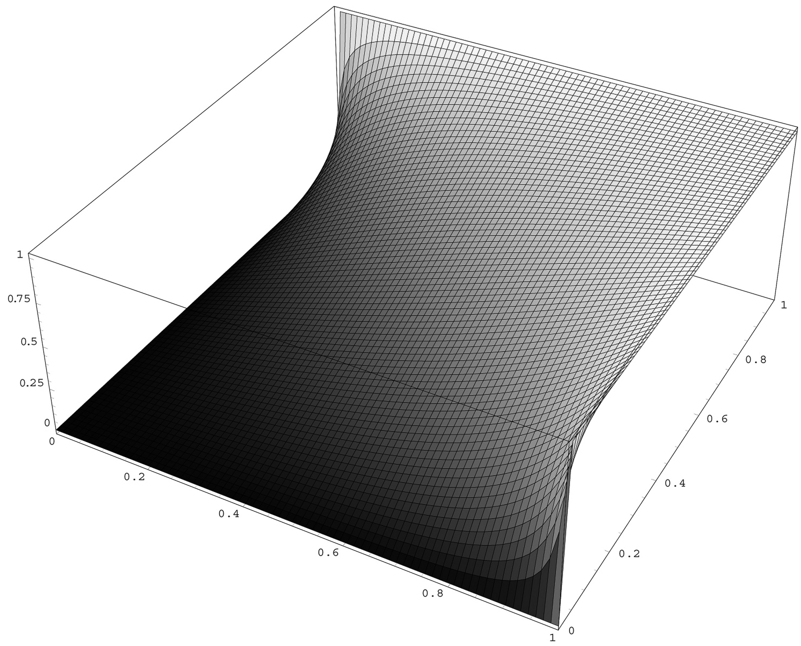

Figure A.1: The Cross Ratio uninorm.

It is easy to see that for each uninorm $*_u$, $(0 *_u 1)$ is a null element, and in particular $(0 *_u 1) \in \{0, 1\}$. Whenever $0 *_u 1 = 0$ we call $*_u$ a *conjunctive* uninorm, while if $0 *_u 1 = 1$, $*_u$ is said to be a *disjunctive* uninorm. Here, we deal with conjunctive uninorms only.

Notice that a uninorm $*_u$ admits a residual implication $\Rightarrow_{*_u}$(and hence it is called *residuated*) iff it is conjunctive and left-continuous (see [63]). In that case $\langle [0, 1], *_u, \Rightarrow_{*_u}, \leq, e, f, 0, 1 \rangle$ is a commutative bounded pointed residuated lattice (see below and [141]).

In the following we recall the basic properties of some remarkable classes of conjunctive uninorms.

Conjunctive uninorms where $x \mapsto x *_u 1$ is continuous on $[0, e[$ can be seen as an ordinal sum of a t-norm and a t-conorm, as shown in the following theorem. The class of such uninorms is denoted by $\mathcal{U}_{\min}$.

Theorem A.1.1 ([61]) *A binary operator $*_u$ is a conjunctive uninorm with neutral element $e \in]0, 1]$ such that $x \mapsto x *_u 1$ is continuous on $[0, e[$ iff there* *exist a t-norm $*$ and a t-conorm $\diamond$ such that*

$$x *_u y = \begin{cases} e \cdot \left(\frac{x}{e} * \frac{y}{e}\right) & \text{if } x, y \in [0, e]^2 \\ e + (1 - e) \cdot \left(\frac{x-e}{1-e} \diamond \frac{y-e}{1-e}\right) & \text{if } x, y \in [e, 1]^2 \\ \min(x, y) & \text{otherwise} \end{cases}.$$

Another remarkable class is given by *representable uninorms*, i.e. uninorms that can be represented by means of a one-variable function $h : [0, 1] \to \overline{\mathbb{R}}$, with $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, $h(0) = -\infty$, $h(e) = 0$, and $h(1) = +\infty$ such that:

$$x *_u y = h^{-1}(h(x) + h(y)).$$

These uninorms are called *almost-continuous* being continuous on $(0, 1)$.

Theorem A.1.2 ([61, 129]) *Given a uninorm $*_u$ with neutral element $e \in (0, 1)$, the following are equivalent:*

(*i*) *$*_u$ is representable,*

(*ii*) *$*_u$ is strictly increasing and continuous on $(0, 1)$.*

Any two conjunctive representable uninorms are order isomorphic, and in particular they are isomorphic to the Cross Ratio uninorm.

A special kind of representable uninorms is that of *rational uninorms* (studied by Fodor [60]), i.e. uninorms which can be represented in the following form

$$x *_u y = \frac{P_n(x,y)}{P_m(x,y)},$$

where $P_n(x, y)$, and $P_m(x, y)$ are polynomials of order n and m, respectively.

Theorem A.1.3 ([60]) *Rational uninorms are given by the following parametric form, for $x, y \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\}$, and $e \in]0, 1[$*

$$x *_u y = \frac{(1-e)xy}{(1-e)xy + e(1-x)(1-y)}.$$

$[0, 1)$-*continuous uninorms*, i.e. uninorms continuous on the whole right-open unit interval, where studied in [86] by Hu and Li. This class of uninorms enjoys the following representation theorem (see also [63]).

Theorem A.1.4 ([86]) *For a conjunctive uninorm $*_u$, the following are equivalent:*

(*i*) *For some continuous t-norm $*$, $u \in [0, e]$, and strictly increasing function $h : [u, 1] \to \overline{\mathbb{R}}$:*

$$x *_u y = \begin{cases} e \cdot \left(\frac{x}{e} * \frac{y}{e}\right) & x, y \in [0, u] \\ h^{-1}(h(x) + h(y)) & x, y \in [u, 1] \\ \min(x, y) & \text{otherwise} \end{cases},$$

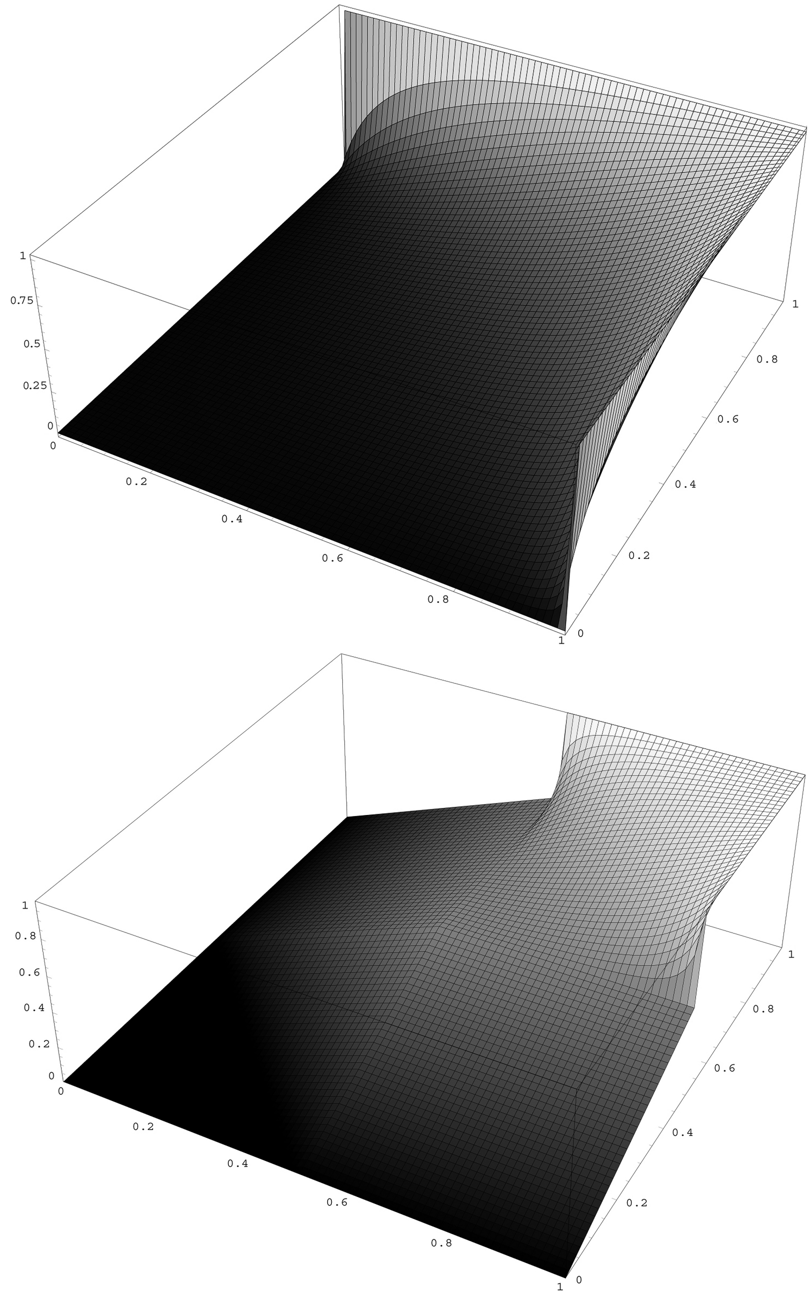

Figure A.2: A rational uninorm with identity element $e = \frac{8}{9}$, and a $[0, 1)$-continuous uninorm obtained from the Łukasiewicz t-norm and the Cross Ratio uninorm.

