Theoretical foundations (under a logical and computational point of view) of Fuzzy Description Logics and their application as ontology representation languages

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Chapter 1

Introduction

Chapter 2

Preliminaries

This work is the result of applying results in Mathematical Fuzzy Logic in order to generalize the framework of Classical Description Logics to the fuzzy and many-valued cases. Even though these two formalisms are, strictly speaking, not part of the central subject of this work, nevertheless we need to define them, in order to develop the rest of the work and improve its understandability. In the present chapter we present these two formalisms and provide the results that we need for the overall development of the main matter of this work. On the one hand, in Section 2.1 we provide an overview on Mathematical Fuzzy Logic. On the other hand, in Section 2.2 we provide an overview on Classical Description Logics.

2.1 Mathematical Fuzzy Logic

What we call *Mathematical Fuzzy Logic* is a recent paradigm that aims to treat vague reasoning by means of formally defined many-valued and fuzzy logic systems. This paradigm has not been thought as we nowadays know it, rather it is the result of a process that began in ancient times.

The suspect that vague sentences and predicates can lead to an unusual behavior of the reasoning process was already present since the IV century b.C. when, according to the tradition, the greek philosopher Eubulides from Mileto proposed what is known as the *sorite paradox* (or, in modern english, the *heap paradox*). There exist several formulations of such paradox (at least two for each vague predicate), we report here a version that is quite close to the original one:

- 10.000 sand grains are a heap.
- If we take a sand grain away from a sand heap,
  the result keeps being a heap.

- 1 sand grain is a heap.

At first sight it can seem a linguistic trick. Nevertheless, this paradox can be formalized. If we consider, for every natural n such that $1 \leq n \leq 10.000$, the proposition:

$$p_n = \text{"}n \text{ sand grains are a heap,"}$$

we can formalize the paradox in the following form:

$$\begin{array}{ll} \vdash & p_{10.000} \\ \vdash & p_{10.000} \to p_{9.999} \\ \vdash & p_{9.999} \\ \vdash & p_{9.999} \to p_{9.998} \\ \vdash & p_{9.998} \\ & \vdots \\ \vdash & p_2 \\ \vdash & p_2 \to p_1 \\ \hline \vdash & p_1 \end{array}$$

As we can easily see, when we try to express it formally, if we consider the above sentences as either true or false, it keeps being a paradox.

As happened with other ancient paradoxes, also for sorite paradox there had been to wait a long time before a solution could be found and this solution needed a widening of the classical bi-valued framework. In modern times, the first person who thought in terms of three-valued logic has been C. S. Peirce in a manuscript of year 1909, but he did not make it public. The birth of many-valued logics is, indeed, attributed to J. Łukasiewicz, who, in 1920 starting from philosophical considerations about the problem of *contingent future events*, defined the first three-valued logic and published the result in [Łukasiewicz, 1920]. Subsequently and jointly with A. Tarski, in [Łukasiewicz and Tarski, 1930] he defined a logic whose propositions are valued in the real unit interval $[0, 1]$. A couple of years after, in order to prove that Intuitionistic Propositional Logic has no finite lineal model, K. Gödel defined in [Gödel, 1932] a class of lineal algebras of every finite cardinality. Considering the class of semantics defined by Gödel in his paper, M. Dummet, in [Dummett, 1959] defined axiomatically a logical calculus and proved its completeness with respect to that class of semantics. The last one of the basic many-valued logics had to wait until year 1996 to be defined. In that year, in fact, P. Hájek, L. Godo and F. Esteva proposed in [Hájek et al., 1996] an axiomatic system whose semantics is the product between real numbers in the real unit interval $[0, 1]$ and called it Product Logic.

Beyond the context of many-valued logics, L. A. Zadeh defined, in [Zadeh, 1965], the notion of *fuzzy set*. Zadeh's definition of fuzzy set is based on a generalization of the range of the set characteristic function to the real unit interval $[0, 1]$. Set operations are also generalized to the operations of $\min\{x, y\}$, $\max\{x, y\}$ and $1 - x$ for intersection, union and complementation respectively, where x and y are the images of the generalized characteristic functions of two different sets over the same individual element. Following the intuition, the subset relation has been defined as true between two fuzzy sets when the instantiation of the subset by means of every domain element is less or equal than the instantiation of the superset by means of the same element.

In the logic system behind Zadeh's set theory the semantics chosen for the implication operator was the so-called *Kleene-Dienes implication*. This operation is the straightforward generalization of the semantics of the classical material implication, i.e. $\max\{1-x, y\}$, where x and y are the evaluations of the antecedent and the consequent of the implication respectively. Nevertheless, this semantics for the implication gives rise to the lack of the classical correspondence between implication and subset/superset relationship. Indeed, given two fuzzy sets C and D, with such semantics, the fact that $C \subseteq D$ is no more equivalent to the fact that, for every individual element x belonging to the domain, $C(x) \Rightarrow D(x) = 1$, as it would be desirable. A way to overcome this shortcoming is, as it began to become clear during the '80s (see, for example [Alsina et al., 1983]), the use of a class of operations over the real unit interval coming from the theory of probabilistic metric spaces, called *t-norms*. Using a t-norm, as a semantics for the intersection, and its residuum, as a semantics for the implication, in fact, solves the above mentioned problem and provides a mathematically well-founded background for fuzzy set theory.

At the end of the '90s P. Hájek, considering all these researches, in his work [Hájek, 1998b], defined what nowadays is known as Mathematical Fuzzy Logic, that is, fuzzy logic with a strong mathematical background based on t-norms. Thanks to this new framework, great advancements on the subject have been done until the present day. The interested reader can find in [Cintula et al., 2011] an exhaustive survey on the matter.

In the rest of this Section we will provide the definitions of fuzzy logics under the propositional, modal and first-order points of view as well as the results that we need in order to easily develop the central subject of this work.

2.1.1 Propositional logic

In this section we introduce the fuzzy propositional logics we are going to use and their underlying semantics. Even though propositional logic is not a central matter in this work it is important to introduce them for two reasons. The first is that modal logic, which we will introduce in a subsequent section and which is a notational variant of $\mathcal{ALC}$-like languages, is defined as an expansion of propositional logic. The second reason is that the algebraic semantics of the fuzzy propositional logics here considered, called *MTL-chains*, are the algebras of truth values of the FDLs considered in our framework and, therefore, they deserve special attention in order to understand the central part of this work.

Syntax

There are several ways to provide a syntactical definition of a logic. In this work we consider one among the most common, that is the so-called *Hilbert style calculus*. Given a finite set of propositional connectives $\mathbf{s} = \{\star_1, \dots \star_i\}$, called *signature*, each one with its arity, and a denumerable set of propositional variable $Var = \{p_1, p_2, \dots\}$, the set of $\mathbf{s}$-formulas, denoted by $Fm_{\mathbf{s}}$ is built inductively in the following way:

1. each propositional variable is a formula,
2. each 0-ary connective is a formula,
3. if $\varphi_1, \dots, \varphi_j$ are formulas and $\star \in \mathbf{s}$ is a j-ary connective, then $\star(\varphi_1, \dots, \varphi_j)$ is a formula.

Notice that, since in our context we will make use of propositional connectives that are at most binary, we will, in general, use the notation $\varphi \star \psi$, instead of the above prenex notation, in which the same formula should be denoted by $\star(\varphi, \psi)$. Given two formulas φ, ψ and a propositional variable p (not necessarily occurring in φ) the *substitution* of p for ψ in φ, denoted by $\varphi[\psi/p]$ is the formula obtained by replacing every occurrence of p in φ for formula ψ. Given a set of formulas T and a formula φ, an *inference rule*, denoted by:

$$\frac{T}{\varphi}$$

is a rule that allows to obtain formula $\varphi[\psi/p]$ from the set of formulas $\{\chi[\psi/p] \mid \chi \in T\}$, for every formula ψ and propositional variable p. A *deduction* of a formula φ from a set of formulas T by means of a finite set *Rules* of inference rules is a finite sequence $\langle\varphi_1, \dots, \varphi_k\rangle$ of formulas where $\varphi = \varphi_k$ and, for each i with $1 \leq i \leq k$, either $\varphi_i \in T$ or φ_i is obtained from a subset of $\{\varphi_1, \dots, \varphi_{i-1}\}$ by applying an inference rule from *Rules*. We say that a formula φ is *deducible* from a given set of formulas T (denoted $T \vdash \varphi$) by means of a finite set of inference rules *Rules* if there exists a deduction of φ from T by means of *Rules*.

Given a set of formulas T, called *axioms* and a finite set of inference rules *Rules*, a *propositional logic* $\mathcal{L}$ is usually defined in the literature in two different ways:

- either as the set of formulas that are deducible from T by means of *Rules*, which is usually called the set of *theorems* of $\mathcal{L}$,
- or as the set of pairs $\langle T', \varphi\rangle$, where T' is a set of formulas and φ is a formula deducible from $T \cup T'$ by means of *Rules*; this set of pair is usually called the *deducibility relation* of $\mathcal{L}$.

Since both the above sets[1] are infinite and uniquely identified by means of T and *Rules* (even though they are not always decidable), it is usual to define a propositional logic through a set of axioms and a set of inference rules. In the remainder of this section we are going to define in this way the propositional versions of the logics that fall within the scope of the present work.

MTL and its axiomatic extensions The logic MTL has been defined in [Esteva and Godo, 2001] over the signature $\mathbf{s}_{MTL} = \{\otimes, \wedge, \rightarrow, \bot\}$, where $\otimes$ is called *strong conjunction*, $\wedge$ *weak conjunction*, $\rightarrow$ *implication* and $\bot$ is a constant symbol. This logic has been axiomatized with the following set of axioms:

$(A1)$ $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$

$(A2)$ $(\varphi \otimes \psi) \rightarrow \varphi$

$(A3)$ $(\varphi \otimes \psi) \rightarrow (\psi \otimes \varphi)$

$(A4)$ $\varphi \otimes (\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \psi)$

$(A5)$ $(\varphi \wedge \psi) \rightarrow \varphi$

$(A6)$ $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$

$(A7a)$ $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \otimes \psi) \rightarrow \chi)$

$(A7b)$ $((\varphi \otimes \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$

$(A8)$ $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$

$(A9)$ $\bot \rightarrow \varphi$

And its unique rule of inference is Modus Ponens (MP):

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi}$$

From the primitive connectives it is possible to define more, in particular:

$$\begin{aligned} \varphi \vee \psi &:= ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\ \varphi \equiv \psi &:= (\varphi \rightarrow \psi) \otimes (\psi \rightarrow \varphi) \\ \neg\varphi &:= \varphi \rightarrow \bot \\ \top &:= \bot \rightarrow \bot \end{aligned}$$

In this work we are going to deal with axiomatic extensions of MTL.

The logic SMTL[2] is the axiomatic extension of MTL by axiom:

(S) $\varphi \wedge \neg\varphi \rightarrow \bot$ (strictness)

[1] Clearly, the first definition is a particular case of the second, namely when $T' = \emptyset$.

[2] SMTL means *strict MTL* in the sense that $(\varphi \wedge \neg\varphi) \leftrightarrow 0$ is a theorem. Algebraically this property is called "pseudo-complementation" and denoted as (PC) in some more algebraic works like [Galatos et al., 2007].

The logic IMTL (*Involutive* MTL) is the axiomatic extension of MTL by axiom:

(Inv) $\neg\neg\varphi \to \varphi$ (involutive negation)

The logic BL has been defined in [Hájek, 1998a] as the axiomatic extension of MTL by axiom:

(D) $\varphi \wedge \psi \to \varphi \otimes (\varphi \to \psi)$ (divisibility)

Note that, in presence of divisibility, weak conjunction turns out to be a definable connective and is defined as:

$$\varphi \wedge \psi \quad := \quad \varphi \otimes (\varphi \to \psi)$$

So, BL formulas, as well as the formulas of each among its axiomatic extensions, can be defined over the signature $\mathbf{s}_{BL} = \{\otimes, \to, \bot\}$.

The logic SBL is the axiomatic extension of BL by axiom (S), or, equivalently, it is the axiomatic extension of SMTL by axiom (D).

Product logic has been defined in [Hájek et al., 1996] and it can be seen as the axiomatic extension of SBL by axiom:

(Π) $\neg\neg\chi \to (((\varphi \otimes \chi) \to (\psi \otimes \chi)) \to (\varphi \to \psi))$ (simplification)

Hence Product Logic is the axiomatic extension of SMTL by axioms (D) and (Π).

Gödel logic is the axiomatic extension of BL (or either SBL or SMTL) by axiom:

(Id) $\varphi \to (\varphi \otimes \varphi)$ (idempotence)

Finally, Łukasiewicz logic is the axiomatic extension of BL by axiom (Inv) or, equivalently, the axiomatic extension of IMTL by axiom (D).

Each one of the above mentioned propositional calculus, except for Product Logic, has finitely valued extensions. In the literature does not exist any work which provides an axiomatic system for these logics. Nevertheless, in [Cintula et al., 2011] (pp. 404-407) is reported an equational characterization of the varieties generated by finite BL-chains that can give an hint on how these logic should be axiomatized. We are not reporting these results here, since we are rather interested in their semantics.

Language expansions Besides the axiomatic extensions in this work we are going to consider some *language expansions* of MTL and its extensions. Language expansions of a given propositional logic $\mathcal{L}$ are logics obtained by adding new connectives to its signature. Clearly, once a new connective has been introduced, it is necessary to add new formulas to the set of axioms of $\mathcal{L}$ in order to settle down the behavior of the new connective.

Involutive negation In the case that the definable negation is not involutive, that is, when the logic considered is not an extension of IMTL, an

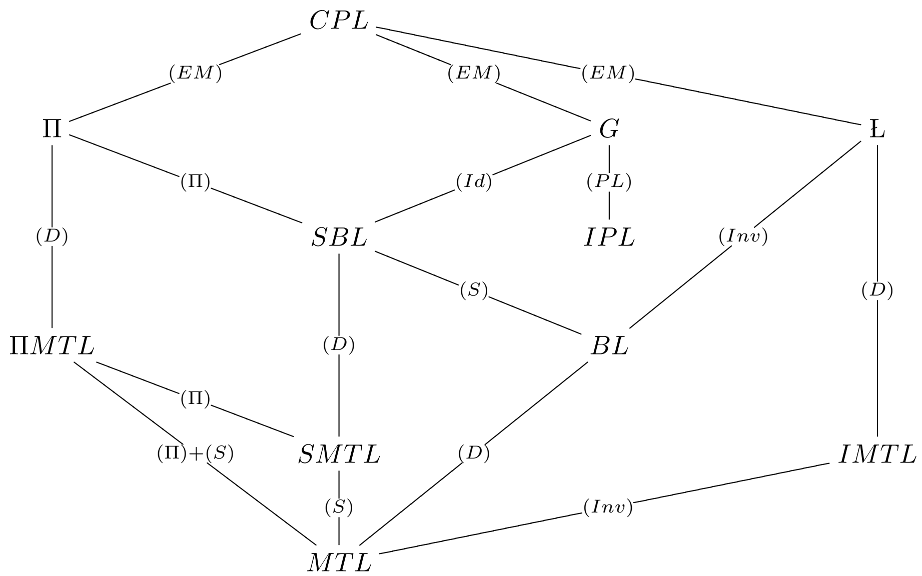


Figure 2.1: The hierarchy of fuzzy logics

interesting expansion is the one obtained by adding an involutive negation as an extra connective. This logic, which we will denote by $\mathcal{L}_\sim$, is obtained from $\mathcal{L}$, as is done in the context of Intuitionistic logic (see [Monteiro, 1980]) or in the context of Gödel logic (see [Esteva et al., 2000]), by adding a new unary connective $\sim$ and the axioms:

($A_\sim 1$) $\sim\sim\varphi \leftrightarrow \varphi$

($A_\sim 2$) $\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$

($A_\sim 3$) $\neg\varphi \to \sim\varphi$

Notice that in these logics we can define a connective of *strong disjunction* in this way:

$$\varphi \oplus \psi := \sim(\sim\varphi \& \sim\psi)$$

Baaz Delta operator Another interesting connective that can be added to the signature of a given logic $\mathcal{L}$ is the unary connective $\triangle$. The resulting propositional calculus, which we will denote by $\mathcal{L}_\triangle$, were introduced in [Hájek, 1998b] as the expansions of $\mathcal{L}$ by the unary connective $\triangle$, satisfying the following axioms, introduced in [Baaz, 1996] in the framework of Gödel Logic:

($A_\triangle 1$) $\triangle\varphi \vee \neg\triangle\varphi$,

($A_\triangle 2$) $\triangle(\varphi \vee \psi) \rightarrow (\triangle\varphi \vee \triangle\psi)$,

($A_\triangle 3$) $\triangle\varphi \rightarrow \varphi$,

($A_\triangle 4$) $\triangle\varphi \rightarrow \triangle \triangle \varphi$,

($A_\triangle 5$) $\triangle(\varphi \rightarrow \psi) \rightarrow (\triangle\varphi \rightarrow \triangle\psi)$.

and the necessitation inference rule:

$$\frac{\varphi}{\triangle\varphi}$$

Truth constants An expansion that is particularly interesting in relation with Fuzzy Description Logics is the one obtained by adding truth constants to the signature of a given logic $\mathcal{L}$. The resulting logic, which we denote by $\mathcal{L}_c$, is defined by adding the following sets of formulas and inference rules to the set of axioms that define $\mathcal{L}$ (for more information about t-norm based fuzzy logics with truth constants see for instance [Esteva et al., 2007]).

The *Book-keeping axioms*:

($A_c 1a$) $\overline{r \circ s} \leftrightarrow \bar{r} \star \bar{s}$

($A_c 1b$) $\overline{\circ r} \leftrightarrow \star \bar{r}$

for $\star$ being any binary or unary connective and $\circ$ its corresponding truth function from the algebra of truth values that is the semantics of $\mathcal{L}$ (see Section 2.1.1).

The *Witnessing axiom*:

($A_c 2$) $\bigvee_{i=1,\ldots,n}(\varphi \leftrightarrow \bar{r}_i)$

The rule:

$$\frac{\bar{r}_{n-1} \vee \varphi}{\varphi}$$

Semantics

Given a logic $\mathcal{L}$, a fundamental task is that of deciding whether a given set of formulas T and a formula φ in the language of $\mathcal{L}$ belong to its deducibility relation. Even though deducing a formula from a set of formulas is a quite simple task, deciding whether there is such a deduction is not. Defined like it has been done in the previous section it is indeed an undecidable problem. In order to prove its decidability in the case of propositional logic it has been necessary to introduce the concept of *semantics*.

Since G. Boole in the XIX century, semantics and, in particular, algebraic semantics, has become a fundamental part of the study of propositional logic. Providing a logic $\mathcal{L}$ with an algebraic semantics means finding a class of algebras that "works like" the set of formulas of $\mathcal{L}$. In order to achieve this result a class of algebras has to be defined whose operations can be "translated" into the signature of $\mathcal{L}$ and into whose operations the signature of $\mathcal{L}$ can be translated. Moreover, it has to be proven that such translation "well relates" the sets of formulas of $\mathcal{L}$ in which we are interested (namely the theorems and the deducibility relation) with a distinct subset of the domain of the algebra of truth values. The proof of this last fact is usually called *Completeness Theorem* in the case of the set of theorems and *Strong Completeness Theorem* in the case of the deducibility relation. Algebraic semantics are nowadays a very powerful tool in the study of logic, until the point that, sometimes, logics are defined from classes of algebras and not the contrary.

Within the framework of propositional fuzzy logic two kinds of algebraic semantics have been considered, namely the *general* and the *standard* semantics. Considering the general semantics means working with a given class of algebras, while considering the standard semantics means working with the algebra of that class whose domain is the real unit interval $[0, 1]$. Nevertheless, it has been proved that, in the case of the propositional calculus, these two kinds of semantics are indeed complete with respect to the same logical calculus.

In this section we are going to introduce the classes of algebras that are the semantics of the logics defined in the previous section and to report some interesting results. As we will see later on, the structures that we are going to define are fundamental for the understanding of the present work because they turn out to be the algebras of truth values in which description concepts will take their values.

General semantics: the class of MTL-algebras and its subclasses An *algebra* $\mathbf{A}$ is a structure composed by a nonempty set A, called the *domain* or *universe* of $\mathbf{A}$ and a set of operations $\mathbf{s}$ called (as in the case of propositional logic) signature, such that, for every $x_1, \ldots, x_j \in A$ and each j-ary operation $\circ \in \mathbf{s_A}$, it holds that $\circ(x_1, \ldots, x_j) \in A$. A class of algebras $\mathbb{K}$ is usually defined through a finite set of equations that are supposed to be true in every algebra belonging to $\mathbb{K}$ and in no algebra not belonging to $\mathbb{K}$. Definitorial equations are expressions of the form:

$$(\forall x)(\forall y)(x = y)$$

where $x, y \in A$ and $\circ \in \mathbf{s_A}$. Following the tradition in MFL, we will use the expression:

$$x \approx y$$

in order to abbreviate the former one.

A *lattice* is an algebra[3] $\mathbf{A} = \langle A, \vee, \wedge \rangle$ with two binary operations $\vee$ and $\wedge$, called *join* and *meet*, which satisfies the following equations:

[3]For further information about lattices, the interested reader can find in [Burris and Sankappanavar, 1981] a clear and exhaustive overview.

$(E1)$ $x \vee y \approx y \vee x$

$(E2)$ $x \wedge y \approx y \wedge x$ (commutativity)

$(E3)$ $x \vee (y \vee z) \approx (x \vee y) \vee z$

$(E4)$ $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$ (associativity)

$(E5)$ $x \vee x \approx x$

$(E6)$ $x \wedge x \approx x$ (idempotence)

$(E7)$ $x \approx x \vee (x \wedge y)$

$(E8)$ $x \approx x \wedge (x \vee y)$ (absorption)

In a lattice $\mathbf{A}$ an *order relation* can be defined between every two elements $a, b \in A$, in the following way:

$$a \leq b \iff a \wedge b = a \iff a \vee b = b$$

For every subset $X \subseteq A$, an *upper bound* of X is an element $x \in A$ such that, for every element $y \in X$, it holds that $y \leq x$; a *lower bound* of X is an element $x \in A$ such that, for every element $y \in X$, it holds that $x \leq y$; the *least upper bound* or *supremum* of X (denoted $\sup(X)$) is an element $x \in A$ such that $x \leq y$ for every upper bound y of X, if, moreover, $\sup(X) \in X$, we call it the maximum of X (denoted $\max X$); the *greatest lower bound* or *infimum* of X (denoted $\inf(X)$) is an element $x \in A$ such that $x \geq y$ for every lower bound y of X, if, moreover, $\inf(X) \in X$, we call it the minimum of X (denoted $\min X$).

A lattice $\mathbf{A}$ is *bounded* if $\inf(A)$ and $\sup(A)$ always exist; it is *complete* if, for every subset $X \subseteq A$, $\inf(X)$ and $\sup(X)$ always exist. Clearly, if a lattice is complete, it is bounded as well.

A *monoid* is an algebra $\mathbf{A} = \langle A, *, 1\rangle$, where:

- $*$ is an associative binary operation,
- $1 \in A$ is the neutral element of operation $*$, in the sense that, for every $x \in A$, it holds that $x * 1 = 1 * x = x$.

A monoid $\mathbf{A}$ is commutative if operation $*$ is.

We say that an algebra $\mathbf{A} = \langle A, \wedge, \vee * \Rightarrow, 0, 1\rangle$ is a *bounded commutative integral residuated lattice* if:

- $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice where $0 = \inf(A)$ and $1 = \sup(A)$,
- $\langle A, *, 1\rangle$ is a commutative monoid,
- there exists a unique binary operation $\Rightarrow$ satisfying for all $a, b, c \in [0, 1]$ the following condition (called *residuation*):

$$a * b \leq c \text{ if and only if } a \leq b \Rightarrow c,$$

the operator $\Rightarrow$ is called the *residuum* of the operation $*$ and it is defined as

$$x \Rightarrow y = \max\{z \in [0,1] \mid x * z \leq y\}.$$

An *MTL-algebra* $\mathbf{A} = \langle A, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a bounded commutative integral residuated lattice which satisfies the equation:

(PL) $(x \Rightarrow y) \vee (y \Rightarrow x) \approx 1$ (pre-linearity)

An *IMTL-algebra* $\mathbf{A}= \langle A, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a MTL-algebra which satisfies the equation:

(Inv) $x \approx (x \Rightarrow 0) \Rightarrow 0$ (involutive negation)

An *SMTL-algebra* $\mathbf{A}= \langle A, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a MTL-algebra which satisfies the equation:

(S) $x \wedge (x \Rightarrow 0) \approx 0$ (strictness)

A *BL-algebra* $\mathbf{A}= \langle A, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is an MTL-algebra which satisfies the equation:

(D) $x \wedge y \approx x * (x \Rightarrow y)$ (divisibility)

A Π-*algebra* $\mathbf{A}= \langle A, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is an SMTL-algebra which satisfies the equations (D) and:

(Π) $((z \Rightarrow 0) \Rightarrow 0) \Rightarrow (((x * z) \Rightarrow (y * z)) \Rightarrow (x \Rightarrow y)) \approx 1$ (simplification)

A *Gödel-algebra* $\mathbf{A}= \langle A, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a BL-algebra which satisfies the equation:

(Id) $x \approx x * x$ (idempotence)

An *MV-algebra* $\mathbf{A}= \langle A, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a BL-algebra which satisfies (Inv) or, equivalently, an IMTL-algebra which satisfies (D).

Moreover, if any of these algebras is linearly ordered, we say that it is an *MTL-chain* (respectively *SMTL-chain*, Π-*chain* and so on).

All the logics defined in these preliminaries are algebraizable in the sense of Blok and Pigozzi (see [Garcìa–Cerdaña et al., 2005]) and their algebraic semantics are the varieties of the corresponding MTL-algebras. Moreover all of these logics are chain-complete (what is called "semilinear" in [Cintula and Noguera, 2010]) in the sense that they are strong complete for evaluations over the chains of the corresponding variety.

A *propositional evaluation* is a mapping $e\colon Fm_{\mathcal{L}} \to A$ such that:

- $e(\bot) = 0$,
- $e(\top) = 1$,
- for every pair of formulas $\varphi, \psi \in Fm_{\mathcal{L}}$, every logical connective $\star \in \mathbf{s}_{\mathcal{L}}$ and its respective algebraic operation $\circ \in \mathbf{s}_{\mathbf{A}}$, it holds that:

$$e(\varphi \star \psi) = e(\varphi) \circ e(\psi)$$

Under a semantic point of view, the notions of theorems and deducibility relation are substituted by those of tautologies and consequence relation, but can be defined the concept of *r-satisfiability* as well. Given a set of formulas Γ and a formula φ, we say that:

- φ is an $\mathcal{L}$-*tautology* (denoted $\models_{\mathcal{L}} \varphi$) if, for every $\mathcal{L}$-algebra $\mathbf{A}$ and every propositional evaluation $e\colon Fm_{\mathcal{L}} \to A$, it holds that $e(\varphi) = 1$,
- φ is a *logical consequence* of Γ (denoted $\Gamma \models_{\mathcal{L}} \varphi$) if, for every $\mathcal{L}$-algebra $\mathbf{A}$ and every propositional evaluation $e\colon Fm_{\mathcal{L}} \to A$ such that $e(\psi) = 1$, for every $\psi \in \Gamma$, it holds that $e(\varphi) = 1$,
- φ is *r-satisfiable* if there exists a $\mathcal{L}$-algebra $\mathbf{A}$, a value $r \in A$ and a propositional evaluation $e\colon Fm_{\mathcal{L}} \to A$ such that $e(\varphi) = r$.

Standard semantics A natural semantics for the MTL logic and its axiomatic extensions is the evaluation over the real unit interval, i.e. over the MTL-chains whose lattice reduct is $[0,1]$ with the usual order. These chains, called "standard chains" are related to a special kind of operation called "t-norms".

Definition 1. A *t-norm* is a binary operation $*$ on the real unit interval $[0,1]$ that is associative, commutative, non-decreasing in both arguments and having 1 as neutral (unit) element.

The property:

$$x * \vee(X) \approx \bigvee_{y \in X}\{x * y\}$$

called "left continuity" of a t-norm is a sufficient and necessary condition for the existence of the residuum of the t-norm $*$. Using this residuum, the following result characterizes standard chains.

Proposition 2. *A structure $\langle [0,1], \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a standard MTL-chain if and only if $*$ is a left-continuous t-norm and $\Rightarrow$ is its residuum. This structure will be denoted from now on by $[0,1]_*$. Moreover a standard chain satisfies divisibility (Hence it is a BL-chain) if and only if the t-norm is continuous.*

In [Jenei and Montagna, 2002] it is proved that MTL is *strong standard complete* (strong complete for evaluations over the standard chains), i.e. for any set of formulas $\Gamma \cup \{\varphi\}$ and any evaluation e over a standard chain,

$\Gamma \vdash_{[0,1]_{MTL}} \varphi$ iff $e(\varphi) = 1$ for any evaluation e such that $e(\gamma) = 1$ for all $\gamma \in \Gamma$.

This result is not automatically translatable to axiomatic extensions of MTL. It is easily extended to SMTL and the standard SMTL-chains but not to BL and the standard BL-chains (hence neither to its axiomatic extensions). If $\mathcal{L}$ is either BL or SBL or Łukasiewicz or Product or Gödel logic only the finite strong standard completeness results are valid, i.e. for any *finite* set of formulas $\Gamma \cup \{\varphi\}$ and any evaluation e over a standard $\mathcal{L}$-chain,

$\Gamma \vdash_{[0,1]_\mathcal{L}} \varphi$ iff $e(\varphi) = 1$ for any evaluation e such that $e(\gamma) = 1$ for all $\gamma \in \Gamma$,

An interesting result for Łukasiewicz, Product and Gödel logics is that the corresponding standard-chains are all isomorphic[4]. The most used representative of standard chains of these three logics (unique up to isomorphisms), are the ones defined by the so-called Łukasiewicz, product and minimum t-norms and their residua (collected in Table 1).

$*$	Minimum (Gödel)	Product (of real numbers)	Łukasiewicz
$x * y$	$\min(x, y)$	$x \cdot y$	$\max(0, x + y - 1)$
$x \Rightarrow y$	$\begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise} \end{cases}$	$\min(1, 1 - x + y)$
$x \Rightarrow 0$	$\begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$	$1 - x$

Table 2.1: The three main continuous t-norms.

Let I be a bounded set and, for every $i \in I$, let $\mathbf{A}_i = \langle A_i, \wedge, \vee, *_i, \Rightarrow_i, 0, 1\rangle$ be chain. Suppose that, for every $i, j \in I$, $A_i \cap A_j = \emptyset$. We define the *ordinal sum* of these chains as the chain:

$$\bigoplus_{\mathbf{i} \in \mathbf{I}} \mathbf{A_i} = \langle \bigoplus_{i \in I} A_i, *, \wedge, \vee, 0, 1\rangle$$

where the domain is:

$$\bigoplus_{i \in I} A_i = \bigcup_{i \in I} \{A_i \setminus \{1\}\} \cup \{\top\}$$

with lineal order defined by the condition:

$$a \leq b \iff \begin{cases} \text{either } a, b \in A_i \quad \text{and} \quad a \leq_i b \\ \text{or } a \in A_i, \quad b \in A_j \quad \text{and} \quad i < j \end{cases}$$

and, for every $a, b \in \bigoplus_{i \in I} A_i$,

$$a * b = \begin{cases} a *_i b, & \text{if } a, b \in A_i \\ a \wedge b, & \text{otherwise} \end{cases}$$

Once defined what an ordinal sum is, we can report an important result about continuous t-norms, the *Mostert-Shields Theorem*, proved in [Mostert and Shields, 1957].

Theorem 3. *Every continuous t-norm is the ordinal sum of either Łukasiewicz, product or Gödel t-norm.*

[4]In fact for Gödel logic there is only one standard chain while for Łukasiewicz and Product there are infinite different but isomorphic ones.

From the previous results seems natural the definition of the logic of a (continuous) t-norm.

Definition 4. We say that a logic (called $\mathcal{L}(*)$) is the logic of a continuous t-norm $*$ if it is an axiomatic extension of BL which is finite strong standard complete with respect to evaluations over the standard chain $[0,1]_*$, i.e. for any *finite* set of formulas $\Gamma \cup \{\varphi\}$ and any evaluation e over $[0,1]_*$,

$$\Gamma \vdash_{\mathcal{L}(*)} \varphi \text{ iff } e(\varphi) = 1 \text{ for any evaluation } e \text{ such that } e(\gamma) = 1 \text{ for all } \gamma \in \Gamma.$$

2.1.2 Modal logic

In this section we introduce the framework of modal logic. It is important because, as we will see, modal language is a notational variant of the description language mainly considered in this work. For this reason, we will consider the general framework of *multi-modal language* of which the usual modal language is a particular case.

Modal Logic was already known and studied in ancient times by the Aristotele's school. In its modern version it has been defined by C. I. Lewis and C. H. Langford, who, in [Lewis and Langford, 1932] established a modern notation, gave sets of axioms for some logical system and provided a matrix-like truth-functional semantics for those systems. Further studies, due, above all, to E. J. Lemmon (see [Lemmon, 1957, Lemmon, 1966a, Lemmon, 1966b]) introduced an algebraic semantics for Lewis and Langford Systems.

Nevertheless, the real cornerstone in the study of Modal Logic has been the work of S. Kripke, who, in [Kripke, 1963, Kripke, 1965], defined what is nowadays known as *Kripke-style semantics*, based on a particular kind of relational structures, called *Kripke frames.* This kind of structures gave a clear and well-defined semantics to modal systems, allowing great advancements in the study of Modal Logic, also under the syntactic and computational points of view.

Syntax

Given a propositional signature $\mathbf{s}$, a multi-modal signature $\mathbf{s}_\Box$ is obtained by adding a non-empty finite subset of the set of unary modal connectives $\{\Box_n, \Diamond_n \mid n \in \mathbb{N}\}$. From this new signature, the set of modal formulas $Fm_{\mathbf{s}_\Box}$ is built recursively applying the same rules as for propositional formulas, but in the new multi-modal language.

So, a modal system $\mathcal{L}_\Box$ is defined as an expansion of a given propositional logic $\mathcal{L}$ by means of a set of modal connectives. The set of axioms that defines $\mathcal{L}_\Box$ is built from the set of axioms that defines $\mathcal{L}$ by adding the axioms that define the behavior of the modal connectives and the inference rules of $\mathcal{L}_\Box$ are the inference rules of $\mathcal{L}$ plus some other inference rule.

Within the classical framework, several modal system are known that expand the Classical Propositional Calculus and, for many of them it is known whether they are axiomatic extension of other modal systems. An exhaustive study of Modal Logic in the classical framework can be found in [Blackburn et al., 2001].

Within the framework of many valued logics, the study of modal expansions is much more recent and some advances have been done in [Bou et al., 2011b] for the case of finite-valued logics.

Semantics

As we said, a notion that has become fundamental for the study of Modal Logic, under a semantic point of view, is that of Kripke frames and Kripke models. In this work we consider a many-valued generalization of the classical notion of Kripke model following the one provided in [Bou et al., 2011b].

Definition 5 (Kripke frames and models)**.** Given an algebra **T**and $m \in \mathbb{N}$, a **T**-*valued Kripke frame* is a tuple $\mathfrak{F} = \langle W, R_1, \ldots, R_m \rangle$, where

- W is a non-empty crisp set, called *domain* or *set of possible worlds*,
- for every $1 \leq i \leq m$, R_i is a binary relation (called *accessibility relation*) valued in **T**; i.e., it is a mapping $R_i \colon W \times W \to A$.

A Kripke frame is said to be *crisp* if, for every $1 \leq i \leq m$, the range of R_i is included in $\{0, 1\}$. The class of all **T**-valued frames will be denoted by Fr and the class of crisp frames by CFr. A Kripke **T**-*model* is a pair $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, where $\mathfrak{F}$ is an **T**-valued Kripke frame and V is a mapping $V \colon Var \times W \to A$ assigning to each propositional variable and each world in W a value in A. The map V can be uniquely extended to a map, which we also denote by V, assigning to each pair formed by a formula $\varphi \in Fm_{\mathbf{s}_\Box}$ and a world $w \in W$ an element of **T**in such a way that:

- $V(\star(\varphi_1, \ldots, \varphi_n), w) = \circ(V(\varphi_1, w), \ldots, V(\varphi_n, w))$, for every n-ary propositional connective $\star \in \mathbf{s}$ and its truth function $\circ \in \mathbf{s_T}$;
- $V(\overline{r}, w) = r \in A$ for each truth constant $\overline{r} \in \mathbf{s}$;
- for each $1 \leq i \leq m$, $V(\Box_i \varphi, w) = \inf_{w' \in W} \{R_i(w, w') \Rightarrow V(\varphi, w')\}$;
- for each $1 \leq i \leq m$, $V(\Diamond_i \varphi, w) = \sup_{w' \in W} \{R_i(w, w') * V(\varphi, w')\}$.

In the following definition we define a list of problem that can be asked about a many-valued modal logic, some of them are equivalent to reasoning tasks in Fuzzy Description Logic.

Definition 6. Consider a formula $\varphi \in Fm_{\mathbf{s}_\Box}$ an algebra **T**and $r \in A$, then:

- given a Kripke **T**-model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ and $w \in W$, we say that w *r-satisfies* φ (denoted $\mathfrak{M}, w \Vdash^r \varphi$) if $V(\varphi, w) = r$;
- we say that a Kripke **T**-model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ *locally r-satisfies* φ (denoted $\mathfrak{M} \models^r_l \varphi$) if there exists $w \in W$ such that $\mathfrak{M}, w \Vdash^r \varphi$; in this sense we say that φ is *locally r-satisfiable* if there is a Kripke **T**-model $\mathfrak{M}$ which locally r-satisfies it;

- we say that a Kripke $\mathbf{T}$-model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ *globally r-satisfies* φ (denoted $\mathfrak{M} \models_g^r \varphi$) if $\inf_{w \in W}\{V(\varphi, w)\} \geq r$; in this sense we say that φ is *globally r-satisfiable* if there is a Kripke $\mathbf{T}$-model $\mathfrak{M}$ which globally r-satisfies it;
- we say that φ is a *local consequence* of a set of formulas $\Gamma \subseteq Fm_{\mathbf{s}_\Box}$ (denoted $\Gamma \models_l \varphi$) if, for every Kripke $\mathbf{T}$-model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ and $w \in W$, if w 1-satisfies every formula in Γ, then w 1-satisfies φ;
- we say that φ is a *global consequence* of a set of formulas $\Gamma \subseteq Fm_{\mathbf{s}_\Box}$ (denoted $\Gamma \models_g \varphi$) if, every Kripke $\mathbf{T}$-model which globally 1-satisfies every formula in Γ, globally 1-satisfies φ as well; $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ and $w \in W$, if w 1-satisfies every formula in Γ, then w 1-satisfies φ;
- we say that φ is *r-valid in the frame* $\mathfrak{F}$ (denoted $\mathfrak{F} \models^r \varphi$) if it is globally r-satisfied in every Kripke $\mathbf{T}$-model based on $\mathfrak{F}$; in this sense, given a class K of frames, we write $\mathsf{K} \models^r \varphi$ to mean that φ is r-valid in every frame in that class.

The set of the formulas that are 1-valid in every frame of a class K is denoted, in [Bou et al., 2011b] by $\mathbf{\Lambda}(\mathsf{K}, \mathbf{T})$. In [Bou et al., 2011b, Section 4.2] axiomatizations for the sets $\mathbf{\Lambda}(\mathsf{K}, \mathbf{Ł}_n)$ and $\mathbf{\Lambda}(\mathsf{K}, \mathbf{Ł}_n^c)$ (the expansion of the previous language by truth constants) are given for $\mathsf{K} = \mathsf{Fr}$, and $\mathsf{K} = \mathsf{CFr}$.

Besides these sets of valid formulas, we will consider, for each $m \in L_n$, the sets $\mathbf{Sat_m}(\mathtt{Fr}, Ł_n)$, $\mathbf{Sat_m}(\mathtt{Fr}, Ł_{n,\Delta})$, etc. of modally m-satisfiable formulas.

2.1.3 First order predicate logic

Syntax

In order to define what a predicate logic is, we have, previously, to define what a *predicate language* is.

Definition 7. A *predicate signature* $\mathbf{s}$ is compound by a countable set of relation symbols (also called *predicates*) $P_1, \ldots, P_n, \ldots$, each one with arity ≥ 1, a countable set of function symbols $f_1, \ldots, f_n, \ldots$, each one with its arity, a countable set of constant symbols $c_1, \ldots, c_n, \ldots$, that are 0-ary function symbols.

Given a countable set Var of individual variables, the set of *Terms* over a predicate signature is defined inductively as follows:

- every variable $x \in Var$ is a term,
- every constant $c \in \mathbf{s}$ is a term,
- if $t_1, \ldots, t_n$ are terms and $f \in \mathbf{s}$ is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term.

Now, let $\mathbf{l}$ be a propositional language, as defined in Section 2.1.1, then the set of symbols $\mathbf{l}\forall := \mathbf{l} \cup \{\forall, \exists\}$ is a first order language. The set $Fm_{\mathbf{l}\forall,\mathbf{s}}$ of *Formulas* over a first order language $\mathbf{l}\forall$ and a predicate signature $\mathbf{s}$ is defined inductively as follows:

- $\bot$ and $\top$ are formulas,
- if $t_1, \ldots, t_n$ are terms and $P \in \mathbf{s}$ is an n-ary predicate, then $P(t_1, \ldots, t_n)$ is a formula (called *atomic formula*),
- if $\varphi_1, \ldots, \varphi_n$ are formulas and $\star \in \mathbf{l}$ is an n-ary logical operator, then $\star(\varphi_1, \ldots, \varphi_n)$ is a formula,
- if $\varphi(x)$ is a formula, then $(\forall x)\varphi(x)$ and $(\exists x)\varphi(x)$ are formulas.

As usual a variable that does not fall within the scope of a quantifier is said to be *free*, otherwise, it is said to be *bound*. The notation $\varphi(x_1, \ldots, x_n)$ means that the variables that are free in φ are among $x_1, \ldots, x_n$. We say that a formula that has no free variable is *closed*, otherwise it is *open*. Given a term t and a formula $\varphi(x_1, \ldots, x_n)$, we denote by $\varphi(x_1, \ldots, x_n)[t/x_1]$ the result of substituting every occurrence of variable x_1 for t in $\varphi(x_1, \ldots, x_n)$.

Following [Hájek, 1998b], given a propositional residuated logic $\mathcal{L}$, we define the first order logic associated with $\mathcal{L}$ (denoted by $\mathcal{L}\forall$), as follows:

Definition 8. $\mathcal{L}\forall$ is the first order logic such that:

1. its set of formulas is $Fm_{\mathcal{L}\forall} := Fm_{\mathbf{l}_{\mathcal{L}\forall},\mathbf{s}}$ where $\mathbf{s}$ is an arbitrary predicate signature,
2. it is axiomatized by means of the following set of axiom schemata:
   - (P) the axioms resulting from the axioms of $\mathcal{L}$ after the substitution of propositional formulas by formulas of the new predicate language.
   - ($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable for x in φ.
   - ($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ.
   - ($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi(x))$, where x is not free in χ.
   - ($\exists$2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi(x) \to \chi)$, where x is not free in χ.
   - ($\forall$3) $(\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi(x))$, where x is not free in χ.
3. its rules of inference are Modus Ponens (MP) and generalization (G): From φ infer $(\forall x)\varphi(x)$.

All the logics considered so far enjoy two important properties.

Definition 9.
1. We say that a logic $\mathcal{L}$ enjoys the *Local Deduction Theorem* (*LDT*, for short) if for each theory T and formulas φ, ψ, it holds that $T, \varphi \vdash \psi$ iff there exists a natural number n such that $T \vdash \varphi^n \to \psi$, where $\varphi^n = \varphi \star \ldots \star \varphi$, n times.
2. We say that a logic $\mathcal{L}$ enjoys *Delta Deduction Theorem* (ΔDT, for short) if, for each theory T and formulas φ, ψ, it holds that $T, \varphi \vdash \psi$ iff $T \vdash \Delta\varphi \to \psi$.

3. We say that a logic $\mathcal{L}$ enjoys *Invariance under Substitution* (*Sub*, for short) if, for every formulas φ, ψ, χ and every formula ζ occurring in χ, it holds that $\varphi \equiv \psi \vdash \chi[\varphi/\zeta] \equiv \chi[\psi/\zeta]$.

Next we recall the definition of *core fuzzy logic* given in [Cintula and Hájek, 2006] and that of *strict core fuzzy logic*, given in [Cerami and Esteva, 2011].

Definition 10. 1. We say that a logic $\mathcal{L}$ is a *core fuzzy logic* if it is finitary, enjoys LDT, Sub and expands MTL.

2. We say that a logic $\mathcal{L}$ is a *strict core fuzzy logic* if it is finitary, enjoys LDT, Sub and expands SMTL.

3. We say that a logic $\mathcal{L}_\Delta$ is a Δ-*core fuzzy logic* if it enjoys ΔDT, Sub and expands MTL_Δ.

Semantics

From a semantic point of view first order models are compound of a set of elements, an algebra of truth values and an assignation function.

Definition 11. A *first order structure* for a given predicate language Γ is a pair $(\mathbf{A}, \mathbf{M})$, where $\mathbf{A}$ is an $\mathcal{L}$-chain and $\mathbf{M}$=$(M, (P_\mathbf{M})_{P\in\Gamma}, (f_\mathbf{M})_{f\in\Gamma}, (c_\mathbf{M})_{c\in\Gamma})$, where:

1. The set M, called *domain*, is a non-empty set,

2. for each predicate symbol $P \in \Gamma$ of arity n, $P_\mathbf{M}$ is an n-ary $\mathbf{A}$-fuzzy relation on M, i.e. an n-ary function $P_\mathbf{M} \colon M^n \to A$,

3. for each function symbol $f \in \Gamma$ of arity n, $f_\mathbf{M}$ is an n-ary (crisp) function on M and

4. for each constant symbol $c \in \Gamma$, $c_\mathbf{M}$ is an element of M.

The truth value $\|\varphi\|_v^{\mathbf{A},\mathbf{M}}$ of a predicate formula φ in a given model v is defined as follows.

Definition 12. Let Γ be a predicate language, $\mathbf{A}$ an $\mathcal{L}$-chain and $(\mathbf{A}, \mathbf{M})$ a first order structure, then a first order assignation v is a mapping $v : Var \to M$. As usual each assignation, defined on the set of individual variables, extends univocally to a first order assignation (that we will denote by v as well) satisfying, for every terms $t_1, \ldots, t_n$ and each n-ary function $f \in \Gamma$, that $v(f(t_1, \ldots, t_n)) = f_\mathbf{M}(v(t_1), \ldots, v(t_n))$. To denote that assignation$v$ assigns objects $a_1, \ldots, a_n$ to variables $x_1, \ldots, x_n$, we will write $v([a_1/x_1], \ldots, [a_n/x_n])$.
Moreover, each assignation v, defined on the set of individual variables yields a first order model $\|\cdot\|_v^{(\mathbf{A},\mathbf{M})} : Fm_{\mathcal{L}\forall} \to \mathbf{A}$ such that:

1. for each n-tuple of terms $t_1, \ldots, t_n$ and each n-ary relation $P \in \Gamma$, it holds that $\|P(t_1, \ldots, t_n)\|_v^{(\mathbf{A},\mathbf{M})} = P_{\mathbf{M}}(v(t_1), \ldots, v(t_n)) \in A$,
2. if $\varphi_1, \ldots, \varphi_n$ are formulas, $\star \in \mathbf{s}_{\mathcal{L}\forall}$ an n-ary logical connective and $\circ \in \mathbf{s}_{\mathbf{A}}$ its truth function, then $\|\star(\varphi_1, \ldots, \varphi_n)\|_v^{(\mathbf{A},\mathbf{M})} = \circ(\|\varphi_1\|_v^{(\mathbf{A},\mathbf{M})}, \ldots, \|\varphi_n\|_v^{(\mathbf{A},\mathbf{M})})$.
3. if $\varphi(x_1, \ldots, x_n)$ is a formula with n free variables and v is a first order assignation such that $v(x_i) = a_i$ and $a_i \in M$, for $1 < i \leq n$, then we have that $\|(\forall x_1)\varphi(x_1, x_2, \ldots, x_n)\|_v^{(\mathbf{A},\mathbf{M})} = \inf_{a \in M}\{\|\varphi(a, a_2, \ldots, a_n)\|^{(\mathbf{A},\mathbf{M})}\}$,
4. if $\varphi(x_1, \ldots, x_n)$ is a formula with n free variables and v is a first order assignation such that $v(x_i) = a_i$ and $a_i \in M$, for $1 < i \leq n$, then we have that $\|(\exists x_1)\varphi(x_1, x_2, \ldots, x_n)\|_v^{(\mathbf{A},\mathbf{M})} = \sup_{a \in M}\{\|\varphi(a, a_2, \ldots, a_n)\|^{(\mathbf{A},\mathbf{M})}\}$.

Clearly, depending on the model, the infimum and supremum of a set of values of formulas do not necessarily exist and, in this case we will say that a given quantified formula has an undefined truth value. Following [Hájek, 1998b], we will say that if, for a given model v, both infima and suprema of sets of values are defined for every formula, then v is a *safe* model. Moreover, if, for a given first order structure $(\mathbf{A}, \mathbf{M})$, each assignation v, defined in it, is safe, we will say that $(\mathbf{A}, \mathbf{M})$ is a safe structure.

From now on and for simplicity, we will omit the name "safe" before the first order structures, i.e., when we speak about a first order structure $(\mathbf{A}, \mathbf{M})$, we implicitly mean a *safe* first order structure $(\mathbf{A}, \mathbf{M})$.

The notions of *tautologies*, *logical consequence* and *r-satisfiability* are defined in the usual way.

The witnessed model property In recent times first order Fuzzy Logic has been deeply studied. Generalizing the classical case, the value of a universally (existentially) quantified formula is defined as the infimum (supremum) of the values of the results of replacing the quantified variable by the interpretation of a term of the language in a first-order model. Notice that in the context of Classical Logic, as well as every finitely valued logic, infima and suprema turn out to be minima and maxima, respectively. However, when we move to infinitely valued logics, this is not the case, the infimum or supremum of a set of values C may be an element $c \notin C$, i.e., a quantified formula may have no *witness*. Following these ideas, Hájek introduced in [Hájek, 2007a], [Hájek, 2007b] the notion of *witnessed model*, i.e., a model in which each quantified formula has a witness and proved that this is an important property because it implies a limited form of finite model property for certain fragments of predicate fuzzy logic (see [Hájek, 2005]). Moreover, Cintula and Hájek introduce in [Cintula and Hájek, 2006] the so-called witnessed axioms that, added to any first-order core fuzzy logic, give a logic complete with respect to witnessed models. Subsequently they prove that these axioms are derivable in Łukasiewicz first-order Logic, showing that Ł∀ is complete with respect to witnessed models (we will say that Ł∀ has the witnessed model property).

Witnessed models have been firstly defined in [Hájek, 2005] in the following way:

Definition 13. For any structure $(\mathbf{A}, \mathbf{M})$, a formula $(\forall y)\varphi(y, x_1, \ldots, x_n)$ is $\mathbf{A}$-witnessed in $\mathbf{M}$ if, for each assignation $c_1, \ldots, c_n \in M$, to $x_1, \ldots, x_n$, there is $c \in M$ such that $\|(\forall y)\varphi(y, c_1, \ldots, c_n)\|^{\mathbf{A},\mathbf{M}} = \|\varphi(c, c_1, \ldots, c_n)\|^{\mathbf{A},\mathbf{M}}$. Similarly for $(\exists y)\varphi(y, x_1, \ldots, x_n)$. $\mathbf{M}$ is $\mathbf{A}$-witnessed if all quantified formulas are $\mathbf{A}$-witnessed in $\mathbf{M}$.

Within the framework of classical predicate logic, where the first order structures are evaluated on a two element chain, there is no need of making a difference between witnessed and non witnessed models, because every model is indeed witnessed, and the same holds for every finite-valued logic. The need of speaking about witnessed models arises when we move to infinite-valued logics, since we can meet sets of truth values whose infima (resp. suprema) is not an element of the set. Later on, in [Cintula and Hájek, 2006], Hájek and Cintula consider the following couple of axioms (called witnessed axioms) already given by Baaz in [Baaz, 1996]:

(C∃) $(\exists y)((\exists x)\varphi(x) \to \varphi(y))$,

(C∀) $((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

They prove that each first order core fuzzy logic $\mathcal{L}\forall$, extended with this couple of axioms (denoted $\mathcal{L}\forall^w$), is complete with respect to the witnessed models evaluated over $\mathcal{L}$-chains. Moreover, in [Cintula and Hájek, 2006] it is proved that Łukasiewicz predicate logic is the only logic of a continuous t-norm equivalent to its witnessed axiomatic extension, i.e., (C∃) and (C∀) are theorems of Łukasiewicz predicate Logic. As a consequence of this fact Łukasiewicz is the only infinite-valued logic of a continuous t-norm which is complete with respect to witnessed models, i.e. it satisfies the *witnessed model property.*

The quasi-witnessed model property Neither Gödel, nor Product first-order Logic share witnessed model property because witnessed axioms are not theorems of these logics. In fact no other first-order logic of a continuous t-norm enjoys this property, since it is related to continuity of the truth functions, a property that only Łukasiewicz logic has. Nevertheless, in [Laskowski and Malekpour, 2007] it is proved that Product Predicate Logic enjoys a weaker property, what we call *quasi-witnessed model property.* Quasi-witnessed models[5] are models in which, whenever the value of a universally quantified formula is strictly greater than 0, then it has a witness, while existentially quantified formulas are always witnessed.

[5] These models are called "closed models" in [Laskowski and Malekpour, 2007] but we decided, after some discussions with colleagues, to use the more informative name of "quasi-witnessed models". We take into account the fact that the name "closed" is used in mathematics and logic in different contexts with different meanings and could induce some confusion.

Definition 14. Let Γ be a predicate language and $(\mathbf{A}, \mathbf{M})$ a first-order structure, then we say that a Γ-formula $\varphi(x, y_1, \ldots, y_n)$ is **A**-*quasi-witnessed* in **M** if:

1. For each tuple $c_1, \ldots, c_n$ of elements in M there exists an element $a \in M$ such that $\|(\exists x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = \|\varphi(a, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})}$.
2. For each tuple $c_1, \ldots, c_n$ of elements in M either $\|(\forall x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = 0$, or there exists an element $b \in M$ such that $\|(\forall x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = \|\varphi(b, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})}$.

We say that a first-order structure $(\mathbf{A}, \mathbf{M})$ is quasi-witnessed if for each formula and for every assignation v of the variables on **M** the formula is quasi-witnessed.

In [Cerami and Esteva, 2011] we introduced both the so-called *strict core* fuzzy logics and the following *quasi-witnessed* axioms (generalizations of the witnessed axioms of Hájek-Cintula to cope with quasi-witnessed models):

Definition 15. Let $\mathcal{L}\forall$ be any strict core first-order logic, we denote by $\mathcal{L}\forall^{qw}$ the axiomatic extension of $\mathcal{L}\forall$ by the following axiom schemata called, from now on, "quasi-witnessed axioms":

(C$\exists$) $(\exists y)((\exists x)\varphi(x) \to \varphi(y))$,

(ΠC$\forall$) $\neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

These quasi-witnessed axioms are a modification of the witnessed axioms given above. The first one, $(C\exists)$, is a witnessed axiom and the second one says that the witnessed axiom $(C\forall)(\exists y)(\varphi(y) \to (\forall x)\varphi(x))$ is valid in a structure $(\mathbf{A}, \mathbf{M})$ only when the truth value of $(\forall x)\varphi(x)$ is different from 0, i.e., when $\|\neg\neg(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 1$. In the same paper, we proved, following the style of [Cintula and Hájek, 2006] that, if we add quasi-witnessed axioms to any first-order strict core fuzzy logic, the resulting logic enjoys the quasi-witnessed model property. From this result, the one in [Laskowski and Malekpour, 2007] about the completeness of Product first-order Logic with respect to quasi-witnessed models, follows as a corollary. Moreover, we proved that quasi-witnessed axioms are tautologies in no logic of a continuous t-norm, but Product and Łukasiewicz predicate logics.

Δ-strict fuzzy logics Again in [Cerami and Esteva, 2011] we studied the expansion of first order strict core fuzzy logics by Δ operator. We gave the following so-called Δ-*quasi-witnessed axioms*[6]:

Definition 16. We denote by $\mathcal{L}_\Delta\forall^{\Delta qw}$ the axiomatic extension of $\mathcal{L}_\Delta\forall$ by the following axiom schemata called, from now on, "Δ-quasi-witnessed axioms":

(C$_\Delta\exists$) $(\exists y)\Delta((\exists x)\varphi(x) \to \varphi(y))$,

[6]Throughout this section $\mathcal{L}_\Delta$ will denote the extension of a Δ-core fuzzy logic by the strictness axiom (S).

($\Pi C_\Delta \forall$) $\neg\neg(\forall x)\varphi(x) \to ((\exists y)\Delta(\varphi(y) \to (\forall x)\varphi(x)))$.

We proved, as in [Cintula and Hájek, 2006], that the extension of a logic $\mathcal{L}_\Delta\forall$ by means of these axioms, is complete with respect to quasi-witnessed models, but not with respect to models that are embeddable into a quasi-witnessed model (like the extension of a strict core fuzzy logic by the usual quasi-witnessed axioms). So, it makes sense to say that these extensions are the logics of quasi-witnessed models.

2.2 (Classical) Description Logic

This work proposes a generalization of Classical Description Logic (DL) to the many-valued and fuzzy case. In order to make a confrontation of the new generalized framework with the old one, we briefly introduce in this section the classical framework on DL. For an exhaustive presentation of the subject, the reader is invited to read the general *Handbook of Description Logic* [Baader et al., 2003] and the more recent paper [Baader et al., 2008].

2.2.1 A little bit of history

Description Logics, as we nowadays know them, are the result of at least 30 years of research on the field of knowledge representation. This research did not begin within the DL framework, rather arrived to this framework through an evolution process of older formalisms such as *frame-based systems* and *KL-ONE based systems*.

Frame-based systems

DLs are considered an evolution of frame-based systems that were systems based on the old idea that human mind can be represented in its totality by a more or less comprehensive program. For this fact one of the main features (and, as it became evident later, weaknesses) of these old systems was that of considering formal logic as a useless limitation. The main examples of frame-based systems are Quillian's *Semantic networks* and Minsky's *Frame systems*.

Semantic networks Semantic networks have been defined in the '60s by M. R. Quillian in [Quillian, 1967] with the aim of giving a model of the way human memory works. The program that Quillian defines can be roughly divided into three parts:

- The first part is a *memory model* that works like a linked vocabulary. For each word definition in the vocabulary there is a *plane* containing a *type node* for the word defined by the definition and a *token node* for every other word appearing in the definition. The plane contains as well links between the type node and the token nodes and between token nodes that describe the structure of the definition. These links are of different types, depending on which kind of word (noun, verb, etc.) the linked token node represents and are conditioned by the adverb that introduce them. Between planes there are other kinds of links that relate the token nodes in a given plane to the type nodes of the planes where the definitions of the token nodes in the first plane lie. Once a significative amount of of definition have been introduced, a net-like structure is obtained. Within this structure, for every word can be identified what Quillian denotes by *full word concept*, that is, the set of type and token nodes that can be reached following all the links from the type node representing a given word. Clearly two words can lie within the full word concept of each other and there can have circular link paths.

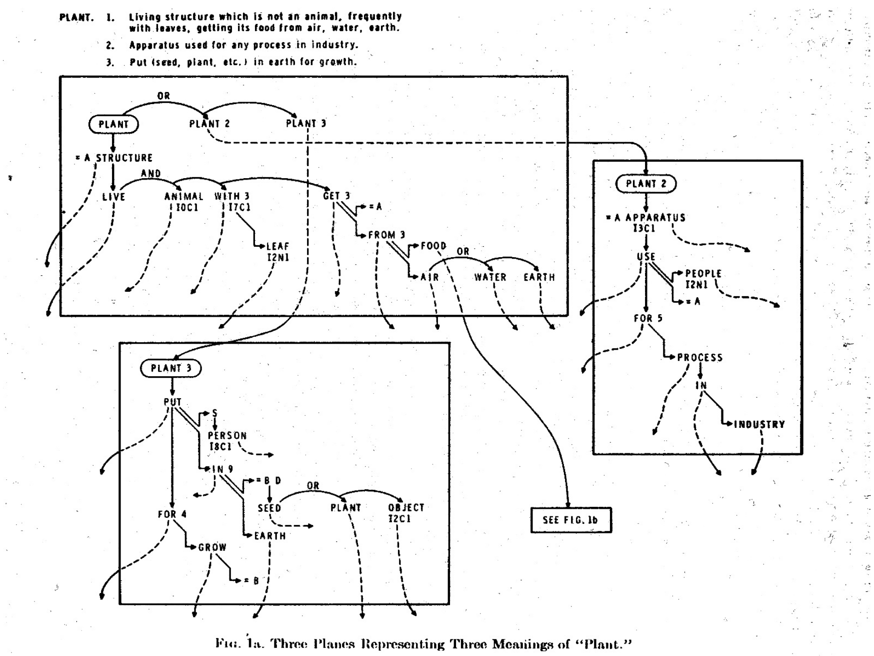


Figure 2.2: Example of Quillian's plane

- Semantic networks are not just a linked vocabulary since the second part of the program, called *search program* allow to look for hidden relations between words. Among the queries that can be done to the search program there are (a) the set of words that lie in the full word concept of two given words, (b) whether a word appears in the full word concept of another word or (c) which modifier a word appears in the definition of another word with.

- The third part of the program is a sentence generator. The sentence generator utilizes the work done by the search program on the data provided by the memory model. It was thought to express sentences as close as it was possible to the natural language, for this reason its aim was not that of infer hidden logical consequences from the data provided in the memory model.

Frame Systems Frame systems have been defined in the '80s by M. Minsky in [Minsky, 1981] with the aim of explaining the way people face new and known challenges by using *mental frames*, that in natural language can be denoted as nothing more than mental stereotypes. Frames, according to Minsky, are data structures that represent stereotyped situations. At the higher levels of a frame there are nodes that do not changes with the instantiation of a situation, while at the lower levels there are empty nodes that can be filled up either with contingent information or with other frames. People use mental frames not only in order to act in a given situation, but, above all, to act fast, since there is no need to compute how to behave every time the same situation (or a similar one) is faced. When either a new situation or an old one with new features is faced, is when preexisting frames are either modified or substituted by new ones. Since, when a frame has to be used, it is important that the nodes in the lower levels have already been filled up, Minsky's frame systems are often considered an example of default reasoning.

Formally a *frame system* is a set of frames that consider the same situation seen from different points of view. Among the reasoning services of frame systems there are: (a) subsumption between frames, in order to give specific situations a more general meaning, (b) search of slot fillers, in order to add information to a given situation. there is no standard semantics, but a number of expert system based on this formalism have been done. As an example we provide the notation used in the KEE system.

Frame systems had procedural and descriptive aspects. The semantics of procedural aspects was not very clear. For this reason, it is difficult to compare them with other formalisms. About the descriptive aspects, despite the fact that Minsky presented his formalism as an alternative to logic-based formalisms, already since the '70s there were ways to express frames in first order sentences.

KL-ONE based systems

Frame-based systems were formalisms based on researches about human cognitive behavior. In this sense, given a memory model, their goal was to obtain a program that imitates human mental skills, e.g. natural language understanding. For this reason these systems were thought in a way that they could support language ambiguity and this fact made them far from based on formal logic, when their authors were not explicitly against the use of logic.

During the second half of '70s began to have researches that evidenced the limits of frame-based systems. Among those limits we can find the following ones:

- it was not so clear what the systems had to compute (see [Woods, 1975]),

```
Frame: Course in KB University
  MemberSlot: enrolls
    ValueClass: Student
    Cardinality.Min: 2
    Cardinality.Max: 30
  MemberSlot: taughtby
    ValueClass: (UNION GradStudent
                       Professor)
    Cardinality.Min: 1
    Cardinality.Max: 1

Frame: AdvCourse in KB University
  SuperClasses: Course
  MemberSlot: enrolls
    ValueClass: (INTERSECTION
                  GradStudent
                  (NOT Undergrad))
    Cardinality.Max: 20

Frame: BasCourse in KB University
  SuperClasses: Course
  MemberSlot: taughtby
    ValueClass: Professor

Frame: Professor in KB University

Frame: Student in KB University

Frame: GradStudent in KB University
  SuperClasses: Student
  MemberSlot: degree
    ValueClass: String
    Cardinality.Min: 1
    Cardinality.Max: 1

Frame: Undergrad in KB University
  SuperClasses: Student
```

Figure 2.3: Example of KEE Knowledge Base

- there was not a simple way to give these system a clear formal semantics,
- most aspects of these systems can be formalized by means of first order logic and it seems that the contributions of frame-based systems is not so novel (see [Hayes, 1977]).

Despite neither the first version of KL-ONE, developed by R. J. Brachman in [Brachman, 1979], was based on formal logic, nevertheless this new representation system brought some significative novelty with respect to the old framework of frame-based systems. We report some of them:

- it includes the skill of extract implicit conclusions from given knowledge,
- it gives the user the possibility of defining new complex concepts and roles,
- it introduces the difference between *individual concepts* and *generic concepts*,
- the difference between the concept definitions with sufficient and necessary condition and those with just necessary ones is studied ,
- *classification* (computation of the hierarchy of subsumptions) and *realization* (computation of the more specific atomic concept) are added to the reasoning tasks,

Besides these novelties, KL-ONE had some weaknesses that became evident quite early. Among those weaknesses we can find the lack of a clear formal semantics and the fact that the algorithms for deciding classification and realization were incomplete. In order to overcome the weaknesses of KL-ONE it has been proposed, as guidelines for new systems, (a) the fact of thinking the system under the point of view of *functionality*, i.e. the reasoning services provided to the user, more than under the point of view of the mere concept representation;

(b) a clearer distinction between the knowledge representing relations among concepts and that representing assertion about individuals. These new guidelines were taken into account to build the systems KRIPTON and KANDOR.

Description Logics

Besides the weaknesses that KL-ONE-like system presented, they brought a new way to see knowledge representation systems. On the one hand, in fact it has been adopted the so-called *functional approach*, that consisted in putting the attention on the services provided by the KR systems, more than on the way it represents knowledge. This change of perspective can be seen at the origin of the growing interest that, since the '80s, researchers put on decision algorithms and their complexity. On the other hand, the need of a clear semantics can be seen at the origin of the fact that systems began to be more and more logic-based and an unambiguous Tarsky-style semantics was adopted.

The fact of putting attention on the reasoning tasks and on the logical language of the systems allowed to think about those systems in a more abstract way as clearly defined *description languages*, even though no necessarily there was actual program behind the language. This means, as well, that the languages are now quantitatively comparable, mainly under two points of view: the computational complexity of reasoning, on the one side, and the expressivity of the language, on the other. Since the '80s, the history of proper DL systems is, indeed, characterized by the tradeoff between complexity and expressivity of the language and the search of a fair equilibrium between these two features has been the main fuel of the great advancements that research in DL have seen since then.

The DL systems of the '80s, like BACK and LOOM, used so-called *structural subsumption algorithms*. These kinds of algorithm perform a comparison in the syntactic structure of two given concept description after having transformed them in a suitable normal form. Structural subsumption algorithms are relatively efficient when applied to very inexpressive languages, as proven in [Brachman and Levesque, 1984]. Nevertheless, in more expressive languages these algorithms turn out to be incomplete. Further researches of the same period, like [Brachman and Levesque, 1985], allowed by the use of abstract languages, revealed that expressivity improvements increase intractability of the reasoning tasks. In particular, [Nebel, 1990] revealed that reasoning in presence of a Terminological Box is a computationally intractable problem in itself.

The '90s saw the introduction of a new kind of algorithm: the *tableau based algorithms* (see, for example, [Hollunder et al., 1990]). These kind of algorithm revealed to be complete also for quite expressive DLs and allowed a systematic study of complexity of reasoning in various DLs, in particular, those related with logical languages (see [Donini et al., 1992, Schmidt-Schauss and Smolka, 1991]). Moreover, they are suitable to be highly optimized in such a way that they can lead to a good practical behavior of the system. In the same period the relationships between DLs and classical modal logics ([Schild, 1991]), on the one hand and with fragments of classical first order logic ([**?**]) are investigated.

Nowadays very expressive DL systems are utilized as the reasoning engines of the Semantic Web and for knowledge representation in medical and bio-informatic data bases.

2.2.2 Syntax

Knowledge is represented in DL systems through the construction of *concept descriptions* by means of a machinery that consists of a set N_A of *concepts names*, a set N_R of *role names*, a set N_I of *individual names*, and a set of *concept* and *role constructors.* The difference between description languages consists in the set of concept and role constructors utilized to build up concept descriptions and each of these sets is denote by a sequence of letters. In what follows we briefly introduce, by means of syntactic rules, the symbology used to denote each constructor, the name of the constructor and the letter used to denote the language that utilizes that constructor.

Concept constructors

Given a variable for atomic concept $A \in N_A$, a variable for role name $R \in N_R$ and variables for complex concepts C, D, a *concept description* is inductively built in accordance with the following syntactic rules:

C, D	$\longrightarrow$	$\bot$	empty concept	$\mathcal{FL}_0$
		$\top$	universal concept	$\mathcal{FL}_0$
		A	atomic concept	$\mathcal{FL}_0$
		$C \sqcap D$	conjunction	$\mathcal{FL}_0$
		$\forall R.C$	value restriction	$\mathcal{FL}_0$
		$\exists R.\top$	restricted existential quantif.	$\mathcal{FL}^-$
		$\neg A$	atomic complementation	$\mathcal{AL}$
		$\neg C$	complementation	$\mathcal{C}$
		$C \sqcup D$	disjunction	$\mathcal{U}$
		$\exists R.C$	existential quantification	$\mathcal{E}$
		$\geq n\, R$	unqualified	
		$\leq n\, R$	number	$\mathcal{N}$
		$= n\, R$	restriction	
		$\geq n\, R.C$	qualified	
		$\leq n\, R.C$	number	$\mathcal{Q}$
		$= n\, R.C$	restriction	
		$\{a\}$	nominals	$\mathcal{O}$
		d	concrete domains	(D)

Role constructors

Given variables for complex concepts R, S and a variable for functional role name $f \in N_R$, a *role description* is inductively built in accordance with the following syntactic rules:

R, S	$\longrightarrow$	R	atomic role	$\mathcal{FL}_0$
		R^+	transitive role	$_{R^+}$
		R^*	reflexive-transitive role	$\mathcal{S}$
		U	universal role	$\mathcal{S}$
		R^-	inverse role	$\mathcal{I}$
		$R \sqcap S$	role intersection	$\mathcal{R}$
		$\neg R$	role complementation	$\mathcal{H}$
		$R \sqcup S$	role union	$\mathcal{H}$
		$R \circ S$	role composition	
		f	functional role (feature)	$\mathcal{F}$

Note that the language name $\mathcal{S}$ denotes $\mathcal{ALC}$ plus reflexive-transitive roles.

2.2.3 Semantics

An *interpretation* is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consisting of a nonempty set $\Delta^\mathcal{I}$ (called *domain*) and of an *interpretation function* $\cdot^\mathcal{I}$ that assigns:

1. to each individual a an element $a^\mathcal{I} \in \Delta^\mathcal{I}$ such that $a^\mathcal{I} \neq b^\mathcal{I}$ if $a \neq b$ (*Unique Name Assumption*, different individuals denote different objects of the domain),

2. to each atomic concept A a subset $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ of the domain set,

3. to each role R a binary relation $R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$ on the domain set.

Moreover, the interpretation function is inductively extended to complex concepts and roles as follows:

$$
\begin{aligned}
\bot^\mathcal{I} &= \emptyset \\
\top^\mathcal{I} &= \Delta^\mathcal{I} \\
(\neg C)^\mathcal{I} &= \Delta^\mathcal{I} \setminus C^\mathcal{I} \\
(C \sqcap D)^\mathcal{I} &= C^\mathcal{I} \cap D^\mathcal{I} \\
(C \sqcup D)^\mathcal{I} &= C^\mathcal{I} \cup D^\mathcal{I} \\
(\forall R.C)^\mathcal{I} &= \{a \in \Delta^\mathcal{I} \mid \forall b \in \Delta^\mathcal{I}, R^\mathcal{I}(a,b) \rightarrow C^\mathcal{I}(b)\} \\
(\exists R.C)^\mathcal{I} &= \{a \in \Delta^\mathcal{I} \mid \exists b \in \Delta^\mathcal{I} \text{ such that } R^\mathcal{I}(a,b) \wedge C^\mathcal{I}(b)\} \\
(\geq n\, R)^\mathcal{I} &= \{a \in \Delta^\mathcal{I} \mid |\{b \in \Delta^\mathcal{I} \mid R^\mathcal{I}(a,b)\}| \geq n\} \\
(\leq n\, R)^\mathcal{I} &= \{a \in \Delta^\mathcal{I} \mid |\{b \in \Delta^\mathcal{I} \mid R^\mathcal{I}(a,b)\}| \leq n\} \\
(= n\, R)^\mathcal{I} &= \{a \in \Delta^\mathcal{I} \mid |\{b \in \Delta^\mathcal{I} \mid R^\mathcal{I}(a,b)\}| = n\} \\
(\geq n\, R.C)^\mathcal{I} &= \{a \in \Delta^\mathcal{I} \mid |\{b \in \Delta^\mathcal{I} \mid R^\mathcal{I}(a,b) \wedge C^\mathcal{I}(b)\}| \geq n\} \\
(\leq n\, R.C)^\mathcal{I} &= \{a \in \Delta^\mathcal{I} \mid |\{b \in \Delta^\mathcal{I} \mid R^\mathcal{I}(a,b) \wedge C^\mathcal{I}(b)\}| \leq n\} \\
(= n\, R.C)^\mathcal{I} &= \{a \in \Delta^\mathcal{I} \mid |\{b \in \Delta^\mathcal{I} \mid R^\mathcal{I}(a,b) \wedge C^\mathcal{I}(b)\}| = n\} \\
\{a\}^\mathcal{I} &= \{a^\mathcal{I}\} \subseteq \Delta^\mathcal{I} \\
U^\mathcal{I} &= \Delta^\mathcal{I} \times \Delta^\mathcal{I} \\
(R^-)^\mathcal{I} &= \{(b,a) \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid (a,b) \in R^\mathcal{I}\} \\
(\neg R)^\mathcal{I} &= \Delta^\mathcal{I} \times \Delta^\mathcal{I} \setminus R^\mathcal{I} \\
(R \sqcap S)^\mathcal{I} &= R^\mathcal{I} \cap S^\mathcal{I} \\
(R \sqcup S)^\mathcal{I} &= R^\mathcal{I} \cup S^\mathcal{I} \\
(R \circ S)^\mathcal{I} &= R^\mathcal{I} \circ S^\mathcal{I}
\end{aligned}
$$

Moreover, allowing transitive, transitive-reflexive or functional roles, means that:

- there exists a subset $N_{R^+} \subseteq N_R$ of the set of role names such that, for each $R \in N_{R^+}$ and $a, b, c \in \Delta^{\mathcal{I}}$, it holds that, if $(a, b), (b, c) \in R^{\mathcal{I}}$, then $(a, c) \in R^{\mathcal{I}}$ as well,
- there exists a subset $N_{R^*} \subseteq N_R$ of the set of role names such that, for each $R \in N_{R^*}$ and $a, b, c \in \Delta^{\mathcal{I}}$, it holds both that, if $(a, b), (b, c) \in R^{\mathcal{I}}$, then $(a, c) \in R^{\mathcal{I}}$ and $(a, a) \in R^{\mathcal{I}}$,
- there exists a subset $N_F \subseteq N_R$ of the set of role names such that, for each $f \in N_F$ and $a \in \Delta^{\mathcal{I}}$, it holds that $|\{b \in \Delta^{\mathcal{I}} \mid (a, b) \in f^{\mathcal{I}}\}| \leq 1$.

Some straightforward consequences of the semantics of constructors are that:

- $\mathcal{ALE}, \mathcal{ALU} \subseteq \mathcal{ALC}$,
- if $\mathcal{DL}$ is a description language, then $\mathcal{DLF} \subseteq \mathcal{DLN} \subseteq \mathcal{DLQ}$.
- if $\mathcal{DL}$ is a description language, then $\mathcal{DLR} \subseteq \mathcal{DLH}$.

2.2.4 Reasoning

As said before, besides the description of the world, a fundamental service provided by DL systems is that of inferring hidden conclusions from known premises. In this section we give an account of the syntax and semantics of the premises and the types of conclusions that can be inferred from those premises.

Knowledge bases

Given a description language $\mathcal{DL}$ and two $\mathcal{DL}$ concepts C, D, a *general concept inclusion* (GCI) (or *inclusion axiom*) is an expression of the form:

$$C \sqsubseteq D$$

An interpretation $\mathcal{I}$ satisfies an inclusion axiom $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
An *equivalence axiom* is an expression of the form:

$$C \equiv D$$

which, in the classical case, is an abbreviation for the pair of axioms $C \sqsubseteq D$ and $D \sqsubseteq C$. An interpretation $\mathcal{I}$ satisfies an equivalence axiom $C \equiv D$ if $C^{\mathcal{I}} = D^{\mathcal{I}}$.

A finite set $\mathcal{T}$ of GCIs is called a *terminology* or *TBox*. An axiom of the form $A \equiv C$, where A is a concept name, is called a *definition*. It is said that a concept name A *directly uses* a concept name B in a TBox $\mathcal{T}$ if there is a definition $A \sqsubseteq C \in \mathcal{T}$ such that B occurs in C. Furthermore, it is said that a concept name A *uses* a concept name B if B is in the transitive closure of the relation of directly using with respect to A. A TBox $\mathcal{T}$ is called *definitorial* or *acyclic* if:

- it contains only definitions,
- it contains at most one definition for each concept name occurring in it,
- no concept name occurring in it uses itself.

Given a description language $\mathcal{DL}$, a $\mathcal{DL}$ concept C, a role R and two individuals $a, b \in N_I$, a *concept assertion axiom* (or *assertion*) is an expression of the form:

$$C(a)$$

An interpretation $\mathcal{I}$ satisfies an assertion $C(a)$ if $C^{\mathcal{I}}(a^{\mathcal{I}}) \neq \emptyset$.

A *role assertion axiom* is an expression of the form:

$$R(a, b)$$

An interpretation $\mathcal{I}$ satisfies a role assertion $R(a, b)$ if $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \neq \emptyset$.

A finite set of concept and role assertion axioms is called *ABox*. An ABox is said to be *local* if just an individual a appears in each assertion.

Given a description language $\mathcal{DLH}$ and two $\mathcal{DLH}$ roles R, S, a *role inclusion axiom* is an expression of the form:

$$R \sqsubseteq S$$

An interpretation $\mathcal{I}$ satisfies a role inclusion axiom $R \sqsubseteq S$ if $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$. A finite set of role inclusion axioms is called *RBox*.

Finally a *knowledge base* $\mathcal{K}$ consists of a TBox, an ABox and an RBox, each one possibly empty.

Main inference problems

Consider a knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{R})$, a pair of concepts C, D, a pair of roles R, S and a pair of individuals a, b, then we can define the main reasoning tasks considered in the literature.

- C is *satisfiable* if there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$.
- $\mathcal{K}$ is *consistent* if there is an interpretation $\mathcal{I}$ that satisfies every axiom in assertion axiom in $\mathcal{A}$ and every inclusion axiom in $\mathcal{T} \cup \mathcal{R}$. In this case we say that $\mathcal{I}$ is a *model* of $\mathcal{K}$
- C is *satisfiable with respect to* $\mathcal{K}$ if there exists a model $\mathcal{I}$ of $\mathcal{K}$ such that $C^{\mathcal{I}} \neq \emptyset$.
- Concept D *subsumes* concept C with respect to $\mathcal{K}$ (in symbols $\mathcal{K} \models C \sqsubseteq D$) if, in every model $\mathcal{I}$ of $\mathcal{K}$, it holds that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- Two concepts C, D are *equivalent* with respect to $\mathcal{K}$ (in symbols $\mathcal{K} \models C \equiv D$) if, in every model $\mathcal{I}$ of $\mathcal{K}$, it holds that $C^{\mathcal{I}} = D^{\mathcal{I}}$.

- An individual a is an *instance* of C with respect to $\mathcal{K}$ (in symbols $\mathcal{K} \models a\colon C$) if, in every model $\mathcal{I}$ of $\mathcal{K}$, it holds that $a^{\mathcal{I}} \in C^{\mathcal{I}}$.

- A pair of individuals a, b is an *instance* of R with respect to $\mathcal{K}$ (in symbols $\mathcal{K} \models (a,b)\colon R$) if, in every model $\mathcal{I}$ of $\mathcal{K}$, it holds that $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$.

Note that, due to the classical semantics, in DL languages where all the boolean operators are present, each one of the above reasoning problems can be reduced to knowledge base consistency.

Complexity

The study of the computational complexity of the reasoning tasks is fundamental In Description Logics and it has worked, since the beginning of the research on DL, as an engine for the improvements made on this subject. For many languages the complexity classes they belong to have been identified and often a systematic study of what causes the increment of complexity has been undertaken. Here we summarize some important results in the literature.

	Sat.	Unsat.	Sat. acyclic KB	Sat. w.r.t. KB	Subs.
$\mathcal{FL}^-$					PTIME
$\mathcal{AL}$			co-NP	EXPTIME	PTIME
$\mathcal{ALI}$					PTIME
$\mathcal{ALN}$			PSPACE		PTIME
$\mathcal{ALNI}$	PTIME				co-NP
$\mathcal{ALE}$		NP	co-NP	PSPACE	NP
$\mathcal{FL}^-\mathcal{E}$		NP			NP
$\mathcal{ALR}$		NP			NP
$\mathcal{ALER}$		NP			NP
$\mathcal{ALU}$		co-NP			co-NP
$\mathcal{ALC}$	PSPACE	PSPACE			PSPACE
$\mathcal{ALEN}$	PSPACE				
$\mathcal{ALUR}$	PSPACE				
$\mathcal{ALNR}$	PSPACE				
$\mathcal{ALCNR}$	PSPACE				
$\mathcal{ALCH}$	NEXPTIME				
$\mathcal{ALCNO}$	NEXPTIME				
$\mathcal{ALCNR}$				NEXPTIME	

Table 2.2: Important complexity results in classical DL

The result about $\mathcal{FL}^-$ refers just to a PTIME hardness and is from [Brachman and Levesque, 1984]. The results about $\mathcal{AL}$ from [Schmidt-Schauss and Smolka, 1991]. The results about $\mathcal{ALI}$ and $\mathcal{ALNI}$ are from available in [Donini et al., 1999]. The ones about $\mathcal{ALN}$, $\mathcal{ALU}$, $\mathcal{ALUR}$,

$\mathcal{ALNR}$ and $\mathcal{ALCNR}$ are from [Donini et al., 1997]. The result about $\mathcal{ALE}$ is from [Donini et al., 1992]. The NP-hardness results for $\mathcal{ALR}$, $\mathcal{ALER}$, $\mathcal{FL}^{-}\mathcal{E}$, are proven in [Donini et al., 1997], while the NP membership results for the same languages are proven in [Donini et al., 1992]. Complexity of concept satisfiability w.r.t. acyclic axioms for language $\mathcal{AL}$ can be found in [Buchheit et al., 1994], [Calvanese, 1996] and [Buchheit et al., 1998] and the same result for the language $\mathcal{ALE}$ in [Calvanese, 1996] both with respect to acyclic and general axioms. The result for $\mathcal{ALC}$ is from [Schmidt-Schauss and Smolka, 1991]. The result for $\mathcal{ALEN}$ is from [Hemaspaandra, 1999]. The result for $\mathcal{ALCH}$ is from [Lutz and Sattler, 2001]. The complexity of concept satisfiability w.r.t. general KBs refers to cyclic definitions and it is found in [Küsters, 1998]. The result for $\mathcal{ALCNO}$ is from [Tobies, 2001]. The result for $\mathcal{ALCNR}$ is from [Buchheit et al., 1993].

Chapter 3

Fuzzy Description Logic

In this chapter we introduce the fuzzy generalization of Description Logics. We introduce our proposal for a syntax and semantics of Fuzzy Description Logics up to the language that, in the classical case, would correspond to $\mathcal{ALC}$. We discuss the consequences that these choices have on the hierarchy of basic FDL languages. Moreover, we provide a translation from $\mathcal{ALC}$-like concepts to fuzzy first order formulas and prove that it preserves the meaning of the involved concepts. We also provide a translation from $\mathcal{ALC}$-like concepts to Fuzzy multi-Modal Logic formulas and vice-versa and, again, prove that it preserves the meaning of the expressions involved.

3.1 Syntax

In this section we introduce the syntax of concepts and fuzzy axioms. In what folloes, we assume that the algebra of truth values $\mathbf{T}$ is a complete chain.

3.1.1 Concepts

In the present work we will keep the symbols $\sqcup$, $\sqcap$, $\neg$, $\bot$, and $\top$ to denote the constructors of weak disjunction, weak conjunction, residuated negation, empty and universal concepts, respectively. Moreover, as in [Cerami et al., 2010b], the following symbols for the new propositional constructors will be adopted:

- $\boxplus$ for strong disjunction,
- $\boxtimes$ for strong conjunction,
- $\sqsupset$ for residuated implication,
- $\sim$ for strong complementation,
- a symbol $\overline{r}$ for each $r \in T \setminus \{0, 1\}$,
- $\blacktriangle$ for Monteiro-Baaz operator.

The classical signature can be maintained in a multi-valued framework, however here they will denote fuzzy sets and fuzzy relations. A *description signature* is a tuple $\mathcal{D} = \langle N_I, N_A, N_R \rangle$, where:

- $N_I = \{a, b, \dots\}$ is a countable set of *individual names*,
- $N_A = \{A, B, \dots\}$ is a countable set of *atomic concepts* or *concept names*,
- $N_R = \{R, S, \dots\}$ is a countable set of *atomic roles* or *role names.*

The logical symbols are: a subset of the propositional constructors considered above, plus the quantifiers $\forall, \exists$. We will keep using the point "." and parenthesis "(" and ")" as auxiliary symbol with the usual syntax.

The language considered in the present work will be called $\mathfrak{I}\mathcal{ALCE}$. Given $A \in N_A$, and $R \in N_R$, a *complex concept* in the language $\mathfrak{I}\mathcal{ALCE}$ is inductively defined in accordance to the following syntactic rules:

C, D	$\longrightarrow$	$\bot$	empty concept	$\mathcal{FL}_0$
		$\top$	universal concept	$\mathcal{FL}_0$
		$\overline{r}$	constant concept	$\mathcal{FL}_0$
		A	atomic concept	$\mathcal{FL}_0$
		$C \boxtimes D$	strong conjunction	$\mathcal{FL}_0$
		$\forall R.C$	value restriction	$\mathcal{FL}_0$
		$\exists R.\top$	restricted existential quantif.	$\mathcal{FL}^-$
		$\sim A$	atomic complementation	$\mathcal{AL}$
		$\blacktriangle C$	delta operator	$\mathcal{D}$
		$C \sqsupset D$	implication	$\mathfrak{I}$
		$C \sqcap D$	weak conjunction	$\mathfrak{I}$
		$\sim C$	complementation	$\mathcal{C}$
		$C \boxplus D$	strong disjunction	$\mathcal{U}$
		$\exists R.C$	existential quantification $\mathcal{E}$	

The notation proposed here is thought in order to maintain, as much as possible, the similarity with classical DL notation while, at the same time, introducing the notation used in the framework of MFL. So:

- The language $\mathcal{FL}_0$ is, as in the classical case, with
  - the empty concept $\bot$,
  - the universal concept $\top$,
  - the strong conjunction $\boxtimes$,
  - the value restriction $\forall$,
  - the constant concepts $\overline{r}$,

  as concept constructors. The choice of the symbol $\boxtimes$ for strong conjunction is due to the aim of maintaining a notation that is sometimes in MFL. The presence of constant concepts $\overline{r}$ in language $\mathcal{FL}_0$ is due to the fact that it contains already the classical constant concepts $\bot$ and $\top$.

- The language $\mathcal{FL}^-$ is built, as in the classical case, by adding the restricted existential quantification $\exists R.\top$ to $\mathcal{FL}_0$.

- The language $\mathcal{AL}$ is built, again, as in the classical case, by adding the atomic complementation $\sim A$ to language $\mathcal{FL}^-$. In this case, we will use the symbol $\sim$ for complementation, as for languages that include $\mathcal{ALC}$, because it is traditionally used in MFL to denote the involutive negation.

- We introduce the symbol $\mathcal{D}$ for languages that have Delta operator $\blacktriangle$.

- We prefix the symbol $\mathfrak{I}$ in those that have implication $\sqsupset$.

- Languages $X\mathcal{U}$ are those that contain strong disjunction $\boxplus$.

- The name for languages that include the unrestricted existential quantification $\exists R.C$ will be maintained in $X\mathcal{E}$ as in the classical case.

Note that residuated negation $\neg$, weak conjunction $\sqcap$ and weak disjunction $\sqcup$ are present in languages that include $\mathfrak{I}\mathcal{AL}$ because, if the algebra of truth values $\mathbf{T}$ is a BL chain, these operators are definable from the implication $\sqsupset$ and either the strong conjunction $\boxtimes$ or the empty concept $\bot$. In fact:

- The constructor of weak conjunction $\sqcap$ is definable from the implication and the strong conjunction in the following way:

$$C \sqcap D := C \boxtimes (C \sqsupset D).$$

  In MTL logic weak conjunction is a primitive connective, for this reason we present it as a primitive concept constructor. Nevertheless, in order to maintain the uniformity of the naming system for FDL languages, we can adopt the convention that the weak conjunction constructor $\sqcap$ is introduced in languages that include $\mathfrak{I}\mathcal{AL}$.

- The constructor of weak disjunction $\sqcup$ is definable from the implication and the weak conjunction in the following way:

$$C \sqcup D := ((C \sqsupset D) \sqsupset D) \sqcap ((D \sqsupset C) \sqsupset C).$$

- The constructor of residuated negation $\neg$ is definable from the implication and the empty concept in the following way:

$$\neg C := C \sqsupset \bot.$$

Hence, these operators are abbreviations in language $\mathfrak{I}\mathcal{AL}$ and in every language expanding it.

3.1.2 Knowledge bases

In the classical framework it is enough to state or deny a given axiom $C \sqsubseteq D$ in order to express that this formula is true or not. In our framework we are rather interested on reasoning on partial truth of formulas. With truth constants in the language we can handle graded inclusion axioms in addition to graded assertion axioms, as usually done in the literature on FDL (see, for example [Straccia, 2004a], [Cerami et al., 2010b]). A graded axiom, either inclusion, like $\langle C \sqsubseteq D \geq r \rangle$, or assertion, like $\langle C(a) \geq r \rangle$, is meant to state either a bound (either upper or lower) or an exact value for the inclusion or the assertion involved. Moreover the bound can be meant to be strict or not.

The first place where fuzzy axioms have been defined this way is [Straccia, 1998], since in previous papers the notion of fuzzy axioms was not considered. In [Straccia, 1998], just non-strict lower bound axioms are considered. In [Straccia, 2001] have been introduced axioms stating a non-strict upper bound as well, but strict bound axioms are not considered. Since then, some works on FDL consider strict bound axioms, like [Straccia, 2004a, Stoilos et al., 2005a, Straccia, 2006, Bobillo and Straccia, 2010] and some others do not consider strict bound axioms, like [Straccia, 2004b, Straccia, 2005b, Bobillo and Straccia, 2007, Cerami et al., 2010b, Bobillo et al., 2011, Baader and Peñaloza, 2011a, Borgwardt and Peñaloza, 2011c, Cerami et al., 2012].

A *fuzzy concept inclusion axiom* (or *fuzzy inclusion*) is an expression of one of the following four forms:

$$\langle C \sqsubseteq D \geq r \rangle \quad \text{non-strict lower bound inclusion axioms} \tag{3.1}$$
$$\langle C \sqsubseteq D \leq r \rangle \quad \text{non-strict upper bound inclusion axioms} \tag{3.2}$$
$$\langle C \sqsubseteq D > r \rangle \quad \text{strict lower bound inclusion axioms} \tag{3.3}$$
$$\langle C \sqsubseteq D < r \rangle \quad \text{strict upper bound inclusion axioms} \tag{3.4}$$

where C, D are concepts and $r \in T$.

A *fuzzy concept assertion axiom* (or *fuzzy assertion*) is an expression of one of the following four forms:

$$\langle C(a) \geq r \rangle \quad \text{non-strict lower bound assertion axioms} \tag{3.5}$$
$$\langle C(a) \leq r \rangle \quad \text{non-strict upper bound assertion axioms} \tag{3.6}$$
$$\langle C(a) > r \rangle \quad \text{strict lower bound assertion axioms} \tag{3.7}$$
$$\langle C(a) < r \rangle \quad \text{strict upper bound assertion axioms} \tag{3.8}$$

where C is a concept, a is an individual constant and $r \in T$.

Finally, a *fuzzy role assertion axioms* (or *fuzzy role assertion*) is an expression of the form:

$$\langle R(a, b) \geq r \rangle \quad \text{role assertion axioms} \tag{3.9}$$

where R is an atomic role, a, b are individual constants and $r \in T$. Note that for roles we only consider non-strict lower bound role assertions $\langle R(a,b) \geq r \rangle$. This is due to the fact that in the literature are not considered other kind of axioms but these.

As in the classical case, a KB for the languages that fall within the scope of this work has two components: TBox and ABox.

A fuzzy TBox for an FDL language is a finite set of fuzzy inclusions. A fuzzy ABox is a finite set of fuzzy assertions and role assertions. A fuzzy KB is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where the first component is a fuzzy TBox and the second one is a fuzzy ABox.

3.2 Semantics

A *fuzzy interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a nonempty (crisp) set $\Delta^{\mathcal{I}}$ (called *domain*) and of a *fuzzy interpretation function* $\cdot^{\mathcal{I}}$ that assigns:

1. to each concept name $A \in N_C$ a fuzzy set, that is, a function $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to \mathbf{T}$,

2. to each role name $R \in N_R$ a fuzzy relation, that is, a function $R^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to \mathbf{T}$,

3. to each individual name $a \in N_I$ an object $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ such that $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ if $a \neq b$ (*Unique Name Assumption*, different individuals denote different objects of the domain).

The semantics of complex concepts is inductively defined as follows:

$$\begin{array}{rcl}
\bot^{\mathcal{I}}(x) & := & 0 \\
\top^{\mathcal{I}}(x) & := & 1 \\
\overline{r}^{\mathcal{I}}(x) & := & r \\
(\sim C)^{\mathcal{I}}(x) & := & 1 - C^{\mathcal{I}}(x) \\
(\blacktriangle C)^{\mathcal{I}}(x) & := & \triangle C^{\mathcal{I}}(x) \\
(C \boxtimes D)^{\mathcal{I}}(x) & := & C^{\mathcal{I}}(x) * D^{\mathcal{I}}(x) \\
(C \sqcap D)^{\mathcal{I}}(x) & := & \min\{C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\} \\
(C \boxplus D)^{\mathcal{I}}(x) & := & C^{\mathcal{I}}(x) \veebar D^{\mathcal{I}}(x) \\
(C \sqsupset D)^{\mathcal{I}}(x) & := & C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \\
(\forall R.C)^{\mathcal{I}}(x) & := & \inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\} \\
(\exists R.C)^{\mathcal{I}}(x) & := & \sup_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) * C^{\mathcal{I}}(y)\}
\end{array}$$

Hence, for every complex concept C we get a function $C^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to [0,1]$.