(*ii*) $*_u$ *is continuous on* $[0,1)$ *and* $e \in (0,1)$.

It follows that every conjunctive $[0,1)$-continuous uninorm is left-continuous and can be represented as an ordinal sum of (0 or 1 each) isomorphic copies of a continuous t-norm and the Cross Ratio uninorm.

To conclude we mention *idempotent uninorms* (see [36]) which form a special class of uninorms where for all $x \in [0,1]$, $x *_u x = x$. Those operators generalize both idempotent t-norms and t-conorms. A typical example is given by

$$x *_u y = \begin{cases} \max(x,y) & x, y \in [e,1]^2 \\ \min(x,y) & \text{otherwise} \end{cases} .$$

Left-continuous idempotent uninorms have been investigated by De Baets in [36], where their structure is characterized w.r.t. a quasi-weak negation.

Theorem A.1.5 ([36]) *A binary operator* $*_u$ *is a conjunctive left-continuous idempotent uninorm with neutral element* $e \in]0,1]$ *iff there exists a quasi-weak negation* g *with fixpoint* e*, such that* $*_u$ *is given by*

$$x *_u y = \begin{cases} \min(x,y) & y \leq g(x) \\ \max(x,y) & \text{otherwise} \end{cases} .$$

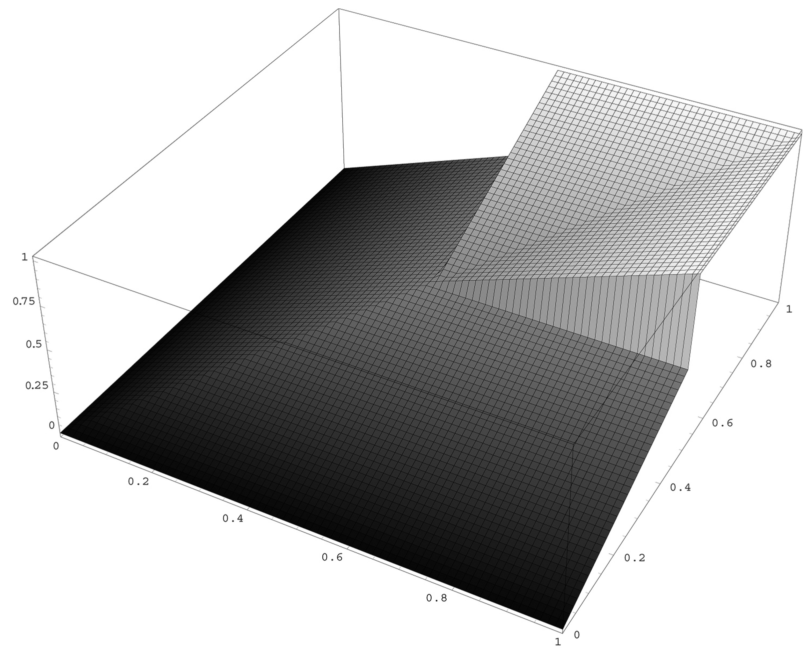

Figure A.3: An idempotent uninorm with identity element $e = \frac{1}{2}$.

A.2 Logics based on uninorms

Logics based on uninorms were first introduced by Metcalfe in [109], and further studied by Metcalfe and Montagna in [110] and by Gabbay and Metcalfe in [63]. They are clearly more general than t-norm based logics, and they can be seen as substructural logics (see [124]), since they lack the Contraction axiom $\varphi \to \varphi \& \varphi$ and the Weakening axiom $\varphi \to (\psi \to \varphi)$.

The language of the Uninorm based Logic UL includes the binary connectives $\&, \to, \wedge, \vee$ and the constants $\overline{0}, \overline{1}, t, f$. Other definable connectives are:

$$\neg\varphi \text{ is } \varphi \to f, \qquad \varphi \leftrightarrow \psi \text{ is } (\varphi \to \psi) \wedge (\psi \to \varphi).$$

The axiomatic system for UL is given by the Hilbert-style calculus defined by the following axiom schemata:

(U1) $\varphi \to \varphi$
(U2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(U3) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(U4) $((\varphi \& \psi) \to \chi) \leftrightarrow (\varphi \to (\psi \to \chi))$
(U5) $(\varphi \wedge \psi) \to \varphi$
(U6) $(\varphi \wedge \psi) \to \psi$
(U7) $((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$
(U8) $\varphi \to (\varphi \vee \psi)$
(U9) $\psi \to (\varphi \vee \psi)$
(U10) $((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$
(U11) $\varphi \leftrightarrow (t \to \varphi)$
(U12) $\overline{0} \to \varphi$
(U13) $\varphi \to \overline{1}$
(U14) $((\varphi \to \psi) \wedge t) \vee ((\psi \to \varphi) \wedge t)$

The inference rules of UL are Modus Ponens (MP) and Adjunction (Adj):

(MP) from φ and $\varphi \to \psi$, derive ψ,

(Adj) from φ and ψ, derive $\varphi \wedge \psi$.

The algebraic semantics for the logic UL is given by prelinear pointed bounded commutative residuated lattices (see [110]).

Definition A.2.1 A pointed bounded commutative residuated lattice (see [141]) is a structure $\mathcal{A} = \langle A, \sqcap, \sqcup, *, \Rightarrow, t, f, 0, 1 \rangle$ such that

- $\langle A, \sqcap, \sqcup, 0, 1 \rangle$ is a bounded lattice with top element 1 and bottom element 0.

- $\langle A, *, t \rangle$ is a commutative monoid.

- the operations $*$ and $\Rightarrow$ form an *adjoint pair*:

$$x * y \leq z \text{ iff } x \leq y \Rightarrow z,$$

A UL-algebra is a pointed bounded commutative residuated lattice satisfying the *prelinearity* condition

$$t \leq ((x \Rightarrow y) \sqcap t) \sqcup ((y \Rightarrow x) \sqcap t).$$

UL standard algebras are structures $\langle [0,1], *, \Rightarrow, \min, \max, e, f, 0, 1\rangle$, where $*$ is a left-continuous conjunctive uninorm, $\Rightarrow$ its residuum, e its neutral element and $f \in [0,1]$.

The notion of evaluation is obviously defined. Given an UL-algebra $\mathcal{A}$, an $\mathcal{A}$-tautology is a formula φ such that $v(\varphi) \geq t$ for all $\mathcal{A}$-evaluations v. Given a theory Γ, an $\mathcal{A}$-evaluation v is a model of Γ if $v(\gamma) \geq t$ for all $\gamma \in \Gamma$.

UL is complete w.r.t. the variety of prelinear pointed bounded commutative residuated lattices. Moreover, being each UL-algebra a subdirect product of linearly ordered UL-algebras, UL clearly is complete w.r.t. to UL-chains.

The Involutive Uninorm Logic IUL [110] is obtained from UL by adding the axiom schema of *involution*

(Inv) $\neg\neg\varphi \rightarrow \varphi$.

The variety of IUL-algebras is similarly defined by requiring UL-algebras to satisfy

$$\neg\neg x = x. \tag{A.1}$$

The Uninorm Mingle Logic UML [110] is obtained from UL by adding the axiom schema of Contraction

(Con) $\varphi \rightarrow \varphi \& \varphi$.