From the semantics of concepts we can define the semantics of fuzzy axioms. We say that a $\mathbf{T}$-interpretation $\mathcal{I}$ *satisfies* the axioms 3.1, 3.1, 3.3 and 3.4 respectively, if

$$\inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \geq r \tag{3.10}$$
$$\inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \leq r \tag{3.11}$$
$$\inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} > r \tag{3.12}$$
$$\inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} < r \tag{3.13}$$

We say that a **T**-interpretation $\mathcal{I}$ *satisfies* the axioms 3.5, 3.6, 3.7 and 3.8 respectively if

$$C^{\mathcal{I}}(a^{\mathcal{I}}) \geq r \tag{3.14}$$
$$C^{\mathcal{I}}(a^{\mathcal{I}}) \leq r \tag{3.15}$$
$$C^{\mathcal{I}}(a^{\mathcal{I}}) > r \tag{3.16}$$
$$C^{\mathcal{I}}(a^{\mathcal{I}}) < r \tag{3.17}$$

We say that a **T**-interpretation $\mathcal{I}$ *satisfies* the axiom 3.9, if

$$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq r \tag{3.18}$$

3.2.1 Witnessed, quasi-witnessed and strongly witnessed interpretations

In the literature have been considered different kinds of interpretations. In the definitions that we propose there is no restriction on the kind of interpretation considered, as it has been done in the first works on FDL. In [Hájek, 2005] the notion of *witnessed interpretation* has been introduced. Since then most researchers preferred to restrict the reasoning tasks to witnessed interpretations because it seems a quite natural restriction and reasoning tasks so restricted have a very good computational behavior.

Definition 17 (*Witnessed interpretation*, [Hájek, 2005])**.** An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is *witnessed* in case that

(wit∃) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\exists R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a, b) * C^{\mathcal{I}}(b),$$

(wit∀) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\forall R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a, b) \Rightarrow C^{\mathcal{I}}(b).$$

Notice that, in Definition 17, we do not ask every formula to be witnessed, just the ones that are the semantics of quantified concepts. In this sense, a witnessed interpretation is not just an FDL interpretation obtained by translating a witnessed model $(\mathbf{M}, \mathbf{T})$ defined for first order logic in Section 2.1.3 to an FDL interpretation $\mathcal{I}_{\mathbf{M}}$, as done in Section 3.6, because it is restricted to certain kinds of formulas.

Following the definition of *closed model* from [Laskowski and Malekpour, 2007], in [Cerami et al., 2010a] the notion of *quasi-witnessed interpretation* has been introduced.

Definition 18 (*Quasi-witnessed interpretation*, [Cerami et al., 2010a]). An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is *quasi-witnessed* when it satisfies condition (**wit**∃) and

(qwit∀) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ either $(\forall R.C)^{\mathcal{I}}(a) = 0$ or there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\forall R.C)^{\mathcal{I}}(a) \;=\; R^{\mathcal{I}}(a, b) \Rightarrow C^{\mathcal{I}}(b).$$

Again, in Definition 18, we do not ask every formula to be quasi-witnessed, just the ones that are the semantics of quantified concepts. In this sense, a quasi-witnessed interpretation is not just an FDL interpretation obtained by translating a closed model $(\mathbf{M}, \mathbf{T})$ defined for first order logic in Section 2.1.3 to an FDL interpretation $\mathcal{I}_{\mathbf{M}}$, as done in Section 3.6. They are rather a generalization of witnessed interpretations.

In [Baader and Peñaloza, 2011b] the notion of *strongly witnessed interpretation* has been introduced.

Definition 19 (*Strongly witnessed interpretation*, [Baader and Peñaloza, 2011b]). An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is *strongly witnessed* when it satisfies conditions (**wit**∃), (**wit**∀) and

(swit∀) for every pair of concepts C, D, there is some $b \in \Delta^{\mathcal{I}}$ such that

$$\inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} = C^{\mathcal{I}}(b) \Rightarrow D^{\mathcal{I}}(b)$$

In this case too, in Definition 19, we do not ask every formula to be strongly witnessed, just the ones that are the semantics of quantified concepts. In this sense, a strongly witnessed interpretation is not just an FDL interpretation obtained by translating a quasi-witnessed model $(\mathbf{M}, \mathbf{T})$ defined for first order logic in Section 2.1.3 to an FDL interpretation $\mathcal{I}_{\mathbf{M}}$, as done in Section 3.6. They are rather a generalization of witnessed interpretations.

This further restriction to the notion of witnessed interpretation is not so much used because, if $\mathbf{T}$ is a continuous chain, it imposes too strict constraints to the interpretations considered. Moreover, as said in [Baader and Peñaloza, 2011b], "it does not capture the spirit of fuzzy concept inclusions", since "it is not really necessary that the infimum of the values for the residuum is indeed reached". On the other hand, if $\mathbf{T}$ is a finite chain, it is straightforward that every interpretation $\mathcal{I}$ is strongly witnessed.

3.3 The Hierarchy of basic FDL languages

Due to the above defined semantics, in our framework the languages $\mathcal{ALE}$ and $\mathfrak{I}\mathcal{AL}$ are not strictly contained in $\mathcal{ALC}$. This is due to the fact that, in most many-valued logics, implication is not definable from conjunction and negation (neither the residuated negation, nor the involutive one). Moreover, the existential quantifier is not definable from the universal one by means of the negation (neither the residuated negation, nor the involutive one) in the same way as it is done in classical DL.

Since in our framework we do not have the same possibility of reducing languages like in the classical case, the hierarchy of basic languages obtained is more cumbersome. The new hierarchy of basic languages can be represented as in Figure 3.1, that shows the partially ordered set of inclusions among the languages obtained by successively adding a basic operator or another. Strong union is definable from strong intersection and strong negation by a De Morgan law, i.e., as

$$C \boxplus D := \sim (\sim C \boxtimes \sim D)$$

Hence, the language $\mathcal{ALU}$ is strictly contained in the language $\mathcal{ALC}$.

Notice that the supremum of the poset in Figure 3.1 will be called in our framework $\mathfrak{I}\mathcal{ALCE}$, instead of $\mathcal{ALC}$, as in the classical case.

Notice that the general hierarchy in Fig. 3.1 can be simplified when we deal with (either finite- or infinite-valued) Łukasiewicz Logic. In this case, indeed, the fact that the residuated negation is involutive implies that $\mathcal{E}$ and $\mathcal{U}$ are definable by duality from value restriction and the strong conjunction, respectively, by means of the involutive negation as is usually done in classical DL and the same holds for the constructor of implication. Thus, in the case of FDLs based on Łukasiewicz Logic, the languages $\mathcal{ALCE}$, $\mathfrak{I}\mathcal{ALC}$, $\mathfrak{I}\mathcal{ALCE}$ coincide with $\mathcal{ALC}$.

3.4 Reasoning tasks

Among the reasoning tasks that can be defined in a multi-valued framework we find the generalization of the ones that are usual in a classical framework. Being the logic many-valued, these tasks can be considered in their graded versions. In addition to these reasoning tasks in the literature have been defined more tasks which are proper of a multi-valued framework and that we report in the following list. In what follows, let $r \in T$.

- Different notions of the **concept satisfiability** task have been considered in the literature:
  - it can be meant as *lower bound* r-satisfiability, that is, the problem whether, for concept C, there exists a $\mathbf{T}$-interpretation $\mathcal{I}$ and an object $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) \geq r$, in this case we say that concept C is $\geq r$*-satisfiable*;

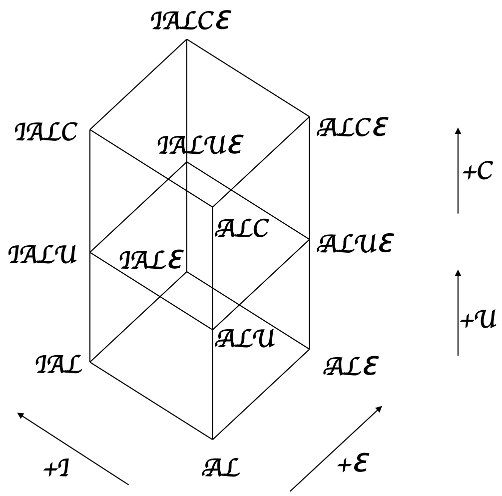


Figure 3.1: Hierarchy of basic languages

- it can be meant as *exact value* r-satisfiability, that is, the problem whether, for concept C, there exists a **T**-interpretation $\mathcal{I}$ and an object $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = r$, in this case we say that concept C is *r-satisfiable.* In the particular case when $r = 1$, we will simply say that C is *satisfiable.*
- it can be meant as *positive* satisfiability, that is, the problem whether, for concept C, there exists a **T**-interpretation $\mathcal{I}$, an object $a \in \Delta^{\mathcal{I}}$ and a truth value $s \in T$ with $s > 0$, such that $C^{\mathcal{I}}(a) = s$, in this case we say that concept C is *positively satisfiable* or *consistent.*

Notice, however, that depending on the chain **T** considered, the notions of $\geq r$-satisfiability and r-satisfiability may not make sense when $r < 1$. It is well known (see [Hájek, 1998b]) that, if there is a valuation e from the set of propositional formulas to a Gödel or product chain **T** such that, for a formula φ, it holds that $e(\varphi) = r < 1$, then it can be found a valuation e' such that $e'(\varphi) = r'$ with $1 > r' \neq r$. So, in what follows, we will speak about 1-satisfiability and positive satisfiability in the case of Gödel and product chains. The notions of $\geq r$- and r-satisfiability will be used only when **T** is a Łukasiewicz chain.

- **Concept r-subsumption** is the problem whether, given concepts C, D, for every **T**-interpretation $\mathcal{I}$ and every $a \in \Delta^{\mathcal{I}}$, it holds that $C^{\mathcal{I}}(a) \Rightarrow$

$D^{\mathcal{I}}(a) \geq r$, in this case we say that concept D *subsumes* concept C *in a degree greater or equal to* r (or that D r*-subsumes* C).

- **Fuzzy knowledge base consistency** is the problem whether, for a given fuzzy KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ there is a **T**-interpretation $\mathcal{I}$ such that, for every fuzzy inclusion of type 3.1, 3.2, 3.3 and 3.4 occurring in $\mathcal{T}$, every fuzzy assertion of type 3.5, 3.6, 3.7 and 3.8 and every fuzzy role assertion of type 3.9 occurring in $\mathcal{A}$, in-equations 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17 and 3.18 respectively hold; in this case we say that the KB is *consistent* and that $\mathcal{I}$ *satisfies* $\mathcal{K}$, in symbols $\mathcal{I} \models \mathcal{K}$.
- **Concept r-satisfiability with respect to a knowledge base** is the problem whether, for a given fuzzy concept C and a fuzzy KB $\mathcal{K}$, there is a **T**-interpretation $\mathcal{I}$ which, on the one hand, satisfies $\mathcal{K}$ and, on the other, satisfies C to a degree greater or equal to r; in this case we say that C is *r-satisfiable w.r.t. $\mathcal{K}$*.
- **Entailment of an axiom by a knowledge base** is the problem whether, for a given fuzzy axiom φ and a fuzzy KB $\mathcal{K}$, every **T**-interpretation $\mathcal{I}$ which satisfies $\mathcal{K}$, also satisfies φ; in this case we say that $\mathcal{K}$ *entails* φ, in symbols $\mathcal{K} \models \varphi$.
- The **best satisfiability degree of a concept with respect to a KB** (defined in [Straccia and Bobillo, 2007]) is the problem of determining, for a given fuzzy concept C and a fuzzy knowledge base $\mathcal{K}$, which is the supremum of the $r \in T$ with respect to which C is $\geq r$-satisfiable with respect to $\mathcal{K}$; that is, $bsd(\mathcal{K}, C) = \sup_{\mathcal{I} \models \mathcal{K}}\{\sup_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x)\}\}$.
- The **best entailment degree of an axiom with respect to a KB** (defined in [Straccia, 2001]) is the problem of determining, for a given (non-fuzzy) axiom $\varphi = C \sqsubseteq D$ or $\varphi = C(a)$ and a fuzzy knowledge base $\mathcal{K}$, which is the supremum of the $r \in T$ with respect to which $\langle \varphi \geq r \rangle$ is entailed by $\mathcal{K}$; that is, $bed(\mathcal{K}, \varphi) = \sup\{r \colon \mathcal{K} \models \langle \varphi \geq r \rangle\}$.

3.5 Reductions

In the classical framework it is usual to consider reductions between reasoning tasks in order to apply procedures, that have been designed for a given task, to other tasks that are reducible to the given one. In this section we will study not only what kinds of reduction between reasoning tasks are achievable within the FDL framework, but also when some types of axioms can be reduced to other ones. In this way, in some cases, the rules for writing axioms can be simplified.

3.5.1 Reductions between axioms

Since in FDL there is the possibility of considering a graded notion of subsumption, equivalence and assertion, there are obviously more types of fuzzy axioms in FDL than crisp axioms in classical DL, as we have seen in Section 3.1.2. A question that naturally arises is, then, whether those types can be reduced, that is, whether there are axioms that can be defined in terms of other axioms, as it is done in classical DL for the case e.g. of the equivalence axioms, that can be expressed as a conjunction of inclusion axioms. In this section we will consider some cases that depend neither on the FDL language considered nor on the particular algebra of truth values adopted as a semantics, as well as cases that depend either on the language or on the algebra of truth values.

Reductions that depend neither on the language nor on the algebra of truth values

In the literature are often considered exact value axioms $\langle C \sqsubseteq D = r\rangle$ and $\langle C(a) = r\rangle$ as well as equivalence axioms $\langle C \equiv D \rhd r\rangle$. Here we explain why we are not considering them in our framework.

Exact value axioms Axioms of types:

$$\langle C \sqsubseteq D = r\rangle, \qquad \langle C(a) = r\rangle$$

are abbreviations for the simultaneous presence of non-strict lower and upper bound axioms, i.e. axioms 3.1 and 3.2 in the first case, and axioms 3.5 and 3.6 in the second case. In other words, the knowledge base $\mathcal{K} \cup \{\langle C \sqsubseteq D = r\rangle\}$ can be substituted by the knowledge base $\mathcal{K} \cup \{\langle C \sqsubseteq D \geq r\rangle, \langle C \sqsubseteq D \leq r\rangle\}$. The same holds for axioms $\langle C(a) = r\rangle$.

Fuzzy equivalence axioms As in the classical framework, the constraints expressed by means of equivalence axioms can be expressed by means of the simultaneous presence of two fuzzy inclusion axioms. This means that the knowledge base $\mathcal{K} \cup \{\langle C \equiv D \rhd r\rangle\}$ can be substituted by the knowledge base $\mathcal{K} \cup \{\langle C \sqsubseteq D \rhd r\rangle, \langle D \sqsubseteq C \rhd r\rangle\}$, with $r \in T$ and $\rhd \in \{\geq, >, =\}$. This is due to the convention that the satisfiability of a set of axioms is the same as the satisfiability of the weak conjunction of these axioms. This convention exploits the good behavior that weak conjunction presents with respect to quantifiers. In fact, for every concepts C and D, $r \in T$ and every interpretation $\mathcal{I}$, it holds that:

$$
\begin{aligned}
& \mathcal{I} \models \langle C \equiv D \geq r\rangle \\
\iff\ & \inf_{x \in \Delta^{\mathcal{I}}}\{(C(x) \Rightarrow D(x)) * (D(x) \Rightarrow C(x))\} \geq r \\
\iff\ & \inf_{x \in \Delta^{\mathcal{I}}}\{\min\{C(x) \Rightarrow D(x), D(x) \Rightarrow C(x)\}\} \geq r \\
\iff\ & \min\{\inf_{x \in \Delta^{\mathcal{I}}}\{C(x) \Rightarrow D(x)\}, \inf_{x \in \Delta^{\mathcal{I}}}\{D(x) \Rightarrow C(x)\}\} \geq r \\
\iff\ & \text{both } \inf_{x \in \Delta^{\mathcal{I}}}\{C(x) \Rightarrow D(x)\} \rhd r \text{ and } \inf_{x \in \Delta^{\mathcal{I}}}\{D(x) \Rightarrow C(x)\} \geq r \\
\iff\ & \text{both } \mathcal{I} \models \langle C \sqsubseteq D \rhd r\rangle \text{ and } \mathcal{I} \models \langle D \sqsubseteq C \geq r\rangle
\end{aligned}
$$

and the same holds true with a strict lower bound $>$ or an equality $=$, instead of a non-stricu lower bound.

If $\rhd \in \{\leq, <\}$, then it is enough with adding just one of the two fuzzy inclusions $\langle C \sqsubseteq D \rhd r\rangle$ and $\langle D \sqsubseteq C \rhd r\rangle$ to $\mathcal{K}$.

Notice that, if, for giving a semantics to the simultaneous presence of more axioms, a strong conjunction is used, instead of the weak one, equivalence axioms can not, in general, be re-defined by means of two inclusions. Under Łukasiewicz semantics, for example, the infimum of an equivalence does not need to take the same value of the strong conjunction of the infimums of two inclusions. Consider indeed the signature $\mathcal{D} = \langle N_C \rangle$, where $N_C = \{A, B\}$. Moreover, consider the interpretation $\mathcal{I}$, where:

- $\Delta^{\mathcal{I}} = \{a, b\}$,
- $A^{\mathcal{I}}(a) = 0.5, \qquad A^{\mathcal{I}}(b) = 0.8, \qquad B^{\mathcal{I}}(a) = 0.6, \qquad B^{\mathcal{I}}(b) = 0.5,$

Then, on the one hand, $\inf_{x \in \Delta^{\mathcal{I}}}\{(A^{\mathcal{I}}(x) \Rightarrow B^{\mathcal{I}}(x)) * (B^{\mathcal{I}}(x) \Rightarrow A^{\mathcal{I}}(x))\} = 0.7$. But, on the other hand, $\inf_{x \in \Delta^{\mathcal{I}}}\{A^{\mathcal{I}}(x) \Rightarrow B^{\mathcal{I}}(x)\} * \inf_{x \in \Delta^{\mathcal{I}}}\{B^{\mathcal{I}}(x) \Rightarrow A^{\mathcal{I}}(x)\} = 0.6$.

Reductions that depend on the language

In languages that are enough expressive, the number of axioms proposed in Section 3.1 can be made smaller. Here we see when and how this can be achieved.

Fuzzy assertion axioms In every language that contains language $\mathfrak{I}\mathcal{F}\mathcal{L}_0\mathcal{C}\mathcal{D}$ (that is, every language that contains truth constants, implication concept constructor $\sqsupset$, Delta operator $\blacktriangle$ and involutive negation $\sim$) every fuzzy assertion axiom can be rewritten in terms of a non-strict lower bound assertion axiom $\langle C(a) \geq r\rangle$. We provide the rewriting case by case:

- Axioms of type $\langle C(a) \leq r\rangle$ can be rewritten as $\langle \sim C(a) \geq 1 - r\rangle$.
- Axioms of type $\langle C(a) > r\rangle$ can be rewritten as $\langle \neg \triangle (C \sqsupset \overline{r})(a) \geq 1\rangle$.
- Axioms of type $\langle C(a) < r\rangle$ can be rewritten as $\langle \neg \triangle (\overline{r} \sqsupset C)(a) \geq 1\rangle$.
- Axioms of type $\langle C(a) = r\rangle$ can be rewritten as $\langle((\overline{r} \sqsupset C) \boxtimes (C \sqsupset \overline{r}))(a) \geq 1\rangle$.

This means that we can consider just fuzzy assertion axioms of type $\langle C(a) \geq r\rangle$, because every other type of fuzzy assertion axiom can be rewritten in terms of this type of axioms.

Reductions that depend on the algebra of truth values

Lower bound fuzzy axioms In every language that contains language $\mathfrak{I}\mathcal{FL}_0\mathcal{C}$ (that is, every language that contains truth constants, implication concept constructor $\sqsupset$ and involutive negation $\sim$) and when the algebra of truth values $\mathbf{T}$ is a discrete chain, it is possible to reduce upper bound inclusion axioms to lower bound concept assertions. In fact, for every concepts C and D, $r \in T$, every interpretation $\mathcal{I}$ and an individual name a, it holds that:

$$\begin{aligned} & \mathcal{I} \models \langle C \sqsubseteq D \leq r \rangle \\ \iff & \inf_{x \in \Delta^{\mathcal{I}}} \{C(x) \Rightarrow D(x)\} \leq r \\ \iff & 1 - \inf_{x \in \Delta^{\mathcal{I}}} \{C(x) \Rightarrow D(x)\} \geq 1 - r \\ \iff & \sup_{x \in \Delta^{\mathcal{I}}} \{1 - (C(x) \Rightarrow D(x))\} \geq 1 - r \\ \iff & \mathcal{I} \models \langle \sim (C \sqsupset D)(a) \geq 1 - r \rangle \end{aligned}$$

In the same way the result can be proven for $<$ and $>$. In FDLs does not exist a way to express an upper bound inclusion axiom $\langle C \sqsubseteq D \leq r \rangle$ as a lower bound inclusion axiom $\langle C' \sqsubseteq D' \geq r' \rangle$. This is due to semantic considerations. If the algebra of truth values $\mathbf{T}$ is a discrete chain, an upper bound inclusion axiom states that there is an $\mathbf{T}$-interpretation $\mathcal{I}$ and a domain object $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) \Rightarrow D^{\mathcal{I}}(a) \leq r$. Hence, upper bound inclusion axioms are quite non-standard axioms, because, differently from usual axioms, they have an existential flavour. The exact value inclusion axiom $\langle C \sqsubseteq D = r \rangle$ is the conjunction of the lower bound inclusion axiom $\langle C \sqsubseteq D \geq r \rangle$ and the upper bound inclusion axiom $\langle C \sqsubseteq D \leq r \rangle$. Hence, what it expresses is that, for every interpretation $\mathcal{I}$, not only for every $a \in \Delta^{\mathcal{I}}$ it holds that $(C \sqsupset D)^{\mathcal{I}}(a) \geq r$, but also that there indeed exists one element $b \in \Delta^{\mathcal{I}}$ such that $(C \sqsupset D)^{\mathcal{I}}(b) \geq r$. It is worth notice that the satisfiability of an exact value inclusion axiom $\langle C \sqsubseteq D = r \rangle$ is equivalent to the satisfiability of axiom $\langle C \sqsubseteq D \geq r \rangle$ in a strongly witnessed interpretation. This is indeed different from expressing that a given inclusion $C \sqsubseteq D$ shall have exactly value r for every object of every interpretation. This last constraint can be expressed as the conjunction of the axioms:

- $\langle C \sqsubseteq D \geq r \rangle$,
- $\langle \top \sqsubseteq \sim (C \sqsupset D) \geq 1 - r \rangle$.

Strict upper bound fuzzy axioms From the considerations made about lower bound inclusion axioms and the fact that in general

$$\mathcal{I} \models \langle C \sqsubseteq D > r \rangle \iff \mathcal{I} \not\models \langle C \sqsubseteq D \leq r \rangle$$

we easily obtain that, when $\mathbf{T}$ is a discrete chain, in every language that contains language $\mathfrak{I}\mathcal{FL}_0\mathcal{C}$ it holds that

$$\mathcal{I} \models \langle C \sqsubseteq D > r \rangle \iff \mathcal{I} \not\models \langle \sim (C \sqsupset D)(a) \geq 1 - r \rangle$$

where a is a new individual name. The same is true for strict lower bound assertions.

Strict bound inclusion axioms When the complete chain $\mathbf{T}$ is a finite chain, it is possible to reduce strict bound axioms to non-strict ones. This is due to the fact that, with a finite set of truth values, if a truth value is strictly greater than a given value $r \in T$, then it is greater or equal to the lower truth value greater or equal than r. Note that, when T is a finite linearly ordered set, there is always such a value. So, let $\mathcal{I}$ be an FDL interpretation, $T = \{r_1 < \ldots < r_n\}$ and $1 \leq i \leq n$, then:

- $\mathcal{I} \models \langle C \sqsubseteq D > r_i \rangle \iff \mathcal{I} \models \langle C \sqsubseteq D \geq r_{i+1} \rangle$,
- $\mathcal{I} \models \langle C \sqsubseteq D < r_i \rangle \iff \mathcal{I} \models \langle C \sqsubseteq D \leq r_{i-1} \rangle$,

Smaller syntax for fuzzy axioms

Resuming what explained in the previous subsections, in every language that contains language $\mathfrak{I}\mathcal{FL}_0\mathcal{C}$ and under a discrete chain of truth values, the set of fuzzy axioms needed in order to define every other fuzzy axiom is the following:

$$\langle C \sqsubseteq D \geq r \rangle, \quad \langle C(a) \geq r \rangle, \quad \langle R(a,b) \geq r \rangle \tag{3.19}$$

This means that for every knowledge base $\mathcal{K}$ there exists another knowledge base $\mathcal{K}'$ where just axioms in 3.19 occur.

3.5.2 Reductions between reasoning tasks

Within the classical framework, every reasoning task can be polynomially reduced to knowledge base (in)consistency (see [Baader et al., 2008, pag. 142]). In this section we will see which reductions can be performed in FDLs. In this case, again, we will consider cases that depend neither on the FDL language considered nor on the particular algebra of truth values adopted as a semantics, as well as cases that depend either on the language or on the algebra of truth values. Moreover, we will consider reductions to reasoning tasks other than KB consistency that will be useful in the following chapters.

Reductions to KB consistency that depend neither on the language nor on the algebra of truth values

- Concept r-satisfiability and $\geq r$-satisfiability can be both reduced to knowledge base consistency. Deciding whether a concept C is r-satisfiable ($\geq r$-satisfiable, respectively) is the same as deciding whether the ABox $\mathcal{A} = \{\langle C(a) = r \rangle\}$ (ABox $\mathcal{A} = \{\langle C(a) \geq r \rangle\}$, respectively) is consistent, where a is an individual name. Concept positive satisfiability, as well, can be reduced to knowledge base consistency, but in a different way. Deciding whether a concept C is positively satisfiable, is the same as deciding whether the TBox $\mathcal{T} = \{\langle C \sqsubseteq \perp \geq 1\rangle\}$ is unsatisfiable.

- Concept r-satisfiability with respect to a knowledge base can be reduced to knowledge base consistency. Deciding whether a concept C is r-satisfiable with respect to knowledge base $\mathcal{K}$ is the same as deciding whether knowledge base $\mathcal{K} \cup \{\langle C(a) \geq r\rangle\}$ is consistent, where a is a new individual not appearing in $\mathcal{K}$.

- The best satisfiability degree of a concept with respect to a KB can be reduced to a family of knowledge base consistency problems. In fact, by the previous item, determining the best satisfiability degree of concept C w.r.t. knowledge base $\mathcal{K}$ is the same as determining which is the greater value r such that $\mathcal{K} \cup \{\langle C(a) \geq r\rangle\}$ is consistent, where a is a new individual not appearing in $\mathcal{K}$.

Reductions to KB consistency that depend on the language

- In every language that contains language $\mathfrak{I}\mathcal{F}\mathcal{L}_0\mathcal{C}$ (that is, every language that contains the implication concept constructor $\sqsupset$ and the involutive negation $\sim$), the entailment of an axiom by a knowledge base $\mathcal{K}$ can be reduced to knowledge base consistency. In the case of this reasoning task, the way the reduction is performed, depends on the type of axiom entailed.

  - $\mathcal{K}$ entails axiom $\langle C \sqsubseteq D \geq r\rangle$ iff, for every $s \in T$ such that $s > 0$, $\mathcal{K} \cup \{\langle \neg(\overline{r} \sqsupset (C \sqsupset D))(a) \geq s\rangle\}$ is inconsistent, where a is a new individual not appearing in $\mathcal{K}$.
  - $\mathcal{K}$ entails axiom $\langle C \sqsubseteq D > r\rangle$ iff concept $\sim (C \sqsupset D)$ is not $\geq 1 - r$-satisfiable with respect to knowledge base $\mathcal{K}$ iff $\mathcal{K} \cup \langle \sim (C \sqsupset D)(a) \geq 1 - r\rangle$ is inconsistent.
  - $\mathcal{K}$ entails axiom $\langle C(a) \geq r\rangle$ iff $\mathcal{K} \cup \{\langle \neg(\overline{r} \sqsupset C)(a) \geq 1\rangle\}$ is inconsistent.
  - As we have seen in Section 3.5.1, strict bound fuzzy assertion axioms like $\langle C(a) > r\rangle$ and $\langle C(a) < r\rangle$ need the presence of the Delta operator $\blacktriangle$ in order to be re-written as non-strict lower bound fuzzy assertion axioms. Nevertheless, entailment of such axioms by $\mathcal{K}$, can be reduced to knowledge base consistency without the need of $\blacktriangle$. It is, indeed, easy to see that $\mathcal{K}$ entails axiom $\langle C(a) > r\rangle$ iff $\mathcal{K} \cup \{\langle C(a) \leq r\rangle\}$ is inconsistent, iff $\mathcal{K} \cup \{\langle \sim C(a) \geq 1 - r\rangle\}$ is inconsistent. The same holds for axioms $\langle C(a) < r\rangle$ and $\langle C(a) \geq r\rangle$.

- The best entailment degree of a fuzzy axiom with respect to a KB can be reduced to a family of knowledge base consistency problems. In fact, by the previous item, determining the best entailment degree of an axiom φ w.r.t. knowledge base $\mathcal{K}$ is the same as determining which is the greater value $r \in T$ such that:

- knowledge base $\mathcal{K} \cup \{\langle \neg(\overline{r} \sqsupset (C \sqsupset D))(a) \geq 1 \rangle\}$ is inconsistent, if $\varphi = C \sqsubseteq D$,
- knowledge base $\mathcal{K} \cup \{\langle \neg(\overline{r} \sqsupset C)(a) \geq 1 \rangle\}$ is inconsistent, if $\varphi = C(a)$,

Reductions to KB consistency that depend on the algebra of truth values

- When the complete chain $\mathbf{T} = \{r_1 < \ldots < r_n\}$ is a finite chain, in every language that contains language $\mathfrak{IFL}_0\mathcal{C}$ (that is, every language that contains the implication concept constructor $\sqsupset$ and involutive negation $\sim$), the concept r-subsumption problem is reduced to concept lower bound unsatisfiability. In fact, deciding whether concept D r_i-subsumes concept C is the same as deciding whether concept $\sim (C \sqsupset D)$ is not $\geq 1 - r_{i-1}$-satisfiable.
- When $\mathbf{T}$ is a finite chain deciding whether a concept C is positively satisfiable, is the same as deciding whether the same concept is $\geq r_2$-satisfiable, where r_2 is the smaller element of T strictly greater that 0.
- When $\mathbf{T}$ is a strict chain (i.e., the residuated negation is the Gödel one) deciding whether a concept C is positively satisfiable, is the same as deciding whether the concept $\neg\neg C$ is 1-satisfiable.

Reductions among satisfiability notions

- In a language with truth constants, the $\geq r$-satisfiability can be easily reduced to r-satisfiability, since, for every concept C and $r \in T$, it holds that

$$C \text{ is } \geq r\text{-satisfiable} \iff \overline{r} \sqsupset C \text{ is 1-satisfiable.}$$

- The problem of positive satisfiability, as well, can be reduced to 1-subsumption, that, as we will see later on in Section 4.2, is a decidable problem. In fact, for every concept C it holds that

$$C \text{ is consistent} \iff C \text{ is not 1-subsumed by } \bot.$$

- If the complete chain $\mathbf{T}$ is finite, then both problems can be reduced to r- satisfiability without the help of truth constants or a reduction to subsumption. In fact, for every concept C and $r \in T$, it holds that

$$C \text{ is } \geq r\text{-satisfiable} \iff \text{there is } r' \in T \text{ such that } r' \geq r \text{ and } C \text{ is } r'\text{-satisfiable}$$

and

$$C \text{ is positively satisfiable} \iff C \text{ is} \geq r_{n-1}\text{-satisfiable},$$

where r_{n-1} is the lower truth value strictly greater than 0. The finiteness of T, gives us the decidability of the problem. Nevertheless the previous reductions remains clearly the more efficient, since they need just one 1-satisfiability or 1-subsumption test instead of one r-satisfiability test for each truth value greater than r or 0.

Reducing KB consistency

- Knowledge base consistency can be easily reduced to concept r-satisfiability w.r.t. a knowledge base. Let, in fact, $\mathcal{K}$ be a knowledge base, then:

$$\mathcal{K} \text{ is consistent} \iff \text{concept } \top \text{ is satisfiable w.r.t. } \mathcal{K}.$$

- Knowledge base consistency can be reduced to the entailment of an axiom by a knowledge base. Let, in fact, $\mathcal{K}$ be a knowledge base, then:[1]

$$\begin{aligned} \mathcal{K} \text{ is satisfiable} &\iff \mathcal{K} \not\models \langle \bot(a) \geq 1 \rangle \\ &\iff \mathcal{K} \not\models \langle \top \sqsubseteq \bot \geq 1 \rangle. \end{aligned}$$

3.6 Relation to first order predicate logic

In [Borgida, 1996], Borgida provides a translation of DL concepts into first order classical logic. The relationship between FDL and first order fuzzy logic has been firstly described in [Tresp and Molitor, 1998]. A more systematic investigation on this subject has been undertaken in [García-Cerdaña et al., 2010] and [Cerami et al., 2010b], where it is investigated the idea, presented in [Hájek, 2005] of a Fuzzy Description Logic tightly related to Mathematical Fuzzy Logic. In [Tresp and Molitor, 1998], [García-Cerdaña et al., 2010] and [Cerami et al., 2010b] FDL is indeed presented as a fragment of MFL. Here we will present a quite different way to obtain the same translation and prove that it preserve the meaning of the expressions involved by defining their respective semantics from each other, according to the following schema:

3.6.1 Concepts

Given a description signature $\mathcal{D} = \langle N_I, N_C, N_R \rangle$, we define the first order signature $\mathbf{s}_{\mathcal{D}} = \{c_1, c_2, \ldots\} \cup \{P_1, P_2, \ldots\}$, with $i = |N_I|$, $j = |N_C \cup N_R|$ and where

- $\{c_1, c_2, \ldots\} := N_I$ is a set of constant symbols,
- $\{P_1, P_j, \ldots\} := N_C \cup N_R$ is a set of unary and binary predicates.

[1]The argument is by U. Straccia

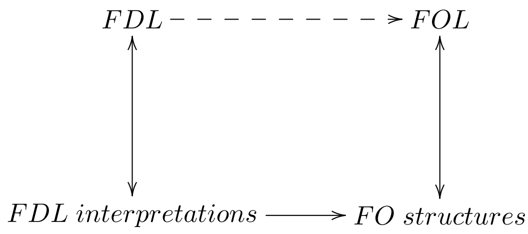


Figure 3.2: Relations to FOL

Let $\mathbf{l}$ be the propositional language of an extension $\mathcal{L}$ of MTL logic and Var a countable set of individual variables. Then, for every concept name $A \in N_C$, every role name $R \in N_R$ and every $x, y \in Var$, we can define the translations

$$\tau^x : N_C \to Fm_{\mathbf{l}\forall,\mathbf{s}_{\mathcal{D}}}$$

and

$$\tau^{x,y} : N_R \to Fm_{\mathbf{l}\forall,\mathbf{s}_{\mathcal{D}}}$$

of concept and role names, respectively, into the set of atomic first order formulas of the logic $\mathcal{L}\forall$, in the following way:

$$\begin{array}{rcl} \tau^x(A) & := & A(x) \\ \tau^x(\bot) & := & \bot, \\ \tau^x(\top) & := & \top, \\ \tau^x(\overline{r}) & := & \overline{r}, \\ \tau^{x,y}(R) & = & R(x,y) \end{array}$$

This translation can be inductively extended over the set of complex concept in the following way:

$$\begin{array}{rcl} \tau^x(\neg C) & := & \neg\tau^x(C), \\ \tau^x(\sim C) & := & \sim \tau^x(C), \\ \tau^x(\blacktriangle C) & := & \triangle\tau^x(C), \\ \tau^x(C \boxtimes D) & := & \tau^x(C) \otimes \tau^x(D) \\ \tau^x(C \sqcap D) & := & \tau^x(C) \wedge \tau^x(D) \\ \tau^x(C \boxplus D) & := & \tau^x(C) \oplus \tau^x(D) \\ \tau^x(C \sqcup D) & := & \tau^x(C) \vee \tau^x(D) \\ \tau^x(C \sqsupset D) & := & \tau^x(C) \to \tau^x(D) \\ \tau^x(\forall R.C) & := & (\forall y)(\tau^{x,y}(R) \to \tau^y(C)), \text{ with } y \neq x \\ \tau^x(\exists R.C) & := & (\exists y)(\tau^{x,y}(R) \otimes \tau^y(C)), \text{ with } y \neq x \end{array}$$

Notice that as a result of such translation, we obtain first order formulas $\tau^x(C)$ with only one free variable, x.

Next we show, in Lemmas 20 and 22 that the translation preserves the same meaning of the original expression through a definition of a first order structure and of an FDL interpretation from each other.

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an FDL interpretation, then we can define the first order structure $\mathbf{M}_{\mathcal{I}} = (M_{\mathcal{I}}, \{P_{\mathbf{M}_{\mathcal{I}}} : P \in N_C \cup N_R\}, \{c_{\mathbf{M}_{\mathcal{I}}} : c \in N_I\})$, where:

- $M_{\mathcal{I}} := \Delta^{\mathcal{I}}$,
- for each concept name $A \in N_C$, $A_{\mathbf{M}_{\mathcal{I}}}$ is the unary function $A_{\mathbf{M}_{\mathcal{I}}} : M_{\mathcal{I}} \to T$, such that, for every $a \in M_{\mathcal{I}}$, it holds that $A_{\mathbf{M}_{\mathcal{I}}}(a) = A^{\mathcal{I}}(a)$,
- for each role name $R \in N_R$, $R_{\mathbf{M}_{\mathcal{I}}}$ is the binary function $R_{\mathbf{M}_{\mathcal{I}}} : M_{\mathcal{I}} \times M_{\mathcal{I}} \to T$, such that, for every $a, b \in M_{\mathcal{I}}$, it holds that $R_{\mathbf{M}_{\mathcal{I}}}(a, b) = R^{\mathcal{I}}(a, b)$,
- for each individual $a \in N_I$, $a_{\mathbf{M}_{\mathcal{I}}}$ is an element of $M_{\mathcal{I}}$, such that $a_{\mathbf{M}_{\mathcal{I}}} = a^{\mathcal{I}}$.

Lemma 20. *Let C be an $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCE}$ concept. Then $\|\tau^x(C)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}_{v([a/x])} = C^{\mathcal{I}}(a)$, for every object $a \in \Delta^{\mathcal{I}}$.*

Proof. The proof is by structural induction on complex concepts.

- For concept names and constant concepts it is straightforward by definition.
- Suppose that the statement holds for concepts C and D. Then $\|\tau^x(C \boxtimes D)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}_{v([a/x])} = \|\tau^x(C) \otimes \tau^x(D)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}_{v([a/x])} = \|\tau^x(C)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}_{v([a/x])} * \|\tau^x(D)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}_{v([a/x])} = C^{\mathcal{I}}(a) * D^{\mathcal{I}}(a) = (C \boxtimes D)^{\mathcal{I}}(a)$. In the same way the statement can be proved also for constructors $\sqcap, \boxplus, \sqcup, \sqsupset, \sim, \blacktriangle$ and $\neg$.
- Suppose that the statement holds for the role name R and for concept C. Then $\|\tau^x(\forall R.C)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}_{v([a/x])} = \|(\forall y)(\tau^{x,y}(R) \to \tau^y(C))\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}_{v([a/x])} = \inf_{y \in M_{\mathcal{I}}}\{\|\tau^{x,y}(R)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}_{v([a/x])} \Rightarrow \|\tau^y(C)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}\}_{v([a/x])} = \inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(a, y) \Rightarrow C^{\mathcal{I}}(y)\} = (\forall R.C)^{\mathcal{I}}(a)$. In the same way the statement can be proved also for concept $\exists R.C$.

So, for every $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCE}$ concept C we have that $\|\tau^x(C)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}_{v([a/x])} = C^{\mathcal{I}}(a)$. □

On the other hand, let $\mathbf{M}$ be a first order structure such that $\mathbf{s}_{\mathcal{D}} = \mathbf{s}_{\mathbf{M}}$, then we can define the interpretation $\mathcal{I}_{\mathbf{M}} = (\Delta^{\mathcal{I}_{\mathbf{M}}}, \cdot^{\mathcal{I}_{\mathbf{M}}})$, where:

- $\Delta^{\mathcal{I}_{\mathbf{M}}} = M$,
- for each concept name $A \in N_C$, $A^{\mathcal{I}_{\mathbf{M}}}$ is the unary function $A^{\mathcal{I}_{\mathbf{M}}} : \Delta^{\mathcal{I}_{\mathbf{M}}} \to T$, such that, for every $a \in \Delta^{\mathcal{I}_{\mathbf{M}}}$, it holds that $A^{\mathcal{I}_{\mathbf{M}}}(a) = A_{\mathbf{M}}(a)$,
- for each role name $R \in N_R$, $R^{\mathcal{I}_{\mathbf{M}}}$ is the binary function $A^{\mathcal{I}_{\mathbf{M}}} : \Delta^{\mathcal{I}_{\mathbf{M}}} \times \Delta^{\mathcal{I}_{\mathbf{M}}} \to T$, such that, for every $a, b \in \Delta^{\mathcal{I}_{\mathbf{M}}}$, it holds that $R^{\mathcal{I}_{\mathbf{M}}}(a, b) = R_{\mathbf{M}}(a, b)$,

- for each individual $a \in N_I$, $a^{\mathcal{I}_\mathbf{M}}$ is an element of $\Delta^{\mathcal{I}_\mathbf{M}}$, such that $a^{\mathcal{I}_\mathbf{M}} = a_\mathbf{M}$.

As a straightforward consequence of the definitions of $\mathbf{M}_\mathcal{I}$ and $\mathcal{I}_\mathbf{M}$, we have the following lemma.

Lemma 21. *For every* $\mathbf{T}$*-interpretation* $\mathcal{I}$ *and every first order structure* $(\mathbf{T}, \mathbf{M})$ *it holds that*

- $\mathcal{I} = \mathcal{I}_{\mathbf{M}_\mathcal{I}}$,
- $\mathbf{M} = \mathbf{M}_{\mathcal{I}_\mathbf{M}}$.

From Lemma 21 and Lemma 20 we can prove a further consequence.

Lemma 22. *Let* C *be an* $\mathbf{T}$-$\mathcal{ALC}$ *concept. Then* $C^{\mathcal{I}_\mathbf{M}}(a) = \|\tau^x(C)\|^{(\mathbf{T},\mathbf{M})}_{v([a/x])}$, *for every object* $a \in M$.

Proof. From Lemma 20 we have that $C^{\mathcal{I}_\mathbf{M}}(a) = \|\tau^x(C)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}_\mathbf{M}})}_{v([a/x])}$. From Lemma 21 we have that $\|\tau^x(C)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}_\mathbf{M}})}_{v([a/x])} = \|\tau^x(C)\|^{(\mathbf{T},\mathbf{M})}_{v([a/x])}$. So, $C^{\mathcal{I}_\mathbf{M}}(a) = \|\tau^x(C)\|^{(\mathbf{T},\mathbf{M})}_{v([a/x])}$. □

Remark 23. The first order language considered could be built by means of a set of just two variables. The limitation to just two variables is enough in order to define the translation only for the kind of first order formulas that correspond to $\mathfrak{I}\mathcal{ALCE}$ concept. In fact, in case of nested quantifier, like in the concept:

$$\forall R.\exists R.\forall R.A$$

we have that the translation is

$$\tau^x(\forall R.\exists R.\forall R.A) = (\forall y)(R(x,y) \rightarrow (\exists x)(R(y,x) \otimes (\forall y)(R(x,y) \rightarrow A(y))))$$

whose meaning, with respect to a structure $(\mathbf{M}, \mathbf{T})$ is

$$\inf_{y \in M}\{R^\mathbf{M}(x,y) \Rightarrow \sup_{x \in M}\{R^\mathbf{M}(y,x) * \inf_{y \in M}\{R^\mathbf{M}(x,y) \Rightarrow A^\mathbf{M}(y)\}\}\}$$

and, since the inner variable "y" is closed, when a value for the outer function "inf" has to be calculated, this variable falls outside its scope.

Moreover, in case of conjugated quantified concepts, like

$$(\forall R.A) \boxtimes (\exists R.B)$$

we have that the translation is

$$\tau^x((\forall R.A) \boxtimes (\exists R.A)) = (\forall y)(R(x,y) \rightarrow A(y)) \otimes (\exists y)(R(x,y) \otimes B(y))$$

whose meaning, with respect to a structure $(\mathbf{M}, \mathbf{T})$ is

$$\inf_{y \in M}\{R^\mathbf{M}(x,y) \Rightarrow A^\mathbf{M}(y)\} * \sup_{y \in M}\{R^\mathbf{M}(x,y) * B^\mathbf{M}(y)\}\}\}$$

where each appearance of variable "y" is closed inside the scope of a different quantifier and, for this reason, it does not fall inside the scope of the other quantifier. □

3.6.2 Fuzzy axioms

First of all, we utilize the translation $\tau^x(\cdot)$, introduced in Section 3.6.1 in order to obtain a corresponding translation τ from fuzzy axioms to first order formulas.

Let x be an individual variable, then, for the fuzzy inclusion axioms, the translation is defined as follows:

$$\tau(\langle C \sqsubseteq D \geq \overline{r}\rangle) := \overline{r} \to (\forall x)(\tau^x(C) \to \tau^x(D)),$$
$$\tau(\langle C \sqsubseteq D \leq \overline{r}\rangle) := (\forall x)(\tau^x(C) \to \tau^x(D)) \to \overline{r},$$
$$\tau(\langle C \sqsubseteq D > \overline{r}\rangle) := \neg \triangle ((\forall x)(\tau^x(C) \to \tau^x(D)) \to \overline{r}),$$
$$\tau(\langle C \sqsubseteq D < \overline{r}\rangle) := \neg \triangle (\overline{r} \to (\forall x)(\tau^x(C) \to \tau^x(D))),$$

For the fuzzy assertion axioms, the translation is defined as follows:

$$\tau(\langle C(a) \geq \overline{r}\rangle) := \overline{r} \to \tau^x(C)[a/x],$$
$$\tau(\langle C(a) \leq \overline{r}\rangle) := \tau^x(C)[a/x] \to \overline{r},$$
$$\tau(\langle C(a) > \overline{r}\rangle) := \neg \triangle (\tau^x(C)[a/x] \to \overline{r}),$$
$$\tau(\langle C(a) < \overline{r}\rangle) := \neg \triangle (\overline{r} \to \tau^x(C)[a/x]),$$

For the fuzzy role assertion axioms, the translation is defined as follows:

$$\tau(\langle R(a,b) \geq \overline{r}\rangle) := \overline{r} \to \tau^{x,y}(R)[a/x, b/y]$$

As for concepts, here again it is possible to show that the translation preserves the meaning of the original expressions, through the same translation between first order structures and FDL interpretations given in Section 3.6.1.

Lemma 24. *Let $\langle \varphi \rhd r\rangle$ be a fuzzy axiom, with $\rhd \in \{\geq, \leq, >, <\}$. Then a $\mathbf{T}$-interpretation $\mathcal{I}$ satisfies $\langle \varphi \rhd r\rangle$ if and only if $(\mathbf{T}, \mathbf{M}_\mathcal{I})$ 1-satisfies $\tau(\langle \varphi \rhd r\rangle)$.*

Proof. Let $\langle \varphi \rhd r\rangle$ be a fuzzy axiom and $\mathcal{I}$ a $\mathbf{T}$-interpretation, then

- If $\langle \varphi \rhd r\rangle = \langle C \sqsubseteq D \geq \overline{r}\rangle$, then $\mathcal{I}$ satisfies $\langle C \sqsubseteq D \geq \overline{r}\rangle$ if and only if $\inf_{x\in\Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} \geq r$. By Lemma 20, we have that $r \leq \inf_{x\in\Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} = \inf_{x\in\Delta^\mathcal{I}}\{\|\tau^x(C)\|^{(\mathbf{T},\mathbf{M}_\mathcal{I})} \Rightarrow \|\tau^x(D)\|^{(\mathbf{T},\mathbf{M}_\mathcal{I})}\} = \inf_{x\in\Delta^\mathcal{I}}\{\|\tau^x(C) \to \tau^x(D)\|^{(\mathbf{T},\mathbf{M}_\mathcal{I})}\} = \|(\forall x)(\tau^x(C) \to \tau^x(D))\|^{(\mathbf{T},\mathbf{M}_\mathcal{I})}$. So, since the residuated implication $\to$ defines an order, we have that structure $(\mathbf{T}, \mathbf{M}_\mathcal{I})$ satisfies $\overline{r} \to (\forall x)(\tau^x(C) \to \tau^x(D)) = \tau(\langle C \sqsubseteq D \geq \overline{r}\rangle)$. In the same way can be proved that the statement holds for axioms of type $\langle C \sqsubseteq D \leq \overline{r}\rangle$.
- If $\langle \varphi \rhd r\rangle = \langle C \sqsubseteq D > \overline{r}\rangle$, then $\mathcal{I}$ satisfies $\langle C \sqsubseteq D > \overline{r}\rangle$ if and only if $\inf_{x\in\Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} > r$. Hence $\inf_{x\in\Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} \not\leq r$ i.e. $\mathcal{I}$ does not satisfy axiom $\langle C \sqsubseteq D \leq \overline{r}\rangle$. By the previous result we have that $(\mathbf{T}, \mathbf{M}_\mathcal{I})$ does not satisfy $\tau(\langle C \sqsubseteq D \leq \overline{r}\rangle) = (\forall x)(\tau^x(C) \to \tau^x(D)) \to \overline{r}$. Then $\|(\forall x)(\tau^x(C) \to \tau^x(D)) \to \overline{r}\|^{(\mathbf{T},\mathbf{M}_\mathcal{I})} < 1$ and, therefore, $\|\neg \triangle (\forall x)(\tau^x(C) \to \tau^x(D)) \to \overline{r}\|^{(\mathbf{T},\mathbf{M}_\mathcal{I})} = 1$. In the same way can be proved that the statement holds for axioms of type $\langle C \sqsubseteq D < \overline{r}\rangle$.

- If $\langle \varphi \rhd r\rangle = \langle C(a) \geq \overline{r}\rangle$, then $\mathcal{I}$ satisfies $\langle C(a) \geq \overline{r}\rangle$ if and only if $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq r$. By Lemma 20, we have that $r \leq C^{\mathcal{I}}(a^{\mathcal{I}}) = \|\tau^x(C)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}_{[a_{\mathbf{M}_{\mathcal{I}}}/x]} = \|\tau^x(C)[a/x]\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}$. So, since the residuated implication $\rightarrow$ defines an order, we have that structure $(\mathbf{T},\mathbf{M}_{\mathcal{I}})$ satisfies $\overline{r} \rightarrow \tau^x(C)[a/x] = \tau(\langle C(a) \geq \overline{r}\rangle)$. In the same way can be proved that the statement holds for axioms of type $\langle C(a) \leq \overline{r}\rangle$.
- If $\langle \varphi \rhd r\rangle = \langle C(a) > \overline{r}\rangle$, then $\mathcal{I}$ satisfies $\langle C(a) > \overline{r}\rangle$ if and only if $C^{\mathcal{I}}(a^{\mathcal{I}}) > r$. Hence $C^{\mathcal{I}}(a^{\mathcal{I}}) \nleq r$ i.e. $\mathcal{I}$ does not satisfy axiom $\langle C(a) \leq \overline{r}\rangle$. By the previous result we have that $(\mathbf{T},\mathbf{M}_{\mathcal{I}})$ does not satisfy $\tau(\langle C(a) \leq \overline{r}\rangle) = \tau^x(C)[a/x] \rightarrow \overline{r}$. Then $\|\tau^x(C)[a/x] \rightarrow \overline{r}\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})} < 1$ and, therefore, $\|\neg \triangle \tau^x(C)[a/x] \rightarrow \overline{r}\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})} = 1$. In the same way can be proved that the statement holds for axioms of type $\langle C(a) < \overline{r}\rangle$.
- If $\langle \varphi \rhd r\rangle = \langle R(a,b) \geq \overline{r}\rangle$, then $\mathcal{I}$ satisfies $\langle R(a,b) \geq \overline{r}\rangle$ if and only if $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq r$. By the definition of $(\mathbf{T},\mathbf{M}_{\mathcal{I}})$, we have that $r \leq R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = \|\tau^{x,y}(R)\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}_{[a_{\mathbf{M}_{\mathcal{I}}}/x, b_{\mathbf{M}_{\mathcal{I}}}/y]} = \|\tau^{x,y}(R)[a/x,b/y]\|^{(\mathbf{T},\mathbf{M}_{\mathcal{I}})}$. So, since the residuated implication $\rightarrow$ defines an order, we have that structure $(\mathbf{T},\mathbf{M}_{\mathcal{I}})$ satisfies $\overline{r} \rightarrow \tau^{x,y}(R)[a/x,b/y] = \tau^{x,y}(\langle R(a,b) \geq \overline{r}\rangle)$.

So, for every fuzzy axiom $\langle \varphi \rhd r\rangle$ it holds that a $\mathbf{T}$-interpretation $\mathcal{I}$ satisfies $\langle \varphi \rhd r\rangle$ if and only if $(\mathbf{T},\mathbf{M}_{\mathcal{I}})$ satisfies $\tau(\langle \varphi \rhd r\rangle)$. □

Remark 25. In FDLs where the residuated negation is Gödel negation, as well as in FDLs based on finite-valued Łukasiewicz Logic there is no need of operator $\triangle$ in order to translate strict axioms, since this operator is definable within the language either as:

$$\triangle x := \neg \sim x$$

or as:

$$\triangle x := x^{n-1}$$

Let φ be an axiom and $r \in T$. If the residuated negation is Gödel, then we have that

$$\tau(\langle C \sqsubseteq D > \overline{r}\rangle) := \neg\neg \sim ((\forall x)(\tau^x(C) \rightarrow \tau^x(D)) \rightarrow \overline{r}),$$
$$\tau(\langle C \sqsubseteq D < \overline{r}\rangle) := \neg\neg \sim (\overline{r} \rightarrow (\forall x)(\tau^x(C) \rightarrow \tau^x(D))),$$
$$\tau(\langle C(a) > \overline{r}\rangle) := \neg\neg \sim (\tau^x(C)[a/x] \rightarrow \overline{r}),$$
$$\tau(\langle C(a) < \overline{r}\rangle) := \neg\neg \sim (\overline{r} \rightarrow \tau^x(C)[a/x]),$$

If n is the cardinality of T, we have that

$$\tau(\langle C \sqsubseteq D > \overline{r}\rangle) := \neg(((\forall x)(\tau^x(C) \rightarrow \tau^x(D)) \rightarrow \overline{r})^{n-1}),$$
$$\tau(\langle C \sqsubseteq D < \overline{r}\rangle) := \neg((\overline{r} \rightarrow (\forall x)(\tau^x(C) \rightarrow \tau^x(D)))^{n-1}),$$
$$\tau(\langle C(a) > \overline{r}\rangle) := \neg((\tau^x(C)[a/x] \rightarrow \overline{r})^{n-1}),$$
$$\tau(\langle C(a) < \overline{r}\rangle) := \neg((\overline{r} \rightarrow \tau^x(C)[a/x])^{n-1}),$$

3.6.3 Reasoning tasks

Now we can utilize the translation $\tau^x(\cdot)$, introduced in Section 3.6.1 and extended to fuzzy axioms in the previous subsection in order to obtain a corresponding translation of the reasoning tasks. In what follows, let C, D be two $\mathfrak{I}\mathcal{ALCE}$-concepts and $r, s \in T$.

- For concept r-satisfiability, we can consider the following two problems of first order logic:
  - C is $\geq r$-satisfiable if and only if formula $\overline{r} \rightarrow \tau^x(C)$ is 1-satisfiable;
  - C is r-satisfiable if and only if formula $\tau^x(C)$ is r-satisfiable if and only if formula $\overline{r} \leftrightarrow \tau^x(C)$ is 1-satisfiable;
  - C is positively satisfiable if and only if formula $\tau^x(\neg C)$ is not a theorem.
- D r-subsumes C if and only if formula $\overline{r} \rightarrow \tau^x(C \sqsupset D)$ is valid.
- A knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ is consistent if and only if formula $\bigwedge_{\langle \varphi \rhd r \rangle \in \mathcal{T} \cup \mathcal{A}} \tau(\langle \varphi \rhd r \rangle)$ is 1-satisfiable.
- C is r-satisfiable w.r.t. $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ if and only if formula $\left(\bigwedge_{\langle \varphi \rhd r \rangle \in \mathcal{T} \cup \mathcal{A}} \tau(\langle \varphi \rhd r \rangle) \right) \wedge (\overline{r} \rightarrow \tau^x(C))$ is 1-satisfiable.
- Knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ entails the fuzzy axiom $\langle \varphi \rhd r \rangle$ if and only if formula $\tau(\langle \varphi \rhd r \rangle)$ is a logical consequence of the set of formulas $\{\tau(\langle \psi \rhd s \rangle) \mid \langle \psi \rhd s \rangle \in \mathcal{T} \cup \mathcal{A}\}$.
- The best satisfiability degree of a concept C with respect to a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, translated to first order logic, is the problem of determining which is the higher value r with respect to which formula $\left(\bigwedge_{\langle \varphi \rhd s \rangle \in \mathcal{T} \cup \mathcal{A}} \tau(\langle \varphi \rhd r \rangle) \right) \wedge (\overline{r} \rightarrow \tau^x(C))$ is 1-satisfiable. With respect to the usual problems in first order logic, this problem can be considered a family of problem, more than a single one, that is, one satisfiability problem for each $r \in T$.
- The best entailment degree of an axiom φ with respect to a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, translated to first order logic, is the problem of determining which is the higher value r with respect to which $\tau(\langle \varphi \rhd r \rangle)$ is a logical consequence of the set of formulas $\{\tau(\langle \psi \rhd s \rangle) \mid \langle \psi \rhd s \rangle \in \mathcal{T} \cup \mathcal{A}\}$. Again, with respect to the usual problems in first order logic, this problem can be considered a family of problem, more than a single one, that is, one logical consequence problem for each $r \in T$.

3.7 Relation to multi-modal logic

In [Schild, 1991] it is provided a translation of DL concepts into classical propositional multi-modal logic. The relationship between FDL and fuzzy multi-modal logic has been described in [Cerami et al., 2012] for the case of finite-valued Łukasiewicz Logic. In that paper the relationship between both formalisms is obtained through their respective relations with first order predicate logic. Here we present a generalization of the result in [Cerami et al., 2012] by means of a more direct translation.

3.7.1 Concepts

Given a description signature $\mathcal{D} = \langle N_I, N_C, N_R \rangle$, we define the multi-modal language $\mathbf{l}_{\Box_\mathcal{D}} := \mathbf{l} \cup \{\Box_R, \Diamond_R \mid R \in N_R\}$ over the set $Var_\mathcal{D} = N_C$ of propositional variables where

- $\mathbf{l}$ is the set of propositional connectives of any extension $\mathcal{L}$ of MTL logic,
- $\{\Box_R, \Diamond_R \mid R \in N_R\}$ is a set of unary modal operators.

For every concept name $A \in N_C$ we can define the translation $\tau : N_C \to Var_\mathcal{D}$ from the set of concept names into the set of propositional variables of the logic $\mathcal{L}_\Box$, in the following way:

$$\begin{array}{rcl} \tau(A) & := & p_A \\ \tau(\bot) & := & \bot, \\ \tau(\top) & := & \top, \\ \tau(\overline{r}) & := & \overline{r}, \end{array}$$

This translation can be inductively extended over the set of complex concepts in the following way:

$$\begin{array}{rcl} \tau(\neg C) & := & \neg\tau(C), \\ \tau(\sim C) & := & \sim \tau(C), \\ \tau(\blacktriangle C) & := & \triangle\tau(C), \\ \tau(C \boxtimes D) & := & \tau(C) \otimes \tau(D) \\ \tau(C \sqcap D) & := & \tau(C) \wedge \tau(D) \\ \tau(C \boxplus D) & := & \tau(C) \oplus \tau(D) \\ \tau(C \sqcup D) & := & \tau(C) \vee \tau(D) \\ \tau(C \sqsupset D) & := & \tau(C) \to \tau(D) \\ \tau(\forall R.C) & := & \Box_R\tau(C) \\ \tau(\exists R.C) & := & \Diamond_R\tau(C) \end{array}$$

Next we show that the translation preserves the meaning of the original expression through a definition of a Kripke model from an FDL interpretation. Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be an FDL interpretation, then we can define the **T**-valued Kripke model $\mathfrak{M}_\mathcal{I} = \langle W_\mathcal{I}, \{R_{\mathfrak{M}_\mathcal{I}} \mid R \in N_R\}, V_\mathcal{I} \rangle$, where:

- $W_\mathcal{I} = \Delta^\mathcal{I}$,
- for each role name $R \in N_R$, $R_{\mathfrak{M}_\mathcal{I}}$ is a binary **T**-fuzzy accessibility relation on $W_\mathcal{I}$, i.e. a binary function $R_{\mathfrak{M}_\mathcal{I}} \colon W_\mathcal{I} \times W_\mathcal{I} \to T$, such that, for every $a, b \in W_\mathcal{I}$, it holds that $R_{\mathfrak{M}_\mathcal{I}}(a, b) = R^\mathcal{I}(a, b)$,

- for each element $a \in W_{\mathcal{I}}$ and for every propositional variable $p_A \in Var_{\mathcal{D}}$, it holds that $V_{\mathcal{I}}(p_A, a) = A^{\mathcal{I}}(a)$.

Lemma 26. *Let C be an $\mathbf{T}$-$\mathcal{ALC}$ concept. Then, for every $x \in \Delta^{\mathcal{I}}$, it holds that $V_{\mathcal{I}}(\tau(C), a) = C^{\mathcal{I}}(a)$, for every object $a \in \Delta^{\mathcal{I}}$.*

Proof. The proof is by structural induction on complex concepts.

- For concept names and constant concepts it is straightforward by definition.

- Suppose that the statement holds for concepts C and D. Then $V_{\mathcal{I}}(\tau(C \boxtimes D), a) = V_{\mathcal{I}}(\tau(C) \otimes \tau(D), a) = V_{\mathcal{I}}(\tau(C), a) * V_{\mathcal{I}}(\tau(D), a) = C^{\mathcal{I}}(a) * D^{\mathcal{I}}(a) = (C \boxtimes D)^{\mathcal{I}}(a)$. In the same way the statement can be proved also for constructors $\sqcap, \boxplus, \sqcup, \sqsupset, \sim, \blacktriangle$ and $\neg$.

- Suppose that the statement holds for concept C. Then $V_{\mathcal{I}}(\tau(\forall R.C), a) = V_{\mathcal{I}}(\Box_R \tau(C), a) = \inf_{y \in W_{\mathcal{I}}}\{R_{\mathfrak{M}_{\mathcal{I}}}(a, y) \Rightarrow V_{\mathcal{I}}(\tau(C), y)\} = \inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(a, y) \Rightarrow C^{\mathcal{I}}(y)\} = (\forall R.C)^{\mathcal{I}}(a)$. In the same way the statement can be proved also for concept $\exists R.C$.

So, for every $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCE}$ concept C and every $a \in \Delta^{\mathcal{I}}$ it holds that $C^{\mathcal{I}}(a) = V_{\mathcal{I}}(\tau(C), a)$. $\Box$

In the case of multi-modal logic it is also possible to provide a translation from multi-modal formulas into description concepts. Given a multi-modal language $\mathbf{l}_{\Box} = \mathbf{l} \cup \{\Box_i, \Diamond_i \mid i \in I\}$, with I countable and a set of propositional variables $Var = \{p_1, p_2, \ldots\}$, we define the description signature $\mathcal{D}_{\mathbf{l}_{\Box}} = \langle N_I^{\mathbf{l}_{\Box}}, N_C^{\mathbf{l}_{\Box}}, N_R^{\mathbf{l}_{\Box}} \rangle$, where

- $N_I^{\mathbf{l}_{\Box}} := \emptyset$,

- $N_C^{\mathbf{l}_{\Box}} := \{A_p \colon p \in Var\}$,

- $N_R^{\mathbf{l}_{\Box}} := \{R_i \colon \Box_i \in \mathbf{l}_{\Box}\}$.

For every propositional variable $p \in Var$ we can define the translation $\rho : Var \to N_C^{\mathbf{l}_{\Box}}$ from the set of propositional variable into the set of concept names of the signature $\mathcal{D}_{\mathbf{l}_{\Box}}$, in the following way:

$$\begin{array}{rcl} \rho(p) & := & A_p \\ \rho(\bot) & := & \bot, \\ \rho(\top) & := & \top, \\ \rho(\overline{r}) & := & \overline{r}, \end{array}$$

This translation can be inductively extended over the set of complex concepts in the following way:

$$\begin{array}{rcl}
\rho(\neg\varphi) & := & \neg\rho(\varphi),\\
\rho(\sim\varphi) & := & \sim\rho(\varphi),\\
\rho(\triangle\varphi) & := & \blacktriangle\rho(\varphi),\\
\rho(\varphi\otimes\psi) & := & \rho(\varphi)\boxtimes\rho(\psi)\\
\rho(\varphi\wedge\psi) & := & \rho(\varphi)\sqcap\rho(\psi)\\
\rho(\varphi\oplus\psi) & := & \rho(\varphi)\boxplus\rho(\psi)\\
\rho(\varphi\vee\psi) & := & \rho(\varphi)\sqcup\rho(\psi)\\
\rho(\varphi\rightarrow\psi) & := & \rho(\varphi)\sqsupset\rho(\psi)\\
\rho(\Box_i\varphi) & := & \forall R_i.\rho(\varphi)\\
\rho(\Diamond_i\varphi) & := & \exists R_i.\rho(\varphi)
\end{array}$$

As a straightforward consequence of the definitions of τ and ρ, we have the following lemma.

Lemma 27. *For every* $\mathbf{T}$-$\mathcal{ALC}$ *concept* C *and every multi-modal formula* φ *it holds that:*

- $\rho(\tau(C)) = C$,
- $\tau(\rho(\varphi)) = \varphi$.

Again, it is possible to show that the translation preserves the meaning of the original expression through a definition of an FDL interpretation from a Kripke model. Let $\mathfrak{M} = \langle W, \{R_1, \ldots, R_n\}, V\rangle$ be a Kripke model, then we can define the interpretation $\mathcal{I}_{\mathfrak{M}} = (\Delta^{\mathcal{I}_{\mathfrak{M}}}, \cdot^{\mathcal{I}_{\mathfrak{M}}})$, where:

- $\Delta^{\mathcal{I}_{\mathfrak{M}}} := W$,
- for each concept name $A_p \in N_C^{\mathbf{l}_\Box}$, $A_p^{\mathcal{I}_{\mathfrak{M}}}$ is the unary function $A_p^{\mathcal{I}_{\mathfrak{M}}} : \Delta^{\mathcal{I}_{\mathfrak{M}}} \rightarrow T$, such that, for every $a \in \Delta^{\mathcal{I}_{\mathfrak{M}}}$, it holds that $A_p^{\mathcal{I}_{\mathfrak{M}}}(a) = V(p, a)$,
- for each role name $R_i \in N_R^{\mathbf{l}_\Box}$, $R_i^{\mathcal{I}_{\mathfrak{M}}}$ is the binary function $R_i^{\mathcal{I}_{\mathfrak{M}}} : \Delta^{\mathcal{I}_{\mathfrak{M}}} \times \Delta^{\mathcal{I}_{\mathfrak{M}}} \rightarrow T$, such that, for every $a, b \in \Delta^{\mathcal{I}_{\mathfrak{M}}}$, it holds that $R_i^{\mathcal{I}_{\mathfrak{M}}}(a, b) = R_i(a, b)$,

As a straightforward consequence of the definitions of $\mathfrak{M}_{\mathcal{I}}$ and $\mathcal{I}_{\mathfrak{M}}$, we have the following lemma.

Lemma 28. *For every* $\mathbf{T}$*-interpretation* $\mathcal{I}$ *and every Kripke* $\mathbf{T}$*-model* $\mathfrak{M}$ *it holds that:*

- $\mathcal{I} = \mathcal{I}_{\mathfrak{M}_{\mathcal{I}}}$,
- $\mathfrak{M} = \mathfrak{M}_{\mathcal{I}_{\mathfrak{M}}}$.

From Lemma 27, Lemma 28 and Lemma 26 we can prove a further consequence.

Lemma 29. *Let* φ *be a multi-modal formula. Then, for every* $w \in W$ *it holds that* $(\rho(\varphi))^{\mathcal{I}_{\mathfrak{M}}}(w) = V(\varphi, w)$.

Proof. From Lemma 26 we have that $(\rho(\varphi))^{\mathcal{I}_{\mathfrak{M}}}(w) = V_{\mathcal{I}_{\mathfrak{M}}}(\tau(\rho(\varphi)), w)$. From Lemma 27 we have that $V_{\mathcal{I}_{\mathfrak{M}}}(\tau(\rho(\varphi)), w) = V_{\mathcal{I}_{\mathfrak{M}}}(\varphi, w)$. From Lemma 28 we have that $V_{\mathcal{I}_{\mathfrak{M}}}(\varphi, w) = V(\varphi, w)$. So, $(\rho(\varphi))^{\mathcal{I}_{\mathfrak{M}}}(w) = V(\varphi, w)$. $\Box$

3.7.2 Fuzzy axioms

First of all, we utilize the translation $\tau(\cdot)$, introduced in Section 3.7 in order to obtain a corresponding translation of the fuzzy inclusion axioms proposed in Section 3.1.2. In this case, however, we need to use a multi-modal language that contains the universal modality $\Box_U$, as well as Delta operator $\triangle$ and truth constants.

$$\tau(\langle C \sqsubseteq D \geq \overline{r}\rangle) := \Box_U(\overline{r} \to (\tau(C) \to \tau(D)))$$
$$\tau(\langle C \sqsubseteq D \leq \overline{r}\rangle) := \Box_U((\tau(C) \to \tau(D)) \to \overline{r})$$
$$\tau(\langle C \sqsubseteq D > \overline{r}\rangle) := \Box_U \neg \triangle ((\tau(C) \to \tau(D)) \to \overline{r})$$
$$\tau(\langle C \sqsubseteq D < \overline{r}\rangle) := \Box_U \neg \triangle (\overline{r} \to (\tau(C) \to \tau(D)))$$

Here again it is possible to show that the translation preserves the meaning of the original expressions. Note that, in presence of the universal modality $\Box_U$ fuzzy axiom satisfiability is obtained as both local and global satisfiability of its multi-modal translation.

Lemma 30. *Let $\langle \varphi \rhd r\rangle$ be a fuzzy inclusion or equivalence axiom, with $\rhd \in \{\geq, \leq, >, <\}$. Then a* **T***-interpretation $\mathcal{I}$ satisfies $\langle \varphi \rhd r\rangle$ if and only if $\mathfrak{M}_\mathcal{I}$ globally satisfies $\tau(\langle \varphi \rhd r\rangle)$ if and only if $\mathfrak{M}_\mathcal{I}$ locally satisfies $\tau(\langle \varphi \rhd r\rangle)$.*

Proof. Let $\langle \varphi \rhd r\rangle$ be a fuzzy axiom and $\mathcal{I}$ a **T**-interpretation, then

- If $\langle \varphi \rhd r\rangle = \langle C \sqsubseteq D \geq \overline{r}\rangle$, then $\mathcal{I}$ satisfies it if and only if $\inf_{x\in\Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} \geq r$. By Lemma 20, we have that $r \leq \inf_{x\in\Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} = \inf_{x\in W_\mathcal{I}}\{V_\mathcal{I}(\tau(C), x) \Rightarrow V_\mathcal{I}(\tau(D), x)\} = \inf_{x\in W_\mathcal{I}}\{V_\mathcal{I}(\tau(C) \to \tau(D), x)\}$. Hence, for every $x \in W_\mathcal{I}$, it holds that $r \leq V_\mathcal{I}(\tau(C) \to \tau(D), x)$ and, therefore, $1 = \inf_{x\in W_\mathcal{I}}\{r \Rightarrow V_\mathcal{I}(\tau(C) \to \tau(D), x)\} = \inf_{w\in W_\mathcal{I}}\{R_U(w, x) \Rightarrow V_\mathcal{I}(r \to (\tau(C) \to \tau(D), x))\} = V_\mathcal{I}(\Box_U(\overline{r} \to (\tau(C) \to \tau(D))), w) = V_\mathcal{I}(\tau(\langle C \sqsubseteq D \geq \overline{r}\rangle), w)$, for every $w \in W_\mathcal{I}$. So, $\mathfrak{M}_\mathcal{I}$ both globally and locally 1-satisfies $\tau(\langle \varphi \rhd r\rangle)$. In the same way it can be proved that the statement holds for axioms of type $\langle C \sqsubseteq D \leq \overline{r}\rangle$.
- If $\langle \varphi \rhd r\rangle = \langle C \sqsubseteq D > \overline{r}\rangle$, then $\mathcal{I}$ satisfies it if and only if $\inf_{x\in\Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} > r$. Hence $\inf_{x\in\Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} \not\leq r$ i.e. $\mathcal{I}$ does not satisfy axiom $\langle C \sqsubseteq D \leq \overline{r}\rangle$. By the previous result we have that $\mathfrak{M}_\mathcal{I}$ satisfies $\tau(\langle C \sqsubseteq D \leq \overline{r}\rangle) = \Box_U((\tau(C) \to \tau(D)) \to \overline{r})$, neither globally, nor locally. Then $V_\mathcal{I}(\Box_U((\tau(C) \to \tau(D)) \to \overline{r}), w) < 1$, for every $w \in W_\mathcal{I}$ and, therefore, $V_\mathcal{I}(\Box_U \neg \triangle ((\tau(C) \to \tau(D)) \to \overline{r}), w) = 1$. In the same way can be proved that the statement holds for axioms of type $\langle C \sqsubseteq D < \overline{r}\rangle$,.

So, for every fuzzy axiom $\langle \varphi \rhd r\rangle$ it holds that a **T**-interpretation $\mathcal{I}$ satisfies $\langle \varphi \rhd r\rangle$ if and only if $\mathfrak{M}_\mathcal{I}$ satisfies $\tau(\langle \varphi \rhd r\rangle)$. □

In a language without a universal modality $\Box_U$ (but with Delta operator $\triangle$ and truth constants) we can not obtain multi-modal formulas as a translation of fuzzy axioms. Nevertheless, their satisfiability with respect to an interpretation $\mathcal{I}$ can be translated to either global or local satisfiability of certain multi-modal formulas with respect to model $\mathfrak{M}_\mathcal{I}$, depending on what kind of axiom has to be translated. So, in such a language, a translation of the fuzzy inclusion axioms can be obtained as follows:

Lemma 31. *For every* **T***-interpretation* $\mathcal{I}$ *the following equivalences hold:*

1. $\mathcal{I} \models \langle C \sqsubseteq D \geq \overline{r} \rangle \iff \mathfrak{M}_\mathcal{I} \models^r_g \tau(C) \to \tau(D)$
   $\iff \mathfrak{M}_\mathcal{I} \models^1_g \overline{r} \to (\tau(C) \to \tau(D))$,

2. $\mathcal{I} \models \langle C \sqsubseteq D \leq \overline{r} \rangle \iff \mathfrak{M}_\mathcal{I} \models^1_l (\tau(C) \to \tau(D)) \to \overline{r}$,

3. $\mathcal{I} \models \langle C \sqsubseteq D > \overline{r} \rangle \iff \mathfrak{M}_\mathcal{I} \models^1_g \neg \triangle ((\tau(C) \to \tau(D)) \to \overline{r})$,
   $\iff \mathfrak{M}_\mathcal{I} \not\models^1_l (\tau(C) \to \tau(D)) \to \overline{r}$

4. $\mathcal{I} \models \langle C \sqsubseteq D < \overline{r} \rangle \iff \mathfrak{M}_\mathcal{I} \models^1_l \neg \triangle (\overline{r} \to (\tau(C) \to \tau(D)))$,
   $\iff \mathfrak{M}_\mathcal{I} \not\models^r_g \tau(C) \to \tau(D)$

Proof. Let $\mathcal{I}$ be a **T** interpretation, then:

1. We have that $\mathcal{I}$ satisfies $\langle C \sqsubseteq D \geq \overline{r} \rangle$ if and only if $\inf_{x \in \Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} \geq r$. By Lemma 20, we have that $r \leq \inf_{x \in \Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} = \inf_{x \in W_\mathcal{I}}\{V_\mathcal{I}(\tau(C), x) \Rightarrow V_\mathcal{I}(\tau(D), x)\} = \inf_{x \in W_\mathcal{I}}\{V_\mathcal{I}(\tau(C) \to \tau(D), x)\}$. Hence, on the one hand, for every $x \in W_\mathcal{I}$, it holds that $r \leq V_\mathcal{I}(\tau(C) \to \tau(D), x)$, that is, $\mathfrak{M}_\mathcal{I} \models^r_g \tau(C) \to \tau(D)$. On the other hand, for every $x \in W_\mathcal{I}$, it holds that $V_\mathcal{I}(\overline{r} \to (\tau(C) \to \tau(D)), x) = 1$, that is, $\mathfrak{M}_\mathcal{I} \models^1_g \overline{r} \to (\tau(C) \to \tau(D))$.

2. We have that $\mathcal{I}$ satisfies $\langle C \sqsubseteq D \leq \overline{r} \rangle$ if and only if $\inf_{x \in \Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} \leq r$. By Lemma 20, we have that $r \geq \inf_{x \in \Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} = \inf_{x \in W_\mathcal{I}}\{V_\mathcal{I}(\tau(C), x) \Rightarrow V_\mathcal{I}(\tau(D), x)\} = \inf_{x \in W_\mathcal{I}}\{V_\mathcal{I}(\tau(C) \to \tau(D), x)\}$. Hence, there exists $x \in W_\mathcal{I}$ such that $V_\mathcal{I}(\overline{r} \to (\tau(C) \to \tau(D)), x) = 1$, that is, $\mathfrak{M}_\mathcal{I} \models^1_l \overline{r} \to (\tau(C) \to \tau(D))$.

3. We have that $\mathcal{I}$ satisfies $\langle C \sqsubseteq D > \overline{r} \rangle$ if and only if $\inf_{x \in \Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} > r$. Hence $\inf_{x \in \Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} \not\leq r$ i.e. $\mathcal{I}$ does not satisfy axiom $\langle C \sqsubseteq D \leq \overline{r} \rangle$. So, by item 2, we have that $\mathfrak{M}_\mathcal{I} \not\models^1_l (\tau(C) \to \tau(D)) \to \overline{r}$. Moreover, since $\inf_{x \in \Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} \not\leq r$, then, by Lemma 20, we have that $r \not\geq \inf_{x \in \Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} = \inf_{x \in W_\mathcal{I}}\{V_\mathcal{I}(\tau(C), x) \Rightarrow V_\mathcal{I}(\tau(D), x)\} = \inf_{x \in W_\mathcal{I}}\{V_\mathcal{I}(\tau(C) \to \tau(D), x)\}$. Hence, for every $x \in W_\mathcal{I}$, it holds that $V_\mathcal{I}(\tau(C) \to \tau(D)) \to \overline{r}, x) < 1$, that is, $V_\mathcal{I}(\neg \triangle (\tau(C) \to \tau(D)) \to \overline{r}, x)) = 1$. So, $\mathfrak{M}_\mathcal{I} \models^1_g \neg \triangle ((\tau(C) \to \tau(D)) \to \overline{r})$.

4. We have that $\mathcal{I}$ satisfies $\langle C \sqsubseteq D < \overline{r} \rangle$ if and only if $\inf_{x \in \Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} < r$. Hence $\inf_{x \in \Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} \not\geq r$ i.e. $\mathcal{I}$ does not satisfy

axiom $\langle C \sqsubseteq D \geq \overline{r} \rangle$. So, by item 1, we have that $\mathfrak{M}_{\mathcal{I}} \not\models^r_g \tau(C) \to \tau(D)$. Moreover, since $\inf_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} < r$, then, by Lemma 20, we have that $r > \inf_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} = \inf_{x \in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C), x) \Rightarrow V_{\mathcal{I}}(\tau(D), x)\} = \inf_{x \in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C) \to \tau(D), x)\}$. Hence there exists $x \in W_{\mathcal{I}}$ such that $V_{\mathcal{I}}(\overline{r} \to (\tau(C) \to \tau(D)), x) < 1$, that is, $V_{\mathcal{I}}(\neg \triangle (\overline{r} \to (\tau(C) \to \tau(D)), x)) = 1$. So, $\mathfrak{M}_{\mathcal{I}} \models^1_l \neg \triangle (\overline{r} \to (\tau(C) \to \tau(D)))$. □

Differently from the case of first order logic and despite the fact that FDL interpretations can be a semantics for fuzzy assertions, within the multi-modal language it is not possible to translate fuzzy assertions like $\langle C(a) \geq r \rangle$. This is due to the fact that in multi-modal languages there is not a syntactic entity that can work as a translation for FDL individuals.

3.7.3 Reasoning tasks

Now we can utilize the translation $\tau(\cdot)$, introduced in Section 3.7 and extended to fuzzy axioms in the previous subsection in order to obtain a corresponding translation of the reasoning tasks. Nevertheless, due to the fact that we can not obtain a corresponding translation of fuzzy assertion axioms, in the same way, we can not obtain a translation to multi-modal logic of the problems related to knowledge bases where the ABox is non-empty. For this reason we will not consider the knowledge base consistency problem when the ABox is not empty and will consider problems related to knowledge bases $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{A} = \emptyset$. In what follows, let C, D be two concepts and $r, s \in T$.

- For concept r-satisfiability, we can consider the following two problems of first order logic:
  - C is $\geq r$-satisfiable if and only if formula $\tau(C)$ is locally r-satisfiable, for some $s \geq r$, if and only if formula $\overline{r} \to \tau(C)$ is locally 1-satisfiable;
  - C is r-satisfiable if and only if formula $\tau(C)$ is locally r-satisfiable if and only if formula $\overline{r} \leftrightarrow \tau(C)$ is locally 1-satisfiable.
  - C is positively satisfiable if and only if formula $\tau(\neg C)$ is not a theorem.
- D r-subsumes C if and only if formula $\overline{r} \to \tau(C \sqsupset D)$ is valid.
- A knowledge base $\mathcal{K} = \langle \mathcal{T} \rangle$ is consistent if and only if formula $\bigwedge_{\langle \varphi \rhd r \rangle \in \mathcal{T}} \tau(\langle \varphi \rhd r \rangle)$ is globally 1-satisfiable.
- C is r-satisfiable w.r.t. $\mathcal{K} = \langle \mathcal{T} \rangle$ if and only if there exists a **T**-valued Kripke model $\mathfrak{M}$ such that both $\mathfrak{M} \models^1_g \left(\bigwedge_{\langle \varphi \rhd r \rangle \in \mathcal{T}} \tau(\langle \varphi \rhd r \rangle) \right)$ and $\mathfrak{M} \models^r_l \tau(C)$.
- Knowledge base $\mathcal{K} = \langle \mathcal{T} \rangle$ entails the fuzzy axiom $\langle \varphi \rhd r \rangle$ if and only if formula $\tau(\langle \varphi \rhd r \rangle)$ is a global consequence of the set of formulas $\{\tau(\langle \psi \rhd s \rangle) \mid \langle \psi \rhd s \rangle \in \mathcal{T}\}$.

- The best satisfiability degree of a concept C with respect to a KB $\mathcal{K} = \langle \mathcal{T} \rangle$, translated to multi-modal logic, is the problem of determining which is the higher value r with respect to which there exists a **T**-valued Kripke model $\mathfrak{M}$ such that both $\mathfrak{M} \models_g^1 \left(\bigwedge_{\langle \varphi \triangleright r \rangle \in \mathcal{T}} \tau(\langle \varphi \triangleright s \rangle) \right)$ and $\mathfrak{M} \models_l^1 \overline{r} \to \tau(C)$. With respect to the usual problems in multi-modal logic, this problem can be considered a family of problem, more than a single one, that is, one satisfiability problem for each $r \in T$.

- The best entailment degree of an axiom φ with respect to a KB $\mathcal{K} = \langle \mathcal{T} \rangle$, translated to multi-modal logic, is the problem of determining which is the higher value r with respect to which $\tau(\langle \varphi \triangleright r \rangle)$ is a global consequence of the set of formulas $\{\tau(\langle \psi \triangleright s \rangle) \mid \langle \psi \triangleright s \rangle \in \mathcal{T}\}$. Again, with respect to the usual problems in first order logic, this problem can be considered a family of problem, more than a single one, that is, one logical consequence problem for each $r \in T$.

Chapter 4

Decidability

Decidability is a fundamental topic in classical DL. In FDL it is a very important topic as well. The study of decidability in FDL has brought to the generalization of classical algorithms, as well as to the thinking of new ones. In the fuzzy framework, however, an important role play also the results on undecidability for the infinite-valued case in presence of general concept inclusions.

Results and proofs that we published will be exhaustively reported in order to give an example of how some results have been achieved and what kind of procedures and proof strategies have been employed in order to achieve those results. More results existing in the literature will be reported in Section 6.2.

4.1 Witnessed satisfiability and Łukasiewicz logic

Concept satisfiability is one of the simplest reasoning tasks in FDL and the one that is more studied in the logical counterpart. Since [Hájek, 2005] the attention of most researchers focussed on the satisfiability with respect to witnessed interpretations, that we will call, from now on, *witnessed satisfiability*.

In [Hájek, 2005] it is proven that concept witnessed r-satisfiability is a decidable problem for a Fuzzy Description Logic restricted to language $\mathfrak{I}\mathcal{ALCE}$, based on any t-norm. In order to achieve such result, in [Hájek, 2005] is defined an algorithm that, given a concept C_0, obtains a propositional theory T_{C_0}. We report the algorithm from [Hájek, 2005] in Definition 33. Before introducing the algorithm, we need some previous definitions from [Hájek, 2005].

Definition 32. 1. *Nesting degree of quantifiers* in C (or $C(a)$) is defined inductively: $nest(A) = 0$, if A is an atomic concept name; if C and D are concepts, then $nest(C \boxtimes D) = nest(C \sqsupset D) = max(nest(C), nest(D))$; finally, if C is a concept and R a role name, then $nest(\forall R.C) = nest(\exists R.C) = nest(C) + 1$.

2. *Generalized atoms* are quantified concepts, i.e. concepts of the form $\forall R.C$ or $\exists R.C$, where C is a propositional combination of concepts and generalized atoms; the latter will be called *generalized atoms* of C. We will also use the term generalized atom for instances of quantified concepts, the context will clarify the precise meaning.

Definition 33 ([Hájek, 2005]Definition 3)**.** Given $C_0(a_0)$, step 0 just transfers it to further processing in step 1; the constant a_0 has level 0. For $i > 0$ step i processes generalized atoms of formulas transferred from step $i-1$; they have the form $(QR.C)(b)$, where Q is $\forall$ or $\exists$, R is a role, C a concept with nesting degree $\leq n-1$ and b is a constant of level i. For each generalized atom α in question, do the following:

If α is $(\forall R.C)(b)$ then produce a new constant d_α and the axiom

$$(\forall R.C)(b) \equiv (R(b, d_\alpha) \to C(d_\alpha))$$

If α is $(\exists R.C)(b)$ then produce d_α and the axiom

$$(\exists R.C)(b) \equiv (R(b, d_\alpha) \otimes C(d_\alpha))$$

In both cases call the generated axioms *witnessing axioms* for α and d_α a constant *belonging to* R, b.

After this is done for all α in question (in the present step) consider each α once more and do the following:

If α is $(\forall R.C)(b)$ and d_β is any constant belonging to R, b and different from d_α, produce the axiom

$$(\forall R.C)(b) \to (R(b, d_\beta) \to C(d_\beta))$$

Similarly for α being $(\exists R.C)(b)$, produce

$$(R(b, d_\beta) \otimes C(d_\beta)) \to (\exists R.C)(b)$$

After proving that the provided algorithm is correct and complete with respect to the problems considered, Hájek proves that satisfiability, validity and subsumption with respect to witnessed interpretations coincide with the same problems with respect to finite interpretations. Hence, when these problems are restricted to witnessed interpretations, the FDLs considered enjoy the finite model property and are decidable.

Depending on the complete chain **T** considered, the witnessed satisfiability does not need to coincide with the same problem with respect to unrestricted interpretations, but both notions indeed coincide in some cases:

- If **T** is a finite t-norm, then, trivially, witnessed satisfiability coincides with unrestricted satisfiability, for each notion of satisfiability considered.

- In [Hájek, 1998b, Theorem 5.4.30] it is proven that, if a formula φ is not true in a $[0,1]_{Ł}$-model, then there exists an integer n such that φ is not true in a $Ł_n$-model. Using this result and the fact that a formula φ is positively satisfiable iff $\neg\varphi$ is not a theorem, it is easily obtained that unrestricted positive satisfiability coincides with witnessed positive satisfiability in the standard Łukasiewicz chain $[0,1]_{Ł}$.
- In [Hájek, 2005, Lemma 3] Hájek proves that, for each $r \in [0,1]$, if a formula φ is r-satisfiable in a $[0,1]_{Ł}$-model, then it is r-satisfiable in a witnessed $[0,1]_{Ł}$-model.
- In [Cintula and Hájek, 2006] is proven that, with the general Łukasiewicz semantics, witnessed satisfiability coincides with unrestricted satisfiability, for each notion of satisfiability considered.

As remarked in [Hájek, 2005] and [Cintula and Hájek, 2006] by means of counter-examples, in the case of infinite-valued Gödel and product t-norms, there exist formulas that are satisfiable, but not in a witnessed interpretation.

4.1.1 Quasi-witnessed satisfiability and product logic

In [Cerami et al., 2010a] it is proven that concept quasi-witnessed positive satisfiability is a decidable problem for a Fuzzy Description Logic restricted to language $\mathfrak{I}\mathcal{ALE}$, based on product t-norm.

We will use the notations Sat_r and Sat_1 to denote the sets of positively satisfiable and 1-satisfiable concepts, respectively; and we will write QSat_r and QSat_1 to denote the same sets restricted to quasi-witnessed interpretations. In particular, in [Cerami et al., 2010a] has been proven the following theorem.

Theorem 34. *The sets* QSat_r *and* QSat_1 *are decidable.*

By the completeness result of product first order logic with respect to quasi-witnessed models reported in Section 2.1.3, we obtain that Sat_r is a decidable problem under product general semantics.

In the case of the standard semantics, the landscape is not the same. In Appendix A it is proven that quasi-witnessed positive satisfiability and unrestricted positive satisfiability indeed coincide under the standard product semantics. Hence unrestricted positive satisfiability is a decidable problem in $[0,1]_{\Pi}$. For the 1-satisfiability problem under standard product semantics, completeness with respect to quasi-witnessed models and, thus, decidability of language $\mathfrak{I}\mathcal{ALE}$ based on product t-norm are still open problems.

The proof follows the one provided in [Hájek, 2005] for the case of witnessed interpretations. We report here the whole proof.

The Reduction to the Propositional Case

In order to prove that positive satisfiability in a quasi-witnessed interpretation is decidable we are going to codify quasi-witnessed interpretations by some finite number of formulas in the propositional product logic; using this finite codification we will not know how to recover the same initial interpretation, but we will be able to build an interpretation with the same associated truth value.

First of all, let us a fix an infinite set $\text{Ind} = \{a_i : i \in \omega\}$, whose elements will be called *individuals* or *constants*. With a little language abuse, throughout Section 4.1.1, an *assertion* will denote any propositional combination of expressions of the forms $C(a)$ and $R(a, b)$ where C is a concept, R is a role name and a, b are individuals. The definitions of the notions of nesting degree and generalized atoms can be found in Definition 32.

Definition 35 (Labelling)**.** Let C_0 be a concept. The *labelling function* is the function which associates to every occurrence D of a subconcept in C_0 an element of $\mathbb{N}^{\leq k}$ (where $k = nest(C_0)$) defined by the conditions:

1. $l(C_0)$ is the empty sequence $\emptyset$,
2. if D is a propositional combination of concepts $D_1, \ldots, D_n$, then $l(D_i) := l(D)$ for every $i \leq n$.
3. if D is $\forall R.D'$ or $\exists R.D'$, then $l(D')$ is the concatenated sequence $l(D), n$, where n is the minimum number m such that the sequence $l(D), m$ has not been used to label any occurrence in C_0.

In order to illustrate the notions defined in the last definitions, as well as further definitions, we propose an example that will be used throughout the paper.

Example 36. *Consider the concept*

$$\texttt{Example} := \forall R.\exists R.A \sqcap \neg\forall R.(\exists R.A \boxtimes \exists R.A)$$

where A is an atomic concept. Then,

1. *concept* `Example` *has nesting degree 2.*
2. *the generalized atoms in* `Example` *are: $\forall R.\exists R.A$, $\forall R.(\exists R.A \boxtimes \exists R.A)$ and $\exists R.A$.*
3. *the labelling function associated with occurrences in* `Example` *is given by the genealogical tree*

$$\dfrac{\dfrac{\dfrac{A : 1,1}{\exists R.A : 1}}{\forall R.\exists R.A : \emptyset} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A : 2,1}{\exists R.A : 2} \quad \dfrac{A : 2,2}{\exists R.A : 2}}{\exists R.A \boxtimes \exists R.A : 2}}{\forall R.(\exists R.A \boxtimes \exists R.A) : \emptyset}}{\neg\forall R.(\exists R.A \boxtimes \exists R.A) : \emptyset}}{Example : \emptyset}$$

Here we have used the notation $D : \sigma$ to indicate that the labelling of occurrence D is the sequence σ.

Next, for every concept C_0 we are going to recursively associate two finite sets T_{C_0} and Y_{C_0} of assertions.

Definition 37 (Algorithm)**.** Given a concept C_0, we construct finite sets T_{C_0} and Y_{C_0} of assertions. The construction takes steps $0, \ldots, n$ where n is the nesting degree of the concept C_0. At each step some generalized atoms are processed; and at each step we add some new constants from Ind and some new formulas to T_{C_0} and Y_{C_0} and we transfer some assertions of concepts for processing in the next step. The assertions produced in step i will have nesting degree $n-i$; after step n is completed the algorithm stops.

At step 0, we simply transfer the assertion $C_0(d)$ to be further processed in step 1; and we say that constant d has level 0. For $i > 0$, step i selects the generalized atoms in formulas transferred from step $i-1$ and processes them. We know that the generalized atoms just selected have the form $QR.C(d_\sigma)$, where $Q \in \{\forall, \exists\}$, R is a role, C a concept with nesting degree $\leq n-i$, d_σ is a constant produced in the previous step and σ is the label of the generalized atom we are considering. For each generalized atom α, at step i we firstly do the following:

- If α is $\forall R.C(d_\sigma)$, then produce a new constant $d_{\sigma,n}$ and add to T_{C_0} the assertion

$$(\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n}))) \sqcup \neg\forall R.C(d_\sigma)$$

- If α is $\exists R.C(d_\sigma)$, then produce a new constant $d_{\sigma,n}$ and add to T_{C_0} the assertion:

$$(R(d_\sigma, d_{\sigma,n}) \boxtimes C(d_{\sigma,n})) \equiv \exists R.C(d_\sigma)$$

We will say that $d_{\sigma,n}$ is a *constant associated to* R, d_σ. Now, we consider each α of the present step and do the following:

- If α is $(\forall R.C)(d_\sigma)$ and $d_{\sigma,m}$ is any constant associated to R, d_σ, then add to T_{C_0} the assertion

$$\forall R.C(d_\sigma) \sqsupset (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))$$

- If α is $(\exists R.C)(d_\sigma)$ and $d_{\sigma,m}$ is any constant associated to R, d_σ, then add to T_{C_0} the assertion

$$(R(d_\sigma, d_{\sigma,m}) \boxtimes C(d_{\sigma,m})) \sqsupset \exists R.C(d_\sigma)$$

- If α is $(\forall R.C)(d_\sigma)$, then add to Y_{C_0} the assertion

$$\neg\forall R.C(d_\sigma) \boxtimes (R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n}))$$

Example 38. *Following Definition 37, the assertions belonging to* $T_{\texttt{Example}}$ *are:*

- $(\forall R.\exists R.A(d) \equiv (R(d, d_1) \sqsupset \exists R.A(d_1))) \sqcup \neg\forall R.\exists R.A(d)$,

- $(\forall R.(\exists R.A \boxtimes \exists R.A)(d) \equiv (R(d,d_2) \sqsupset (\exists R.A \boxtimes \exists R.A)(d_2))) \sqcup \neg\forall R.(\exists R.A \boxtimes \exists R.A)(d)$,
- $\forall R.\exists R.A(d) \sqsupset (R(d,d_2) \sqsupset A(d_2))$,
- $\forall R.(\exists R.A \boxtimes \exists R.A)(d) \sqsupset (R(d,d_1) \sqsupset (\exists R.A \boxtimes \exists R.A)(d_1))$,
- $\exists R.A(d_1) \equiv (R(d_1,d_{1,1}) \boxtimes A(d_{1,1}))$,
- $\exists R.A(d_2) \equiv (R(d_2,d_{2,1}) \boxtimes A(d_{2,1}))$,
- $\exists R.A(d_2) \equiv (R(d_2,d_{2,2}) \boxtimes A(d_{2,2}))$,
- $(R(d_2,d_{2,2}) \boxtimes A(d_{2,2})) \sqsupset \exists R.A(d_2)$,
- $(R(d_2,d_{2,1}) \boxtimes A(d_{2,1})) \sqsupset \exists R.A(d_2)$.