A UML-*algebra* is a UL-algebra satisfying the idempotence law:

$$x * x = x. \tag{A.2}$$

The Involutive Uninorm Mingle Logic IUML [110] is IUL plus (Con) and $f \leftrightarrow t$, and an IUML-algebra is an IUL-algebra satisfying (A.2) and $f = t$. A standard IUML-algebra is a structure

$$\langle [0,1], *, \Rightarrow, \min, \max, \tfrac{1}{2}, \tfrac{1}{2}, 0, 1\rangle,$$

where $*$ is the following idempotent conjunctive uninorm:

$$x * y = \begin{cases} \max(x,y) & x + y > 1 \\ \min(x,y) & \text{otherwise} \end{cases}.$$

The Basic Uninorm Logic BUL (see [63]) is the logic of uninorms continuous on $[0,1)$, and is obtained by adding to UL the restricted divisibility axiom

[RDiv] $(\overline{1} \rightarrow \varphi) \vee (\varphi \rightarrow (\psi \wedge u)) \wedge (\psi \rightarrow (\varphi \& (\varphi \rightarrow \psi)))$, and

[U] $u \leftrightarrow (u \& u)$,

where u is $\overline{1} \to t$. The variety of BUL-algebras is constituted by UL-algebras satisfying the restricted divisibility condition

$$t \leq (1 \Rightarrow x) \sqcup (x \Rightarrow (y \sqcap u)) \sqcup (y \Rightarrow (x * (x \Rightarrow y))),$$

and $u * u = u$, where u is an abbreviation for $1 \Rightarrow t$. Standard BUL-algebras are structures where the monoidal operation $*$ is a $[0,1)$-continuous residuated uninorm.

The Involutive Basic Uninorm Logic IBUL [63] and the related variety of IBUL-algebras are defined by adding to BUL and to BUL-algebras, respectively, (Inv) and (A.1).

The Cross Ratio Logic CRL is an extension of IBUL by means of $f \leftrightarrow t$. A CRL-algebra is an IBUL-algebra satisfying $f = t$. CRL-algebras are obviously based of the Cross Ratio Uninorm.

The Cancellative Basic Uninorm Logic CBUL [63] is defined by adding to BUL the restricted cancellation axiom:

$$\text{[RCan]}\ ((\varphi \to \overline{0}) \vee (\overline{1} \to \varphi)) \vee ((\varphi \to (\varphi \& \psi)) \to \psi).$$

The variety of CBUL-algebras is constituted by BUL-algebras satisfying the restricted cancellation condition:

$$t \leq ((x \Rightarrow 0) \sqcup (1 \Rightarrow x)) \sqcup ((x \Rightarrow (x * y)) \Rightarrow y). \tag{A.3}$$

Notice that MTL can be obtained as an extension of UL by adding the weakening axiom $\varphi \to (\psi \to \varphi)$.

All the above logics enjoy the following completeness theorem.

Theorem A.2.2 ([110]) *Let $\mathcal{L}$ be a schematic extension of* UL. *Let Γ be a theory over $\mathcal{L}$, and φ be a formula. Then the following are equivalent:*

1. $\Gamma \vdash_{\mathcal{L}} \varphi$,
2. *for each $\mathcal{L}$-chain $\mathcal{A}$ and each $\mathcal{A}$-model v of Γ, $v(\varphi) \geq t$,*
3. *for each $\mathcal{L}$-algebra $\mathcal{A}$ and each $\mathcal{A}$-model v of Γ, $v(\varphi) \geq t$.*

In [110], Metcalfe and Montagna proved standard completeness for UL, UML, and IUML by exploiting proof-theoretic techniques, while in [63] Gabbay and Metcalfe proved standard completeness for BUL, IBUL, CBUL, and CRL by exploiting a connection with linearly ordered Abelian groups.

Theorem A.2.3 ([110, 63]) *Let $\mathcal{L}$ be any logic among* UL, UML, IUML, BUL, IBUL, CBUL, *and* CRL. *Let Γ be a theory over $\mathcal{L}$, and φ be a formula. Then the following are equivalent:*

1. $\Gamma \vdash_{\mathcal{L}} \varphi$,
2. *for every $\mathcal{L}$-standard algebra $\mathcal{A}$ and every $\mathcal{A}$-model v of Γ, $v(\varphi) \geq t$.*

Standard completeness for IUL still is an open problem.

A.3 Definability of uninorms

We now focus on uninorms and try to characterize their definability. Some of the results presented below easily follow from the characterization of definability of t-norms given in Chapter 5.

Theorem A.3.1 *Suppose that a left-continuous conjunctive uninorm $*_u$ is implicitly definable. Then every subset of $[0,1]^n$ that is first-order definable (without parameters) in the language $\{*_u, +, \cdot, \leq, 0, 1\}$ is $\mathbb{Q}$-semialgebraic. In particular:*

- (a) *The set of idempotent elements of $*_u$ is $\mathbb{Q}$-semialgebraic. If the underlying t-norm $*$ (t-conorm $\diamond$, resp.) is an ordinal sum of infinitely many t-norms (t-conorms, resp.) then all of them but a finite number are isomorphic to the Gödel t-norm (to the maximum t-conorm, resp.).*

- (b) *For any constant $r \in [0,1]$, the operation $\neg_r(x) = x \Rightarrow_{*_u} r$ is a quasi-weak negation such that $[0,1]$ can be partitioned into a finite number of intervals $I_0, ..., I_m$ so that, in each I_i, $\neg_r$ is continuous and either constant or strictly increasing. In particular, $\neg_r$ has only finitely many discontinuity points.*

Proof.

- (a) This follows by an easy adaptation of the argument in Theorem 5.1.3(b).

- (b) Take any constant $r \in [0,1]$. From the properties of the residuum it is easy to see that $0 \Rightarrow_{*_u} r = 1$, $x \leq (x \Rightarrow_{*_u} r) \Rightarrow_{*_u} r$, and that if $x \leq y$, then $y \Rightarrow_{*_u} r \leq x \Rightarrow_{*_u} r$. This means that $\neg_r(x) = x \Rightarrow_{*_u} r$ is a quasi-weak negation.

  Now, the residuum of $*_u$ is implicitly definable. Consequently $\neg_r$ is implicitly definable as well. By Theorem 4.3.9 it follows that its graph is $\mathbb{Q}$-semialgebraic. Now, by reasoning as in the case of weak negations in Theorem 5.1.3, it is easily seen that the set of discontinuities is finite, and that $[0,1]$ can be partitioned in finitely many subintervals in which $\neg_r$ is either involutive or constant.

■

Recall now that a conjunctive uninorm belonging to $\mathcal{U}_{\min}$ and having e as a neutral element can be represented as an ordinal sum having a t-norm and a t-conorm as summands defined over $[0,e]$ and $[e,1]$, respectively. Take then a term-definable t-norm $*$, a term-definable t-conorm $\diamond$, and let e be rational. The term ℓ^* defines the linear transformation of $*$ into $[0,e]$

$$\ell^*(x,y) = e \cdot [(e \Rightarrow_\pi x) * (e \Rightarrow_\pi y)],$$

and the term $\ell^\diamond$ is the linear transformation of $\diamond$ into $[e,1]$

$$\ell^\diamond(x,y) = e \oplus (\neg e) \cdot [(\neg e \Rightarrow_\pi (x \ominus e)) \diamond (\neg e \Rightarrow_\pi (y \ominus e))].$$

Then, we can define the term corresponding to a conjunctive uninorm:

$$u^*(x,y) = [\ell^*(x,y) \sqcap \delta((x \sqcup y) \Rightarrow e)] \sqcup [\ell^\diamond(x,y) \sqcap \delta(e \Rightarrow (x \sqcap y)))] \sqcup \\ [(x \sqcap y) \sqcap \neg\delta((x \sqcup y) \Rightarrow e) \sqcap \neg\delta(e \Rightarrow (x \sqcap y))].$$

From the previous construction we immediately obtain:

Proposition A.3.2 *Let $*$ and $\diamond$ be a term-definable t-norm and a term-definable t-conorm, respectively. Then the uninorm obtained as ordinal sum of $*$ and $\diamond$ (belonging to $\mathcal{U}_{\min}$) is term-definable (up to isomorphism).*

It is now easy to prove the following theorem.

Theorem A.3.3 *Let $*_u$ be any uninorm belonging to the class of conjunctive uninorms $\mathcal{U}_{\min}$. Then*

i. *$*_u$ is term-definable (up to isomorphism) iff the underlying t-norm and t-conorm are term-definable.*

ii. *$*_u$ is implicitly definable (up to isomorphism) iff the underlying t-norm and t-conorm are implicitly definable.*

Proof. To prove (i), just notice that the left-to-right direction is obvious, while the right-to-left direction corresponds to Proposition A.3.2.

Now, if both the underlying t-norm and t-conorm are implicitly definable, then it is easy to see from Theorem A.1.1 that the graph of $*_u$ is definable in the reals, and so, by Theorem 4.3.9 it is $\mathbb{Q}$-semialgebraic. Therefore, $*_u$ is implicitly definable. The converse is obvious. ∎

By Theorem A.3.1 and Theorem A.3.3, we immediately obtain:

Corollary A.3.4 *Let $*_u$ be any uninorm belonging to the class of conjunctive uninorms $\mathcal{U}_{\min}$, such that the underlying t-norm and t-conorm are continuous. Then, the following are equivalent:*

i. *Up to isomorphism, $*_u$ is implicitly definable in $\text{Ł}\Pi\frac{1}{2}$.*

ii. *Up to isomorphism, $*_u$ is term-definable in $\text{Ł}\Pi\frac{1}{2}$.*

iii. *$*_u$ is representable as a finite ordinal sum.*

As for representable uninorms, notice that there are examples of uninorms which are not definable, like for instance:

$$x *_u y = \begin{cases} 1 - \exp\left(-\frac{1}{2}\log(1-x)\cdot\log(1-y)\right) & x,y \in [0,1]\backslash\{(0,1),(1,0)\} \\ 0 & \text{otherwise} \end{cases}.$$

However, each member of the class of representable uninorms is order-isomorphic to the Cross Ratio uninorm, which, being piecewise rational, is definable by the following term:

$$u_c(x,y) = ((x \cdot y) \Rightarrow_\pi ((x \cdot y) \oplus (\neg x \cdot \neg y))) \sqcap \neg\delta(x \sqcup \neg x) \sqcap \neg\delta(y \sqcup \neg y).$$

Hence we can state the following result:

Theorem A.3.5 *Representable uninorms are term-definable up to isomorphism.*

As for representable uninorms which are rational uninorms we can easily obtain a complete characterization. Indeed, recall that such operators all have this parametric form, depending on $e \in]0,1[$:

$$x *_u y = \frac{(1-e)xy}{(1-e)xy + e(1-x)(1-y)}.$$

It can be immediately seen that whenever e is rational, a rational uninorm exactly is a piecewise rational function and hence it is term-definable in $Ł\Pi\frac{1}{2}$.

Theorem A.3.6 *Every conjunctive rational uninorm having a rational idempotent is term-definable.*

As for $[0,1)$-continuous uninorms, recall that every $[0,1)$-continuous uninorm is order isomorphic to the ordinal sum of the a continuous t-norm and the Cross Ratio uninorm. Then we have:

Theorem A.3.7 *Let $*_u$ be a uninorm continuous on $[0,1)$. The following are equivalent:*

i. *Up to isomorphism, $*_u$ is implicitly definable in $Ł\Pi\frac{1}{2}$.*

ii. *Up to isomorphism, $*_u$ is term-definable in $Ł\Pi\frac{1}{2}$.*

iii. *$*$ is representable as a finite ordinal sum.*

Proof. We prove (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (i) is trivial; (i) $\Rightarrow$ (iii) follows by Theorem A.3.1 and the fact that every uninorm continuous on $[0,1)$ can be seen as an ordinal sum of a continuous t-norm followed by the Cross Ratio uninorm. To prove (iii) $\Rightarrow$ (ii) note that if $*_u$ is representable as a finite ordinal sum it is definable by the term

$$u(x,y)_{[0,1)} = [\ell^* \sqcap \delta((x \sqcup y \Rightarrow s))] \sqcup [\ell^{*_c} \sqcap \delta(s \Rightarrow (x \sqcap y))] \sqcup \\ [(x \sqcap y) \sqcap \neg\delta((x \sqcup y \Rightarrow s)) \sqcap \neg\delta(s \Rightarrow (x \sqcap y))],$$

where ℓ^{*_c} is the linear transformation of the Cross Ratio uninorm in $[s,1]$, and ℓ^* is the linear transformation of a continuous t-norm (with finitely many components, and consequently term-definable by Theorem 5.2.3) in $[0,s]$ (being e and s rationals). ■

Finally, as for left-continuous idempotent conjunctive uninorms, recall the representation theorem given above (i.e. Theorem A.1.5). Let g be a term-definable quasi-weak negation. Then, the representation of those operators is given by the following term:

$$u^*_{id}(x,y) = [(x \sqcap y) \sqcap \delta(y \Rightarrow g(x))] \sqcup [(x \sqcup y) \sqcap \neg\delta(y \Rightarrow g(x))].$$

We then obtain the following result.

Theorem A.3.8 *Let $*_u$ be a conjunctive idempotent uninorm with identity element $e \in [0,1]$. The following are equivalent:*

i. *Up to isomorphism, $*_u$ is implicitly definable in $Ł\Pi\frac{1}{2}$.*

ii. *Up to isomorphism, $*_u$ is term-definable in $Ł\Pi\frac{1}{2}$.*

iii. *The function $\neg_e x = x \Rightarrow_{*_u} e$ is a quasi-weak negation with a finite number of discontinuity points.*

Proof. We prove (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (i) is trivial, while (i) $\Rightarrow$ (iii) follows from Theorem A.3.1(b). Then we prove (iii) $\Rightarrow$ (ii).

As seen in Chapter 1, if a quasi-weak negation g has finitely many discontinuity points, then $[0,1]$ can be divided into finitely many intervals $I_1 = [0 = a_0, a_1]$, $I_i = (a_{i=1}, a_i]$, $I_{r+1} = (a_r, a_{r+1} = 1]$, such that g is either continuous and involutive or constant on each I_i. Then, proceed exactly as in Theorem 5.1.5 recalling that if $g(x) = 1$ in I_1, then it behaves like a weak negation in $[a_1, 1]$. We can then define a quasi-weak negation g' isomorphic to g which is term-definable in $Ł\Pi\frac{1}{2}$. Hence, we obtain an idempotent uninorm $*'_u$, that is defined by the term $[(x \sqcap y) \sqcap \delta(y \Rightarrow g'(x))] \sqcup [(x \sqcup y) \sqcap \neg\delta(y \Rightarrow g'(x))]$, as shown above.

To see that the $*'_u$ is isomorphic to $*_u$, just recall that, as shown in [37] for each conjunctive idempotent uninorm $*_u$ defined by a quasi-weak negation g, the residuum is given by

$$x \Rightarrow_{*_u} y = \begin{cases} \max(g(x), y) & \text{if } x \leq y \\ \min(g(x), y) & \textit{otherwise} \end{cases}.$$

Hence, $\neg_e x = x \Rightarrow_{*_u} e = g(x)$, meaning that the quasi-weak negation defined as $\neg_e x = x \Rightarrow_{*_u} e$ exactly coincides with the quasi-weak negation used in the construction of the uninorm. Being g and g' isomorphic, the claim immediately follows. ■

A.4 Decidability and complexity

In this section we investigate the complexity of logics associated to left-continuous conjunctive uninorms implicitly definable in $Ł\Pi\frac{1}{2}$. The below results generalize those given in Chapter 5 for left-continuous t-norms. We start from the following theorem.

Theorem A.4.1 *If a left-continuous conjunctive uninorm $*$ is implicitly definable in $Ł\Pi\frac{1}{2}$, then $\mathcal{L}_*$ is in* PSPACE.

Proof. The proof exactly coincides with the one given in Theorem 5.5.1. ■

Theorem A.4.2 *Let $\mathcal{K}$ be a class of left-continuous conjunctive uninorms implicitly definable in* $\text{Ł}\Pi\frac{1}{2}$ *and let $\mathcal{L}_{\mathcal{K}}$ be its associated logic. If $\mathcal{L}_{\mathcal{K}}$ is finitely axiomatizable, then it is decidable.*

Proof. Under the above assumption, $\mathcal{L}_*$ is recursively enumerable, hence it is sufficient to show that the complement of $\mathcal{L}_*$ is recursively enumerable. For every formula $\Phi(x, y, z)$ in the language of $\mathcal{R}$, let $U(\Phi)$ denote the conjunction of the following formulas (expressing that the set defined by Φ is the graph of a left-continuous conjunctive uninorm; being left-continuous and conjunctive is equivalent to the existence of a residuum, which is expressed by (U_5); the existence of a neutral element is encoded in (U_3)):

(U_1) $\forall x\forall y\forall z\forall u\forall v\forall w(\Phi(x, y, z) \wedge \Phi(z, u, v) \wedge \Phi(y, u, w) \rightharpoondown \Phi(x, w, v))$.

(U_2) $\forall x\forall y\forall z(\Phi(x, y, z) \rightharpoondown \Phi(y, x, z))$.

(U_3) $\exists t\forall x(\Phi(x, t, x))$.

(U_4) $\forall x\forall y\forall z((\Phi(x, y, z) \wedge \Phi(x, u, w) \wedge y \leq u) \rightharpoondown z \leq w)$.