While assertions belonging to $Y_{\texttt{Example}}$ *are:*

- $\neg\forall R.\exists R.A(d) \boxtimes (R(d,d_1) \sqsupset \exists R.A(d_1))$,
- $\neg\forall R.(\exists R.A \boxtimes \exists R.A)(d) \boxtimes (R(d,d_2) \sqsupset (\exists R.A \boxtimes \exists R.A)(d_2))$.

As it is said above our aim is to reduce our problem to one in the corresponding propositional calculus. Here we will consider this propositional logic using as variables the set

$$Prop := \{p_{R(a,b)} : R \text{ is a role name and } a,b \in \text{Ind}\} \ \cup$$

$$\{p_{C(a)} : C \text{ atomic or quantified concept and } a \in \text{Ind}\}.$$

We stress that we are taking a concrete fix set as variables. Nevertheless, for a particular concept C_0 it is clear that a finite subset $Prop_{C_0}$ of $Prop$ would be enough. Using that all concepts are indeed propositional combinations of expressions of the form $C(a)$ and $R(a,b)$, the following definition is meaningful. This definition tells us that we can look at assertions as propositional formulas with variables in $Prop$.

Definition 39. The map pr associates to every assertion a formula in the propositional logic (with the variables given above) according to the following clauses:

1. $pr(A(a)) = p_{A(a)}$ if A is an atomic or a quantified concept,
2. $pr(R(a,b)) = p_{R(a,b)}$ if R is a role name and $a,b \in \text{Ind}$,
3. $pr(\bot(a)) = \bot$,
4. $pr(\top(a)) = \top$
5. $pr((C \boxtimes D)(a)) = pr(C(a)) \otimes pr(D(a))$,
6. $pr((C \sqsupset D)(a)) = pr(C(a)) \to pr(D(a))$.

If T is a set of assertions, then $pr(T)$ is $\{pr(\alpha) \mid \alpha \in T\}$.

Example 40. *If $T_{\texttt{Example}}$ is the set defined in the Example 38, then, following Definition 39, propositional formulas belonging to $pr(T_{\texttt{Example}})$ are:*

- $(p_{\forall R.\exists R.A(d)} \equiv (p_{R(d,d_1)} \to p_{\exists R.A(d_1)})) \vee \neg p_{\forall R.\exists R.A(d)}$,
- $(p_{\forall R.(\exists R.A \boxtimes \exists R.A)(d)} \equiv (p_{R(d,d_2)} \to (p_{\exists R.A \boxtimes \exists R.A)(d_2)})) \vee \neg p_{\forall R.(\exists R.A \boxtimes \exists R.A)(d)}$,
- $p_{\forall R.\exists R.A(d)} \to (p_{R(d,d_2)} \to p_{A(d_2)})$,
- $p_{\forall R.(\exists R.A \boxtimes \exists R.A)(d)} \to (p_{R(d,d_1)} \to p_{(\exists R.A \boxtimes \exists R.A)(d_1)})$,
- $p_{\exists R.A(d_1)} \equiv (p_{R(d_1,d_{1,1})} \otimes p_{A(d_{1,1})})$,
- $p_{\exists R.A(d_2)} \equiv (p_{R(d_2,d_{2,1})} \otimes p_{A(d_{2,1})})$,
- $p_{\exists R.A(d_2)} \equiv (p_{R(d_2,d_{2,2})} \otimes p_{A(d_{2,2})})$,
- $(p_{R(d_2,d_{2,2})} \otimes p_{A(d_{2,2})}) \to p_{\exists R.A(d_2)}$,
- $(p_{R(d_2,d_{2,1})} \otimes p_{A(d_{2,1})}) \to p_{\exists R.A(d_2)}$.

On the other hand, propositional formulas belonging to $pr(Y_{\texttt{Example}})$ are:

- $\neg p_{\forall R.\exists R.A(d)} \otimes (p_{R(d,d_1)} \to p_{\exists R.A(d_1)})$,
- $\neg p_{\forall R.(\exists R.A \boxtimes \exists R.A)(d)} \otimes (p_{R(d,d_2)} \to p_{(\exists R.A \boxtimes \exists R.A)(d_2)})$.

The next and crucial step in the proof is the following result. We leave the proofs of each one of the directions for the future two sections.

Proposition 41. *Let C_0 be a concept, and let T_{C_0} and Y_{C_0} be the two finite sets associated by Definition 37. For every $r \in [0,1]$, the following statements are equivalent:*

1. *C_0 is satisfiable with truth value r in a quasi-witnessed interpretation,*
2. *there is some propositional evaluation e over the set $Prop$ such that $e(pr(C(d_0))) = r$, $e[pr(T_{C_0})] = 1$, and $e[\psi] \neq 1$ for every $\psi \in pr(Y_{C_0})$.*

From now on we will say that a propositional evaluation e is *quasi-witnessing relatively to C_0* (*quasi-witnessing*, for short) when it satisfies that $e[pr(T_{C_0})] = 1$, and $e[\psi] \neq 1$ for every $\psi \in pr(Y_{C_0})$.

As a consequence of this last proposition we are now able to prove Theorem 34. This is so because by Proposition 41 we know that $C \in \mathrm{QSat}_1$ iff $\bigvee pr(Y_{C_0})$ is not derivable, in the corresponding propositional calculus, from the set $\{pr(C(d_0))\} \cup pr(T_{C_0})$.

Hence, we have a reduction of this problem to the semantic consequence problem, with a finite number of hypothesis, in the corresponding propositional calculus. This problem can be formalized as the problem of deciding, given two propositional formulas φ and ψ, whether ψ is a semantic consequence of φ, i.e., whether, each propositional evaluation which gives value 1 to φ, also gives value 1 to ψ. In [Hájek, 2006, Theorem 3] it is proved that such problem is in $PSPACE$ for the expansion of product logic with truth constants, but, since a formula without truth constants can be considered as a formula of the expanded language in which do not appear truth constants, this result also holds for the product logic without truth constants. Thus, the proof of Proposition 41 is the only missing step in order to prove Theorem 34.

From DL interpretations to propositional evaluations

The purpose of this section is to show the downwards implication of Proposition 41. Let us assume that for a given concept C_0, there is a quasi-witnessed interpretation $\mathcal{I}$ and an object a such that $C_0^{\mathcal{I}}(a) = r$ for some $r \in [0,1]$. The following definition tells us a way to obtain a propositional evaluation satisfying the requirements in Proposition 41.

Definition 42. Let $\mathcal{I}$ be a quasi-witnessed interpretation, a an object of the domain and C_0 a concept. Let us consider T_{C_0}, Y_{C_0} as the sets of assertions obtained from the concept C_0 by applying Definition 37. We assume that the individual a_0 has been interpreted in $\mathcal{I}$ as the object a; for each step, assume that constants in previous steps have been interpreted in $\mathcal{I}$. For each generalized atom α processed in that step, do the following:

($\forall$1) If $\alpha = \forall R.C(d_\sigma)$ and there exists $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then interpret the constant $d_{\sigma,n}$ as u (calling the expansion of $\Delta^{\mathcal{I}}$ by these constants again $\Delta^{\mathcal{I}}$).

($\forall$2) If $\alpha = \forall R.C(d_\sigma)$ and there is no $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d \in \Delta^{\mathcal{I}}}\ R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then choose an element $u \in \Delta^{\mathcal{I}}$ such that $0 < R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) < 1$. Such an element exists since, being $\mathcal{I}$ a quasi-witnessed interpretation, we have, on the one hand, that, for each $u \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) > 0$ and, on the other hand, if there was no element $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) < 1$, then $\inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\} = 1 = R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u)$ against the supposition. Once chosen the element u, interpret the constant $d_{\sigma,n}$ as u (calling the expansion of $\Delta^{\mathcal{I}}$ by these constants again $\Delta^{\mathcal{I}}$).

($\exists$) If $\alpha = \exists R.C(d_\sigma)$, then choose an element $u \in \Delta^{\mathcal{I}}$ witnessing α and interpret the constant $d_{\sigma,n}$ as u (calling the expansion of $\Delta^{\mathcal{I}}$ by these constants again $\Delta^{\mathcal{I}}$).

Finally, for every generalized atom and every atomic formula α, occurring in T, define $e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}$.

Using and modifying an example reported in [Bobillo and Straccia, 2009c], we provide the following instance of the above definition.

Example 43. *Consider the interpretation $\mathcal{I}$ such that:*

1. $\Delta^{\mathcal{I}} = \{a, b, c, e, f\} \cup \{e_i \mid i \in \omega \backslash \{0\}\}$,
2. *there is a binary relation* r *such that* $r(b,c) = r(e,f) = 1$, $r(a,b) = r(a,e) = 0.5$, $r(a,e_i) = 0.5^i$, *and* $R(x,y) = 0$, *when* x, y *is any other pair of elements of the domain.*
3. *there is a unary predicate* s *such that* $s(c) = s(f) = 0.5$ *and* $s(x) = 0$ *for any other element* x *of the domain.*

So, if we take $R^{\mathcal{I}} = r$, $A^{\mathcal{I}} = s$, $d^{\mathcal{I}} = a$, $d_1^{\mathcal{I}} = b$, $d_{1,1}^{\mathcal{I}} = c$, $d_2^{\mathcal{I}} = e$ *and* $d_{2,1}^{\mathcal{I}} = d_{2,2}^{\mathcal{I}} = f$, *then it is easy to check that:*

1. $\mathcal{I}$ *is a quasi-witnessed model of concept* `Example`,
2. $e_{\mathcal{I}}(pr(\varphi)) = 1$, *for each* $\varphi \in T_{\texttt{Example}}$,
3. $e_{\mathcal{I}}(pr(\psi)) < 1$, *for each* $\psi \in Y_{\texttt{Example}}$.

With the following Lemma and Proposition, we are going to prove that all propositional evaluations obtained in this way are quasi-witnessing.

Lemma 44. *Let* $\mathcal{I}$ *be a quasi-witnessed interpretation,* C_0 *a concept, and let us consider* T_{C_0}, Y_{C_0} *as the sets of assertions obtained from the concept* C_0 *by applying Definition 37. Then, the propositional evaluation* $e_{\mathcal{I}}$ *is quasi-witnessing relatively to* C_0.

Proof. We will show the result considering, case by case, the five kinds of proposition we can find in $pr(T_{C_0})$ and $pr(Y_{C_0})$.

1. Consider the assertion $(\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n}))) \sqcup \neg\forall R.C(d_\sigma)$, then:

   (∀1) if, following Definition 42, we have interpreted the constant $d_{\sigma,n}$ as an element $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then we have that $e_{\mathcal{I}}(pr((\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n}))) \sqcup \neg\forall R.C(d_\sigma))) = (e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) \equiv (e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,n}))) \Rightarrow e_{\mathcal{I}}(pr(C(d_{\sigma,n}))))) \vee \neg e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = ((\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) \equiv (R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,n}^{\mathcal{I}}) \Rightarrow C^{\mathcal{I}}(d_{\sigma,n}^{\mathcal{I}}))) \vee \neg(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = (\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) \equiv (R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,n}^{\mathcal{I}}) \Rightarrow C^{\mathcal{I}}(d_{\sigma,n}^{\mathcal{I}})) = 1$.

   (∀2) if there is no $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then, since $\mathcal{I}$ is a quasi-witnessed interpretation, $e_{\mathcal{I}}(pr((\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n}))) \sqcup \neg\forall R.C(d_\sigma))) = (e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) \equiv e_{\mathcal{I}}((pr(R(d_\sigma, d_{\sigma,n}))) \Rightarrow e_{\mathcal{I}}(pr(C(d_{\sigma,n}))))) \vee \neg e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = ((\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) \equiv (R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,n}^{\mathcal{I}}) \Rightarrow C^{\mathcal{I}}(d_{\sigma,n}^{\mathcal{I}}))) \vee \neg(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = \neg(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = 1$. In every case we have that $e_{\mathcal{I}}(pr((\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n}))) \vee \neg\forall R.C(d_\sigma))) = 1$.

2. Consider the assertion $\exists R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \boxtimes C(d_{\sigma,n}))$. Then, by Definition 42, we have that $e_{\mathcal{I}}(pr(\exists R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \boxtimes C(d_{\sigma,n})) = e_{\mathcal{I}}(pr(\exists R.C(d_\sigma))) \equiv e_{\mathcal{I}}((pr(R(d_\sigma, d_{\sigma,n}))) * e_{\mathcal{I}}(pr(C(d_{\sigma,n})))) = \exists R.C^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) \equiv (R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,n}^{\mathcal{I}}) * C^{\mathcal{I}}(d_{\sigma,n}^{\mathcal{I}})) = 1$.

3. Consider the assertion $\forall R.C(d_\sigma) \sqsupset (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))$. Since $(\forall R.C(d_\sigma))^{\mathcal{I}} = \inf_{d\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then, by Definition 42 we have that $e_{\mathcal{I}}(pr(\forall R.C(d_\sigma) \sqsupset (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m})))) = e_{\mathcal{I}}(pr(\forall R.C(d_\sigma)) \Rightarrow (e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,m})) \Rightarrow e_{\mathcal{I}}(pr(C(d_{\sigma,m}))) = (\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) \Rightarrow (R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,m}^{\mathcal{I}}) \Rightarrow C(d_{\sigma,m}^{\mathcal{I}})) = 1$.

4. Consider the assertion $(R(d_\sigma, d_{\sigma,m}) \boxtimes C(d_{\sigma,m})) \sqsupset \exists R.C(d_\sigma)$. Since $(\exists R.C(d_\sigma))^{\mathcal{I}} = \sup_{d\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) * C^{\mathcal{I}}(d)\}$, then, by Definition 42 we have that $e_{\mathcal{I}}(pr((R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))) \sqsupset \exists R.C(d_\sigma)) = (e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,m})) * e_{\mathcal{I}}(pr(C(d_{\sigma,m}))) \Rightarrow e_{\mathcal{I}}(pr(\exists R.C(d_\sigma)) = (R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,m}^{\mathcal{I}}) * C(d_{\sigma,m}^{\mathcal{I}})) \Rightarrow \exists R.C^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = 1$.

5. Consider the assertion $\neg\forall R.C(d_\sigma) \boxtimes (R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n}))$, then:

   ($\forall$1) if, following Definition 42, we have interpreted the constant $d_{\sigma,n}$ as an element $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then we have that if, on the one hand, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma))) = 1$, then $e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = 0$ and, therefore, by Definition 37, $e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n}))) = 0$. Hence $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma) \boxtimes (R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n})))) = 0 < 1$. If, on the other hand, $e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n}))) = 1$, then, by assumption, $e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = 1$ and, therefore, again, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma) \boxtimes (R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n})))) = 0 < 1$.

   ($\forall$2) if there is no $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then, since $\mathcal{I}$ is a quasi-witnessed interpretation, we have that, necessarily, $\forall R^{\mathcal{I}}.C^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = 0$ and, hence, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma))) = 1$. However, since, by Definition 42, , we have interpreted the constant $d_{\sigma,n}$ as an element $u \in \Delta^{\mathcal{I}}$ such that $0 < R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) < 1$ and, therefore, we have that $e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n}))) < 1$. So, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma) \boxtimes (R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n})))) < 1$.

Hence, for every proposition $pr(\varphi) \in pr(T_{C_0})$, it holds that $e_{\mathcal{I}}(pr(\varphi)) = 1$ and for every proposition $pr(\psi) \in pr(Y_{C_0})$, it holds that $e_{\mathcal{I}}(pr(\psi)) < 1$ and, therefore, $e_{\mathcal{I}}$ is a quasi-witnessing propositional evaluation. □

Proposition 45. *Let $\mathcal{I}$ be a quasi-witnessed interpretation, $C_0(a_0)$ a $\mathfrak{I}\mathcal{ALE}$-assertion and T_{C_0}, Y_{C_0} the sets of assertions produced from $C_0(a_0)$ applying Definition 37, then, for every $\alpha \in T_{C_0} \cup Y_{C_0}$, it holds that $e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}$.*

Proof. We will prove the Lemma by induction on the costruction of α.

1. If α is an atomic formula, it is straightforward from Definition 42.

2. If α is a generalized atom, it is straightforward from Lemma 44.
3. If α is of the form $\delta \star \gamma$ where δ, γ are either atomic formulas or generalized atoms, $\star$ is a concept constructor and $\circ$ is the respective algebraic operation, suppose, by inductive hypothesis, that $e_{\mathcal{I}}(pr(\delta)) = \delta^{\mathcal{I}}$ and $e_{\mathcal{I}}(pr(\gamma)) = \gamma^{\mathcal{I}}$. Hence, $e_{\mathcal{I}}(pr(\alpha)) = e_{\mathcal{I}}(pr(\delta \star \gamma)) = e_{\mathcal{I}}(pr(\delta)) \circ e_{\mathcal{I}}(pr(\gamma)) = \delta^{\mathcal{I}} \circ \gamma^{\mathcal{I}} = (\delta \star \gamma)^{\mathcal{I}} = \alpha^{\mathcal{I}}$.

Hence, for every proposition $pr(\alpha)$ in $pr(T_{C_0} \cup Y_{C_0})$, it holds that $e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}$. In particular, $e_{\mathcal{I}}(pr(C_0(a_0))) = C_0^{\mathcal{I}}(a_0^{\mathcal{I}})$. □

This finishes the proof of the downwards implication of Proposition 41.

From propositional evaluations to DL interpretations

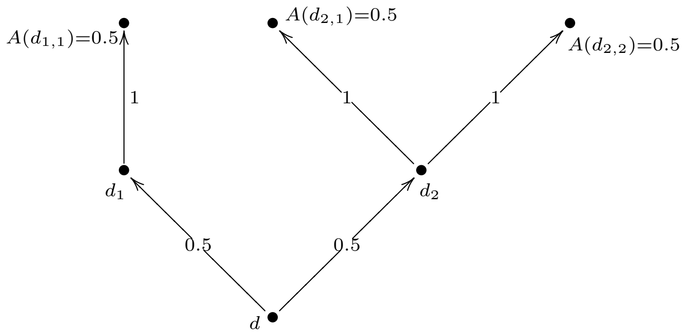
The aim of this section is to prove the upwards implication of Proposition 41. Let us assume that there is a propositional evaluation quasi-witnessing relatively to C_0 such that $e(pr(C_0(d))) = r$ for some $r \in [0, 1]$. First of all, we provide a way to obtain a quasi-witnessed interpretation from a quasi-witnessing propositional evaluation with the above features.

Definition 46. Let $C_0(a_0)$ be an assertion, T_{C_0} and Y_{C_0} the sets of concepts and axioms produced from $C_0(a_0)$ applying Definition 37, let $pr(T_{C_0})$, $pr(Y_{C_0})$ be the sets of propositions obtained by applying Definition 39 and let e be a quasi-witnessing propositional evaluation. Then we define the witnessed part $\mathcal{I}_e^w$ of our interpretation $\mathcal{I}_e$ as follows:

1. $\Delta^{\mathcal{I}_e^w}$ is the set of all constants d_σ occurring in formulas of T.
2. For each atomic concept A, let:
   (a) $A^{\mathcal{I}_e^w}(d_\sigma) = e(pr(A(d_\sigma)))$, where $\sigma = l(A)$, if $pr(A(d_\sigma))$ occurs in $pr(T_{C_0})$,
   (b) $A^{\mathcal{I}_e^w}(d_\sigma) = 0$, otherwise.
3. For each role R let:
   (a) $R^{\mathcal{I}_e^w}(d_\sigma, d_{\sigma,n}) = e(pr(R(d_\sigma, d_{\sigma,n})))$, if $pr(R(d_\sigma, d_{\sigma,n}))$ occurs in $pr(T_{C_0})$,
   (b) $R^{\mathcal{I}_e^w}(d_\sigma, d_{\sigma,n}) = 0$, otherwise.

In order to illustrate Definition 46, we provide an example of the witnessed interpretation arising from $pr(T_{\texttt{Example}})$ and $pr(Y_{\texttt{Example}})$.

Example 47. *Let e be a propositional evaluation such that $p_{R(d,d_1)} = p_{R(d,d_2)} = 0.5$, $p_{R(d_1,d_{1,1})} = p_{R(d_2,d_{2,1})} = p_{R(d_2,d_{2,2})} = 1$, $p_{A(d_{1,1})} = p_{A(d_{2,1})} = p_{A(d_{2,2})} = 0.5$. As we have seen in the previous section, this is indeed a quasi-witnessing propositional evaluation. Moreover, following Definition 46, we obtain the following interpretation:*



We point out that this interpretation, however, is not a model of the concept `Example`.

The structure defined in Definition 46 is a witnessed interpretation which would be enough in case we were only interested on witnessed interpretations. But in order to encompass all quasi-witnessed interpretations we need the following extension of the above interpretation.

Definition 48. Let $C_0(a_0)$ be an assertion, T_{C_0} and Y_{C_0} the sets of first order formulas produced from $C_0(a_0)$ applying Definition 37, let $pr(T_{C_0})$, $pr(Y_{C_0})$ be the sets of propositions obtained by applying Definition 39 and let e be a quasi-witnessing propositional evaluation; finally let $\mathcal{I}_e^w$ be the interpretation defined in Definition 46. Then we define the first order interpretation $\mathcal{I}_e$ as the following expansion of $\mathcal{I}_e^w$:

1. The domain $\Delta^{\mathcal{I}_e}$ is obtained by adding to $\Delta^{\mathcal{I}_e^w}$ an infinite set of new individuals $\{d_\sigma^i | i \in \omega\backslash\{0\}\}$, for each $d_\sigma \in \Delta^{\mathcal{I}_e^w}$, but not for d.

2. if A is an atomic concept, and $pr(A(d_\sigma^i))$ occurs in $pr(T_{C_0})$, then $A^{\mathcal{I}_e}(d_\sigma^i) = (A^{\mathcal{I}_e}(d_\sigma))^i$,

3. For each role R:

   (a) if R appears in an universally quantified formula, then:

      i. if $e(pr(\forall R.C(d_\sigma))) \neq e(pr(R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_\alpha)))$, then:

         A. $R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}^i) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^i$, for every $i \in \omega\backslash\{0\}$,

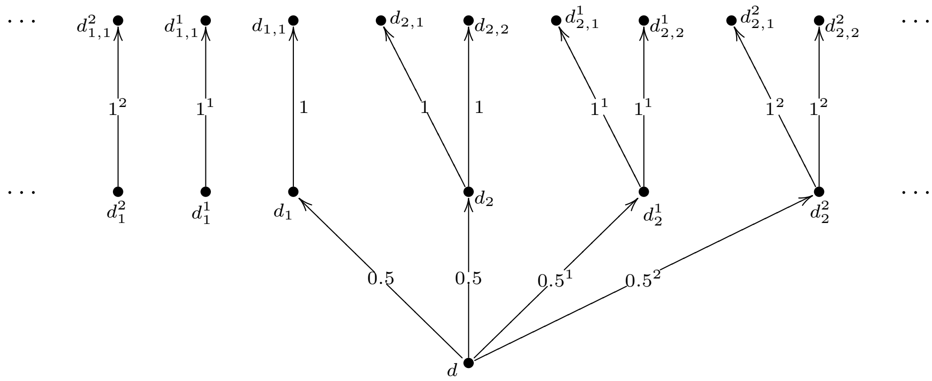
         B. $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,n}^j) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^j$, for every $i, j \in \omega\backslash\{0\}$,

      ii. if $e(pr(\forall R.C(d_\sigma))) = e(pr(R(d_\sigma, d_{\sigma,n}) \sqsupset D(d_{\sigma,n})))$, then $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,n}^j) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^j$, for every $i, j \in \omega\backslash\{0\}$, if $i = j$ and $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,n}^j) = 0$, otherwise,

   (b) if R appears in an existentially quantified formula, then $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,n}^j) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^j$, for every $i, j \in \omega\backslash\{0\}$, if $i = j$ and $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,n}^j) = 0$, otherwise.

In order to illustrate Definition 48, we provide an example of the quasi-witnessed interpretation arising from $pr(T_{\texttt{Example}})$ and $pr(Y_{\texttt{Example}})$.

Example 49. *Let e be the same propositional evaluation as in the previous example, then, following Definition 46, we obtain the following interpretation:*



In this case it is worth pointing out that this interpretation is indeed a model of `Example`*.*

Lemma 50. *Let $D(d_\sigma) \in Sub(C_0)$ and e a quasi-witnessing propositional evaluation, then, for each $i \in \omega \backslash \{0\}$, it holds that $D^{\mathcal{I}_e}(d^i_\sigma) = (D^{\mathcal{I}_e}(d_\sigma))^i$.*

Proof. The proof is by induction on the nesting degree of C_0.

(0) An assertion with nesting degree equal to 0 is either an atomic concept or a propositional combination of atomic concepts:

1. If C_0 is an atomic concept, then it is straightforward from Definition 48.
2. Let $C_0 = E \circ F$, where E, F are atomic concepts and $\circ \in \{\Rightarrow, *\}$. Suppose, by inductive hypothesis, that the claim holds for two concepts E, F, then:

$$\begin{array}{rcl}(E^{\mathcal{I}_e} \circ F^{\mathcal{I}_e})(d^i_\sigma) & = & E^{\mathcal{I}_e}(d^i_\sigma) \circ F^{\mathcal{I}_e}(d^i_\sigma) \\ & = & (E^{\mathcal{I}_e}(d_\sigma))^i \circ (F^{\mathcal{I}_e}(d_\sigma))^i \\ & = & (E^{\mathcal{I}_e}(d_\sigma) \circ F^{\mathcal{I}_e}(d_\sigma))^i \\ & = & (E^{\mathcal{I}_e} \circ F^{\mathcal{I}_e}(d_\sigma))^i\end{array}$$

(k+1) Let $D(d_\sigma)$ be a generalized atom with nesting degree equal to $k+1$ and suppose, by inductive hypothesis, that, for each generalized atom $E(d_{\sigma,n})$ with nesting degree equal to k, it holds that $E^{\mathcal{I}_e}(d^i_{\sigma,n}) = (E^{\mathcal{I}_e}(d_{\sigma,n}))^i$, then:

1. If $D(d_\sigma) = \exists R.E(d_\sigma)$, then, by Definition 48, $D^{\mathcal{I}_e}(d^i_\sigma) = \sup_{d \in \Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d^i_\sigma, d) * E^{\mathcal{I}_e}(d) = R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,n}) * E^{\mathcal{I}_e}(d^i_{\sigma,n})$ and, by inductive hypothesis, Definition 37 and Definition 48, $R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,n}) *$

$E^{\mathcal{I}_e}(d^i_{\sigma,n}) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^i * (E^{\mathcal{I}_e}(d_{\sigma,n}))^i = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) * E^{\mathcal{I}_e}(d_{\sigma,n}))^i = (D^{\mathcal{I}_e}(d_\sigma))^i$.

2. If $D(d_\sigma) = \forall R.E(d_\sigma)$, and $e(pr(\forall R.E(d_\sigma))) = (R(d_\sigma, d_{\sigma,n}) \sqsupset E(d_{\sigma,n}))$, then, by Definition 48, $E^{\mathcal{I}_e}(d^i_\sigma) = \inf_{d\in\Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d^i_\sigma, d) \Rightarrow E^{\mathcal{I}_e}(d) = R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,n}) \Rightarrow E^{\mathcal{I}_e}(d^i_{\sigma,n})$ and, by inductive hypothesis, Definition 37 and Definition 48, $R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,n}) \Rightarrow E^{\mathcal{I}_e}(d^i_{\sigma,n}) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^i \Rightarrow (E^{\mathcal{I}_e}(d_{\sigma,n}))^i = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow E^{\mathcal{I}_e}(d_{\sigma,n}))^i = (D^{\mathcal{I}_e}(d_\sigma))^i$.
3. If $D(d_\sigma) = \forall R.E(d_\sigma)$, and $e(pr(\forall R.E(d_\sigma))) \neq (R(d_\sigma, d_{\sigma,n}) \sqsupset E(d_{\sigma,n}))$, then, by Definition 48, $D^{\mathcal{I}_e}(d_\sigma) = 0$ and, therefore, by Definition 48, $D^{\mathcal{I}_e}(d^i_\sigma) = \inf_{d\in\Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d^i_\sigma, d) \Rightarrow E^{\mathcal{I}_e}(d) = \inf_{j\in\omega\setminus\{0\}}\{R^{\mathcal{I}_e}(d^i_\sigma, d^j_{\sigma,n}) \Rightarrow E^{\mathcal{I}_e}(d^j_{\sigma,n}) = 0 = (D^{\mathcal{I}_e}(d_\sigma))^i$. □

Proposition 51. *Let e be a quasi-witnessing propositional evaluation, then, for every assertion α, $e(pr(\alpha)) = \alpha^{\mathcal{I}_e}$.*

Proof. The proof is by induction on the nesting degree of α.

(0) An assertion with nesting degree equal to 0 is either an atomic concept or a propositional combination of atomic concepts:

1. If α is an atomic concept, then it is straightforward from Definition 46.
2. Let $\alpha = C \star D$, where C, D are concepts and $\star \in \{\sqsupset, \boxtimes\} \cup \{\rightarrow, \otimes\}$ and let $\circ \in \{\Rightarrow, *\}$. Suppose that the inductive hypothesis holds for two concepts C, D, then, by Definition 39 we have that, for each concept constructor $\star$:

$$\begin{aligned}(C \star D)^{\mathcal{I}_e} &= C^{\mathcal{I}_e} \circ D^{\mathcal{I}_e}\\ &= e(pr(C)) \circ e(pr(D))\\ &= e(pr(C) \star pr(D))\\ &= e(pr(C \star D))\end{aligned}$$

(k+1) Let α be a generalized atom with nesting degree equal to $k+1$ and suppose, by inductive hypothesis, that, for each generalized atom β with nesting degree $\leq n$, occurring within the scope of the quantifier of α, it holds that $e(pr(\beta)) = \beta^{\mathcal{I}_e}$.

1. If $\alpha = \exists R.C(d_\sigma)$, then, by Definition 37 we have that $e(pr(\alpha)) = e(pr(R(d_\sigma, d_{\sigma,n}) \boxtimes C(d_{\sigma,n})))$ and, by Definition 46 and inductive hypothesis, we have that $e(pr(R(d_\sigma, d_{\sigma,n}) \boxtimes C(d_{\sigma,n}))) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) * C^{\mathcal{I}_e}(d_{\sigma,n})$. Let $d \in \Delta^{\mathcal{I}_e}$ be any constant different from $d_{\sigma,n}$, then either d is associated to to R, a or not. In the first case, since, by Definition 37, $e(pr(R(d_\sigma, d) \boxtimes C(d))) \Rightarrow e(pr(\alpha)) = 1$, then $R^{\mathcal{I}_e}(d_\sigma, d) * C^{\mathcal{I}_e}(d) \leq e(pr(\alpha))$. In the second case, by Definition 46, $R^{\mathcal{I}_e}(d_\sigma, d) * C^{\mathcal{I}_e}(d) = 0 * C^{\mathcal{I}_e}(d) = 0 \leq e(pr(\alpha))$. So, in each case, $e(pr(\alpha)) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) * C^{\mathcal{I}_e}(d_{\sigma,n}) = \sup_{d\in\Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d_\sigma, d) * C^{\mathcal{I}_e}(d)\} = \alpha^{\mathcal{I}_e}$.

2. If $\alpha = \forall R.C(a)$ and $e(pr(\alpha)) = e(pr(R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n})))$, then, by Definition 46 and inductive hypothesis, we have that $e(pr(\alpha)) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n})$. Let $d \in \Delta^{\mathcal{I}_e}$ be any constant different from $d_{\sigma,n}$, then either d is associated to R, a or not. In the first case, since, by Definition 37, $e(pr(\alpha)) \Rightarrow e(pr(R(d_\sigma, d) \sqsupset C(d))) = 1$, then $e(pr(\alpha)) \leq R^{\mathcal{I}_e}(d_\sigma, d) \Rightarrow C^{\mathcal{I}_e}(d)$. In the second case, by Definition 48, $R^{\mathcal{I}_e}(d_\sigma, d) \Rightarrow C^{\mathcal{I}_e}(d) = 0 \Rightarrow C^{\mathcal{I}_e}(d) = 1 \geq e(pr(\alpha))$. So, in each case, $e(pr(\alpha)) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n}) = \inf_{d \in \Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d_\sigma, d) \Rightarrow C^{\mathcal{I}_e}(d)\} = \alpha^{\mathcal{I}_e}$.
3. If $\alpha = \forall R.C(a)$ and $e(pr(\alpha)) \neq e(pr(R(d_\sigma, d_{\sigma,n}) \sqsupset C(d_{\sigma,n})))$, then, by Definition 37 we have that $0 = e(pr(\alpha))$ and, by Definition 46 and inductive hypothesis, we have that $e(pr(\alpha)) \neq R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n})$. Again by Definition 37 (look at the set Y) we have that $R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n}) < 1$ and, by the above assumptions, we have that $R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n}) > 0$. Since, by Lemma 50, we have that, for each $i \in \omega \backslash \{0\}$, $R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d^i_{\sigma,n}) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n}))^i$, then $e(pr(\alpha)) = 0 = \inf_{i \in \omega \backslash \{0\}}\{(R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n}))^i\} = \inf_{i \in \omega \backslash \{0\}}\{R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d^i_{\sigma,n})\} = \inf_{d \in \Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d_\sigma, d) \Rightarrow C^{\mathcal{I}_e}(d)\} = \alpha^{\mathcal{I}_e}$.

The result is straightforward for propositional combinations of atomic concepts and generalized atoms with nesting degree equal to $k + 1$.

In particular, $e(pr(C_0(a_0))) = C_0^{\mathcal{I}_e}(a_0)$. □

This finishes the last step in the proof of Proposition 41, and so the last step in the proof of Theorem 34.

4.2 Concept subsumption

The same procedures proposed in [Hájek, 2005], for the case of witnessed interpretations and Łukasiewicz semantics is used to prove decidability of concept 1-subsumption[1] for language $\mathfrak{I}\mathcal{ALCE}$ under Łukasiewicz semantics. For the 1-subsumption problem the reduction is to the propositional entailment problem. In [Hájek, 2005, Theorem 1] it is indeed stated that a concept C_0 is valid iff $pr(T_{C_0})$ entails $pr(C_0)$. This, clearly means that

$$\text{concept } C \text{ is 1-subsumed by concept } D \iff pr(T_{C \sqsupset D}) \text{ entails } pr(C \sqsupset D)$$

For the case of quasi-witnessed interpretations and product t-norm the procedure proposed in Section 4.1.1 can be used as well. We will use the notation

[1] [Hájek, 2005] indeed deals with *validity of formulas*, but recall that a concept C is r-subsumed by a concept D if and only if concept $\top$ is r-subsumed by concept $C \sqsupset D$. So, it is straightforward that, concept $C \sqsupset D$ is valid, if and only if $\top$ is 1-subsumed by $C \sqsupset D$, if and only if C is 1-subsumed by D.

Subs to denote the set of concepts that 1-subsume the top concept $\top$; and we will write QSubs to denote the same problem restricted to quasi-witnessed interpretations. The reason is, again, that the logical consequence problem is decidable for Product Logic and, as for positive satisfiability, we can reduce the problem of deciding whether a concept belongs to QSubs to the logical consequence problem in propositional Product Logic. In fact, once obtained the sets T_{C_0} and Y_{C_0} for a given concept C_0, we have that

$$C_0 \in \text{QSubs} \iff pr(T_{C_0}) \text{ entails } pr(C_0(d_0)) \vee \bigvee pr(Y_{C_0})$$

In this way, in [Cerami et al., 2010a] has been proven the following theorem.

Theorem 52. *The set* QSubs *is decidable.*

By the completeness result of product first order logic with respect to quasi-witnessed models reported in Section 2.1.3, we obtain that QSubs is a decidable problem under product general semantics. In Appendix A it is proven that quasi-witnessed validity and unrestricted validity indeed coincide under the standard product semantics. Hence unrestricted validity is a decidable problem in $[0,1]_\Pi$.

4.3 Knowledge base consistency in Łukasiewicz logic

The suspect that general KB consistency was not a decidable problem with an infinite (non-idempotent) set of truth values, began when in [Bobillo et al., 2011] was proved the failure of the finite model property for language $\Im\mathcal{ALCE}$ based on Łukasiewicz and product t-norm and its decidability kept being suspicious with [Baader and Peñaloza, 2011a]. Nevertheless, until then, there was no direct evidence that those problems were undecidable.

The first result on undecidability can be found in [Baader and Peñaloza, 2011b] for language $\Im\mathcal{ALCE}$ based on product t-norm with respect to witnessed interpretations. The proof in [Baader and Peñaloza, 2011b] consists in a reduction of the *Post Correspondence Problem* (PCP) to the general KB consistency for language $\Im\mathcal{ALCE}$ based on product t-norm. Subsequently, using the same methods in [Baader and Peñaloza, 2011b] and [Borgwardt and Peñaloza, 2011c], we achieved an undecidability proof for language $\Im\mathcal{ALCE}$ based on Łukasiewicz t-norm with respect to witnessed interpretations.

Our proof consists of a reduction of the *reverse* of the PCP and follows conceptually the one in [Baader and Peñaloza, 2011a, Baader and Peñaloza, 2011b, Borgwardt and Peñaloza, 2011c]. PCP is well known to be undecidable [Post, 1946], so is the reverse PCP, as shown next.

Definition 53 (PCP)**.** An instance φ of the *Post Correspondence Problem* (PCP) is defined in the following way: let $v_1, \ldots, v_p$ and $w_1, \ldots, w_p$ be two finite lists of words over an alphabet $\Sigma = \{1, \ldots, s\}$. A *solution* to φ is a non-empty sequence $i_1, i_2, \ldots, i_k$, with $1 \le i_j \le p$ such that $v_{i_1} v_{i_2} \ldots v_{i_k} = w_{i_1} w_{i_2} \ldots w_{i_k}$. Given φ, the decision problem then is to decide whether a solution to φ exists or not.

For the sake of our purpose, we will rely on a variant of the PCP, which we call *Reverse* PCP (RPCP). Essentially, words are concatenated from right to left rather than from left to right.

Definition 54 (RPCP)**.** An instance φ of the *Reverse Post Correspondence Problem* (RPCP) is defined in the following way: let $v_1, \ldots, v_p$ and $w_1, \ldots, w_p$ be two finite lists of words over an alphabet $\Sigma = \{1, \ldots, s\}$. A *solution* to φ is a non-empty sequence $i_1, i_2, \ldots, i_k$, with $1 \le i_j \le p$ such that $v_{i_k} v_{i_{k-1}} \ldots v_{i_1} = w_{i_k} w_{i_{k-1}} \ldots w_{i_1}$. Given φ, the decision problem then is to decide whether a solution to φ exists or not.

For a word $\mu = i_1 i_2 \ldots i_k \in \{1, \ldots, p\}^*$ we will use v_μ, w_μ to denote the words $v_{i_k} v_{i_{k-1}} \ldots v_{i_1}$ and $w_{i_k} w_{i_{k-1}} \ldots w_{i_1}$. We denote the empty string as ε and define v_ε as ε. The alphabet Σ consists of the first s positive integers. We can thus view every word in Σ^* as a natural number represented in base $s+1$ in which 0 never occurs. Using this intuition, we will use the number 0 to encode the empty word.

Now we show that the reduction from PCP to RPCP is a very simple matter and it can be done through the transformation of the instance lists to the lists of their palindromes defined as follows: let $\Sigma = \{1, \ldots, s\}$ be an alphabet and $v = t_1 t_2 \ldots t_{|v|}$ a word over Σ, with $t_i \in \Sigma$, for $1 \le i \le |v|$, then the *palindrome* of v is defined as $pal(v) = t_{|v|} t_{|v|-1} \ldots t_1$.

Lemma 55. *Let $v_1, \ldots, v_p$ and $w_1, \ldots, w_p$ be two finite lists of words over an alphabet $\Sigma = \{1, \ldots, s\}$. For every non-empty sequence $i_1, i_2, \ldots, i_k$, with $1 \le i_j \le p$ it holds that*

$$\begin{array}{rcl} v_{i_1} v_{i_2} \ldots v_{i_k} & = & w_{i_1} w_{i_2} \ldots w_{i_k} \\ & \textit{iff} & \\ pal(v_{i_k}) pal(v_{i_{k-1}}) \ldots pal(v_{i_1}) & = & pal(w_{i_k}) pal(w_{i_{k-1}}) \ldots pal(w_{i_1}) \ . \end{array}$$

(*Proof*) First we prove by induction on k, that, for every sequence $v = v_{i_1} v_{i_2} \ldots v_{i_k}$ of words over Σ, it holds that $pal(v) = pal(v_{i_k}) pal(v_{i_{k-1}}) \ldots pal(v_{i_1})$.

- The case $k = 1$ is straightforward.
- Let $v = v_{i_1} v_{i_2} \ldots v_{i_k}$ and suppose, by inductive hypothesis, that $pal(v_{i_1} v_{i_2} \ldots v_{i_{k-1}}) = pal(v_{i_{k-1}}) pal(v_{i_{k-2}}) \ldots pal(v_{i_1})$. It follows that $pal(v) = pal(v_{i_1} v_{i_2} \ldots v_{i_{k-1}}, v_{i_k}) = pal(v_{i_k}) pal(v_{i_{k-1}}) \ldots pal(v_{i_1})$.

Since the palindrome of a word is unique, we have that, if $v_{i_1} v_{i_2} \ldots v_{i_k} = w_{i_1} w_{i_2} \ldots w_{i_k}$, then $pal(v_{i_1} v_{i_2} \ldots v_{i_k}) = pal(w_{i_1} w_{i_2} \ldots w_{i_k})$ and, thus, $pal(v_{i_k}) pal(v_{i_{k-1}}) \ldots pal(v_{i_1}) = pal(w_{i_k})\ pal(w_{i_{k-1}}) \ldots\ pal(w_{i_1})$. □

Corollary 56. *RPCP is undecidable.*

(*Proof*) The proof is based on the reduction of PCP to RPCP. For every instance $\varphi = (v_1, w_1), \ldots, (v_p, w_p)$ of PCP, let f be the function

$$f(\varphi) = (pal(v_1), pal(w_1)), \ldots, (pal(v_p), pal(w_p)) \ .$$

Clearly f is a computable function. Moreover, φ has a solution if and only if there exists a non-empty sequence $i_1, i_2, \ldots, i_k$, with $1 \leq i_j \leq p$ such that $v_{i_1} v_{i_2} \ldots v_{i_k} = w_{i_1} w_{i_2} \ldots w_{i_k}$, that is, by Lemma 55,

$$pal(v_{i_k})pal(v_{i_{k-1}}) \ldots pal(v_{i_1}) = pal(w_{i_k})pal(w_{i_{k-1}}) \ldots pal(w_{i_1})$$

i.e., $f(\varphi)$ has a solution. Therefore, $\varphi \in PCP$ has a solution if and only if $f(\varphi) \in RPCP$ has a solution. □

4.3.1 Undecidability of general KB satisfiability

We show the undecidability by a reduction of RPCP to KB satisfiability problems. Specifically, given an instance φ of RPCP, we will construct a Knowledge Base $\mathcal{O}_\varphi$ that is satisfiable iff φ has no solution.

In order to do this we will encode words v from the alphabet Σ as rational numbers $0.v$ in $[0, 1]$ in base $s+1$; the empty word will be encoded by the number 0.

So, let us define the following TBoxes:

$$\mathcal{T} \ := \{ \quad V \equiv V_1 \boxplus V_2, W \equiv W_1 \boxplus W_2 \quad \}$$

and for $1 \leq i \leq p$

$$\begin{aligned}
\mathcal{T}^i_\varphi \ := \{ \quad & \top \sqsubseteq \exists R_i.\top, \\
\\
& V \sqsubseteq (s+1)^{|v_i|} \cdot \forall R_i.V_1, \\
& (s+1)^{|v_i|} \cdot \exists R_i.V_1 \sqsubseteq V, \\
& W \sqsubseteq (s+1)^{|w_i|} \cdot \forall R_i.W_1, \\
& (s+1)^{|w_i|} \cdot \exists R_i.W_1 \sqsubseteq W \\
\\
& \langle \top \sqsubseteq \forall R_i.V_2, 0.v_i \rangle, \\
& \langle \top \sqsubseteq \forall R_i.\neg V_2, 1 - 0.v_i \rangle, \\
& \langle \top \sqsubseteq \forall R_i.W_2, 0.w_i \rangle, \\
& \langle \top \sqsubseteq \forall R_i.\neg W_2, 1 - 0.w_i \rangle, \\
\\
& A \sqsubseteq (s+1)^{\max\{|v_i|,|w_i|\}} \cdot \forall R_i.A \\
& (s+1)^{\max\{|v_i|,|w_i|\}} \cdot \exists R_i.A \sqsubseteq A \quad \} \ .
\end{aligned}$$

Now, let

$$\mathcal{T}_\varphi = \mathcal{T} \cup \bigcup_{i=1}^{p} \mathcal{T}_\varphi^i \ .$$

Further we define the ABox $\mathcal{A}$ as follows:

$$\mathcal{A} \quad := \quad \{\neg V(a), \neg W(a), \langle A(a), 0.01\rangle, \langle \neg A(a), 0.99\rangle\} \ .$$

Finally, we define

$$\mathcal{O}_\varphi := \langle \mathcal{T}_\varphi, \mathcal{A}\rangle \ .$$

We now define the interpretation

$$\mathcal{I}_\varphi := (\Delta^{\mathcal{I}_\varphi}, \cdot^{\mathcal{I}_\varphi})$$

as follows:

- $\Delta^{\mathcal{I}_\varphi} = \{1, \ldots, p\}^*$
- $a^{\mathcal{I}_\varphi} = \varepsilon$
- $V^{\mathcal{I}_\varphi}(\varepsilon) = W^{\mathcal{I}_\varphi}(\varepsilon) = 0$, $A^{\mathcal{I}_\varphi}(\varepsilon) = 0.01$, and for $1 \le i \le 2$, $V_i^{\mathcal{I}_\varphi}(\varepsilon) = W_i^{\mathcal{I}_\varphi}(\varepsilon) = 0$
- for all $\mu, \mu' \in \Delta^{\mathcal{I}_\varphi}$ and $1 \le i \le p$

$$R_i^{\mathcal{I}_\varphi}(\mu, \mu') = \begin{cases} 1, & \text{if } \mu' = \mu i \\ 0, & \text{otherwise} \end{cases}$$

- for every $\mu \in \Delta^{\mathcal{I}_\varphi}$, where $\mu = i_1 i_2 \ldots i_k \neq \varepsilon$
  - $V^{\mathcal{I}_\varphi}(\mu) = 0.v_\mu$, $W^{\mathcal{I}_\varphi}(\mu) = 0.w_\mu$
  - $A^{\mathcal{I}_\varphi}(\mu) = 0.01 \cdot (s+1)^{-\sum_{j \in \{i_1, i_2, \ldots, i_k\}} \max\{|v_j|, |w_j|\}}$
  - $V_1^{\mathcal{I}_\varphi}(\mu) = 0.v_{\bar{\mu}} \cdot (s+1)^{-|v_{i_k}|}$, $W_1^{\mathcal{I}_\varphi}(\mu) = 0.w_{\bar{\mu}} \cdot (s+1)^{-|w_{i_k}|}$, where $\bar{\mu} = i_1 i_2 \ldots i_{k-1}$ (last index i_k is dropped from μ, and we assume that $0.\varepsilon$ is 0),
  - $V_2^{\mathcal{I}_\varphi}(\mu) = 0.v_{i_k}$, $W_2^{\mathcal{I}_\varphi}(\mu) = 0.w_{i_k}$.