(U_5) $\forall x\forall y\exists z\forall u\forall v(\Phi(x, u, v) \rightharpoondown (u \leq z \rightleftharpoons v \leq y))$.

Now, for every axiom ψ of $\mathcal{L}_*$ (finitely many!) consider the formula $\psi(\Phi)$ defined as in the proof of the previous theorem (see Theorem 5.5.1). We have that $\mathcal{L}_* \not\vdash \varphi$ iff there is a formula $\Phi(x, y, z)$ such that for every axiom ψ of $\mathcal{L}_*$, the formulas $\psi(\Phi)$ and $U(\Phi)$ are true in $\mathcal{R}$, but $\varphi(\Phi)$ is false in $\mathcal{R}$. Since truth in $\mathcal{R}$ is decidable, the claim is proved. ∎

A.5 Completeness

In Section 5.4, we showed that several t-norm based logics are finitely strongly standard complete w.r.t. their related class of term-definable t-norms. It is then interesting to study the case of uninorm based logics. Here we focus on the Uninorm Mingle Logic UML and on the Basic Uninorm Logic BUL.

We begin with UML, and first give some preliminary notions and results. In the following, C will denote any chain bounded by 0 and 1. The concept of quasi-weak negation is generalized to operators over C.

Definition A.5.1 A quasi-weak negation over C is any operator $g : C \to C$ such that $g(0) = 1$, $x \leq g(g(x))$, and if $x \leq y$, then $g(y) \leq g(x)$, for all $x, y \in C$.

Proposition A.5.2 *Let g be a quasi-weak negation on C. Then,*

i. $g(x) = g(g(g(x)))$, *for all* $x \in C$.

ii. $x \leq g(y)$ *iff* $y \leq g(x)$, *for all* $x, y \in C$.

iii. *g is a quasi-weak negation on C iff either $g(x) = 1$ for all $x \in [0, 1]$, or g is a weak negation over $[c, 1]$, where $c = \sup\{x : g(x) = 1\}$.*

Proposition A.5.3 *For any quasi-weak negation g with fixed point t on a bounded chain C, we can define a residuated pair of operations $*_g, \Rightarrow_g$, such that $\langle C, *_g, \Rightarrow_g, \min, \max, t, f, 0, 1\rangle$ is an* UL*-chain where $x \Rightarrow_g t = g(x)$ for all $x \in C$.*

Proof. Given a quasi-negation g with fixed point t, define

$$x *_g y = \begin{cases} \min(x, y) & \text{if } y \leq g(x) \\ \max(x, y) & \text{otherwise} \end{cases}.$$

Then we can prove the following:

- the residuum of $*_g$ is defined by

$$x \Rightarrow_g y = \begin{cases} \max(g(x), y) & \text{if } x \leq y \\ \min(g(x), y) & \text{otherwise} \end{cases}.$$

- $\langle C, *_g, \Rightarrow_g, \min, \max, t, f, 0, 1\rangle$ is an UL-algebra
- $\neg_t(x) = x \Rightarrow_g t$ is a weak negation such that $\neg_t(x) = g(x)$ for all $x \in C$.

■

Theorem A.5.4 UML *is finitely strongly standard complete w.r.t. the class of* UML-*algebras based on a term-definable conjunctive idempotent uninorm.*

Proof. Suppose that $\Gamma \not\vdash_{\text{UML}} \varphi$. Then we know that there are a totally ordered UML-algebra $\mathcal{A}$ and an $\mathcal{A}$-evaluation v such that $v(\psi) \geq t$ for all $\psi \in \Gamma$ and $v(\varphi) < t$. Let X be the finite set of all values of all subformulas γ of $\Gamma \cup \{\varphi\}$, plus the values given by $\neg_t\gamma$ and $\neg_t\neg_t\gamma$, under v, plus $0, 1, t,$ and f (where $\neg_t x$ clearly corresponds to $x \Rightarrow t$, and t is the fixed point). Let

$$X \cap \neg_t(A) = \{a_0 < \cdots < a_m = 1\},$$

where $\neg_t(A)$ is the image of the universe A of $\mathcal{A}$ under $\neg_t$.

Now, if $a_0 = 0$ (this means that 0 is an involutive element) let $h : X \to [0, 1]$ be the order-preserving mapping such that $h(a_i) = \frac{i}{m}$. Then, we define the following operation:

$$g(x) = \begin{cases} 1 - x & \text{if } x \in \{\frac{i}{m}\} \cup \left(\bigcup\limits_{I \in \mathcal{I}} I\right) \\ \frac{m-i-1}{m} & \text{if } x \in \left(\frac{i}{m}, \frac{i+1}{m}\right) \text{ and } \left[\frac{i}{m}, \frac{i+1}{m}\right] \notin \mathcal{I} \end{cases},$$

where

$$\mathcal{I} = \left\{\left[\tfrac{i}{m}, \tfrac{i+1}{m}\right] \mid \left(\left(\tfrac{i}{m}, \tfrac{i+1}{m}\right) \cup \left(\tfrac{m-i-1}{m}, \tfrac{m-i}{m}\right)\right) \cap h(X) = \emptyset\right\},$$

with $0 \leq i \leq m - 1$.

It is easy to see that g is a quasi-weak negation over $[0, 1]$ (more precisely it is a weak negation).

If $a_0 \neq 0$ (this means that 0 is not an involutive element) let $h' : X \to [0, 1]$ be the order-preserving mapping such that $h'(a_i) = \frac{i+1}{m+1}$.
Then, we define the following operation:

$$g'(x) = \begin{cases} 1 & \text{if } x \leq \frac{1}{m+1} \\ \frac{m+2}{m+1} - x & \text{if } x \in \{\frac{i+1}{m+1}\} \cup \left(\bigcup_{I \in \mathcal{I}} I\right) \\ \frac{m-i}{m+1} & \text{if } x \in \left(\frac{i+1}{m+1}, \frac{i+2}{m+1}\right) \text{ and } \left[\frac{i+1}{m+1}, \frac{i+2}{m+1}\right] \notin \mathcal{I}' \end{cases},$$

where

$$\mathcal{I}' = \left\{\left[\frac{i+1}{m+1}, \frac{i+2}{m+1}\right] \mid \left(\left(\frac{i+1}{m+1}, \frac{i+2}{m+1}\right) \cup \left(\frac{m-i}{m+1}, \frac{m-i+1}{m+1}\right)\right) \cap h'(X) = \emptyset\right\},$$

with $0 \leq i \leq m-1$.

It is easy to see that g' is a quasi-weak negation over $[0,1]$.

Now, let either $q = g$ and $\mu = h$ or $q = g'$ and $\mu = h'$. Define now an idempotent conjunctive uninorm $*_u$ from q. Clearly, μ becomes a morphism from $\mathcal{A}$ into $\langle [0,1], *_u, \Rightarrow_{*_u}, \min, \max, 0, t', f', 1 \rangle$, where t' and f' are the images under μ of t and f, respectively. Thus, we can define an evaluation $w = \mu \circ v$, such that $w(\gamma) \geq t'$ for all $\gamma \in \Gamma$ and $w(\varphi) < t'$.

To conclude the proof, notice that the quasi-weak negation g clearly is term-definable. Hence, the claim follows. ■

We now focus on BUL. Recall that every $[0,1)$-continuous residuated uninorm is an ordinal sum of (0 or 1 each) isomorphic copies of the Cross Ratio uninorm and a continuous t-norm. The Cross Ratio uninorm is definable by the following term:

$$((x \cdot y) \Rightarrow_\pi ((x \cdot y) \oplus (\neg x \cdot \neg y))) \sqcap \neg\delta(x \sqcup \neg x) \sqcap \neg\delta(y \sqcup \neg y).$$

Furthermore, as shown above, a continuous t-norm is definable iff it is representable as a finite ordinal sum. Consequently:

Theorem A.5.5 *The logic* BUL *is finitely strongly standard complete w.r.t. the class of definable* $[0,1)$*-continuous uninorms.*

Proof. Let Γ be a finite set of sentences, and suppose that ϕ cannot be derived from Γ in BUL. Then by [63] there are a BUL-algebra $\mathcal{A}$ and an $\mathcal{A}$-evaluation v such that $v(\psi) \geq e$ for all $\psi \in \Gamma$ and $v(\phi) < e$. Now, let X be the finite set of all values under v of all subformulas occurring in $\Gamma \cup \{\phi\}$. X can be partially embedded into a standard BUL-algebra with a finite number of components $C_0, ..., C_k$. Indeed, for each operation $\circ$, and for all $x, y \neq 1, t$, if x and y belong to C_i, C_j, respectively, then $x \circ y$ belongs either to C_i or C_j. Such a BUL-algebra is a finite ordinal sum of k Łukasiewicz components followed by the Cross Ratio uninorm. Clearly, both the Łukasiewicz t-norm and the Cross Ratio uninorm are definable. Since $Ł\Pi\frac{1}{2}$-definability is preserved (up to isomorphism) under finite ordinal sums, as proved in Proposition 5.2.2, the claim easily follows. ■

Corollary A.5.6 UML *and* BUL *are decidable and* BUL *is in* PSPACE.