It is easy to see that $\mathcal{I}_\varphi$ is a witnessed model of $\mathcal{O}_\varphi$ (note that *e.g.*, $(\forall R_i.V_1)^{\mathcal{I}_\varphi}(\mu) = V_1^{\mathcal{I}_\varphi}(\mu i)$). [2]

Moreover, as in [Baader and Peñaloza, 2011a] it is possible to prove that, for every witnessed model $\mathcal{I}$ of $\mathcal{O}_\varphi$, there is a mapping g from $\mathcal{I}_\varphi$ to $\mathcal{I}$.

Lemma 57. *Let $\mathcal{I}$ be a witnessed model of $\mathcal{O}_\varphi$. Then there exists a function $g : \Delta^{\mathcal{I}_\varphi} \to \Delta^{\mathcal{I}}$ such that, for every $\mu \in \Delta^{\mathcal{I}_\varphi}$, $C^{\mathcal{I}_\varphi}(\mu) = C^{\mathcal{I}}(g(\mu))$ holds for every concept name C and $R_i^{\mathcal{I}_\varphi}(\mu, \mu i) = R_i^{\mathcal{I}}(g(\mu), g(\mu i))$ holds for every i, with $1 \le i \le p$.*

[2]However, $\mathcal{I}_\varphi$ is not a strongly witnessed model of $\mathcal{O}_\varphi$.

(*Proof*) Let $\mathcal{I}$ be a witnessed model of $\mathcal{O}_\varphi$. We will build the function g inductively on the length of μ.

(ε) Since $\mathcal{I}$ is a model of $\mathcal{O}_\varphi$, then there is an element $\delta \in \Delta^{\mathcal{I}}$ such that $a^{\mathcal{I}} = \delta$. Since $\mathcal{I}$ is a model of $\mathcal{A}_\varphi$, setting $g(\varepsilon) = \delta$, we have that $V^{\mathcal{I}_\varphi}(\varepsilon) = 0 = V^{\mathcal{I}}(g(\varepsilon))$ and the same holds for concept W. Moreover, since $\mathcal{I}$ is a model of $\mathcal{T}_\varphi$, we have that $V^{\mathcal{I}}(\delta) = (V_1 \boxplus V_2)^{\mathcal{I}}(\delta)$ and, therefore $V_1^{\mathcal{I}_\varphi}(\varepsilon) = 0 = V_1^{\mathcal{I}}(g(\varepsilon))$ and the same holds for V_2, W_1 and W_2. On the other hand, we have that $A^{\mathcal{I}_\varphi}(\varepsilon) = 0.01 = A^{\mathcal{I}}(g(\varepsilon))$, as well. So, $g(\varepsilon) = \delta$ satisfies the condition of the lemma.

(μi) Let now μ be such that $g(\mu)$ has already been defined. Now, since $\mathcal{I}$ is a witnessed model and satisfies axiom $\top \sqsubseteq \exists R_i.\top$, then for all i, with $1 \leq i \leq p$, there exists a $\gamma \in \Delta^{\mathcal{I}}$ such that $R_i^{\mathcal{I}}(g(\mu), \gamma) = 1$. So, setting $g(\mu i) = \gamma$ we get $1 = R_i^{\mathcal{I}_\varphi}(\mu, \mu i) = R_i^{\mathcal{I}}(g(\mu), g(\mu i))$. Furthermore, by inductive hypothesis, we can assume that $V^{\mathcal{I}}(g(\mu)) = 0.v_\mu$ and $W^{\mathcal{I}}(g(\mu)) = 0.w_\mu$.

Since $\mathcal{I}$ satisfies axiom $V \sqsubseteq (s+1)^{|v_i|} \cdot \forall R_i.V_1$, then $0.v_\mu = V^{\mathcal{I}}(g(\mu)) \leq (s+1)^{|v_i|} \cdot (\forall R_i.V_1)^{\mathcal{I}}(g(\mu)) = (s+1)^{|v_i|} \cdot \inf_{\gamma \in \Delta^{\mathcal{I}}}\{R_i^{\mathcal{I}}(g(\mu), \gamma) \Rightarrow V_1^{\mathcal{I}}(\gamma)\} \leq (s+1)^{|v_i|} \cdot R_i^{\mathcal{I}}(g(\mu), \mu i) \Rightarrow V_1^{\mathcal{I}}(\mu i) = (s+1)^{|v_i|} \cdot V_1^{\mathcal{I}}(g(\mu i))$.

Since $\mathcal{I}$ satisfies axiom $(s+1)^{|v_i|} \cdot \exists R_i.V_1 \sqsubseteq V$, then $0.v_\mu = V^{\mathcal{I}}(g(\mu)) \geq (s+1)^{|v_i|} \cdot (\exists R_i.V_1)^{\mathcal{I}}(g(\mu)) = (s+1)^{|v_i|} \cdot \sup_{\gamma \in \Delta^{\mathcal{I}}}\{R_i^{\mathcal{I}}(g(\mu), \gamma) \otimes V_1^{\mathcal{I}}(\gamma)\} \geq (s+1)^{|v_i|} \cdot R_i^{\mathcal{I}}(g(\mu), \mu i) \otimes V_1^{\mathcal{I}}(\mu i) = (s+1)^{|v_i|} \cdot V_1^{\mathcal{I}}(g(\mu i))$.

Therefore, $(s+1)^{|v_i|} \cdot V_1^{\mathcal{I}}(g(\mu i)) = 0.v_\mu$ and $V_1^{\mathcal{I}}(g(\mu i)) = 0.v_\mu \cdot (s+1)^{-|v_i|} = V_1^{\mathcal{I}_\varphi}(\mu i)$.

Similarly, it can be shown that $W_1^{\mathcal{I}}(g(\mu i)) = 0.w_\mu \cdot (s+1)^{-|w_i|} = W_1^{\mathcal{I}_\varphi}(\mu i)$.

Since $\mathcal{I}$ satisfies axioms $\langle \top \sqsubseteq \forall R_i.V_2, 0.v_i \rangle$ and $\langle \top \sqsubseteq \forall R_i.\neg V_2, 1 - 0.v_i \rangle$, it follows that $(\forall R_i.V_2)^{\mathcal{I}}(g(\mu)) \geq 0.v_i$ and $(\forall R_i.\neg V_2)^{\mathcal{I}}(g(\mu)) \geq 1 - 0.v_i$. Therefore, for $R_i^{\mathcal{I}}(g(\mu), g(\mu i)) = 1$ we have $V_2^{\mathcal{I}}(g(\mu i)) = 0.v_i = V_2^{\mathcal{I}_\varphi}(\mu i)$. Similarly, it can be shown that $W_2^{\mathcal{I}_\varphi}(\mu i) = 0.w_i = W_2^{\mathcal{I}}(g(\mu i))$.

Now, since $\mathcal{I}$ satisfies axiom $V \equiv V_1 \boxplus V_2$, then, $V^{\mathcal{I}}(g(\mu i)) = V_1^{\mathcal{I}}(g(\mu i)) + V_2^{\mathcal{I}}(g(\mu i)) = 0.v_\mu \cdot (s+1)^{-|v_i|} + 0.v_i = 0.v_i v_\mu = V^{\mathcal{I}_\varphi}(\mu i)$.

Finally, by inductive hypothesis, assume that

$$A^{\mathcal{I}}(g(\mu)) = A^{\mathcal{I}_\varphi}(\mu) = 0.01 \cdot (s+1)^{-\sum_{j \in \{i_1, i_2, \ldots, i_k\}} \max\{|v_j|, |w_j|\}},$$

where $\mu = i_1 i_2 \ldots i_k$.

Since $\mathcal{I}$ satisfies axioms $A \sqsubseteq (s+1)^{\max\{|v_i|,|w_i|\}} \cdot \forall R_i.A$, we have that

$$\begin{array}{ll} & A^{\mathcal{I}}(g(\mu)) \\ \leq & (s+1)^{\max\{|v_i|,|w_i|\}} \cdot (\forall R_i.A)^{\mathcal{I}}(g(\mu)) \\ \leq & (s+1)^{\max\{|v_i|,|w_i|\}} \cdot A^{\mathcal{I}}(g(\mu i)) . \end{array}$$

Likewise, since $\mathcal{I}$ satisfies axioms $(s+1)^{\max\{|v_i|,|w_i|\}} \cdot \exists R_i.A \sqsubseteq A$, we have that

$$\begin{array}{rl} & A^{\mathcal{I}}(g(\mu)) \\ \geq & (s+1)^{\max\{|v_i|,|w_i|\}} \cdot (\exists R_i.A)^{\mathcal{I}}(g(\mu)) \\ \geq & (s+1)^{\max\{|v_i|,|w_i|\}} \cdot A^{\mathcal{I}}(g(\mu i)) \end{array}$$

and, thus,

$$A^{\mathcal{I}}(g(\mu)) = (s+1)^{\max\{|v_i|,|w_i|\}} \cdot A^{\mathcal{I}}(g(\mu i)) \ .$$

Therefore,

$$\begin{array}{rl} & A^{\mathcal{I}}(g(\mu i)) \\ = & (s+1)^{-\max\{|v_i|,|w_i|\}} \cdot A^{\mathcal{I}}(g(\mu)) \\ = & (s+1)^{-\max\{|v_i|,|w_i|\}} \cdot A^{\mathcal{I}_\varphi}(\mu) \\ = & (s+1)^{-\max\{|v_i|,|w_i|\}} \cdot 0.01 \cdot (s+1)^{-\sum_{j\in\{i_1,i_2,\ldots,i_k\}} \max\{|v_j|,|w_j|\}} \\ = & 0.01 \cdot (s+1)^{-(\max\{|v_i|,|w_i|\}+\sum_{j\in\{i_1,i_2,\ldots,i_k\}} \max\{|v_j|,|w_j|\})} \\ = & 0.01 \cdot (s+1)^{-\sum_{j\in\{i_1,i_2,\ldots,i_k,i\}} \max\{|v_j|,|w_j|\}} \\ = & A^{\mathcal{I}_\varphi}(\mu i) \ , \end{array}$$

which completes the proof. □

From the last Lemma it follows that if the RPCP instance φ has a solution μ, for some $\mu \in \{1,\ldots,p\}^+$, then $v_\mu = w_\mu$ and, thus, $0.v_\mu = 0.w_\mu$. Therefore, every witnessed model $\mathcal{I}$ of $\mathcal{O}_\varphi$ contains an element $\delta = g(\mu)$ such that $V^{\mathcal{I}}(\delta) = V^{\mathcal{I}_\varphi}(\mu) = 0.v_\mu = 0.w_\mu = W^{\mathcal{I}_\varphi}(\mu) = W^{\mathcal{I}}(\delta)$. Conversely, from the definition of $\mathcal{I}_\varphi$, if φ has no solution, then there is no μ such that $0.v_\mu = 0.w_\mu$, *i.e.*, there is no μ such that $V^{\mathcal{I}_\varphi}(\mu) = W^{\mathcal{I}_\varphi}(\mu)$.

However, as $\mathcal{O}_\varphi$ is always satisfiable, it does not yet help us to decide the RPCP. We next extend $\mathcal{O}_\varphi$ to $\mathcal{O}'_\varphi$ in such a way that an instance φ of the RPCP has a solution iff the ontology $\mathcal{O}'_\varphi$ is not witnessed satisfiable and, thus, establish that the KB satisfiability problem is undecidable. To this end, consider

$$\mathcal{O}'_\varphi := \langle \mathcal{T}'_\varphi, \mathcal{A} \rangle \ ,$$

where

$$\mathcal{T}'_\varphi := \mathcal{T}_\varphi \cup \bigcup_{1\leq i\leq p} \{\top \sqsubseteq \forall R_i.(\neg(V \leftrightarrow W) \boxplus \neg A)\} \ .$$

The intuition here is the following. If there is a solution for RPCP then, by the observation before, there is a point δ in which the value of V and W coincide under $\mathcal{I}$. That is, the value of $\neg(V \leftrightarrow W)$ is 0 and, thus, the one of $\neg(V \leftrightarrow W) \boxplus \neg A$ is less than 1. So, $\mathcal{I}$ cannot satisfy the new GCI in $\mathcal{T}'_\varphi$ and, thus, $\mathcal{O}'_\varphi$ is not satisfiable. On the other hand, if there is no solution to the RPCP then in $\mathcal{I}_\varphi$ there is no point in which V and W coincide and, thus, $\neg(V \leftrightarrow W) > 0$. Moreover, we will show that the value of $\neg(V \leftrightarrow W)$ in all points is strictly greater than A and, as $A \boxplus \neg A$ is 1, so also $\neg(V \leftrightarrow W) \boxplus \neg A$ will be 1 in any point. Hence, $\mathcal{I}_\phi$ is a model of the additional axiom in $\mathcal{T}'_\varphi$, *i.e.*, $\mathcal{O}'_\varphi$ is satisfiable.

Proposition 58. *The instance φ of the RPCP has a solution iff the ontology $\mathcal{O}'_\varphi$ is not witnessed satisfiable.*

(*Proof*) Assume first that φ has a solution $\mu = i_1 \ldots i_k$ and let $\mathcal{I}$ be a witnessed model of $\mathcal{O}_\varphi$. Let $\bar{\mu} = i_1 i_2 \ldots i_{k-1}$ (last index i_k is dropped from μ). Then by Lemma 57 it follows that there are nodes $\delta, \delta' \in \Delta^\mathcal{I}$ such that $\delta = g(\mu)$, $\delta' = g(\bar{\mu})$, with $V^\mathcal{I}(\delta) = V^{\mathcal{I}_\varphi}(\mu) = W^{\mathcal{I}_\varphi}(\mu) = W^\mathcal{I}(\delta)$ and $R^\mathcal{I}_{i_k}(\delta', \delta) = 1$. Then $(V \leftrightarrow W)^\mathcal{I}(\delta) = 1$. Since $(\neg A)^\mathcal{I}(\delta) < 1$, then $(\neg(V \leftrightarrow W) \boxplus \neg A)^\mathcal{I}(\delta) < 1$. Hence there is i, with $1 \leq i \leq p$, such that $(\forall R_i.(\neg(V \leftrightarrow W) \boxplus \neg A))^\mathcal{I}(\delta') < 1$. So, axiom $\top \sqsubseteq \forall R_i.(\neg(V \leftrightarrow W) \boxplus \neg A)$ is not satisfied and, therefore, $\mathcal{O}'_\varphi$ is not satisfiable.

For the converse, assume that φ has no solution. On the one hand we know that $\mathcal{I}_\varphi$ is a model of $\mathcal{O}_\varphi$. On the other hand, since φ has no solution, then there is no $\mu = i_1 \ldots i_k$ such that $v_\mu = w_\mu$ (*i.e.*, $0.v_\mu = 0.w_\mu$) and, therefore, there is no $\mu \in \Delta^{\mathcal{I}_\varphi}$ such that $V^{\mathcal{I}_\varphi}(\mu) = W^{\mathcal{I}_\varphi}(\mu)$. Consider $\mu \in \Delta^{\mathcal{I}_\varphi}$ and i, with $1 \leq i \leq p$ and assume, without loss of generality, that $V^{\mathcal{I}_\varphi}(\mu i) < W^{\mathcal{I}_\varphi}(\mu i)$. Then

$$\begin{aligned}
(V \leftrightarrow W)^{\mathcal{I}_\varphi}(\mu i) &= (V^{\mathcal{I}_\varphi}(\mu i) \Rightarrow W^{\mathcal{I}_\varphi}(\mu i)) \otimes (W^{\mathcal{I}_\varphi}(\mu i) \Rightarrow V^{\mathcal{I}_\varphi}(\mu i)) \\
&= 1 \otimes (W^{\mathcal{I}_\varphi}(\mu i) \Rightarrow V^{\mathcal{I}_\varphi}(\mu i)) \\
&= W^{\mathcal{I}_\varphi}(\mu i) \Rightarrow V^{\mathcal{I}_\varphi}(\mu i) \\
&= 1 - W^{\mathcal{I}_\varphi}(\mu i) + V^{\mathcal{I}_\varphi}(\mu i) \\
&= 1 - (W^{\mathcal{I}_\varphi}(\mu i) - V^{\mathcal{I}_\varphi}(\mu i)) \\
&= 1 - (0.w_{\mu i} - 0.v_{\mu i}) \\
&\leq 1 - 0.01 \cdot (s+1)^{-\max\{|v_{\mu i}|, |w_{\mu i}|\}} \\
&\leq 1 - 0.01 \cdot (s+1)^{-\sum_{j \in \{i_1, i_2, \ldots, i_k, i\}} \max\{|v_j|, |w_j|\}} \\
&= (\neg A)^{\mathcal{I}_\varphi}(\mu i) \ .
\end{aligned}$$

Therefore, $(\neg(V \leftrightarrow W))^{\mathcal{I}_\varphi}(\mu i) \geq A^{\mathcal{I}_\varphi}(\mu i)$. As $A^{\mathcal{I}_\varphi}(\mu i) \oplus (\neg A)^{\mathcal{I}_\varphi}(\mu i) = 1$, it follows that for every $\mu \in \Delta^{\mathcal{I}_\varphi}$ and i, with $1 \leq i \leq p$, it holds that $(\forall R_i.(\neg(V \leftrightarrow W) \boxplus \neg A))^{\mathcal{I}_\varphi}(\mu) = 1$ and, therefore, $\mathcal{I}_\varphi$ is a witnessed model of $\mathcal{O}'_\varphi$. □

By Proposition 58, we have a reduction of a RPCP to a KB satisfiability problem. Note that all roles are crisp. Therefore,

Proposition 59. *The knowledge base satisfiability problem is undecidable for L-$\mathcal{ALC}$ with GCIs. The result holds also if crisp roles are assumed.*

4.3.2 Knowledge Base consistency w.r.t. finite models

In this section we address a sub problem of the previous one. That is, deciding whether a KB has a finite interpretation.

In [Baader and Peñaloza, 2011b] is provided a proof of undecidability for language $\mathfrak{I}\mathcal{ALCE}$ based on product t-norm with respect to strongly witnessed interpretations. Using the same methods in [Baader and Peñaloza, 2011b] and [Borgwardt and Peñaloza, 2011c], we achieved an undecidability proof for for language $\mathfrak{I}\mathcal{ALCE}$ based on Łukasiewicz t-norm with respect to finite interpretations.

As in [Baader and Peñaloza, 2011b], given an instance φ of RPCP, we provide an ontology $\tilde{\mathcal{O}}_\varphi$ and prove that it has a finite model iff φ has a solution. We now define a TBox $\tilde{\mathcal{T}}$ as follows:

$$\begin{aligned} \tilde{\mathcal{T}} \quad := \{ \quad & V \equiv V_1 \boxplus V_2, W \equiv W_1 \boxplus W_2, \\ & \neg(V \leftrightarrow W) \sqsubseteq \max\{C_1, \dots, C_p\} \quad \} \ , \end{aligned}$$

and TBoxes $\tilde{\mathcal{T}}_\varphi^i$ as follows:

$$\begin{aligned} \tilde{\mathcal{T}}_\varphi^i \quad := \{ \quad & C_i \equiv \exists R_i.\top, \\ & \top \sqsubseteq \max\{C_i, \neg C_i\}, \\ & \\ & (C_i \sqsupset V) \sqsubseteq (s+1)^{|v_i|} \cdot \forall R_i.V_1, \\ & (s+1)^{|v_i|} \cdot \exists R_i.V_1 \sqsubseteq (C_i \sqsupset V), \\ & (C_i \sqsupset W) \sqsubseteq (s+1)^{|w_i|} \cdot \forall R_i.W_1, \\ & (s+1)^{|w_i|} \cdot \exists R_i.W_1 \sqsubseteq (C_i \sqsupset W), \\ & \\ & \langle \top \sqsubseteq \forall R_i.V_2, 0.v_i \rangle, \\ & \langle \top \sqsubseteq \forall R_i.\neg V_2, 1 - 0.v_i \rangle, \\ & \langle \top \sqsubseteq \forall R_i.W_2, 0.w_i \rangle, \\ & \langle \top \sqsubseteq \forall R_i.\neg W_2, 1 - 0.w_i \rangle \quad \} \end{aligned}$$

Now, let

$$\tilde{\mathcal{T}}_\varphi = \tilde{\mathcal{T}} \cup \bigcup_{i=1}^{p} \tilde{\mathcal{T}}_\varphi^i \ .$$

Further we define the ABox $\tilde{\mathcal{A}}_\varphi$ as follows:

$$\tilde{\mathcal{A}}_\varphi \quad := \quad \{\neg V(a), \neg W(a), \max\{C_1, \dots, C_p\}(a)\} \ .$$

Finally,

$$\tilde{\mathcal{O}}_\varphi := \langle \tilde{\mathcal{T}}_\varphi, \tilde{\mathcal{A}}_\varphi \rangle \ .$$

Proposition 60. *The instance φ of the RPCP has a solution iff the ontology $\tilde{\mathcal{O}}_\varphi$ has a finite model.*

(*Proof*) ($\Rightarrow$) Let $\mu = i_1 \ldots i_k$ be a solution of φ and let $suf(\mu)$ be the set of all suffixes of μ [3]. We build the finite interpretation $\tilde{\mathcal{I}}_\varphi$ as follows:

- $\Delta^{\tilde{\mathcal{I}}_\varphi} := suf(\mu)$,
- $a^{\tilde{\mathcal{I}}_\varphi} = \varepsilon$,
- $V^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = W^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = 0$, and for $1 \le i \le 2$, $V_i^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = W_i^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = 0$
- for all $\nu \in \Delta^{\tilde{\mathcal{I}}_\varphi}$, $V^{\tilde{\mathcal{I}}_\varphi}(\nu) = 0.v_\nu$, $W^{\tilde{\mathcal{I}}_\varphi}(\nu) = 0.w_\nu$
- for all $\nu, \nu' \in \Delta^{\tilde{\mathcal{I}}_\varphi}$ and $1 \le i \le p$

$$R_i^{\tilde{\mathcal{I}}_\varphi}(\nu, \nu') = \begin{cases} 1, & \text{if } \nu' = i\nu \\ 0, & \text{otherwise} \end{cases}$$

- for all $\nu \in \Delta^{\tilde{\mathcal{I}}_\varphi}$ and $1 \le i \le p$,

$$C_i^{\tilde{\mathcal{I}}_\varphi}(\nu) = \begin{cases} 1, & \text{if } i\nu \in suf(\mu) \\ 0, & \text{otherwise} \end{cases}$$

- for all $\nu \in \Delta^{\tilde{\mathcal{I}}_\varphi}$ and $1 \le i \le p$ such that $i\nu \in suf(\mu)$
  - $V_1^{\tilde{\mathcal{I}}_\varphi}(i\nu) = 0.v_\nu \cdot (s+1)^{-|v_i|}$, $W_1^{\tilde{\mathcal{I}}_\varphi}(i\nu) = 0.w_\nu \cdot (s+1)^{-|w_i|}$,
  - $V_2^{\tilde{\mathcal{I}}_\varphi}(i\nu) = 0.v_i$, $W_2^{\tilde{\mathcal{I}}_\varphi}(i\nu) = 0.w_i$.

We show now that $\tilde{\mathcal{I}}_\varphi$ is a model $\tilde{\mathcal{O}}_\varphi$. Since $V^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = 0.v_\varepsilon = 0$ and $W^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = 0.w_\varepsilon = 0$, then the first two axioms in $\tilde{\mathcal{A}}_\varphi$ are satisfied. Since there is $1 \le i \le p$ such that $i\varepsilon = i \in suf(\mu)$, then $C_i^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = 1$ and, therefore, the third axiom in $\tilde{\mathcal{A}}_\varphi$ is satisfied.

We now show that the axioms in $\tilde{\mathcal{T}}$ and each $\tilde{\mathcal{T}}_\varphi^i$, with $1 \le i \le p$ are satisfied for every $\nu \in suf(\mu)$. So, let $\nu \in suf(\mu) \setminus \{\mu\}$. Then there is $1 \le i \le p$ such that $i\nu \in suf(\mu)$ and, therefore, by the definition of $\tilde{\mathcal{I}}_\varphi$, $C_i^{\tilde{\mathcal{I}}_\varphi}(\nu) = 1$ and $R_i^{\tilde{\mathcal{I}}_\varphi}(\nu, i\nu) = 1$. Therefore, $(C_i \sqsupset V)^{\tilde{\mathcal{I}}_\varphi}(\nu) = V^{\tilde{\mathcal{I}}_\varphi}(\nu)$ from which it follows that every axiom in $\tilde{\mathcal{T}}_\varphi^i$ is satisfied by $\tilde{\mathcal{I}}_\varphi$ (the proof is the same as for $\mathcal{I}_\varphi$ satisfying $\mathcal{T}_\varphi^i$). E.g., note that $V^{\tilde{\mathcal{I}}_\varphi}(\nu) = 0.v_\nu = (s{+}1)^{|v_i|} \cdot V_1^{\tilde{\mathcal{I}}_\varphi}(i\nu)$ and, thus, both $(C_i \sqsupset V) \sqsubseteq (s{+}1)^{|v_i|} \cdot \forall R_i.V_1$ and $(s+1)^{|v_i|} \cdot \exists R_i.V_1 \sqsubseteq (C_i \sqsupset V)$ are satisfied.

Moreover, for every $j \neq i$ and $\nu' \in suf(\mu)$, it holds that $C_j^{\tilde{\mathcal{I}}_\varphi}(\nu) = 0$ and $R_j^{\tilde{\mathcal{I}}_\varphi}(\nu, \nu') = 0$ and, therefore every axiom in $\tilde{\mathcal{T}}_\varphi^j$ is satisfied as well (note that *e.g.*, $(\forall R_j.V_1)^{\tilde{\mathcal{I}}_\varphi}(\nu) = 1$). This last argument holds for μ as well.

[3] A *suffix* of a string $t_1 t_2 \ldots t_n$ is a string $t_{n-m+1} \ldots t_n$ $(0 \le m \le n)$, which is the empty string ε for $m = 0$.

Finally, consider $\tilde{\mathcal{T}}_\varphi$. It is easy to check that the first two axioms are satisfied in every $\nu \in suf(\mu)$. For the third axiom, if $\nu \in suf(\mu)\setminus\{\mu\}$, then there is $1 \leq i \leq p$ such that $C_i^{\tilde{\mathcal{I}}_\varphi}(\nu) = 1$ and, then, the axiom is trivially satisfied. Otherwise, if $\nu = \mu$, since μ is a solution for φ, then $(\neg(V \leftrightarrow W))^{\tilde{\mathcal{I}}_\varphi}(\mu) = 0$ and, then, the axiom is trivially satisfied as well.

($\Leftarrow$) For the converse, suppose that φ has no solution and let $\mathcal{I}$ be a model of $\tilde{\mathcal{O}}_\varphi$. By absurd, let us assume that $\mathcal{I}$ is finite and, thus, *witnessed*. Now, since $\mathcal{I}$ is a model of axioms $\neg V(a)$ and $\neg W(a)$, then there is a node $a^{\mathcal{I}} = \delta \in \Delta^{\mathcal{I}}$, such that $V^{\mathcal{I}}(\delta) = W^{\mathcal{I}}(\delta) = 0$.

Moreover, since $\mathcal{I}$ is a model of axioms $V \equiv V_1 \boxplus V_2$ and $W \equiv W_1 \boxplus W_2$, then $V_1^{\mathcal{I}}(\delta) = V_2^{\mathcal{I}}(\delta) = W_1^{\mathcal{I}}(\delta) = W_2^{\mathcal{I}}(\delta) = 0$ as well.

Next, we prove by induction that for every $n \in \mathbb{N}$ there is an element $\delta_{i_n} \in \Delta^{\mathcal{I}}$ such that:

- $V^{\mathcal{I}}(\delta_{i_n}) = 0.v_{i_n} \ldots v_{i_1}$,
- $W^{\mathcal{I}}(\delta_{i_n}) = 0.w_{i_n} \ldots w_{i_1}$,

and $|\{\delta, \delta_{i_1}, \ldots, \delta_{i_n}\}| = n + 1$ (all elements are distinct). As a consequence, $\Delta^{\mathcal{I}}$ cannot be finite, contrary to the assumption that $\mathcal{I}$ is finite.

Case $n = 1$. Since $\mathcal{I}$ is a witnessed model, it satisfies axiom $\max\{C_1, \ldots, C_p\}(a)$. So, there is i, such that $C_i^{\mathcal{I}}(\delta) = 1$. Let $i_1 = i$. Since $\mathcal{I}$ satisfies axiom $C_{i_1} \equiv \exists R_{i_1}.\top$, then there is $\delta' \in \Delta^{\mathcal{I}}$ such that $R_{i_1}^{\mathcal{I}}(\delta, \delta') = 1$. Let $\delta_{i_1} = \delta'$. Since $\mathcal{I}$ satisfies axiom $(s+1)^{|v_{i_1}|} \cdot \exists R_{i_1}.V_1 \sqsubseteq (C_{i_1} \sqsupset V)$, then $0 = (1 \Rightarrow 0) = (C_{i_1}(\delta) \Rightarrow V)^{\mathcal{I}}(\delta) \geq (s+1)^{|v_{i_1}|} \cdot \sup_{\delta' \in \Delta^{\mathcal{I}}}\{R_{i_1}^{\mathcal{I}}(\delta, \delta') \otimes V_1^{\mathcal{I}}(\delta')\} \geq R_{i_1}^{\mathcal{I}}(\delta, \delta_{i_1}) \otimes V_1^{\mathcal{I}}(\delta_{i_1}) = 1 \otimes V_1^{\mathcal{I}}(\delta_{i_1}) = V_1^{\mathcal{I}}(\delta_{i_1})$. Hence, $V_1^{\mathcal{I}}(\delta_{i_1}) = 0$. In the same way it can be proved that $W_1^{\mathcal{I}}(\delta_{i_1}) = 0$.
Since $\mathcal{I}$ satisfies axiom $\langle \top \sqsubseteq \forall R_{i_1}.V_2, 0.v_{i_1}\rangle$, we have that $0.v_{i_1} \leq (R_{i_1}^{\mathcal{I}}(\delta, \delta_{i_1}) \Rightarrow V_2^{\mathcal{I}}(\delta_{i_1})) = (1 \Rightarrow V_2^{\mathcal{I}}(\delta_{i_1})) = V_2^{\mathcal{I}}(\delta_{i_1})$.
Since $\mathcal{I}$ satisfies axiom $\langle \top \sqsubseteq \forall R_{i_1}.\neg V_2, 1 - 0.v_{i_1}\rangle$, it follows that $1 - 0.v_{i_1} \leq (R_{i_1}^{\mathcal{I}}(\delta, \delta_{i_1}) \Rightarrow \neg V_2^{\mathcal{I}}(\delta_{i_1})) = (1 \Rightarrow \neg V_2^{\mathcal{I}}(\delta_{i_1})) = \neg V_2^{\mathcal{I}}(\delta_{i_1}) = 1 - V_2^{\mathcal{I}}(\delta_{i_1})$ and therefore, $V_2^{\mathcal{I}}(\delta_{i_1}) \leq 0.v_{i_1}$. So, $V_2^{\mathcal{I}}(\delta_{i_1}) = 0.v_{i_1}$. In the same way it can be proved that $W_2^{\mathcal{I}}(\delta_{i_1}) = 0.w_{i_1}$.
Finally, since $\mathcal{I}$ satisfies axiom $V \equiv V_1 \boxplus V_2$, then $V^{\mathcal{I}}(\delta_{i_1}) = V_1^{\mathcal{I}}(\delta_{i_1}) \oplus V_2^{\mathcal{I}}(\delta_{i_1}) = 0 \oplus 0.v_{i_1} = 0.v_{i_1}$. In the same way it can be proved that $W^{\mathcal{I}}(\delta_{i_1}) = 0.w_{i_1}$. Moreover, since $V^{\mathcal{I}}(\delta) = 0 \neq 0.v_{i_1} = V^{\mathcal{I}}(\delta_{i_1})$, then $\delta \neq \delta_{i_1}$ and, thus, $|\{\delta, \delta_{i_1}\}| = 2$, which completes the case.

Induction step $n + 1$. Let $n > 1$ and suppose, by inductive hypothesis, that, for every $j \leq n$, the above conditions hold.
Since φ has no solution, then $v_{i_n} \ldots v_{i_1} \neq w_{i_n} \ldots w_{i_1}$ and, therefore, by inductive hypothesis, $V^{\mathcal{I}}(\delta_{i_n}) = 0.v_{i_n} \ldots v_{i_1} \neq$

$0.w_{i_n} \dots w_{i_1} = W^{\mathcal{I}}(\delta_{i_n})$. Hence $(V \leftrightarrow W)^{\mathcal{I}}(\delta_{i_n}) < 1$ and, therefore, $\neg(V \leftrightarrow W)^{\mathcal{I}}(\delta_{i_n}) > 0$. So, since $\mathcal{I}$ satisfies axiom $\neg(V \leftrightarrow W) \sqsubseteq \max\{C_1, \dots, C_p\}$, $(\max\{C_1, \dots, C_p\})^{\mathcal{I}}(\delta_{i_n}) > 0$ follows and, thus, there is i such that $C_i^{\mathcal{I}}(\delta_{i_n}) > 0$. Therefore, as $\mathcal{I}$ satisfies axiom $\top \sqsubseteq \max\{C_i, \neg C_i\}$, we have that $C_i^{\mathcal{I}}(\delta_{i_n}) = 1$. Now, let $i_{n+1} = i$.

Since $\mathcal{I}$ satisfies axiom $C_{i_{n+1}} \equiv \exists R_{i_{n+1}}.\top$, then there is $\delta' \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}_{i_{n+1}}(\delta_{i_n}, \delta') = 1$. So, let $\delta_{i_{n+1}} = \delta'$.

Since $\mathcal{I}$ satisfies axiom $(C_{i_{n+1}} \sqsupset V) \sqsubseteq (s+1)^{|v_{i_{n+1}}|} \cdot \forall R_{i_{n+1}}.V_1$, then $0.v_{i_n} \dots v_{i_1} = (1 \Rightarrow 0.v_{i_n} \dots v_{i_1}) = (C_{i_n} \Rightarrow V)^{\mathcal{I}}(\delta_{i_n})) \leq (s+1)^{|v_{i_{n+1}}|} \cdot \inf_{\delta' \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}_{i_{n+1}}(\delta_{i_n}, \delta') \Rightarrow V_1^{\mathcal{I}}(\delta')\} \leq (R^{\mathcal{I}}_{i_{n+1}}(\delta_{i_n}, \delta_{i_{n+1}}) \Rightarrow V_1^{\mathcal{I}}(\delta_{i_{n+1}})) = V_1^{\mathcal{I}}(\delta_{i_{n+1}})$. On the other hand, since $\mathcal{I}$ satisfies axiom $(s+1)^{|v_{i_{n+1}}|} \cdot \exists R_{i_{n+1}}.V_1 \sqsubseteq (C_{i_{n+1}} \sqsupset V)$, then $0.v_{i_n} \dots v_{i_1} = (1 \Rightarrow 0.v_{i_n} \dots v_{i_1}) = (C_{i_n} \Rightarrow V)^{\mathcal{I}}(\delta_{i_n}) \geq (s+1)^{|v_{i_{n+1}}|} \cdot \sup_{\delta' \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}_{i_{n+1}}(\delta_{i_n}, \delta') \otimes V_1^{\mathcal{I}}(\delta')\} \geq (s+1)^{|v_{i_{n+1}}|} \cdot (R^{\mathcal{I}}_{i_{n+1}}(\delta_{i_n}, \delta_{i_{n+1}}) \otimes V_1^{\mathcal{I}}(\delta_{i_{n+1}})) = (s+1)^{|v_{i_{n+1}}|} \cdot V_1^{\mathcal{I}}(\delta_{i_{n+1}})$. So, $0.v_{i_n} \dots v_{i_1} = (s+1)^{|v_{i_{n+1}}|} \cdot V_1^{\mathcal{I}}(\delta_{i_{n+1}})$ and, thus, $V_1^{\mathcal{I}}(\delta_{i_{n+1}}) = (s+1)^{-|v_{i_{n+1}}|} \cdot 0.v_{i_n} \dots v_{i_1}$. In the same way it can be proved that $W_1^{\mathcal{I}}(\delta_{i_{n+1}}) = 0.w_{i_n} \dots w_{i_1} \cdot (s+1)^{-|w_{i_{n+1}}|}$. Since $\mathcal{I}$ satisfies axiom $\langle \top \sqsubseteq \forall R_{i_{n+1}}.V_2, 0.v_{i_{n+1}} \rangle$, we get $0.v_{i_{n+1}} \leq R^{\mathcal{I}}_{i_{n+1}}(\delta_{i_n}, \delta_{i_{n+1}}) \Rightarrow V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = 1 \Rightarrow V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = V_2^{\mathcal{I}}(\delta_{i_{n+1}})$. Similarly, since $\mathcal{I}$ satisfies axiom $\langle \top \sqsubseteq \forall R_{i_{n+1}}.\neg V_2, 1 - 0.v_{i_{n+1}} \rangle$, we get $1 - 0.v_{i_{n+1}} \leq R^{\mathcal{I}}_{i_{n+1}}(\delta_{i_n}, \delta_{i_{n+1}}) \Rightarrow \neg V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = 1 \Rightarrow \neg V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = \neg V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = 1 - V_2^{\mathcal{I}}(\delta_{i_{n+1}})$ and therefore, $V_2^{\mathcal{I}}(\delta_{i_{n+1}}) \leq 0.v_{i_{n+1}}$. So, $V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = 0.v_{i_{n+1}}$. In the same way it can be proved that $W_2^{\mathcal{I}}(\delta_{i_{n+1}}) = 0.w_{i_{n+1}}$.

Finally, since $\mathcal{I}$ satisfies axiom $V \equiv V_1 \boxplus V_2$, then $V^{\mathcal{I}}(\delta_{i_{n+1}}) = V_1^{\mathcal{I}}(\delta_{i_{n+1}}) \oplus V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = ((s+1)^{-|v_{i_{n+1}}|} \cdot 0.v_{i_n} \dots v_{i_1}) \oplus 0.v_{i_{n+1}} = 0.v_{i_{n+1}} \dots 0.v_{i_1}$. In the same way it can be proved that $W^{\mathcal{I}}(\delta_{i_{n+1}}) = 0.w_{i_{n+1}} \dots 0.w_{i_1}$.

Moreover, since, by inductive hypothesis, for every $j \leq n$, $V^{\mathcal{I}}(\delta_{i_j}) = 0.v_{i_j} \dots v_{i_1} \neq 0.v_{i_{n+1}} \dots v_{i_j} \dots v_{i_1} = V^{\mathcal{I}}(\delta_{i_{n+1}})$, then $\delta_{i_j} \neq \delta_{i_{n+1}}$. Furthermore, as $V^{\mathcal{I}}(\delta) = 0 \neq V^{\mathcal{I}}(\delta_{i_{n+1}})$, then $\delta \neq \delta_{i_{n+1}}$ and, thus, $|\{\delta, \delta_{i_1}, \dots, \delta_{i_{n+1}}\}| = n+2$, which completes the case.

So, $\tilde{\mathcal{O}}_\varphi$ has no finite model.

□

By Proposition 60, we have a reduction of a RPCP to a finite satisfiability problem. Again, note that all roles are crisp. Therefore,

Proposition 61. *The knowledge base finite satisfiability problem is undecidable for L-$\mathcal{ALC}$ with GCIs. The result holds also if crisp roles are assumed.*

Chapter 5

Computational complexity

Dealing with computational complexity of FDLs means, mainly, dealing with FDLs based on a finite chain $\mathbf{T}$ of truth values. This is due to the fact that the reasoning tasks for FDLs based on infinite algebras either have been proved to be undecidable problems, or the algorithms used to prove the decidability are not useful in order to prove complexity bounds.

Results and proofs that we published will be exhaustively reported in order to give an example of how some results have been achieved and what kind of procedures and proof strategies have been employed in order to achieve those results. More results existing in the literature will be reported in Section 6.2.

5.1 Concept satisfiability

In this section we report the existing results on the computational complexity of the concept satisfiability problem. We will explain why, despite this problem is decidable in the case of infinite-valued FDLs, its complexity is still an open problem. For the finite-valued case, we will report the results and expose two methods for proving such results.

5.1.1 The infinite-valued case

Despite the fact that concept satisfiability is a problem that has been proved to be decidable for languages $\mathfrak{I}\mathcal{ALCE}$ under infinite-valued Łukasiewics semantics and $\mathfrak{I}\mathcal{ALE}$ under infinite-valued product semantics, its complexity in those cases is still an open problem. The algorithms used in [Hájek, 2005] for the first case and in Section 4.1.1 for the second one, utilize a reduction to propositional logic that is not polynomial. As an example we will take the formula $\varphi^{\mathcal{B}}(m)$, that is reported in [Blackburn et al., 2001, p. 384] in order to prove that classical Modal Logic lacks the *polysize model property* and will translate it to FDL language by means of the translation $\rho(\cdot)$ from modal formulas to FDL concepts provided in Section 3.7.1. For every natural number m, consider a description signature containing the sets of atomic concepts $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_m\}$ and the atomic role R. For any $0 \leq i \leq m-1$ define the concepts:

$$C_i \quad := \quad A_i \sqsupset (\exists R.(A_{i+1} \boxtimes B_{i+1}) \boxtimes \exists R.(A_{i+1} \boxtimes \neg B_{i+1})) \tag{5.1}$$

and

$$D_i \quad := \quad (B_i \sqsupset \forall R.B_i) \boxtimes (\neg B_i \sqsupset \neg \forall R.B_i) \tag{5.2}$$

Moreover, define $\forall R^i.E$ as a shorthand for $\overbrace{\forall R.\ldots\forall R}^{i \text{ times}}.E$ and $\forall R^{(m)}.E$ as a shorthand for $E \boxtimes \forall R.E \boxtimes \forall R.^2 E \boxtimes \ldots \boxtimes \forall R.^m E$

Now, concept $\rho(\varphi^{\mathcal{B}}(m))$ is the conjunction of the following concepts:

$$\begin{array}{llllll}
\text{(i)} & A_0 & & & & \\
\text{(ii)} & \forall R^{(m)}.(A_i \sqsupset (\boxtimes_{i \neq j} \neg A_j)) & (0 \leq i \leq m) & & & \\
\text{(iii)} & C_0 \quad \boxtimes \forall R.C_1 & \boxtimes \forall R^2.C_2 & \boxtimes \forall R^3.C_3 & \boxtimes \ldots \boxtimes \forall R^{m-1}.C_{m-1} \\
\text{(iv)} & \quad\quad \forall R.D_1 & \boxtimes \forall R^2.D_1 & \boxtimes \forall R^3.D_1 & \boxtimes \ldots \boxtimes \forall R^{m-1}.D_1 \\
 & & \boxtimes \forall R^2.D_2 & \boxtimes \forall R^3.D_2 & \boxtimes \ldots \boxtimes \forall R^{m-1}.D_2 \\
 & & & \boxtimes \forall R^3.D_2 & \boxtimes \ldots \boxtimes \forall R^{m-1}.D_2 \\
 & & & & \vdots \\
 & & & & \boxtimes \forall R^{m-1}.D_{m-1}
\end{array}$$

Now, as pointed out in [Blackburn et al., 2001], the size of $\rho(\varphi^{\mathcal{B}}(m))$ is quadratic in m. If we apply the algorithm provided in [Hájek, 2005] or the one explained in Section 4.1.1, the cardinality of the set $T_{\rho(\varphi^{\mathcal{B}}(m))}$ of sentences that are produced by the algorithm at each step can be calculated by means of the following observations. Let us denote by $|i|$ the number of generalized atoms of $\rho(\varphi^{\mathcal{B}}(m))$ whose label has cardinality i.

1. At the first step we have that, for each generalized atom $E(d)$ (with $l(E) = \emptyset$) of $\rho(\varphi^{\mathcal{B}}(m))$ the algorithm deterministically produces a new constant d_a and a sentence which says that d_a is a witness for such $D(d)$. So, for each generalized atom, we have a new element in $T_{\rho(\varphi^{\mathcal{B}}(m))}$.

2. Subsequently, for each new produced constant d_b which is not a witness for E, the algorithm deterministically produces a new sentence which says that d_b is not a witness for E. Since d_b turns out to be the witness of a generalized atom $E'(d)$ of $T_{\rho(\varphi^{\mathcal{B}}(m))}$ which share the same label with E, another sentence will be produced which says that d_a is not a witness for $E'(d)$. So, we have that $|0|^2$ new elements are in $T_{\rho(\varphi^{\mathcal{B}}(m))}$.

3. After steps 1 and 2 the algorithm produced:

   (a) an amount of $|0|$ new constants $d_1, \dots, d_{|0|}$,

   (b) an amount of $|1|$ new generalized atoms for each new constant d_a, with $1 \leq a \leq |0|$.

   So, since each set of generalized atoms identified by the same new constant is processed by the algorithm as in step 2, but this does not happen when two generalized atoms do not share the same new constant, an amount of $|0| \cdot |1|^2$ is added to $T_{\rho(\varphi^{\mathcal{B}}(m))}$.

4. Again, after step 3 the algorithm produced:

   (a) an amount of $|0| \cdot |1|$ new constants $d_{1,1}, \dots, d_{1,|1|}, d_{2,1}, \dots, d_{2,|1|}, \dots, d_{|0|,1}, \dots, d_{|0|,|1|}$,

   (b) an amount of $|2|$ new generalized atoms for each new constant d_a, with $1 \leq a \leq |0| \cdot |1|$.

   So, applying again the process in step 3, we obtain that an amount of $|0| \cdot |1| \cdot |2|^2$ is added to $T_{\rho(\varphi^{\mathcal{B}}(m))}$, and so on until the algorithm processes the whole concept $\rho(\varphi^{\mathcal{B}}(m))$.

So, at the end of the process, when no more generalized atoms are produced to be further processed, the size of the resulting propositional theory $T_{\rho(\varphi^{\mathcal{B}}(m))}$ can be described by function $f(m) \colon \mathbb{N} \to \mathbb{N}$:

$$
\begin{aligned}
f(m) \quad := \quad & \Big(\sum_{i=1}^{m} i\Big)^2 + \\
& \sum_{i=1}^{m} i \cdot \Big(\Big(\sum_{i=1}^{m} i\Big) - 2\Big)^2 + \\
& \sum_{i=1}^{m} i \cdot \Big(\Big(\sum_{i=1}^{m} i\Big) - 2\Big) \cdot \Big(\Big(\Big(\sum_{i=1}^{m} i\Big) - 2\Big) - 3\Big)^2 + \\
& \vdots \\
& \sum_{i=1}^{m} i \cdot \Big(\Big(\sum_{i=1}^{m} i\Big) - 2\Big) \cdot \ldots \cdot \Big(\Big(\ldots \Big(\Big(\Big(\sum_{i=1}^{m} i\Big) - 2\Big) - 3\Big) \ldots\Big) - m\Big)^2
\end{aligned}
$$

which can be shown to be strictly greater than function $m!$, for each m. We will proof this by induction on m:

1. Consider, as base case, $m = 2$. In this case we have that $m! = 2 \cdot 1 = 2$, while $f(m) = (1 + 2)^2 + (1 + 2) \cdot ((1 + 2) - 2)^2 = 12$.