Proof. The results of decidability immediately follows from Theorem A.4.2. The fact that BUL is in PSPACE is easy to check. Indeed, given a formula φ in n variables, we have that ϕ is provable in BUL iff it is valid in the ordinal sum $*$ of $n+1$ Łukasiewicz components followed by one cross-ratio component. Since $*$ is definable, we know that checking validity of φ in the related BUL standard algebra is in PSPACE (however this result is useless, since Gabbay and Metcalfe have shown that BUL is in co-NP). ■

A.6 Open problems

Problem. Is the logic UL finitely strongly standard complete w.r.t. the class of left-continuous conjunctive uninorms definable in $\text{Ł}\Pi\frac{1}{2}$?

An affirmative answer to this open problem would immediately imply that UL is decidable.

Appendix B

Basic Algebraic Notions

We review here some basic algebraic notions needed in this work. The interested reader can find an extensive and deep treatment of those concepts in the monograph [15] by Burris and Sankappanavar, and in the monograph [107] by McKenzie, McNulty and Taylor.

An *algebra* $\mathcal{A}$ is a structure $\langle A, F\rangle$ such that A is a nonempty set and $F = \langle f_i : i \in I\rangle$, where f_i is a finitary operation on A for each $i \in I$. A is called the *universe* of $\mathcal{A}$, each f_i is referred to as a *basic operation* of $\mathcal{A}$, and I is called the *index set* or *the set of operation symbols*.

Given an algebra $\mathcal{A}$, there is a function ρ from I into the set of natural numbers $\mathbb{N}$ called the *rank function*. The pair $\langle I, \rho\rangle$ is called a *similarity type*. An algebra of type ρ is an algebra $\langle A, F\rangle$ in which f_i is a $\rho(f_i)$-ary operation for every $f_i : i \in I$. Two algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be *similar* iff they have the same similarity type. We say that $\mathcal{A}$ is a *reduct* of $\mathcal{B}$ (and that $\mathcal{B}$ is an *expansion of* $\mathcal{A}$) iff $\mathcal{A}$ and $\mathcal{B}$ have the same universe, and the rank function of $\mathcal{A}$ is a subset of the rank function of $\mathcal{B}$ for all operation symbols of $\mathcal{A}$.

Examples. We now provide examples of algebras which are mentioned in the text. First, recall that a binary operation $\cdot$ defined over a universe A is called *commutative* if for all $x, y \in A$

$$x \cdot y = y \cdot x.$$

A *semigroup* is an algebra $\mathcal{A} = \langle A, \cdot\rangle$ such that for all $x, y, z \in A$:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

A *monoid* is an algebra $\mathcal{A} = \langle A, \cdot, e\rangle$ such that $\mathcal{A} = \langle A, \cdot\rangle$ is a semigroup and for all $x \in A$:

$$x \cdot e = e \cdot x = x.$$

A *group* is an algebra $\mathcal{A} = \langle A, \cdot, ^{-1}, e\rangle$ such that $\mathcal{A} = \langle A, \cdot, e\rangle$ is a monoid and for all $x \in A$:

$$x \cdot x^{-1} = x^{-1} \cdot x = e.$$

Recall that a commutative group is also called an *Abelian group*.

A *ring* is an algebra $\mathcal{A} = \langle A, +, \cdot, -, 0, 1\rangle$ such that $\mathcal{A} = \langle A, +, -, 0\rangle$ is an Abelian group, $\mathcal{A} = \langle A, \cdot\rangle$ is a monoid and for all $x, y, z \in A$:

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z),$$

$$(y + z) \cdot x = (y \cdot x) + (z \cdot x).$$

A ring is called a *commutative ring* if $\mathcal{A} = \langle A, \cdot\rangle$ is a commutative monoid.

A *field* is an algebra $\mathcal{A} = \langle A, +, \cdot, -, ^{-1}, 0, 1\rangle$ where $\mathcal{A} = \langle A, +, \cdot, -, 0, 1\rangle$ is a commutative ring and $\mathcal{A} = \langle A, \cdot, ^{-1}, 1\rangle$ is an Abelian group.

A *lattice* is a structure $\mathcal{A} = \langle A, \sqcap, \sqcup\rangle$ such that both $\mathcal{A} = \langle A, \sqcap\rangle$ and $\mathcal{A} = \langle A, \sqcup\rangle$ are commutative semigroups, both $\sqcap$ and $\sqcup$ are idempotent, i.e.

$$x \sqcap x = x \quad \text{and} \quad x \sqcup x = x,$$

and the absorption laws are satisfied, i.e.

$$x \sqcap (x \sqcup y) = x \quad \text{and} \quad x \sqcup (x \sqcap y) = x.$$

Lattices can be also seen as partially ordered sets. A structure $\langle A, \leq\rangle$ is called a *partially ordered set* (poset), if $\leq$ is a partial order over A, i.e. a *binary relation* such that for all $x, y, z \in A$:

- Reflexivity: $x \leq x$;
- Anti-symmetry: if $x \leq y$ and $y \leq x$, then $x = y$;
- Transitivity: if $x \leq y$ and $y \leq z$, then $x \leq z$.

$\langle A, \leq\rangle$ is called a *totally ordered set* (linearly ordered set, chain) if $\leq$ is a total order over A, i.e. a partial order such that for all $x, y \in A$, either $x \leq y$ or $y \leq x$.

Let $\langle A, \leq\rangle$ be a poset and let $X \subseteq A$ be a non-emptyset. An element $a \in A$ is an *upper bound* for X if $x \leq a$ for every $x \in X$. An element $a \in A$ is the *least upper bound* (or *supremum*) of X if a is an upper bound of X, and $x \leq y$ for every $x \in X$ implies $a \leq y$, i.e.: a is the smallest among the upper bounds of X. Similarly we can define what it means for a to be a *lower bound* of A, and for a to be the *greatest lower bound* (or *infimum*) of A.

A poset is a lattice if and only if for every $x, y \in A$ both supremum $\sup\{x, y\}$ and infimum $\inf\{x, y\}$ exist in A.

Lattice-ordered groups (ℓ-groups for short) are structures $\mathcal{A} = \langle A, \sqcap, \sqcup, \cdot, ^{-1}, e\rangle$ such that $\mathcal{A} = \langle A, \sqcap, \sqcup\rangle$ is a lattice, $\mathcal{A} = \langle A, \cdot, ^{-1}, e\rangle$ is a group, and, for all $x, y, z \in A$,

$$x \cdot (y \sqcap z) = (x \cdot y) \sqcap (x \cdot z).$$

An element $u \in A$ is called a *strong unit* if for every $x \in A$, there exists an $n \in \mathbb{N}$ such that $nu \geq x$.

A lattice $\mathcal{A} = \langle A, \sqcap, \sqcup \rangle$ is called *complete* iff every $X \subseteq A$ admits an infimum and a supremum. A element a of a lattice is called *compact* if, whenever $a \leq \sup X$, there is a finite $Y \subseteq X$ such that $a \leq \sup Y$. A lattice is called *algebraic* iff it is complete and every element is a join of compact elements.

A *distributive lattice* is a lattice satisfying the following identities (notice that one implies the other):

$$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z),$$

$$x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z).$$

A binary operation $*$ is said *isotone* in the first variable whenever, if $x \leq y$, then

$$x * z \leq y * z.$$

$*$ is said *antitone* in the second variable whenever, if $x \leq y$, then

$$y * z \leq x * z.$$

The concepts of isotone operation in the second variable and antitone operation in the first variable are defined in a similar way.

A residuated lattice is a structure $\mathcal{A} = \langle A, \sqcap, \sqcup, *, 1, \rightarrow, \rightsquigarrow \rangle$, such that

- $\langle A, *, 1 \rangle$ is a monoid,
- $\langle A, \sqcap, \sqcup \rangle$ is a lattice,
- $\rightarrow$ and $\rightsquigarrow$ are two binary operations antitone in the first variable and isotone in the second, such that for all $x, y, z \in A$

$$x * y \leq z \text{ iff } x \leq y \rightarrow z \text{ iff } y \leq x \rightsquigarrow z.$$

A residuated lattice is called:

- *bounded* if A has both a top element and a bottom element,
- *integral* if for each $x \in A$, $x \sqcap 1 = x$,
- *commutative*, if $*$ is a commutative operation,
- *prelinear* if it satisfies the equation

$$(x \rightarrow y) \sqcup (y \rightarrow x) = 1 = (x \rightsquigarrow y) \sqcup (y \rightsquigarrow x).$$

In any commutative residuated lattice, the operations $\rightarrow$ and $\rightsquigarrow$ coincide, then:

$$x * y \leq z \text{ iff } x \leq y \rightarrow z \text{ [adjointness property]}.$$

A *Heyting algebra* is a bounded commutative integral residuated lattice in which the monoidal operation is idempotent, i.e.:

$$x * x = x.$$

It is easy to see that in every Heyting algebra $x \sqcap y = x * y$.

A *Boolean algebra* is a structure $\langle A, \sqcap, \sqcup, \neg, 0, 1\rangle$ such that $\langle A, \sqcap, \sqcup\rangle$ is a distributive lattice, and the following equations hold:

$$\begin{array}{ll} x \sqcap 0 = 0, & x \sqcup 1 = 1, \\ x \sqcap \neg x = 0, & x \sqcup \neg x = 1. \end{array}$$

Let $\mathcal{A}$ be an algebra. A subset B of the universe A of $\mathcal{A}$ which is closed under the basic operations of $\mathcal{A}$ is called a *subuniverse* of $\mathcal{A}$. An algebra $\mathcal{B}$ is said to be a *subalgebra* of $\mathcal{A}$ iff $\mathcal{A}$ and $\mathcal{B}$ are similar, B is a subuniverse of $\mathcal{A}$, and every operation of $\mathcal{B}$ is the restriction of the operations of $\mathcal{A}$. Let $\mathbf{Sub}(\mathcal{A})$ be the set of subalgebras of $\mathcal{A}$. $\mathbf{Sub}(\mathcal{A})$ is an algebraic lattice.

Let $\mathcal{A}$ and $\mathcal{B}$ be similar algebras and let h be a mapping from A into B such that it *respects the operations*, i.e., for every f_i and every $a_1, \ldots, a_n$,

$$h(f_i(a_1, \ldots, a_n)) = f_i(h(a_1), \ldots, h(a_n)).$$

Such a mapping h is called an *homomorphism*. An injective (one-to-one) homomorphism is called an *embedding*. A surjective (onto) homomorphism is called an *epimorphism*. An *isomorphism* between $\mathcal{A}$ and $\mathcal{B}$ is a one-to-one homomorphism from A onto B; in that case $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic*, denoted by $\mathcal{A} \cong \mathcal{B}$. An homomorphism from A into A is called an *endomorphism*, while an isomorphism from A onto A is called an *automorphism*.

Let I be any set and let A_i be a set for each $i \in I$. The system $A = \langle A_i : i \in I\rangle$ is called a *system of sets indexed by* I. A choice function, is a mapping g over I such that $g(i) \in A_i$ for all $i \in I$. The *direct product* of the system A is the set of all choice functions for A, and it is denoted by $\prod_{i \in I} A_i$.

Let $\mathcal{A} = \langle \mathcal{A}_i : i \in I\rangle$ be a system of similar algebras. The *direct product* of $\langle \mathcal{A}_i : i \in I\rangle$ is the algebra, denoted by

$$\prod \mathcal{A},$$

with the same similarity type, with universe $\prod_{i \in I} A_i$, such that for each operation f and all $g^0, \ldots, g^{m-1} \in \prod_{i \in I} A_i$ (where m is the rank of f)

$$(f(g^0, \ldots, g^{m-1}))_i = f(g_i^0, \ldots, g_i^{m-1})$$

for all $i \in I$.

Given a homomorphism from $\mathcal{A}$ into $\mathcal{B}$, define a binary relation

$$\theta = \{(a, b) : h(a) = h(b)\}.$$

The above relation is an equivalence relation, i.e. reflexive, transitive and symmetric (if $x \leq y$ then $y \leq x$). Being h an homomorphism, θ has the *substitution property*, i.e. for every basic operation f and $a_1, b_1, \ldots, a_n, b_n \in A$ if $a_i \theta b_i$, then

$$f(a_i, \ldots, b_n) \theta f(b_1, \ldots, b_n).$$

We call the relation θ just described the *kernel* of h, denoted by ker h.

In general, a *congruence* of an algebra $\mathcal{A}$ is an equivalence relation with the substitution property. Let $\mathbf{Con}(\mathcal{A})$ be the set of congruences of $\mathcal{A}$. $\mathbf{Con}(\mathcal{A})$ is an algebraic lattice, bounded by the the trivial congruences $\mathbf{0}_{\mathcal{A}}$ and $\mathbf{1}_{\mathcal{A}}$, which correspond to the reflexive relation on $\mathcal{A}$ and to the Cartesian product $A \times A$ (i.e. the direct product of A with itself), respectively.

Given a congruence relation θ we call the set

$$a/\theta = \{b : a\theta b \text{ and } b \in A\}$$

the *congruence class* of a modulo θ, and we denote by A/θ the set of all congruence classes of $\mathcal{A}$ given θ.

There is a natural map h from A onto A/θ. defined by

$$h(a) = a/\theta.$$

For each basic operation f, we can define an operation f_θ on A/θ as follows:

$$h(f(a_i, \ldots, b_n)) = f_\theta(h(a_i), \ldots, h(b_n)).$$

The previous definition makes h into an homomorphism.

Let $\mathcal{A}$ be an algebra and θ be a congruence relation on A. The *quotient algebra* $\mathcal{A}/\theta$ is the algebra similar to $\mathcal{A}$ with universe A/θ in which for each f, there is a corresponding operation f_θ.

Given that the congruence θ is obviously the kernel of the quotient map from $\mathcal{A}$ onto $\mathcal{A}/\theta$, it immediately follows that the congruence relations on $\mathcal{A}$ are exactly the kernels of the homomorphisms with domain A.

Let C be a set. A filter $\mathcal{F}$ is a family of subsets of C such that:

- $C \in \mathcal{F}$,
- if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$, and
- if $X \in \mathcal{F}$, and $X \subseteq Y \subseteq C$, then $Y \in \mathcal{F}$.

A filter $\mathcal{F}$ is called *proper* if $\emptyset \notin \mathcal{F}$.

Let $\mathcal{A} = \langle \mathcal{A}_i : i \in I \rangle$ be a system of similar algebras, and let $\mathcal{F}$ be a proper filter on I. Define the following binary relation $\theta_{\mathcal{F}}$ on $\prod_{i \in I} A_i$ by:

$$\langle a, b \rangle \in \theta_{\mathcal{F}} \text{ iff } \{i \in I : a(i) = b(i)\} \in \mathcal{F},$$

with $a, b \in \prod_{i \in I} A_i$. $\theta_{\mathcal{F}}$ is a congruence on $\prod_{i \in I} \mathcal{A}_i$. The *reduced product* $\prod_{i \in I} \mathcal{A}_i/\mathcal{F}$ is an algebra whose universe is $\prod_{i \in I} A_i/\mathcal{F}$, and for each n-ary operation f and for $a_1, \ldots, a_n \in \prod_{i \in I} A_i$,

$$f(a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}) = f(a_1, \ldots, a_n)/\mathcal{F}.$$

Let $\mathcal{B}_i$ and $\mathcal{A}$ be similar algebras and $\mathcal{B} = \prod_{i \in I} \mathcal{B}_i$. There is a natural bijection between $\hom(\mathcal{A}, \mathcal{B})$ (i.e. the set of homomorphisms from $\mathcal{A}$ into $\mathcal{B}$) and $\prod_{i \in I} \hom(\mathcal{A}, \mathcal{B}_i)$. In fact, for every system $\langle h_i : i \in I \rangle \in \prod_{i \in I} \hom(\mathcal{A}, \mathcal{B}_i)$ there is a unique homomorphism $h \in \hom(\mathcal{A}, \mathcal{B})$ satisfying $h_i = p_i \circ h$ for all $i \in I$, where p_i is the projection of $\mathcal{B}$ onto $\mathcal{B}_i$.