2. Suppose, by induction hypothesis, that $f(m) > m!$, we have to show that $f(m+1) > (m+1)!$. We know that each addend in $f(m)$ is a product of a finite number of factors. Let us denote by $F_{f(m)}$ the set of factors in the last addend in $f(m)$, then the function $g(j)\colon \{1,\dots,m\} \to F_{f(m)}$, defined by:

$$g(j) = \begin{cases} \sum_{i=1}^{m} i, & \text{if } j = m, \\ \left(\dots\left(\left(\sum_{i=1}^{m} i\right) - 2\right)\dots\right) - ((m+1) - j), & \text{if } j < m \end{cases}$$

is a bijection between the factors in $m!$ and the factors in $F_{f(m)}$. Since for every $1 < j \leq m$ we have that $j > g(j)$ and, for $j = 1$, we have that $j = g(j)$, then $F_{f(m)} > m!$.

On the other hand, let us consider the first two factors appearing in $F_{f(m)}$ (that is, $g(m)$ and $g(m-1)$): since

$$\sum_{i=1}^{m} i = \tfrac{m \cdot (m+1)}{2}$$

and, for $m > 3$,

$$\left(\sum_{i=1}^{m} i\right) - 2 > 2 \cdot (m-1).$$

Then

$$\begin{aligned} & \sum_{i=1}^{m} i \cdot \left(\left(\sum_{i=1}^{m} i\right) - 2\right) && (5.3) \\ > \; & \frac{m \cdot (m+1)}{2} \cdot 2 \cdot (m-1) && (5.4) \\ = \; & m \cdot (m+1) \cdot (m-1) && (5.5) \\ > \; & m \cdot m \cdot (m-1) && (5.6) \end{aligned}$$

,

Now, 5.6 is the product of the last two factors appearing in $m!$ (that is, $g^{-1}(\sum_{i=1}^{m} i)$ and $g^{-1}\left(\left(\sum_{i=1}^{m} i\right) - 2\right)$), multiplied by m. Then,

$$F_{f(m)} > m! \cdot m$$

and, therefore,

$$\begin{aligned} & f(m+1) \\ > \; & f(m) + F_{f(m)} \\ > \; & m! + m! \cdot m \\ = \; & m! \cdot (m+1) \\ = \; & (m+1)! \end{aligned}$$

Hence neither the algorithm provided in [Hájek, 2005] nor the one explained in Section 4.1.1 are polynomial. On the other hand, the algorithms we are going to provide in the following sections, do not need to halt in finite time, if the set of truth values to be checked is infinite. So, characterizing computational complexity of these algorithms is still an open problem in the case of an infinite set of truth values.

5.1.2 The case of $Ł_n$

A first step towards understanding the complexity of the concept r-satisfiability problem in finite-valued fuzzy description logics is the work undertaken in [Bou et al., 2011a]. The algebra of truth values considered in this work is any finite Łukasiewics chain and the language used is that of Modal Logic. Since, as proved in Section 3.7, there is a close connection between the expressive powers of the minimal n-valued Łukasiewicz Modal Logic and the one of the description logic $Ł_n$-$\mathcal{ALC}$ without knowledge base, the result here reported can be translated to our framework. The result we are going to prove in this section is the following.

Theorem 62. *For every $n \in \mathbb{N}$ and every $r \in L_n$,*

- *the set of modally r-satisfiable formulas over Kripke $Ł_n$-models is PSPACE-complete,*
- *the set of modally valid formulas over Kripke $Ł_n$-models is PSPACE-complete.*

The same complexity result is attained when we add the Delta operator and/or the canonical truth constants. And also we get the same complexity when we only deal with crisp Kripke models.

In the rest of this section we prove this last theorem. Since

- φ is modally valid iff $\varphi \vee \overline{s}$ is not modally s-satisfiable (where s is the penultimate element of $Ł_n$, i.e., $s = \frac{n-2}{n-1}$), and
- φ is modally r-satisfiable iff $\overline{r} \leftrightarrow \varphi$ is modally satisfiable,

it will be enough to prove PSPACE-completeness of the modal satisfiability problems. It may seem that this trick needs the use of canonical constants in the language, but by McNaughton theorem (see [Cignoli et al., 2000, Corollary 3.2.8], we can also reduce r-satisfiability to satisfiability without the help of canonical constants; for example, we notice that

- φ is modally 0.75-satisfiable, iff
- $\varphi^2 \leftrightarrow \neg(\varphi^2)$ is modally satisfiable,

Thus, by the inclusion relationships among the sets of modally satisfiable formulas it will be enough to prove that

- $\mathrm{Sat}_1(\mathtt{Fr}, \text{Ł}^c_{n,\Delta})$ and $\mathrm{Sat}_1(\mathtt{CFr}, \text{Ł}^c_{n,\Delta})$ are in PSPACE,
- $\mathrm{Sat}_1(\mathtt{Fr}, \text{Ł}_n)$ and $\mathrm{Sat}_1(\mathtt{CFr}, \text{Ł}_n)$ are PSPACE-hard.

Hence, we get that all sets introduced in Section 2.1.2 are PSPACE-complete.

Satisfiability is in PSPACE

We start giving a PSPACE algorithm for solving $\mathrm{Sat}_1(\mathtt{Fr}, \text{Ł}^c_{n,\Delta})$, and later we will see that this algorithm can be slightly modified to compute $\mathrm{Sat}_1(\mathtt{CFr}, \text{Ł}^c_{n,\Delta})$. Our algorithm follows a similar approach to the one given in [Blackburn et al., 2001, p. 383–388]. We stress the fact that all formulas considered in this section may contain the Delta operator and truth constants.

Definition 63. Let Γ be a set of modal formulas, and $Sub(\Gamma)$ be the set of its subformulas. We define the *closure* of Γ, in symbols $Cl(\Gamma)$, as the set

$$(Sub(\Gamma) \cup \{\Box\neg\sigma : \Diamond\sigma \in Sub(\Gamma)\} \cup \{\Diamond\neg\sigma : \Box\sigma \in Sub(\Gamma)\})^+,$$

where the superscript $^+$ refers to the process of deleting all occurrences of two consecutive negation symbols (i.e., $\neg\neg$). When $Cl(\Gamma) = \Gamma$ we will say that Γ is *closed*.

Note that if Γ is finite, then so is $Cl(\Gamma)$.

Definition 64. Let Γ be a closed set of modal formulas. We define the sequence $(\Gamma_0, \Gamma_1, \ldots, \Gamma_{deg(\Gamma)})$ by the recurrence

- $\Gamma_0 := \Gamma$,
- $\Gamma_{r+1} := \{\psi : \Diamond\psi \in \Gamma_r\} \cup \{\psi : \Box\psi \in \Gamma_r\}$.

The *family of modal levels* of Γ is the set $\Gamma^\circ := \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{deg(\Gamma)}\}$.

Note that, for every r, $deg(\Gamma_r) \leq deg(\Gamma) - r$. In particular $deg(\Gamma_{deg(\Gamma)}) = 0$.

Definition 65. Let Γ be a closed set of formulas. A *Hintikka function* over some $\Gamma_r \in \Gamma^\circ$ is a mapping $H : \Gamma_r \to L_n$ such that

1. H is a homomorphism of non modal connectives (which includes the Delta operator and truth constants),
2. $H(\Diamond\psi) = \sim H(\Box\neg\psi)$, for each $\Diamond\psi \in \Gamma_r$,
3. $H(\Box\psi) = \sim H(\Diamond\neg\psi)$, for each $\Box\psi \in \Gamma_r$.

It is said that H is an *atom* if there exists a Kripke model $\mathfrak{M} = \langle W, R, V\rangle$ and a world $w \in W$ such that, for each formula $\psi \in \Gamma$, it holds that $H(\psi) = V(\psi, w)$.

Lemma 66. *Let $H : \Gamma_r \to L_n$ and $H' : \Gamma_{r+1} \to L_n$ be two Hintikka functions, then:*

$$\begin{aligned} & \min\{H'(\psi) \Rightarrow H(\Diamond\psi) : \Diamond\psi \in \Gamma_r\} = \\ = \; & \min\{H(\Box\vartheta) \Rightarrow H'(\vartheta) : \Box\vartheta \in \Gamma_r\}. \end{aligned}$$

Proof. For every formula $\Diamond\psi \in \Gamma_r$ it is obvious that

$$\begin{aligned} & H'(\psi) \Rightarrow H(\Diamond\psi) = \sim H(\Diamond\psi) \Rightarrow \sim H'(\psi) = \\ & = H(\neg\Diamond\psi) \Rightarrow H'(\neg\psi) = H(\Box\neg\psi) \Rightarrow H'(\neg\psi). \end{aligned}$$

Then, using that $\Diamond\psi \in \Gamma_r$ iff $\Box\neg\psi \in \Gamma_r$ (by Definition 63), we get that $\min\{H'(\psi) \Rightarrow H(\Diamond\psi) : \Diamond\psi \in \Gamma_r\} = \min\{H(\Box\neg\psi) \Rightarrow H'(\neg\psi) : \Box\neg\psi \in \Gamma_r\}$. From this fact, it easily follows that $\min\{H'(\psi) \Rightarrow H(\Diamond\psi) : \Diamond\psi \in \Gamma_r\} = \min\{H(\Box\vartheta) \Rightarrow H'(\vartheta) : \Box\vartheta \in \Gamma_r\}$. □

Definition 67. Let $H : \Gamma_r \to L_n$ be a Hintikka function, $k \in L_n$ and $\Diamond\psi \in \Gamma_r$. We say that a Hintikka function $H' : \Gamma_{r+1} \to L_n$ is *induced by* $\Diamond\psi$ and *k-related* to H (in symbols, $H' \in H_{\Diamond\psi,k}$) if the following conditions hold:

- $H(\Diamond\psi) = k * H'(\psi)$,
- for each $\Box\vartheta \in \Gamma_r$, $H(\Box\vartheta) \leq k \Rightarrow H'(\vartheta)$.

Lemma 68. *Let Γ be a closed set of formulas, $\Gamma_r \in \Gamma^\circ$ and H a Hintikka function over Γ_r. If H is an atom, then for every $\Diamond\psi \in \Gamma_r$, there is some $k \in L_n$ and some $H' \in H_{\Diamond\psi,k}$ such that H' is an atom.*

Proof. Let H be an atom over Γ_r and $\Diamond\psi \in \Gamma_r$. Then, by Definition 65, there exist a Kripke model $\mathfrak{M} = \langle W, R, V\rangle$ and $w \in W$ such that $V(\Diamond\psi, w) = H(\Diamond\psi)$. Hence there exists $w' \in W$ such that $V(\Diamond\psi, w) = R(w, w') * V(\psi, w')$. Let $H' : \Gamma_{r+1} \to L_n$ be the Hintikka function defined by $H'(\varphi) = V(\varphi, w')$, for every formula $\varphi \in \Gamma_{r+1}$. It is obvious that H' is an atom. Take $k = R(w, w')$, then $H(\Diamond\psi) = V(\Diamond\psi, w) = R(w, w') * V(\psi, w') = k * H'(\psi)$ i.e., H and H' satisfy the first condition of Definition 67. On the other hand, for each $\Box\vartheta \in \Gamma_r$, we have that $V(\Box\vartheta, w) = \min\{R(w, w'') \Rightarrow V(\vartheta, w'') : w'' \in W\}$, and hence $H(\Box\vartheta) = V(\Box\vartheta, w) \leq R(w, w') \Rightarrow V(\vartheta, w') = k \Rightarrow H'(\vartheta)$. So, there is $k \in L_n$ such that $H' \in H_{\Diamond\psi,k}$. □

Definition 69. Let Γ be a finite closed set of formulas, H be a Hintikka function over Γ_0, and $\mathcal{H}$ be a family of Hintikka functions with domains (denoted by *dom*) belonging to Γ°. We say that $\mathcal{H}$ is a *witness set generated by H on Γ* when

1. $H \in \mathcal{H}$,
2. if $I \in \mathcal{H}$ and $\Diamond\psi \in dom(I)$, then there is some $k \in L_n$ and some $J \in I_{\Diamond\psi,k}$ such that $J \in \mathcal{H}$,
3. if $J \in \mathcal{H}$ and $J \neq H$, then there are $I^0, \ldots, I^r \in \mathcal{H}$ satisfying $I^0 = H$, $I^r = J$, and for each $0 \leq i < r$, there are a formula $\Diamond\psi \in dom(I^i)$ and an element $k \in L_n$ such that $I^{i+1} \in I^i_{\Diamond\psi,k}$.

Lemma 70. *Let Γ be a finite closed set of formulas, and H be a Hintikka function over Γ_0 (i.e., Γ). Then, H is an atom iff there is a witness set generated by H on Γ.*

Proof. Let Γ be a finite closed set of formulas, and H a Hintikka function over Γ_0.

($\Rightarrow$) We proceed by induction on the nesting degree of the set $dom(H)$.

(0) If $deg(\Gamma_r) = 0$ and H is an atom, then $\mathcal{H} = \{H\}$ is a witness set generated by H on Γ_0.

(d) Let $deg(\Gamma_r) = d$ and H be an atom over Γ_r. Suppose, by inductive hypothesis, that, for each $\Gamma_s \in \Gamma^\circ$ such that $deg(\Gamma_s) < d$ and each Hintikka function H' over Γ_s, it holds that, if H' is an atom, then there is a witness set generated by H' on Γ_s. Since H is an atom over Γ_r, then, by Lemma 68, for each $\Diamond\psi \in \Gamma_r$ there exist $k \in L_n$ and an atom $I^\psi \in H_{\Diamond\psi,k}$ over Γ_{r+1}. Since the degree of $\Gamma_{r+1} < d$, then, by inductive hypothesis, each atom I^ψ generates a witness set $\mathcal{I}^\psi$ on Γ_{r+1}. So, the set

$$\mathcal{H} = \{H\} \cup \bigcup_{\Diamond\psi \in \Gamma_r} \mathcal{I}^\psi$$

is a witness set generated by H on Γ.

($\Leftarrow$) Suppose now that there is a witness set $\mathcal{H}$ generated by H on Γ, then we have to show that there exists a model which satisfies H. So, define the model $\mathfrak{M} = \langle W, R, V\rangle$, where:

– $W = \mathcal{H}$,

– $$R(I, I') = \begin{cases} \min\{I'(\chi) \Rightarrow I(\Diamond\chi) : \Diamond\chi \in dom(I)\}, & \text{if } I' \in I_{\Diamond\psi,k} \text{ for some } k \in L_n^c \text{ and some } \Diamond\psi \in dom(I) \\ 0, & \text{otherwise,} \end{cases}$$

– for each variable $p \in Var$ and $I \in \mathcal{H}$, let $V(p, I) = I(p)$.

On the one hand, since for each $I \in \mathcal{H}$, $dom(I)$ contains a finite number of formulas of the form $\Diamond\psi$, then, by Definition 69, each element of the model has a finite number of R-successors. On the other hand, whenever $I' \in I_{\Diamond\psi,k}$, then $deg(dom(I')) < deg(dom(I))$ and, therefore, the depth of the model is finite as well (it is indeed equal to $deg(\Gamma)$).

To end the proof, we have to show that, for every formula $\varphi \in \Gamma$, it holds that $V(\varphi, H) = H(\varphi)$. In order to achieve this result we will prove by induction that for each $I \in W$, it holds that $V(\varphi, I) = I(\varphi)$. So, let $I \in W$ and $\varphi \in dom(I)$, then:

- If $\varphi = p$ is a propositional variable, then, by definition of V, we have that $V(p, I) = I(p)$.
- If φ is a propositional combination of variables or modal formulas, since H is a Hintikka function, by Definition 65 it holds that $V(\varphi, H) = H(\varphi)$.
- Let $\varphi = \Diamond\psi$ and suppose, by inductive hypothesis, that for each $J \in W$ such that $deg(dom(J)) < deg(dom(I))$ and for each formula χ, it holds that $V(\chi, J) = J(\chi)$. By Definitions 67 and 69, we have that there exists $J \in I_{\Diamond\psi,k}$, for a $k \in L_n$, such that, for each $\Box\vartheta \in dom(I)$, we have that $I(\Box\vartheta) \leq k \Rightarrow J(\vartheta)$, then, by residuation, $k \leq I(\Box\vartheta) \Rightarrow J(\vartheta)$, for each $\Box\vartheta \in dom(I)$ and, therefore, by Lemma 66 and the construction of $\mathfrak{M}$, $k \leq \min\{I(\Box\vartheta) \Rightarrow J(\vartheta) : \Box\vartheta \in dom(I)\} = \min\{J(\chi) \Rightarrow I(\Diamond\chi) : \Diamond\chi \in dom(I)\} = R(I, J)$. So, by Definition 67 and the inductive hypothesis, $I(\Diamond\psi) = k * J(\psi) \leq R(I, J) * J(\psi) = R(I, J) * V(\psi, J) \leq \max\{R(I, I') * V(\psi, I') : I' \in W\} = V(\Diamond\psi, I)$. On the other hand, let $I' \in W$ be such that $I' \in I_{\Diamond\chi,k'}$ for a $\Diamond\chi \in dom(I)$ and $k' \in L_n$, then, by the construction of $\mathfrak{M}$ and inductive hypothesis, $I(\Diamond\psi) \geq I(\Diamond\psi) \wedge I'(\psi) = (I'(\psi) \Rightarrow I(\Diamond\psi)) * I'(\psi) \geq \min\{I'(\vartheta) \Rightarrow I(\Diamond\vartheta) : \Diamond\vartheta \in dom(I)\} * I'(\psi) = R(I, I') * V(\psi, I')$. Hence $I(\Diamond\psi) \geq \max\{R(I, I') * V(\psi, I') : I' \in W\} = V(\Diamond\psi, I)$. So, $V(\Diamond\psi, I) = I(\psi)$.

So, for each formula φ, $V(\varphi, H) = H(\varphi)$ and, then, H is an atom over Γ. □

Next we consider the algorithm $Witness(H, \Gamma)$ given in Figure 5.3. This algorithm returns a boolean, and is very close to the one given in [Blackburn et al., 2001] for the minimal classical modal logic.

```
if H is a Hintikka function and Γ = dom(H)
  and for each subformula ◇ψ ∈ dom(H) there are
  k ∈ L_n and a Hintikka function I ∈ H_{◇ψ,k} such that
  Witness(I, dom(I))
 then
    return true
else
    return false
end if
```

Figure 5.1: The Algorithm $Witness(H, \Gamma)$

Lemma 71. *Let Γ be a finite closed set of formulas, and $H : \Gamma \to L_n$. Then, $Witness(H, \Gamma)$ returns* **true** *if and only if H is a Hintikka function over Γ that generates a witness set in Γ.*

Proof. Let Γ be a finite closed set of formulas, and $H : \Gamma \to L_n$.

($\Rightarrow$) Suppose that $Witness(H, \Gamma)$ returns **true**, we proceed by induction on the degree of Γ.

(0) If $deg(\Gamma) = 0$ and $Witness(H, \Gamma)$ returns **true** then, H is a Hintikka function over Γ, and hence $\mathcal{H} = \{H\}$ is a witness set generated by H on Γ.

(d) Let $deg(\Gamma) = d$ and suppose, by inductive hypothesis, that for each set Γ' of formulas such that $\Gamma' \subseteq \Gamma$ and $deg(\Gamma') < d$ and each function $H' : \Gamma' \to L_n$, it holds that, if $Witness(H', \Gamma')$ returns **true**, then H' is a Hintikka function over Γ' that generates a witness set in Γ'. If $Witness(H, \Gamma)$ returns **true** then, on the one hand, H is a Hintikka function over Γ. On the other hand, for each formula $\Diamond\psi \in \Gamma$, there are $k \in L_n$ and $I \in H_{\Diamond\psi,k}$ such that $Witness(I, \Gamma')$, where $\Gamma' \in \Gamma^\circ$ is such that $deg(\Gamma') = d - 1$. Since $deg(\Gamma') < d$, and $Witness(I, \Gamma')$ returns **true**, then, by inductive hypothesis, I is a Hintikka function over Γ' that generates a witness set $\mathcal{I}^\psi$ in Γ'. So, the set

$$\mathcal{H} = \{H\} \cup \bigcup_{\Diamond\psi\in\Gamma} \mathcal{I}^\psi$$

is a witness set generated by H on Γ.

($\Leftarrow$) Suppose that H is a Hintikka function over Γ that generates a witness set in Γ, we proceed by induction on the degree of Γ.

(0) If $deg(\Gamma) = 0$ then it is enough that H is a Hintikka function over Γ for $Witness(H, \Gamma)$ to return **true**.

(d) Let $deg(\Gamma) = d$ and suppose, by inductive hypothesis, that for each set Γ' of formulas such that $\Gamma' \subset \Gamma$ and $deg(\Gamma') < d$ and each function $H' : \Gamma' \to L_n$, it holds that, if H' is a Hintikka function over Γ' that generates a witness set in Γ', then $Witness(H', \Gamma')$ returns **true**. So, if H is a Hintikka function over Γ that generates a witness set $\mathcal{H}$ in Γ, then, by Definition 69, we have that, for each formula $\Diamond\psi \in \Gamma$ there are $k \in L_n$ and $I \in H_{\Diamond\psi,k} \cap \mathcal{H}$. Then we have that I is a Hintikka function over Γ' that generates a witness set in Γ', where $\Gamma' \in \Gamma^\circ$ is such that $deg(\Gamma') = d-1$. Hence, since $deg(\Gamma') < d$, then, by inductive hypothesis, $Witness(I, \Gamma')$ returns **true**. So, $Witness(H, \Gamma)$ returns **true**. □

Theorem 72. $\mathbf{Sat_1}(\mathtt{Fr}, L^c_{n,\Delta})$ *is in PSPACE.*

Proof. Let φ be a modal formula. By Lemmas 70 and 71, we have that φ is r-satisfiable iff there is a Hintikka function $H : Cl(\varphi) \to L_n$ such that $H(\varphi) = m$ and $Witness(H, Cl(\varphi))$ returns **true**. Thus we need to prove that $Witness$ can be given a PSPACE implementation. Consider a non-deterministic Turing machine that guesses a Hintikka function H over $Cl(\varphi)$ and runs $Witness(H, Cl(\varphi))$, then we need to prove that this machine runs in NPSPACE and, by an appeal to Savitch's Theorem we will achieve the desired result. The key points of the implementation are the following:

1. As pointed out in [Blackburn et al., 2001], encoding a subset Γ of $Cl(\varphi)$ requires space $\mathcal{O}(|\varphi|)$ (here $|\varphi|$ refers to the length of the encoding of φ). On the one hand, each element of a function $H : \Gamma \to L_n$ can be represented as an ordered pair $\langle \psi, i \rangle \in \Gamma \times L_n$ and, on the other hand, $|H| = |\Gamma|$. Hence, if $j = \max\{|\overline{r}| : r \in L_n\}$, then encoding a Hintikka function requires space bounded above by $|\varphi| + j \cdot |\varphi|$, that is space $\mathcal{O}(|\varphi|)$.

2. For each subformula $\Diamond\psi \in dom(H)$, whether there are $k \in L_n$ and a Hintikka function $I \in H_{\Diamond\psi,k}$, can be checked separately. Given a subformula $\Diamond\psi \in dom(H)$, the value $k \in L_n$ and the Hintikka function $I \in H_{\Diamond\psi,k}$ to be checked can be selected by non-deterministic choice. Note that, although the size of the set $H_{\Diamond\psi,k}$ can be in $\mathcal{O}(n^{|\Diamond\psi|})$, for a given function I, we do not need to check every element of $H_{\Diamond\psi,k}$ to see whether $I \in H_{\Diamond\psi,k}$, since we only need to test if I satisfies the conditions of Definition 67 and this can be done within space linear on the size of I.

Hence, by the previous points, every time algorithm $Witness$ is applied to a function H and its domain $Cl(\varphi)$, a subformula $\Diamond\psi \in dom(H)$ is selected and a $k \in L_n$ and $I \in H_{\Diamond\psi,k}$ are non-deterministically chosen, the space needed is in $\mathcal{O}(|\varphi|)$. So, since $deg(\varphi)$ recursive calls are needed until we meet a Hintikka function I whose domain contains no modal formula and $deg(\varphi) \leq |\varphi|$, the amount of space required to run the algorithm is $\mathcal{O}(|\varphi|^2)$. Moreover, to keep track of the subformulas that have been checked by the algorithm, it is enough to implement two kinds of pointers to the modal operators occurring in the representation of φ: one pointer to indicate that, for a given subformula $\Diamond\psi \in dom(H)$ it has been fully checked whether there is $k \in L_n$ and a Hintikka function $I \in H_{\Diamond\psi,k}$ such that $Witness(I, dom(I))$ and the other pointer when the same has not yet been fully checked. □

Theorem 73. $\mathrm{Sat}_1(\mathtt{CFr}, Ł^c_{n,\Delta})$ *is in PSPACE.*

Proof. It is easy to see that the same algorithm given in Figure 5.3, but replacing $k \in L_n$ with $k \in \{0, 1\}$, computes $\mathrm{Sat}_1(\mathtt{CFr}, Ł^c_{n,\Delta})$. □

Satisfiability is PSPACE-hard

Here we will prove that the four problems pointed out at the end of Section 5.1.2 are PSPACE-hard.

The PSPACE-hardness of the set $\mathrm{Sat}_1(\mathtt{CFr}, Ł_n)$ is proved by using a polynomial reduction into the problem of satisifiability for classical Kripke models.

Theorem 74. $\mathrm{Sat}_1(\mathtt{CFr}, Ł_n)$ *is PSPACE-hard.*

Proof. Let us consider the mapping tr from classical modal formulas into our modal formulas defined by

- $tr(p) = p \& \overset{(n-1)}{\ldots} \& p$, if p is a propositional variable,
- $tr(\bot) = \bot$,
- $tr(\varphi_1 \wedge \varphi_2) = tr(\varphi_1) \wedge tr(\varphi_2)$,
- $tr(\varphi_1 \to \varphi_2) = tr(\varphi_1) \to tr(\varphi_2)$,
- $tr(\Diamond \varphi) = \Diamond\ tr(\varphi)$.

This translation is clearly polynomial (because essentially we are only replacing variables), and by induction on formulas it is easy to check that for all modal formulas φ, it holds that

- φ is modally satisfiable in a classical Kripke model, iff
- $tr(\varphi)$ is modally satisfiable in a crisp Kripke model.

By the PSPACE-hardness of classical modal logic ([Ladner, 1977]) the proof finishes. □

Unfortunately, for the case of the set $\mathrm{Sat}_1(\mathtt{Fr}, Ł_n)$, the authors do not know how to get the PSPACE-hardness by a reduction from the classical case. Such a reduction can be obtained by the mapping tr' defined like tr except for the condition

- $tr'(\Diamond \varphi) = (\Diamond\ tr'(\varphi))^{n-1}$,

but this reduction is not polynomial. Thus, in order to prove our next theorem we need to go into the details of codifying quantified Boolean formulas QBF (it is well known that validity of QBF is PSPACE-complete). Since we essentially use the same ideas that are used in the classical modal case (see the proof given in [Blackburn et al., 2001, Theorem 6.50]), we will not go into all the details of the proof.

Theorem 75. $\mathrm{Sat}_1(\mathtt{Fr}, L_n)$ *is PSPACE-hard.*

Proof. Let us consider β a QBF formula. By the proof given in [Blackburn et al., 2001, Theorem 6.50], it is well known how to define (see [Blackburn et al., 2001, p. 390] a classical modal formula $f(\beta)$ such that

- β is valid, iff
- $f(\beta)$ is modally satisfiable in a classical Kripke model.

The formula $f(\beta)$ can also be seen as one of our modal formulas, and it is quite straightforward to check that for the formulas of the form $f(\beta)$ it happens that

- $f(\beta)$ is modally satisfiable in a classical Kripke model, iff

- $f(\beta)$ is modally satisfiable in a Kripke $Ł_n$-model.

This fact is based on the properties stated at the end of Section 2.1.2. □

To finish this section let us point that when our language has the Delta operator, this last proof can be simplified quite a lot just by realizing that the reduction tr' can be somehow converted into one that is polynomial; this is so because

$$\Delta\varphi \leftrightarrow \varphi^{n-1}$$

is a valid formula.

5.1.3 The general case of finite-valued FDLs

The proof given in Section 5.1.2 can not be straightforwardly generalized to the case of every finite-valued $\mathfrak{I}\mathcal{ALCE}$, because some steps relies on the good behavior of Łukasiewicz negation with respect to the quantifiers. So, in this section we will consider directly the satisfiability problem in the general setting of finite-valued $\mathfrak{I}\mathcal{ALCE}$. This gives us also the possibility of proposing a new procedure based on the one presented in Definition 33.

In rest of the present section, instead of talking about r-satisfiability we will use the terminology "modally r-satisfiable" (see Definition 76) because we will keep the terminology r-satisfiable for the case that we consider propositional assignations.

Definition 76. (Modal r-satisfiability in $\mathfrak{I}\mathcal{ALCE}$) A concept C is said to be *modally r-satisfiable* in case that there is an interpretation $\mathcal{I}$ and an object $a \in I$ such that $C^{\mathcal{I}}(a) = t$.

Next we define the main computational problem we deal with in this paper, together with its parametrized versions. It is worth saying that we are not only considering a different computational problem for every finite MTL-chain $\mathbf{T}$, we also consider one computational problem which can be understood as the uniform version of the ones parametrized by $\mathbf{T}$.

Definition 77. The computational problem **Satisf** is the following one:

INPUT: $(\mathbf{T}, C, r)$ where $\mathbf{T}$ is a finite MTL-chain, C is a concept of $\mathfrak{I}\mathcal{ALCE}$ and $r \in T$.

OUTPUT: Yes/No depending whether C is modally r-satisfiable or not.

Moreover, for every finite MTL-chain $\mathbf{T}$, the computational problem $\mathbf{Satisf_T}$ is the one obtained by fixing the finite MTL-chain in the previous problem.

We can think on the elements of T as truth values. Our interest in the present section is on finite MTL chains, and so we will always assume that the lattice part of $\mathbf{T}$ is fixed in the sense that T is the set $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ (for some natural number $n \geq 2$) and that $0 < \frac{1}{n-1} < \ldots < \frac{n-2}{n-1} < 1$. In particular, n will always refer to the cardinal of T.

The assumption on the lattice part of **T** that we have adopted above makes that the input **T**, in the uniform problem, can be simply codified as its cardinal n and the tables of the t-norm and its residuum.

The main statement on the present section is the following theorem.

Theorem 78. Satisf *is PSPACE-complete.*

The rest of the section is devoted to give the proof of the membership in PSPACE is done. The proof that concept modal r-satisfiability for finite-valued $\mathfrak{I}\mathcal{ALCE}$ is PSPACE hard is the same proof we provided for Theorem 74 for the case of $\mathfrak{I}\mathcal{ALCE}$ based on $Ł_n$. So, we will not repeat it here.

Notation. For the sake of clarification sometimes we will use $\cdot$ to mean the concatenation of strings, but in most cases we will just juxtapose the symbols we want to concatenate.

PSPACE upper bound

In this section we are going to prove that **Satisf** is in PSPACE. In particular, this implies that each one of the parametrized satisfiability problems by a finite MTL-chain **T** also belongs to PSPACE. In order to achieve this result, we will prove that the algorithm given in [Hájek, 2005] can be given a PSPACE implementation. Our proof follows the same pattern as the proofs in [Blackburn et al., 2001] and [Bou et al., 2011a], but here we do not make use of Hintikka sets or functions, like in the cited papers.

Preliminary definitions Several technical definitions will be needed later to prove PSPACE membership. We state these definitions now.

Definition 79.

An *occurrence* of a subconcept D in C is determined by the occurrence of a constructor or of an atomic concept. We will use _ to mark the occurrence considered.

It is worth noticing that every concept is equivalent to a propositional combination of atoms (i.e., generalized atoms and atomic concepts). Here by propositional combination we allow the use of all constructors except $\forall R$ and $\exists R$.

In the next definition we provide a labeling system that is a modification of the one given in Definition 35. It is crucial for the proof to give such a modification because it allows to recursively define the domain of the interpretation that possibly satisfies a given concept in a way that the labeling function provided in Definition 35 does not do. Given a concept description C, our labeling system assigns each occurrence of a subconcept of C a number that gives an account of the syntactic structure of C. It is closely related to what in [Schmidt-Schauss and Smolka, 1991] is called the *skeleton* of a constraint system, and it is worth emphasizing that the labelling is defined on occurrences (not on subconcepts).

Definition 80 (Labeling)**.** Let C be a concept. A *labelling function* (*label* for short) $l_C(\cdot)$ is the function which associates to every occurrence D of a subconcept in C a string of symbols in $N_R \cup \mathbb{N}$ defined by the conditions:

1. $l_C(C)$ is the empty sequence ε,
2. if D is a propositional combination of concepts $D_1, \dots, D_j$, then $l_C(D_i) := l_C(D)$ for every $i \leq j$.
3. if D is $\forall R.D'$ or $\exists R.D'$, then $l_C(D')$ is the concatenated sequence $l_C(D) \cdot Ri$, where i is the minimum non-zero number j such that the sequence $l_C(D) \cdot Rj$ has not been used to label any occurrence in C.

We will denote by Λ_C the set of labels of all occurrences in C. Given $\lambda \in \Lambda_C$, we define $\text{path}(\lambda)$ as the finite sequence of symbols in N_R obtained by deleting in the sequence λ the symbols from $\mathbb{N}$, and we will refer to it as the *role path* of λ. We define the *length* of λ, in symbols $|\lambda|$, as the number of symbols in the sequence $\text{path}(\lambda)$.
For every atomic role $R \in N_R$, we introduce the binary relation $\prec_R$ among labels by the condition $\lambda \prec_R \lambda'$ in case that $\text{path}(\lambda')$ is $\text{path}(\lambda) \cdot R$. And the relation $\prec$ is defined as $\bigcup\{\prec_R : R \in N_R\}$.

It is worth saying that for every concept C, there are labellings l_C. For the sake of simplicity, whenever in the future we have a fixed concept C, we will write l instead of l_C.

Example 81. *Let us consider the concept* `Example_2` *defined as*

$$\exists S.\big(\exists R.A \to \exists R.(\forall R.A \boxtimes \exists S.A)\big).$$

Then, we can consider the following labels:

$$l\Big(\underline{\exists S}.\big(\exists R.A \to \exists R.(\forall R.A \boxtimes \exists S.A)\big)\Big) = \varepsilon$$
$$l\Big(\exists S.\big(\underline{\exists R}.A \to \underline{\exists R}.(\forall R.A \boxtimes \exists S.A)\big)\Big) = S0$$
$$l\Big(\exists S.\big(\exists R.\underline{A} \to \exists R.(\forall R.A \boxtimes \exists S.A)\big)\Big) = S0R0$$
$$l\Big(\exists S.\big(\exists R.A \to \exists R.(\underline{\forall R}.A \boxtimes \underline{\exists S}.A)\big)\Big) = S0R1$$
$$l\Big(\exists S.\big(\exists R.A \to \exists R.(\forall R.\underline{A} \boxtimes \exists S.A)\big)\Big) = S0R1R0$$
$$l\Big(\exists S.\big(\exists R.A \to \exists R.(\forall R.A \boxtimes \exists S.\underline{A})\big)\Big) = S0R1S0$$

We remind the reader that we follow the above convention to use _ to denote occurrences.

From the above introduced labeling system, we are going to define the set of individuals employed to build an interpretation for a given concept C. Thus, from now on we assume that C and a labelling l are fixed. The individuals we have talked about are the ones introduced in the following definition.

Definition 82. The set Σ_C is defined as the set

$$\Sigma_C := \{\lambda_1 \cdot \ldots \cdot \lambda_s : s \in \mathbb{N}, \lambda_1 \prec \lambda_2 \prec \ldots \prec \lambda_s\}$$

formed by sequences of labels. For the case $s = 0$ we have that $\varepsilon \in \Sigma_C$. Given $\sigma = \lambda_1 \cdot \ldots \cdot \lambda_s \in \Sigma_C$, we define the *length* of σ, in symbols $|\sigma|$, as the number s. And we define its *role path* $\mathrm{path}(\sigma)$ as $\mathrm{path}(\lambda_s)$. Following the same pattern, we will write $\sigma \prec_R \sigma'$ and $\sigma \prec \sigma'$ when the corresponding relation holds between λ_s and $\lambda'_{s'}$.

It is straightforward that $|\sigma| = |\lambda_s|$. In the rest of the paper we will sometimes refer to the elements of Σ_C as constants. The underlying idea is that the set Σ_C of constants is indeed the domain of an interpretation that modally satisfies the concept C. Unfortunately, the cardinality of Σ_C might be not polynomial on the size of concept C.

The next two definitions are very similar to ones stated in [Hájek, 2005]. They give an account of how to make a partition of the theory obtained by applying Hájek's algorithm to a given concept. The theory we consider in the following definitions is a propositional one (non-modal) over the set Var of variables defined as

$$\{B(\sigma) : B \text{ occurrence of an atom in } C \text{ and } \sigma \in \Sigma_C\} \cup$$

$$\{R(\sigma, \sigma') : R \in N_R, \sigma, \sigma' \in \Sigma_C \text{ and } \sigma \prec_R \sigma'\}.$$

We will use the notion of *assertion* to denote expressions $B(\sigma)$ where B is an occurrence of an atom in C and $\sigma \in \Sigma_C$. For every formula φ obtained from the set Var of variables using propositional operators (i.e., non-modal) we define Var_φ as the set of variables appearing in φ. Analogously, we can consider Var_Φ for any set of such formulas.

Definition 83. Let $B(\sigma)$ be an assertion such that B is the occurrence of a generalized atom in C. Then the *Hájek set* $H_C(B(\sigma))$ is defined distinguishing the following two cases.

($\forall$) if $B = \forall R.D$, then $H_C(\forall R.D(\sigma))$ is the following set of formulas:

- $\forall R.D(\sigma) \equiv \big(R(\sigma, \sigma \cdot l(D)) \supset D(\sigma \cdot l(D))\big)$,
- $\forall R.D(\sigma) \supset \big(R(\sigma, \sigma \cdot l(E)) \supset D(\sigma \cdot l(E))\big)$, for each occurrence E of a generalized atom occurring in C such that $\mathrm{path}(l(E)) = \mathrm{path}(l(D))$;

($\exists$) if $B = \exists R.D$, then $H_C(\exists R.D(\sigma))$ is the following set of formulas:

- $\exists R.D(\sigma) \equiv \big(R(\sigma, \sigma \cdot l(D)) \boxtimes D(\sigma \cdot l(D))\big)$,
- $\big(R(\sigma, \sigma \cdot l(E)) \boxtimes D(\sigma \cdot l(E))\big) \supset \exists R.D(\sigma)$, for each occurrence E of a generalized atom occurring in C such that $\mathrm{path}(l(E)) = \mathrm{path}(l(D))$.

The formula in $H_C(B(\sigma))$ having the connective $\equiv$ as main connective will be called the *main formula* of $H_C(B(\sigma))$. The formula $B(\sigma)$ will be called the *head* of each one of the elements in $H_C(B(\sigma))$; and we will call the *body* to the formula lying on the opposite side of the head.

Definition 84. Let C be a concept and $\sigma \in \Sigma_C$. Then, the *Hájek theory* $\mathcal{H}_C(\sigma)$ of σ is the set:

$$\mathcal{H}_C(\sigma) := \bigcup \{H_C(D(\sigma)) : \text{path}\,(\sigma) = \text{path}\,(l(D)),$$

$$D \text{ occurrence of a generalized atom}\}$$

The next simple definition will be heavily used in the future in order to give to each Hájek theory of a given concept C a self-standing status as well as to make a bridge between the model that is claimed to satisfy concept C and the algorithm which says that there exists one.

Definition 85. Let $e : Var_{\mathcal{H}_C(\sigma)} \to T$ and $e' : Var_{\mathcal{H}_C(\sigma')} \to T$ be mappings for some $\sigma, \sigma' \in \Sigma_C$. Then, we say that e and e' are *mutually consistent* if they assign the same value to common elements, that is, if $e[Var_{\mathcal{H}_C(\sigma)} \cap Var_{\mathcal{H}_C(\sigma')}] = e'[Var_{\mathcal{H}_C(\sigma)} \cap Var_{\mathcal{H}_C(\sigma')}]$.

Witness sets and satisfiability We now define what is a *Witness set* in the new framework. Since, following [Blackburn et al., 2001], this structure is used as a bridge structure between a model that is supposed to satisfy a given concept C and a procedure that decides whether such a model exists, we will use again the name used in Definition 69, but adapting the notion to the new framework.

Definition 86. Let C be a concept, let $\sigma \in \Sigma_C$, let $e : Var_{\mathcal{H}_C(\sigma)} \to T$ be a mapping such that $e(\mathcal{H}_C(\sigma)) = 1$ and let $\mathcal{W} \subseteq \bigcup \big\{Func\big(Var(\mathcal{H}_C(\sigma')),\ \mathbf{T}\big) : \sigma' \in \Sigma_C\big\}$ (where $Func(A, B)$ refers to the mappings from A into B). We say that $\mathcal{W}$ is a *witness set generated by* e if:

1. $e \in \mathcal{W}$,
2. for every $e' \in \mathcal{W}$ with $e' : Var(\mathcal{H}_C(\sigma')) \to T$, if a generalized atom E appears in the body of a formula in $\mathcal{H}_C(\sigma')$, then there is a mapping $e'' \in \mathcal{W}$ such that $e'' : Var(\mathcal{H}_C(\sigma' \cdot l(E))) \to T$, $e''_{\sigma' \cdot l(e)}(\mathcal{H}_C(\sigma' \cdot l(E))) = 1$, and e' and e'' are mutually consistent.
3. for every $e' \in \mathcal{W}$ with $e' : Var(\mathcal{H}_C(\sigma')) \to T$, there are $e_0, \ldots, e_k \in \mathcal{W}$ such that:
   - $e_0 = e$,
   - $e_k = e'$ and $k = |\sigma'|$,
   - for every $0 < i \leq k$, e_i overlaps with e_{i-1},
   - for every $0 < i \leq n$, there is $\sigma_i \in \Sigma_C$, such that $e_i : Var(\mathcal{H}_C(\sigma_i)) \to T$, $\sigma_{i-1} \prec \sigma_i$, and $e_i(\mathcal{H}_C(\sigma_i)) = 1$ (here we consider $\sigma_0 := \varepsilon$).

The next lemma will allow us to show that the concept C is satisfiable if and only if there exists a mapping e which generates a witness set on it.

Lemma 87. *Let C be a concept. For every $\sigma \in \Sigma_C$, every $D_1, \ldots, D_i$ occurrences in C, and every $r_1, \ldots, r_i \in T$, if* $\mathrm{path}(\sigma) = \mathrm{path}(l(D_1)) = \ldots = \mathrm{path}(l(D_i))$ *then the following statements are equivalent.*

1. *there is an interpretation $\mathcal{I}$ and an individual $a \in I$ such that $D_1^{\mathcal{I}}(a) = r_1$, $\ldots$, $D_i^{\mathcal{I}}(a) = r_i$,*

2. *there is a mapping $e : Var_{\mathcal{H}_C(\sigma) \cup \{D_1(\sigma), \ldots, D_i(\sigma)\}} \to T$, such that $e(D_1(\sigma)) = r_1$, $\ldots$, $e(D_i(\sigma)) = r_i$, and e generates a witness set.*

Proof. $(1 \Rightarrow 2)$: This is rather easy since every interpretation can be considered as a family of mappings satisfying the desired properties.

$(2 \Rightarrow 1)$: Suppose that there is a mapping $e : Var_{\mathcal{H}_C(\sigma) \cup \{D_1(\sigma), \ldots, D_i(\sigma)\}} \to T$, such that $e(D_1(\sigma)) = r_1$, $\ldots$, $e(D_i(\sigma)) = r_i$, and e generates a witness set. Then we have to show that there exists an interpretation $\mathcal{I}$ and an individual $a \in I$ such that $D_1^{\mathcal{I}}(a) = r_1$, $\ldots$, $D_i^{\mathcal{I}}(a) = r_i$. So, define the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where:

- $\Delta^{\mathcal{I}} = \Sigma_C$,
- $R^{\mathcal{I}}(\sigma, \sigma') = \begin{cases} e_\sigma(R(\sigma, \sigma')), & \text{if } R(\sigma, \sigma') \\ & \text{ occurs in } \mathcal{H}_C(\sigma) \\ 0, & \text{otherwise} \end{cases}$
- for every atomic concept A and $\sigma \in \Sigma_C$, define $A^{\mathcal{I}}(\sigma) = e_\sigma(A(\sigma))$, if $A(\sigma)$ occurs in $\mathcal{H}_C(\sigma)$ and $A^{\mathcal{I}}(\sigma) = 0$, otherwise.

Now we have to show by induction on concepts, that, for every occurrence E of a subconcept of C and every $\sigma \in \Delta^{\mathcal{I}}$, $E^{\mathcal{I}}(\sigma) = e_\sigma(E(\sigma))$.

- If E is an atomic concept, it holds by definition of $\mathcal{I}$.
- If E is a propositional combination of concepts, this is trivial.
- Let $E = \forall R.F$. On the one hand, by Definition 69 and Definition 83, it holds that $e_\sigma(\mathcal{H}_C(\sigma)) = 1$ and:

$$\begin{aligned} & e_\sigma(\forall R.F(\sigma)) \\ = \; & e_\sigma(R(\sigma, \sigma \cdot l(F))) \Rightarrow e_\sigma(F(\sigma \cdot l(F))) \\ = \; & e_\sigma(R(\sigma, \sigma \cdot l(F))) \Rightarrow e_{\sigma \cdot l(F)}(F(\sigma \cdot l(F))) \\ = \; & R^{\mathcal{I}}(\sigma, \sigma \cdot l(F)) \Rightarrow F^{\mathcal{I}}(\sigma \cdot l(F)) \end{aligned}$$

  On the other hand, again by Definition 69 and Definition 83, it holds that, for every generalized atom G such that $\mathrm{path}(l(G)) = \mathrm{path}(l(F))$:

$$
\begin{aligned}
& e_\sigma(\forall R.F(\sigma)) \\
\leq \;& e_\sigma(R(\sigma, \sigma \cdot l(G))) \Rightarrow e_\sigma(F(\sigma \cdot l(G))) \\
= \;& e_\sigma(R(\sigma, \sigma \cdot l(G))) \Rightarrow e_{\sigma \cdot l(G)}(F(\sigma \cdot l(G))) \\
= \;& R^{\mathcal{I}}(\sigma, \sigma \cdot l(G)) \Rightarrow F^{\mathcal{I}}(\sigma \cdot l(G))
\end{aligned}
$$

So, $e_\sigma(\forall R.F(\sigma)) = \min_{x \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(\sigma, x) \Rightarrow F^{\mathcal{I}}(x)\} = (\forall R.F)^{\mathcal{I}}(\sigma)$.

- Let $E = \exists R.F$ and suppose that $F^{\mathcal{I}}(\sigma) = e_\sigma(F(\sigma))$, for every $\sigma \in \Sigma_C$. On the one hand, by Definition 69 and Definition 83, it holds that $e_\sigma(\mathcal{H}_D(\sigma)) = 1$ and:

$$
\begin{aligned}
& e_\sigma(\exists R.F(\sigma)) \\
= \;& e_\sigma(R(\sigma, \sigma \cdot l(F)))) * e_\sigma(F(\sigma \cdot l(F))) \\
= \;& e_\sigma(R(\sigma, \sigma \cdot l(F))) * e_{\sigma \cdot l(F)}(F(\sigma \cdot l(F))) \\
= \;& R^{\mathcal{I}}(\sigma, \sigma \cdot l(F)) * F^{\mathcal{I}}(\sigma \cdot l(F))
\end{aligned}
$$

  On the other hand, again by Definition 69 and Definition 83, it holds that, for every generalized atom G such that $\operatorname{path}(l(G)) = \operatorname{path}(l(F))$:

$$
\begin{aligned}
& e_\sigma(\exists R.F(\sigma)) \\
\geq \;& e_\sigma(R(\sigma, \sigma \cdot l(G))) * e_\sigma(F(\sigma \cdot l(G))) \\
= \;& e_\sigma(R(\sigma, \sigma \cdot l(G))) * e_{\sigma \cdot l(G)}(F(\sigma \cdot l(G))) \\
= \;& R^{\mathcal{I}}(\sigma, \sigma \cdot l(G)) * F^{\mathcal{I}}(\sigma \cdot l(G))
\end{aligned}
$$

  So, $e_\sigma(\exists R.F(\sigma)) = \max_{x \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(\sigma, x) * F^{\mathcal{I}}(x)\} = (\exists R.F)^{\mathcal{I}}(\sigma)$.