A *subdirect representation* of $\mathcal{A}$ with factors $\mathcal{A}_i$ is an embedding $h : \mathcal{A} \to \prod_{i \in I} \mathcal{A}_i$ (or the associated system $\langle h_i : i \in I \rangle$) such that each h_i is an epimorphism into $\mathcal{A}_i$. A *subdirect product* of $\langle \mathcal{A}_i : i \in I \rangle$ is a subalgebra $\mathcal{B}$ of $\underset{i \in I_i}{\mathcal{A}}$ such that each projection p_i from $\mathcal{B}$ into $\mathcal{A}_i$ is an epimorphism.

Theorem B.0.1 *A system of congruences $\langle \theta_i : i \in I \rangle$ of an algebra $\mathcal{A}$ gives a subdirect representation iff $\bigcap_{i \in I} = \mathbf{0}_{\mathcal{A}}$.*

An algebra $\mathcal{A}$ is called *subdirectly irreducible* iff for every subdirect embedding $h : \mathcal{A} \to \prod_{i \in I} \mathcal{A}_i$ with associated homomorphisms $h_i : \mathcal{A} \to \mathcal{A}_i$, there exists an i such that $\mathcal{A} \cong \mathcal{A}_i$.

Theorem B.0.2 *The following statements are equivalent:*

i. $\mathcal{A}$ is subdirectly irreducible,

ii. there exists a minimal congruence $\theta \neq \mathbf{0}_{\mathcal{A}}$,

iii. there exist $a, b \in A$ such that $(a, b) \in \upsilon_{\mathcal{A}}(c, d)$ iff $c \neq d$ (where $\upsilon_{\mathcal{A}}(c, d)$ is called principal congruence*, i.e. the smallest congruence containing c and d).*

This implies that being subdirectly irreducible only depends of the lattice of congruences $\mathbf{Con}(\mathcal{A})$.

Theorem B.0.3 (Birkhoff) *Every algebra $\mathcal{A}$ is decomposable in a subdirect product of subdirectly irreducible algebras that are quotient algebras of $\mathcal{A}$.*

Let $\mathbb{K}$ be a class of similar algebras.

- $\mathcal{A} \in \mathbf{I}(\mathbb{K})$ iff $\mathcal{A}$ is isomorphic to some member of $\mathbb{K}$.

- $\mathcal{A} \in \mathbf{H}(\mathbb{K})$ iff $\mathcal{A}$ is an homomorphic image of some member of $\mathbb{K}$.

- $\mathcal{A} \in \mathbf{S}(\mathbb{K})$ iff $\mathcal{A}$ is isomorphic to a subalgebra of some member of $\mathbb{K}$.

- $\mathcal{A} \in \mathbf{P}(\mathbb{K})$ iff $\mathcal{A}$ is isomorphic to a product of a system of algebras in $\mathbb{K}$.

- $\mathcal{A} \in \mathbf{P_r}(\mathbb{K})$ iff $\mathcal{A}$ is isomorphic to a reduced product of a system of algebras in $\mathbb{K}$.

A class of algebras $\mathbb{K}$ is said to be a *variety* iff it is closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$. We denote by $\mathcal{V}(\mathbb{K})$ the variety generated by $\mathbb{K}$, i.e., the smallest variety containing $\mathbb{K}$. It is clear that $\mathcal{V}(\mathbb{K}) = \mathbf{HSP}(\mathbb{K})$.

Let $\mathbb{K}$ be a class of algebras of the same type ρ. We denote by I the set of operation symbols. Let X be a set disjoint from I. For $0 \leq n < \omega$, let $I_n = \{f \in I : \rho(f) = n\}$. By a sequence from $X \cup I$ we mean a finite sequence $\langle r_0, \ldots, r_{n-1} \rangle$, where each r_i belongs to $X \cup I$. Such a sequence is called a *word* of the alphabet $X \cup I$, and written $r_0 \ldots r_{n-1}$.

The set $T_\rho(X)$ of terms of type ρ over X is the smallest set T of words on $X \cup I$ such that

i. $X \cup I_0 \subseteq T$.

ii. If $t_0, \ldots, t_{n-1} \in T$ and $f \in I_n$, then the word $f(t_0, \ldots, t_{n-1}) \in T$.

If $T_\rho(X)$ is not empty, then the *term algebra of type ρ over X*, denoted by $\mathcal{T}_\rho(X)$, is the algebra of type ρ, whose universe is $T_\rho(X)$, and whose fundamental operations satisfy

$$f^{\mathcal{T}_\rho(X)}(t_0, \ldots, t_{n-1}) = f(t_0, \ldots, t_{n-1}),$$

for $f \in I_n$ and $t_i \in T_\rho(X)$, $0 \leq i < n$.

A term of type ρ is an element of the term algebra $\mathcal{T}_\rho(\omega)$. A term v_n with $n \in \omega$ is called a *variable*. Terms of type ρ are elements of the algebra $\mathcal{T}_\rho(\omega)$ generated by the set of variables $\{v_0, \ldots, v_n, \ldots\}$. For each $n \geq 1$, the term algebra $\mathcal{T}_\rho(n)$ is the subalgebra of $\mathcal{T}_\rho(\omega)$ generated by the variables $v_0, \ldots, v_{n-1}$.

Let $\mathcal{A}$ be an algebra of type ρ, and $t \in \mathcal{T}_\rho(n)$. We define an n-ary operation $t^{\mathcal{A}}$ over A by induction on the length of t. If $t = v_i$, then we put $t^{\mathcal{A}}(a_0, \ldots, a_{n-1}) = a_i$. If t is of the form $f(t_0, \ldots, t_{n-1})$, where f is an m-ary operation, then we put

$$t^{\mathcal{A}}(a_1, \ldots, a_{n-1}) = f^{\mathcal{A}}(t_0^{\mathcal{A}}(a_1, \ldots, a_{n-1}), \ldots, t_{m-1}^{\mathcal{A}}(a_1, \ldots, a_{n-1})).$$

The members of $\mathcal{T}_\rho(n)$ are called *n-ary terms of type ρ*.

An equation of type ρ is a word of the form: $t = s$, where t, s are terms. Given an equation $t(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$ and an algebra $\mathcal{A}$ of type ρ, we say that $\mathcal{A}$ satisfies the above equation iff, for all $a_1, \ldots, a_n \in A$, $t^{\mathcal{A}}(a_1, \ldots, a_n) = s^{\mathcal{A}}(a_1, \ldots, a_n)$. We denote it by $\mathcal{A} \models t = s$. A class of algebras $\mathbb{K}$ of the same type satisfies an equation $t = s$ iff $\mathcal{A} \models t = s$ for all $\mathcal{A} \in \mathbb{K}$. We denote it by $\mathbb{K} \models t = s$.

We say that a class of algebras $\mathbb{K}$ is an *equational class* if there exists a set Σ of equations such that $\mathbb{K} = \{\mathcal{A} : \mathcal{A} \models \Sigma\}$.

Theorem B.0.4 (Birkhoff) *Let $\mathbb{K}$ be a class of algebras of the same type. $\mathbb{K}$ is a variety iff it is an equational class.*

A class of algebras $\mathbb{K}$ is called a *quasivariety* iff it is closed under **I**, **S**, and $\mathbf{P_r}$. The quasivariety generated by a class $\mathbb{K}$, denoted by $\mathcal{QV}(\mathbb{K})$, is the smallest quasivariety containing $\mathbb{K}$. Clearly, $\mathcal{QV}(\mathbb{K}) = \mathbf{ISP}_r(\mathbb{K})$.

A quasi-equation is an expression of the form:

$$t_1 = s_1 \& \ldots \& t_n = s_n \Rightarrow t = s.$$

It is clear that an equation also is a quasi-equation.

Given a quasi-equation in the variables $x_1, \ldots, x_m$, and an algebra $\mathcal{A}$ of type ρ, we say that $\mathcal{A}$ satisfies the quasi-equation

$$t_1 = s_1 \& \ldots \& t_n = s_n \Rightarrow t = s$$

iff, for all $a_1, \ldots, a_m \in A$, $t^{\mathcal{A}}(a_1, \ldots, a_m) = s^{\mathcal{A}}(a_1, \ldots, a_m)$, whenever $t_i^{\mathcal{A}}(a_1, \ldots, a_m) = s_i^{\mathcal{A}}(a_1, \ldots, a_m)$, for all $i \leq n$.

We say that a class of algebras $\mathbb{K}$ is a *quasi-equational class* if there exists a set Σ of quasi-equations such that $\mathbb{K} = \{\mathcal{A} : \mathcal{A} \models \Sigma\}$.

Theorem B.0.5 ([103]) *Let $\mathbb{K}$ be a class of algebras of the same type. $\mathbb{K}$ is a quasivariety iff it is a quasi-equational class.*

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