Hence, for every concept E and every $\sigma \in \Delta^{\mathcal{I}}$, it holds that $E^{\mathcal{I}}(\sigma) = e_\sigma(E(\sigma))$. □

Using the occurrence C itself and the constant ε we get the following corollary.

Corollary 88. *Let C be a concept and $r \in T$. The following statements are equivalent.*

1. *C is modally r-satisfiable,*
2. *there is a mapping $e : Var_{\mathcal{H}_C(\varepsilon)} \to T$, such that $e(C(\varepsilon)) = r$ and e generates a witness set on C.*

Write down $\mathcal{H}_C(\sigma) \cup \{D_1(\sigma), \dots, D_i(\sigma)$.
if there is a mapping $e_\sigma : Var_{\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) | 1 \leq j \leq i\}} \to T$ such that $e_\sigma(D_j(\sigma)) = r_j$, for $1 \leq j \leq i$ and $e_\sigma(\mathcal{H}_C(\sigma)) = 1$ **then**
  if for every $j \leq i$ such that $\deg(D_j) = 0$ **then**
    return **true**
  else
    return the following list of strings
      $(\sigma \cdot l(E_1), \langle E_1, r_{1_1}\rangle, \dots, \langle E_k, r_{1_k}\rangle)$,
      $\vdots$
      $(\sigma \cdot l(E_k), \langle E_1, r_{k_1}\rangle, \dots, \langle E_k, r_{k_k}\rangle)$,
    where $\{E_1, \dots, E_k\}$ are the occurrences in the body of $\mathcal{H}_C(\sigma)$ and, for every $1 \leq l, m \leq k$, $e_\sigma(E_m(\sigma \cdot l(E_l))) = r_{l_m}$.
  end if
else
  return **false**
end if

Figure 5.2: Algorithm $Node_C(\sigma, \langle D_1, r_1\rangle, \dots, \langle D_i, r_i\rangle)$

Witness sets and procedures Let us now consider the procedure $Node_C(\sigma, \langle D_1, r_1\rangle, \dots, \langle D_i, r_i\rangle)$ given in Figure 5.2. This procedure takes as input a string made by an element $\sigma \in \Sigma_C$ and a set of pairs $\langle D_1, r_1\rangle, \dots, \langle D_i, r_i\rangle$, where, for $1 \leq j \leq i$, D_j is the occurrence of a concept in C and r_j is a truth value, and has three possible outputs: true, false or a list of strings $(\sigma \cdot l(E_1), \langle E_1, r_{1_1}\rangle, \dots, \langle E_k, r_{1_k}\rangle), \dots, (\sigma \cdot l(E_k), \langle E_1, r_{k_1}\rangle, \dots, \langle E_k, r_{k_k}\rangle)$ each one of these strings having the same nature than the input.

What it is interesting is that in case that $\text{path}(\sigma) = \text{path}(l(D_1)) = \dots = \text{path}(l(D_i))$, then also the strings obtained as output satisfy this equality requirement.

This procedure will be later used as a subroutine by the algorithm $Witness_C(\sigma, \langle D_1, r_1\rangle, \dots, \langle D_i, r_i\rangle)$ (see Figure 5.3) in order to check for the r-satisfiability of a given concept C. For this reason it is parametrized with a concept C which does not appear within the input string.

Now we check that the time needed by algorithm $Node_C(\sigma, \langle D_1, r_1\rangle, \dots, \langle D_i, r_i\rangle)$ to reach an answer is non-deterministically polynomial on the length of the input.

Lemma 89. *Algorithm $Node_C(\sigma, \langle D_1, r_1\rangle, \dots, \langle D_i, r_i\rangle)$ is in NPTIME.*

Proof. Let C be a concept, $\sigma \in \Sigma_C$, $D_1 \dots, D_i$ occurrences of subconcepts of C such that $path(l(D_j)) = path(l(D_{j+1})) = path(\sigma)$, for $1 \leq j < i$ and let us denote, for short, the input $\sigma, \langle D_1, r_1\rangle, \dots, \langle D_i, r_i\rangle$ by φ. First of all we need to see which is the size of $\mathcal{H}_C(\sigma)$ with respect to the size of the input. As we can see from Definition 83, given a generalized atom $E(a)$, the size of a single formula appearing in a Hájek theory $H_C(E(a))$ is at most $2 \cdot |E|$ (here

$|E|$ refers to the length of the encoding of E). Since, for every generalized atom E appearing in the input φ, it holds that $|E| \leq |\varphi|$, then the time needed to write down a single formula appearing in $H_C(E(a))$ is in $\mathcal{O}(|\varphi|)$. Again Definition 83 says us that, for every generalized atom $QR.E$, with $Q \in \{\forall, \exists\}$, appearing in the input φ, such that $|l(QR.E)| = |\sigma|$, the number of formulas in $H_C(QR.E(\sigma))$, is the number of all the generalized atoms $QP.F$ appearing in the input φ, such that $path(l(QP.F)) = path(\sigma)$ and $P = R$, which is less than $|\varphi|$. Hence, for every generalized atom $QR.E$ appearing in the input φ, such that $path(l(QR.E)) = path(\sigma)$, the number of formulas in $H_C(QR.E(\sigma))$, the time needed to write $H_C(QR.E(\sigma))$ is in $\mathcal{O}(|\varphi|^2)$. By Definition 84, we have that, in order to calculate the size of $\mathcal{H}_C(\sigma)$, we need to sum the sizes of theories $H_C(E(\sigma))$, of generalized atoms E appearing in the input such that $|l(E)| = |\sigma|$. Since the number of such generalized atoms is less that $|\varphi|$, then the time needed to write down $\mathcal{H}_C(\sigma)$ is in $\mathcal{O}(|\varphi|^3)$.

Furthermore, it is easy to see that the size of $\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \mid 1 \leq j \leq i\}$ is constant on the size of $\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \mid 1 \leq j \leq i\}$ (the constant factor depending on the encoding of the mapping $\cdot$). Since, as we have seen, the size of $\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \mid 1 \leq j \leq i\}$ is in $\mathcal{O}(|\varphi|^3)$, so is the size of $\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \mid 1 \leq j \leq i\}$.

It is well-known (see [Hähnle, 2001]) that satisfiability for propositional finite-valued logics is an NP-complete problem. Hence, answering whether for a given mapping e_σ from $Var_{\mathcal{H}_C(\sigma)\cup\{D_j(\sigma)|1\leq j\leq i\}}$ to T it holds that $e_\sigma(D_j(\sigma)) = r_j$, for $1 \leq j \leq i$ and $e_\sigma(\mathcal{H}_C(\sigma)) = 1$ is a task that can be accomplished in an amount of time that is polynomial on the cardinality of the set $Var_{\mathcal{H}_C(\sigma)\cup\{D_j(\sigma)|1\leq j\leq i\}}$. Therefore, the time needed to accomplish this task is still polynomial on the size of φ. Moreover, we will need to write down a possible solution to the above problem in the form $e_\sigma(E_1(\sigma_1)) = r_1, \ldots, e_\sigma(E_m(\sigma_m)) = r_m$, where $E_1(\sigma_1), \ldots, E_m(\sigma_m) \in Var_{\mathcal{H}_C(\sigma)\cup\{D_j(\sigma)|1\leq j\leq i\}}$ and $r_1, \ldots, r_m \in T$. It is easy to see that the time needed to write down such a solution is constant in the size of $Var_{\mathcal{H}_C(\sigma)\cup\{D_j(\sigma)|1\leq j\leq i\}}$ (the constant factor depending on the encoding of the truth values).

Finally, when the output is not simply a boolean, it is just part of the above mentioned solution re-written in a different form. That is, instead of writing $e_\sigma(E_1(\sigma_1)) = r_1, \ldots, e_\sigma(E_m(\sigma_m)) = r_m$, it will be written $(\sigma_1, \langle E_1, r_1\rangle, \ldots, \langle E_m, r_m\rangle), \ldots, (\sigma_m, \langle E_1, r_1\rangle, \ldots, \langle E_m, r_m\rangle)$. Hence, the time needed to write down the output, too, is polynomial on the size of φ. □

Next we consider the algorithm $Witness_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ given in Figure 5.3. This algorithm always returns a boolean, and is very close to the one given in [Blackburn et al., 2001] for the minimal classical modal logic and in [Bou et al., 2011a] for the finite-valued Łukasiewicz modal logic.

Lemma 90. *Let C be a concept. For every $\sigma \in \Sigma_C$, every $D_1, \ldots, D_i$ occurrences in C, and every $r_1, \ldots, r_i \in T$, if* $\mathrm{path}(\sigma) = \mathrm{path}(l(D_1)) = \ldots = \mathrm{path}(l(D_i))$ *then the following statements are equivalent.*

1. *$Witness_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns true,*

if $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns **true then**
  return **true**
end if
if $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns a list of strings **and** for each string $\sigma \cdot l(E_j), \langle E_1, r_{j_1}\rangle, \ldots, \langle E_k, r_{j_k}\rangle$ in this list, it holds that $Witness_C(\sigma \cdot l(E_j), \langle E_1, r_{j_1}\rangle, \ldots, \langle E_k, r_{j_k}\rangle)$ returns **true** **then**
  return **true**
else
  return **false**
end if

Figure 5.3: Algorithm $Witness_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$

2. there is a mapping $e : Var_{\mathcal{H}_C(\sigma)\cup\{D_1(\sigma),\ldots,D_i(\sigma)\}} \to T$, *such that* $e(D_1(\sigma)) = r_1, \ldots, e(D_i(\sigma)) = r_i$, *and* e *generates a witness set.*

Proof. The proof is done by induction (but decreasing the step): first of all we consider the case that $|\sigma| = \deg(C)$, and then we show that if we know the statement for all σ' with $|\sigma'| = l + 1$ then we also know it for the case that $|\sigma| = l$.

The initial case is very simple looking at the code of our program. Let us consider now the induction step.

($1 \Rightarrow 2$): Suppose that $Witness(\varepsilon, \langle C, r\rangle)$ returns **true**, we proceed by induction on the degree of C.

(0) If $\deg(C) = 0$, then $\mathcal{H}_C(\varepsilon)$ is empty. Since $Witness(\varepsilon, \langle C, r\rangle)$ returns **true**, then e is a mapping over $C(\varepsilon)$ such that $e(C(\varepsilon)) = r$ and $\mathcal{W} = \{e\}$ is a witness set generated by e on C.

(d) Let $\deg(C) = d$ and suppose, by inductive hypothesis, that, for each $\sigma \in \Sigma_C$, each occurrences $D_1, \ldots, D_i$ of concepts occurring in C such that $|l(D_j)| > |l(C)|$ and each $r_1, \ldots, r_i \in T$, it holds that, if $Witness(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns **true**, then there is a mapping $e_\sigma : Var_{\mathcal{H}_C(d_\sigma)\cup\{D_j(\sigma)|1\le j\le i\}} \to T$ such that $e_\sigma(D_j(\sigma)) = r_j$, for $1 \le j \le i$ and $e_\sigma(\mathcal{H}_C(\sigma)) = 1$ which generates a witness set $\mathcal{W}_\sigma$ on $D_1, \ldots, D_i$. Now, suppose that $Witness(\varepsilon, \langle C, r\rangle)$ returns **true**, then:

1. On the one hand, $Node(\varepsilon, \langle C, r\rangle)$ returns a list of strings $\{l(D_m), \langle D_1, r_{m_1}\rangle, \ldots, \langle D_k, r_{m_k}\rangle \mid 1 \le m \le k\}$ (remember that $\deg(C) > 0$ by hypothesis) and, hence, there is a mapping e on $\mathcal{H}_C(\varepsilon) \cup \{C(\varepsilon)\}$ such that $C(\varepsilon) = r$ and $e(\mathcal{H}_C(\varepsilon)) = 1$.
2. On the other hand, for each string $l(D_m), \langle D_1, r_{m_1}\rangle, \ldots, \langle D_k, r_{m_k}\rangle$ in the output of $Node(\varepsilon, \langle C, r\rangle)$, it holds that $Witness(l(D_m), \langle D_1, r_{m_1}\rangle, \ldots, \langle D_k, r_{m_k}\rangle)$ returns **true**.

Hence, by inductive hypothesis, for each occurrence D_m appearing in the output of $Node(\varepsilon, \langle C, r\rangle)$, there is a mapping $e_{l(D_m)} : Var_{\mathcal{H}_C(l(D_m))\cup\{D_m(l(D_m))|1\leq m\leq k\}} \to T$ such that $e_{l(D_m)}(D_m(l(D_m))) = r_m$, for $1 \leq m \leq k$ and $e_{l(D_m)}(\mathcal{H}_C(l(D_m))) = 1$ which generates a witness set $\mathcal{W}_{l(D_m)}$ on D_m. Moreover, since the truth values $r_{1_1}, \ldots, r_{k_k}$ are those appearing in the output of $Node(\varepsilon, \langle C, r\rangle)$, then e overlaps with each $e_{l(D_m)}$. So, the set:

$$\mathcal{W} = \{e\} \cup \bigcup\{\mathcal{W}_{l(D)} : D \text{ occurrence in } C \text{ with } |l(D)| = |l(C)| + 1\}$$

is a witness set generated by e.

($2 \Leftarrow 1$): Suppose that e is a mapping which generates a witness set $\mathcal{W}$ on C. We proceed by induction on the degree of C.

(0) If $\deg(C) = 0$, then it is enough that e be a mapping over $C(\varepsilon)$, such that $e(C(\varepsilon)) = r$, for $Witness(\varepsilon, \langle C, r\rangle)$ to return **true**.

(d) Let $\deg(C) > 0$ and suppose, by inductive hypothesis, that, for each $\sigma \in \Sigma_C$, each occurrences $D_1, \ldots, D_i$ of concepts occurring in C such that $|l(D_j)| > |l(C)|$ and each $r_1, \ldots, r_i \in T$, if a mapping $e_\sigma : Var_{\mathcal{H}_C(\sigma)\cup\{D_j(\sigma)|1\leq j\leq i\}} \to T$, such that $e_\sigma(D_j(\sigma)) = r_j$, for $1 \leq j \leq i$ and $e_\sigma(\mathcal{H}_C(\sigma)) = 1$, generates a witness set $\mathcal{W}_\sigma$ on $D_1, \ldots, D_i$, then $Witness(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns **true**. Now, if e is a mapping which generates a witness set $\mathcal{W}$ on C, then, by Definition 69, for each occurrence of a generalized atom D occurring in C such that $|l(D)| = |l(C)| + 1$, there is a mapping $e_{l(D)} : Var_{\mathcal{H}_C(l(D))} \to T$ mutually consistent with e and such that $e_{l(D)}(\mathcal{H}_C(l(D))) = 1$.

Let ol be the transitive closure of the mutually consistent relation between mappings, then the set

$$\{e' \in \mathcal{W} \mid e_{l(D)}\ ol\ e'\}$$

is a witness set generated by $e_{l(D)}$ on D. Hence, by inductive hypothesis, for each generalized atom D_j occurring in C such that $|l(D_j)| = |l(C)| + 1$, for $1 \leq j \leq i$, there is a mapping $e_{l(D_j)} : Var_{\mathcal{H}_C(l(D_j))} \to T$ such that $Witness(l(D_j), \langle D_1, e_{l(D_j)}(D_1(l(D_j)))\rangle, \ldots, \langle D_i, e_{l(D_j)}(D_i(l(D_j)))\rangle)$ returns **true**. So, $Witness(\varepsilon, \langle C, r\rangle)$ returns **true**. □

Corollary 91. *Let C be a concept and $r \in T$. Then $Witness(\varepsilon, \langle C, r\rangle)$ returns* **true** *if and only if there is a mapping $e : \mathcal{H}_C(\varepsilon) \cup \{C(\varepsilon)\} \to T$ such that $e(C(\varepsilon)) = r$ that generates a witness set.*

Main result Combining Corollaries 88 and 91 we easily obtain the following result.

Theorem 92. *The problem* **Satisf** *is in PSPACE.*

Proof. Corollaries 88 and 91 tell us that the algorithm in Figure 5.3 does what we want. It only remains to see that this algorithm belongs to PSPACE.

Let C be an $\mathfrak{I}\mathcal{ALCE}$ concept. By Lemma 87 and Lemma 90 we have that C is r-satisfiable if and only if there is a partial propositional evaluation $e : pr(\mathcal{H}_C(\varepsilon)) \to T$ such that $Witness(\varepsilon, \langle C, r\rangle)$ returns **true**. Hence we need to prove that $Witness$ can be given a PSPACE implementation. Consider a non-deterministic Turing machine that guesses a strings $\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle$ and runs $Witness(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$, then we need to prove that this machine runs in NPSPACE and, by an appeal to Savitch's Theorem, we will achieve the desired result.

Algorithm $Witness$ is a recursive algorithm and, at every recursive call, subroutine $Node_C$ is triggered over one of the strings $\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle$ obtained from a previous triggering of $Node_C$. The choice of the string to be processed by $Node_C$ at every successive step can be done by non-deterministic guess.

Due to the overlapping of mappings e_σ, for $\sigma \in \Sigma_C$, at every application of subroutine $Node_C$ on a string $\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle$, the only information needed is the output obtained by subroutine $Node_C$ on strings $\sigma', \langle D'_1, r'_1\rangle, \ldots, \langle D'_i, r'_i\rangle$, for every σ' that is a prefix of σ. So, at each step, the remaining information can be deleted.

Intuitively, Σ_C can be represented as a tree and the only information that is needed at each step is the one lying in the path from the root to the present step. At every successive recursive call the modal degrees of concepts $D_1, \ldots, D_i$ is strictly less than the modal degrees of concepts processed at the previous call and at most $\deg(C)$ recursive calls are needed until we meet a Hájek set without generalized atoms in the bodies of formulas. So, the maximum amount of information to be retained in memory is the output of subroutine $Node_C$ multiplied by $\deg(C)$. On the one hand, $\deg(C)$ is at most lineal on the size of C and, therefore, of the input. On the other hand, by Lemma 89 the space needed to run subroutine $Node_C$ and to write down its input is polynomial on the size of the input. Hence the amount of space needed by algorithm $Witness$ is polynomial on the size of the input. □

Chapter 6

Related works

In this chapter we give report the process that brought to the choice of a t-norm-based semantics and how research on FDL arrived to the kind of syntax here proposed, that has evident differences with the classical case.

Moreover we give a brief account on the main results that traced the limits of decidability in FDLs. For each reasoning task we will report the results existing in the literature for different semantics.

6.1 Historical remarks

Since the first articles on FDL, it was evident that generalizing the formalism of DL to the fuzzy framework consists in generalizing its semantics. A first step in this sense is that of generalizing the semantics of atomic concepts and roles from crisp to fuzzy sets and relations respectively and the semantics of subsumption to the inclusion between fuzzy sets. Nevertheless this does not mean there is a wide agreement on how to generalize the semantics of complex concepts and, since the beginning of the research on FDL, several solutions have been proposed.

The first attempt in this direction is the one of [Yen, 1991]. At that time the notation reported in Section 2.2.2 had not been fully adopted in the DL community and [Yen, 1991] has been thought as a generalization of [Brachman and Levesque, 1984] where the so-called *Term Subsumption Languages* (TSL) is developed. The language studied in [Yen, 1991], denoted "$\mathcal{FTSL}^-$", takes, as concept constructors, conjunction $(:\mathbf{and}\, C_1, \ldots, C_n)$, value restriction $(:\mathbf{all}\, R\, C)$, restricted existential quantification $(:\mathbf{some}\, R\, \top)$, modifiers **:NOT**, **:VERY**, **:SLIGHTLY**, etc. and an ancestor of concrete domains. The semantics underlying this first proposal was called *test score semantics* (see [Zadeh, 1982]). This name just means that *scores* (what we nowadays call "truth values") are assigned to concepts after performing tests to the system. However, what is interesting, under our point of view, in the semantics used in [Yen, 1991], are the truth functions used to calculate the truth values of complex concepts, in particular:

- It is suggested to use the min function to compute the value of a conjunction (: **and** $C_1, \ldots, C_n$) of concepts. Besides this suggestion, the author not only recognizes that any other t-norm can be used as the semantics of conjunction, but also that both lower and upper bounds for conjunctions can be computed considering min and Łukasiewicz t-norms as upper and lower bounds respectively.

- The semantics of value restriction (: **all** $R\, C$) is defined in two alternative ways. Through a fuzzy implication operator, as is done nowadays, and through the notion of conditional necessity from possibility theory. The author, however adopts the second option.

The main result in [Yen, 1991] is the decidability proof for the concept subsumption problem of the FDL presented and, for it, a structural subsumption algorithm has been used.

The semantics defined in [Yen, 1991] was enough general to leave open the adoption of a truth function for conjunction. Nevertheless, [Yen, 1991] was inspired by practical purposes and his goal was providing a more refined tool for knowledge representation.

Later on, [Tresp and Molitor, 1998] has a more theoretic fashion. The evolution of the notation towards a logical-like abstraction, that can be seen in the DL community, influenced [Tresp and Molitor, 1998], which utilizes the same modern notation reported in Section 2.2.2. The language studied in this work was called $\mathcal{ALC}_{F_M}$ (the subindex F_M stands for infinitely many truth values). It presents, as concept constructors, conjunction $\sqcap$, disjunction $\sqcup$, value restriction $\forall R.C$, existential quantifier $\exists R.C$ and *manipulators* (what we call *modifiers*) $M_i C$. The authors of [Tresp and Molitor, 1998] utilize a translation of the FDL language to fuzzy first order logic and provide a semantics to fuzzy first order logic that, through the translation, turns out to be the semantics of the FDL language, in accordance with the following schema in Figure 6.1.

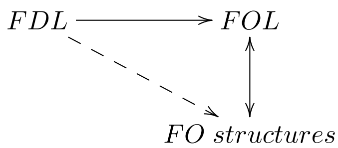


Figure 6.1: Relations to FOL in [Tresp and Molitor, 1998]

The choice of the truth functions for the logical connectives falls on min and max for conjunction and disjunction, respectively. The semantics for the existential quantifier is the one provided in Section 3.1.1 and it is the first place where it has been defined this way. This work is also the first in defining the semantics for value restriction $\forall R.C$ by means of the so-called *Kleene-Dienes implication*, that is defined on $[0, 1]$ in the following way:

$$x \Rightarrow y := \max\{1 - x, y\}$$

which is a straightforward generalization of the classical one. In particular, if $\mathcal{I}$ is an FDL interpretation, the semantics of value restriction $\forall R.C$, based on Kleene-Dienes implication, is defined in the following way:

$$(\forall R.C)^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}}\{\max\{1 - R^{\mathcal{I}}(x, y), C^{\mathcal{I}}(y)\}\}$$

Finally, for the semantics of manipulators M_iC, unary function on $[0, 1]$ were used, as in the framework of *fuzzy edges* (see [Cintula et al., 2011] for details).

The main result in [Tresp and Molitor, 1998] is the decidability proof for the concept subsumption problem of the FDL presented and, for it, a structural subsumption algorithm has been used.

Until [Tresp and Molitor, 1998], the research on FDL had been quite limited, but in the same year Straccia published his first work on FDL, [Straccia, 1998]. The language studied in this work is called (and, indeed, it is) $\mathcal{ALC}$ and the semantics adopted is the same as the one used in [Tresp and Molitor, 1998], plus a unary function that gives the semantics to concept complementation and that is defined as:

$$\neg x := 1 - x$$

The set of operations that includes $\min\{x, y\}, \max\{x, y\}, \max\{1 - x, y\}$ and $1 - x$ on the real unit interval is commonly denoted with the name of *Zadeh's semantics*. The strength of [Straccia, 1998] and of its journal version [Straccia, 2001], is that they set up a clear syntax and semantics, very close to the classical ones and relate each other without the intermediate step of first order logic, like in [Tresp and Molitor, 1998]. In this way Fuzzy Description Logic is set up as an autonomous discipline with a clearly defined syntax an semantics. The main result proved in [Straccia, 2001] is the reduction of fuzzy $\mathcal{ALC}$ to classical $\mathcal{ALC}$ that allows to prove decidability for every reasoning task considered. Later, in [Straccia, 2004b], the same author considers also the more general framework of semantics based on lattices that are supposed to be not necessarily chains. These works, indeed, opened the door to the possibility of expanding the language in order to cover the advances that had been done in the classical framework. A fuzzy semantics for concrete domains was introduced in [Straccia, 2005c]. A semantics for unqualified number restriction, role hierarchies, inverse and transitive roles was introduced in [Stoilos et al., 2005a]. A semantics for nominals was introduced in [Stoilos et al., 2005b]. A semantics for qualified number restriction was introduced in [Bobillo et al., 2007].

However, due to the absence of a residuated implication, an FDL based on Zadeh's semantics is too weak and it can lead to counter-intuitive consequences. This fact has been pointed out in [Hájek, 2005], where the example of the assertion "all hotels near to the main square are expensive" is presented in order to highlight the consequences of using Kleene-Dienes implication in the semantics of value restriction. Such assertion can be formally expressed as $\forall$`hasNear.Expensive(MainSquare)`. Here we will further develop Hájek's example. Consider the following fuzzy ABox $\mathcal{HOTELS}$:

- $\langle$`hasNear(MainSquare,Hotel_1)` $= 0.9\rangle$,
- $\langle$`hasNear(MainSquare,Hotel_2)` $= 0.5\rangle$,
- $\langle$`hasNear(MainSquare,Hotel_3)` $= 0.1\rangle$,
- $\langle$`Expensive(Hotel_1)` $= 0.9\rangle$,
- $\langle$`Expensive(Hotel_2)` $= 0.5\rangle$,
- $\langle$`Expensive(Hotel_3)` $= 0.1\rangle$,

This ABox $\mathcal{HOTELS}$ indeed depicts the ideal situation imagined by Hájek, where "for each hotel the degree of its being near to the main square equals the degree of its being expensive" and where "there is at least one hotel which is near to the main square in degree 0.5". In this ideal situation the truth value of assertion $\forall$`hasNear.Expensive(MainSquare)` should be 1, because hotels are at least as expensive as they lie near the main square. Now, if its truth value is calculated using the truth function of any residuated implication, its value is indeed 1. In spite, using the truth function of Kleene-Dienes implication, the result is different. In fact, in every interpretation $\mathcal{I}$ that is a model of $\mathcal{HOTELS}$, we have that:

$$\begin{aligned} & (\forall\texttt{hasNear.Expensive(MainSquare)})^{\mathcal{I}} \\ = \; & \inf_{x\in\Delta^{\mathcal{I}}}\{\texttt{Near}^{\mathcal{I}}(\texttt{MainSquare}^{\mathcal{I}}, x) \Rightarrow \texttt{Expensive}^{\mathcal{I}}(x)\} \\ \leq \; & \inf\{\max\{1-0.9, 0.9\}, \max\{1-0.5, 0.5\}, \max\{1-0.1, 0.1\}\} \\ = \; & \inf\{0.9, 0.5, 0.9\} \\ = \; & 0.5 \end{aligned}$$

So, the truth value of assertion $\forall$`hasNear.Expensive(MainSquare)`, using Kleene-Dienes implication, is at most 0.5 in a model of $\mathcal{HOTELS}$, against the intuition, reflected in $\mathcal{HOTELS}$, that its truth value should be 1. But the example can go beyond this situation. Consider, in fact, the ABox $\mathcal{HOTELS}'$ obtained by adding to $\mathcal{HOTELS}$ the following set of assertion:

- $\langle$`hasNear(SideSquare,Hotel_1)` $= 0.1\rangle$,
- $\langle$`hasNear(SideSquare,Hotel_2)` $= 0.4\rangle$,
- $\langle$`hasNear(SideSquare,Hotel_3)` $= 0.4\rangle$,

In this new situation the truth value of assertion $\forall$`hasNear.Expensive(SideSquare)` should be no higher than the value of assertion $\forall$`hasNear.Expensive(MainSquare)`, because `SideSquare` lies very from the more expensive hotel (`Hotel_1`) and there is one hotel whose degree of being near is higher than its degree of being expensive (`Hotel_3`).

Indeed, in the situation depicted by ABox $\mathcal{HOTELS}$, the main square is the square that has near the more expensive hotels and the side square is the one that lies nearer to the cheaper hotels. For this reason, it appears counter intuitive the possibility that `SideSquare` can be an instance of concept $\forall$`hasNear.Expensive` in a degree higher than `SideSquare`. Again, if its truth value is calculated with the use of the truth function of any residuated implication, its value is indeed strictly less than 1. In spite, using the truth function of Kleene-Dienes implication, the result can be higher than the truth value of $\forall$`hasNear.Expensive(MainSquare)`. In fact, there is at least an interpretation $\mathcal{I}$ that is a model of $\mathcal{HOTELS}'$, and where we have that:

$$
\begin{aligned}
& (\forall \mathtt{hasNear.Expensive(SideSquare)})^{\mathcal{I}} \\
= \; & \inf_{x \in \Delta^{\mathcal{I}}} \{\mathtt{Near}^{\mathcal{I}}(\mathtt{SideSquare}^{\mathcal{I}}, x) \Rightarrow \mathtt{Expensive}^{\mathcal{I}}(x)\} \\
= \; & \inf\{\max\{1 - 0.1, 0.9\}, \max\{1 - 0.4, 0.5\}, \max\{1 - 0.4, 0.1\}\} \\
= \; & \inf\{0.9, 0.6, 0.6\} \\
= \; & 0.6 > 0.5
\end{aligned}
$$

So, the truth value of assertion $\forall$`hasNear.Expensive(SideSquare)` is greater than that of $\forall$`hasNear.Expensive(MainSquare)` in at least one model of $\mathcal{HOTELS}'$, against the intuition, reflected in $\mathcal{HOTELS}'$, that hotels should be more expensive around the main square.

For this reason, Hájek proposes, in [Hájek, 2005], a more general framework based on Mathematical Fuzzy Logic. With the only exception of [Yen, 1991], the operation min is the only function adopted as a semantics for the conjunction operator until [Hájek, 2005]. In this new framework, not only the semantics of conjunction is a t-norm, but it is also recovered the idea, firstly proposed in [Tresp and Molitor, 1998], of a tight relation between FDL and first order fuzzy logic that, in the meanwhile, had been defined in the general framework of Mathematical Fuzzy Logic (MFL) developing the basic ideas of [Hájek, 1998b]. The main result proved in [Hájek, 2005] is the reduction of fuzzy $\mathcal{ALC}$ to the corresponding propositional calculus, that allows to prove decidability for concept satisfiability and subsumption. The new framework proposed in [Hájek, 2005] inspired several successive works on FDL. Among the ones that consider a t-norm-based semantics we can find [Straccia, 2005c] and, more recently, [Cerami et al., 2010a] and [Baader and Peñaloza, 2011b]. Among the ones that deepen the relationships between FDL and MFL we can find [García-Cerdaña et al., 2010], [Cerami et al., 2010b] and [Cerami et al., 2012].

The new framework proposed in [Hájek, 2005] supposed also a re-thinking about the notation used in FDL. Indeed, the use of the same notation of DL for FDL has been based on the fact that, in order to generalize DLs to the multi-valued framework, it seemed enough to generalize the semantics of concepts and roles to fuzzy sets and fuzzy relations. With this idea it is obvious that the same concept constructors (and, with them, the same formal languages) could be maintained in a multi-valued framework. This formalization worked indeed well when the semantics adopted as underlying truth value algebra was the Zadeh's semantics. But, since [Hájek, 2005], some researchers on FDLs began to adopt a residuated lattice on $[0, 1]$ as algebra of truth values. However, adopting a multi-valued framework and maintaining the same notation as in the classical case, could produce a slight confusion. This is due to several reasons related to differences between the classical and the multi-valued framework. Commonly, with some exceptions, such differences include the following items:

1. two kinds of conjunctions can be considered in the multi-valued framework, with different mathematical properties, and the same holds for disjunction,

2. implication is, in general, not definable from other connectives,

3. the quantifiers are not definable from each other by means of the equivalence $\exists R.C \equiv \neg\forall R.\neg C$,

4. the disjunction is not definable from the residuated negation $\neg C := C \to \bot$ and the conjunction $\sqcap$.

All these items must be taken into account both when choosing the symbols denoting the constructors of our description languages and when building the hierarchy of fuzzy description languages, as we will see later on. As an example recall that, in classical DLs, $\mathcal{ALE}$ is strictly contained in $\mathcal{ALC}$, while within many fuzzy DLs, by item 3 above, this is not the case.

Before moving to basic languages like those already existing in classical DLs, we find worth discussing the case of implication. In classical DL, no language has a primitive concept constructor for implication, even though implication is often implicitly used. This is due to the fact that the implication is definable from conjunction and negation. Nevertheless, in the logic MTL and many of its extensions, implication is in general not definable from other connectives. The first time that the concept constructor $\to$ for the implication is included in the definition of the language as a primitive connective has been in [Hájek, 2005]. Its introduction allows to utilize a concept constructor that is not otherwise definable, even if quite useful in order to define, in BL and its extensions, other concept constructors like those for weak conjunction (whose semantics and symbol are the minimum and $\sqcap$ respectively), weak disjunction (whose semantics and symbol are the maximum and $\sqcup$ respectively) and residuated negation (whose definition is $C \to \bot$).

Another issue that could take great advantage from the use of a residuated implication is the semantics of concept subsumption. Since the first works on FDL, in fact, the semantics for subsumption of concept is defined by means of the inclusion between fuzzy sets, that is, concept C is subsumed by concept D if and only if, for every interpretation $\mathcal{I}$ and every $x \in \Delta^{\mathcal{I}}$, it holds that $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x) = 1$. If the truth function of the implication is the residuum of the truth function of the conjunction, this is equivalent to say that concept $C \to$

D is valid or, equivalently, that concept $\neg(C \to D)$ is not positively satisfiable. If, otherwise, the truth function of the implication is $\max\{1 - C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\}$ the above relation between inclusion and implication does not hold anymore. As an example, consider two concepts A and B. As a matter of fact, their conjunction $A \sqcap B$ is always subsumed by both concepts, that is, $A \sqcap B \sqsubseteq A$. In Zadeh's semantics, in fact, for every interpretation $\mathcal{I}$ and every $x \in \Delta^{\mathcal{I}}$, it holds that $(A \sqcap B)^{\mathcal{I}}(x) = \min\{A^{\mathcal{I}}(x), B^{\mathcal{I}}(x)\} \leq A^{\mathcal{I}}(x)$. Nevertheless, when the truth function of the implication is $\max\{1 - A^{\mathcal{I}}(x), B^{\mathcal{I}}(x)\}$ it is not true that concept $(A \sqcap B) \to A$ is valid. As a counter-example to the relationship between the notion of fuzzy set inclusion based on the order $\leq$ and Kleene-Dienes implication, consider interpretation $\mathcal{I}$ where:

- $\Delta^{\mathcal{I}} = \{a\}$,
- $A^{\mathcal{I}}(a) = B^{\mathcal{I}}(a) = 0.5$,

then,

$$\begin{aligned} & ((A \sqcap B) \to A)^{\mathcal{I}}(a) \\ = \; & \max\{1 - \min\{A^{\mathcal{I}}(a), B^{\mathcal{I}}(a)\}, A^{\mathcal{I}}\} \\ = \; & \max\{1 - \min\{0.5, 0.5\}, 0.5\} \\ = \; & 0.5 \end{aligned}$$

As we can see from the example of interpretation $\mathcal{I}$, for every instance, concept $A \sqcap B$ is less or equal that concept A, but concept $(A \sqcap B) \to A$ is not a valid concept.

The advantage of considering a residuated implication, however not only consists in the fact that it behaves well with the inclusion between fuzzy concepts, but, above all, that by means of a residuated implication it is possible to define a graded notion of subsumption. By defining the semantics of subsumption as:

$$(C \sqsubseteq D)^{\mathcal{I}} := \inf_{x \in \Delta^{\mathcal{I}}}\{(C \to D)^{\mathcal{I}}(x)\}$$

we are not just able to say whether concept C is totally subsumed in concept D, but, when it is not the case, we can also give a truth value to this subsumption. This is indeed a great increment in expressivity.

6.2 Other reasoning tasks

6.2.1 Subsumption

The decidability of concept 1-subsumption for language $\mathfrak{I}\mathcal{ALCE}$ based on finite De Morgan lattices can be easily obtained from the results in [Borgwardt and Peñaloza, 2011b]. Despite in that work it is proved the decidability of the concept 1-subsumption w.r.t. an acyclic[1] TBox, concept 1-subsumption is a particular case concept 1-subsumption w.r.t. a knowledge base, indeed the one in which the knowledge base is empty.

[1]See Section ??.

6.2.2 Knowledge base consistency

As we will see later on, the general KB consistency problem is undecidable under infinite-valued Łukasiewicz and product semantics. Nevertheless, imposing some restriction to the TBox it can be proven decidable. In the following subsection we provide the results existing in the literature about decidability for KBs with empty or acyclic TBoxes and undecidability for general KBs.

The decidability of general knowledge base consistency for language $\mathfrak{I}\mathcal{ALCE}$ based on finite De Morgan lattices can be easily obtained from the results in [Borgwardt and Peñaloza, 2011b]. Despite in that work it is proved the decidability of the concept r-satisfiability w.r.t. a knowledge base problem, knowledge base consistency can be easily reduced to concept r-satisfiability w.r.t. a knowledge base, as we have seen in Section 3.5.2.

6.2.3 Concept satisfiability w.r.t. knowledge bases

The problem of concept r-satisfiability w.r.t. knowledge bases has been recently addressed in [Borgwardt and Peñaloza, 2011a] and [Borgwardt and Peñaloza, 2011c]. In both papers the language considered is the one that here is called $\mathcal{ALCE}$, the algebra of truth values $\mathbf{T}$ is a (not necessarily linear) De Morgan lattice and the knowledge base has empty ABox. In [Borgwardt and Peñaloza, 2011c] it is proved that the problem of concept r-satisfiability w.r.t. a knowledge base with empty ABox is undecidable when the De Morgan lattice $\mathbf{T}$ is infinite and has not been fixed. On the other hand, if $\mathbf{T}$ is finite, the same problems tuns out to be decidable. In[Borgwardt and Peñaloza, 2011b], moreover, it is proved that the problem of concept r-satisfiability w.r.t. a knowledge base with acyclic TBox is decidable when the De Morgan lattice $\mathbf{T}$ is finite.

The decidability of the concept r-satisfiability problem w.r.t. knowledge bases with empty TBoxes for $\mathfrak{I}\mathcal{ALCE}$ under infinite-valued Łukasiewicz t-norm can be easily proved from Theorem **??** and the reduction from the problem of concept r-satisfiability w.r.t. knowledge bases to the problem of knowledge base consistency provided in Section 3.5.2.

The decidability of the concept r-satisfiability problem w.r.t. knowledge bases with acyclic TBoxes for $\mathfrak{I}\mathcal{ALCE}$ under infinite-valued Łukasiewicz t-norm can be easily proved from Theorem **??** and the reduction from the problem of concept r-satisfiability w.r.t. knowledge bases to the problem of knowledge base consistency provided in Section 3.5.2.

The undecidability of the concept r-satisfiability problem w.r.t. knowledge bases with general TBoxes for $\mathfrak{I}\mathcal{ALCE}$ under infinite-valued Łukasiewicz or for $\mathfrak{I}\mathcal{ALE}$ under product t-norm can be easily proved from Proposition 61 and Corollary 5 in [Baader and Peñaloza, 2011b] respectively and the fact that the problem of knowledge base consistency can be reduced to the problem of concept r-satisfiability w.r.t. knowledge bases, as we have seen in Section 3.5.2.

6.2.4 Entailment

As long as we know, the entailment of an axiom by a knowledge base has not yet been directly faced. In[Borgwardt and Peñaloza, 2011b] a similar problem is addressed. There it is, indeed proved that the problem of concept 1-subsumption w.r.t. a knowledge base with acyclic TBox is decidable when the algebra of truth values **T** is a finite De Morgan lattice. In the case of an infinite set of truth values we can obtain some result as consequences of the (un)decidability of knowledge base consistency.

The decidability of the entailment of an axiom by a knowledge base with empty TBoxes for $\mathfrak{I}\mathcal{ALCE}$ under infinite-valued Łukasiewicz t-norm can be easily proved from Theorem **??** and the reduction from the problem of entailment of an axiom by a knowledge base to the problem of knowledge base consistency provided in Section 3.5.2.

The decidability of the entailment of an axiom by a knowledge base with acyclic TBoxes for $\mathfrak{I}\mathcal{ALCE}$ under infinite-valued Łukasiewicz t-norm can be easily proved from Theorem **??** and the reduction from the problem of entailment of an axiom by a knowledge base to the problem of knowledge base consistency provided in Section 3.5.2.

The undecidability of the entailment of an axiom by a knowledge base with general TBoxes for $\mathfrak{I}\mathcal{ALCE}$ under infinite-valued Łukasiewicz or for $\mathfrak{I}\mathcal{ALE}$ under product t-norm can be easily proved from Proposition 61 and Corollary 5 in [Baader and Peñaloza, 2011b] respectively and the fact that the problem of knowledge base consistency can be reduced to the entailment of an axiom by a knowledge base, as we have seen in Section 3.5.2.

6.2.5 Best entailment degree

In the case of the best entailment degree of an axiom by a knowledge base, the reduction from the best entailment degree problem to knowledge base consistency provided in Section 3.5.2 is not useful in order to obtain decidability, since a continuous family of KB consistency problem is obtained this way. Nevertheless, for the case of Łukasiewicz t-norm, as we have seen in Sections **??** and **??** the method to decide KB consistency consists in obtaining a system of in-equations $\mathcal{C}_{\mathcal{F}}$. It is well known that, for a system of in-equations both minimization and maximization problems of a variable are decidible. Hence, the best entailment degree problem of an axiom by a knowledge base with empty or acyclic TBoxes for $\mathfrak{I}\mathcal{ALCE}$ under infinite-valued Łukasiewicz t-norm has to be reduced to an optimization problem for KB consistency. In [Bobillo and Straccia, 2008a] it is showed that given $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, we can compute $bed(\mathcal{K}, C(a))$ as the minimal value of x such that $\langle \mathcal{T}, \mathcal{A} \cup \{\langle \neg C(a), 1-x \rangle\}\rangle$ is satisfiable. Similarly, for $C \sqsubseteq D$, we can compute $bed(\mathcal{K}, C \sqsubseteq D)$ as the minimal value of x such that $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \cup \{\langle C(a), x_1 \rangle\} \cup \{\langle D(a), x_2 \rangle\}\rangle$ is satisfiable under the constraints expressing that $x_1 \Rightarrow x_2 \leq x, x_1 \in [0,1]$ and $x_2 \in [0,1]$, where a is new individual. Hence, the best entailment degree problem of an axiom by a knowledge base with empty or acyclic TBoxes for $\mathfrak{I}\mathcal{ALCE}$ under infinite-valued Łukasiewicz t-norm is decidable.

The undecidability of the best entailment degree of an axiom by a knowledge base with general TBoxes for $\mathfrak{I}\mathcal{ALCE}$ under infinite-valued Łukasiewicz or for $\mathfrak{I}\mathcal{ALE}$ under product t-norm can be easily proved from Proposition 61 and Corollary 5 in [Baader and Peñaloza, 2011b] respectively and the fact that the problem of knowledge base consistency can be reduced to the entailment of an axiom by a knowledge base, as we have seen in Section 6.2.4. The result is, thus, obtained from the fact that the entailment problem is a particular case of the best entailment degree problem.

6.2.6 Best satisfiability degree

The best satisfiability degree of a concept w.r.t. a knowledge base has been proved decidable for language $\mathfrak{I}\mathcal{ALCE}$ based on finite De Morgan lattices in [Borgwardt and Peñaloza, 2011c]. In the case of FDLs based on infinite complete chains, the reduction from the best satisfiability degree problem to knowledge base consistency provided in Section 3.5.2 is not useful in order to obtain decidability, since a continuous family of KB consistency problem is obtained this way. Nevertheless, for the case of Łukasiewicz t-norm, are still worth the considerations made in Section 6.2.5 for the case of the best entailment degree. This time a solution to the best satisfiability degree $bsd(\mathcal{K}, C)$ is determined by the maximal value of x such that $\langle \mathcal{T}, \mathcal{A} \cup \{\langle C(a), x \rangle\} \rangle$ is satisfiable.

The undecidability of the best satisfiability degree of a concept w.r.t. a knowledge base with general TBoxes for $\mathfrak{I}\mathcal{ALCE}$ under infinite-valued Łukasiewicz or for $\mathfrak{I}\mathcal{ALE}$ under product t-norm can be easily proved from Proposition 61 and Corollary 5 in [Baader and Peñaloza, 2011b] respectively and the fact that the problem of knowledge base consistency can be reduced to the satisfiability of a concept w.r.t. a knowledge base, as we have seen in Section 6.2.4. The result is, thus, obtained from the fact that the concept satisfiability w.r.t. a knowledge base is a particular case of the best satisfiability degree problem.

Appendix A

In this appendix we prove, in Theorem 97, that the first order logic given by $[0,1]_\Pi$ coincides with the one given by its one-generated subalgebra, that is quasi-witnessed. The result of Theorem 97 has been previously published in [Cerami et al., 2010a]. It is necessary in order to prove that general tautologies and positively satisfiable formulas coincide with standard tautologies and positively satisfiable formulas.

Recall that an *one-generated subalgebra* of $[0,1]_\Pi$ is the subalgebra of $[0,1]_\Pi$ whose domain is $\{a^0, a^1, a^2, \ldots\} \cup \{0\}$, for $a \in (0,1)$.

In [Hájek, 1998b, Theorem 5.4.30] the author proves that $[0,1]_L$-tautologies coincide with the common $Ł_n$-tautologies for $n \geq 2$, i.e., coincide with the common tautologies of the finite subalgebras of $[0,1]_Ł$. In [Esteva et al., 2010] the authors prove that the result is not valid for a logic of a t-norm different from Łukasiewicz. But Hájek's result can be read in another way since $Ł_n$ are the one-generated subalgebras of $[0,1]_Ł$ whose generator is a rational number. What we prove in this appendix is that this reading of Hájek's result can be generalized to First Order Product Logic.

In order to prove this result we first prove some lemmas and provide some definitions. Firstly we prove the following lemma that uses only residuation condition, and thus it is also true for any MTL-chain (prelinear residuated chain).

Lemma 93. *In any* Π*-chain the following inequalities hold:*

1. $(x \Leftrightarrow x') * (y \Leftrightarrow y') \leq (x \Rightarrow y) \Leftrightarrow (x' \Rightarrow y')$,
2. $(x \Leftrightarrow x') * (y \Leftrightarrow y') \leq (x * y) \Leftrightarrow (x' * y')$,
3. $\inf_{i\in I}\{x_i \Leftrightarrow y_i\} \leq \inf_{i\in I}\{x_i\} \Leftrightarrow \inf_{i\in I}\{y_i\}$,
4. $\inf_{i\in I}\{x_i \Leftrightarrow y_i\} \leq \sup_{i\in I}\{x_i\} \Leftrightarrow \sup_{i\in I}\{y_i\}$.

Proof. The proofs are easy consequences of residuation property

$$x * y \leq z \qquad \text{iff} \qquad x \leq y \Rightarrow z. \tag{res}$$

In particular we point out that $x * (x \Rightarrow y) \leq y$. Next we prove each one of the items.

1. By symmetry it is enough to prove that $(x' \Rightarrow x) * (y \Rightarrow y') \leq (x \Rightarrow y) \Rightarrow (x' \Rightarrow y')$; and this is a consequence of residuation.

2. By symmetry it is enough to prove that $(x \Rightarrow x') * (y \Rightarrow y') \leq (x * y) \Rightarrow (x' * y')$; and this is a consequence of residuation.

3. Since we are considering a chain, we can suppose, without loss of generality, that $\inf_{i\in I}\{y_i\} \leq \inf_{i\in I}\{x_i\}$. Thus, $\inf_{i\in I}\{x_i\} \Leftrightarrow \inf_{i\in I}\{y_i\} = \inf_{i\in I}\{x_i\} \Rightarrow \inf_{i\in I}\{y_i\}$. It is obvious that it is enough to prove that
$$\inf_{i\in I}\{x_i \Rightarrow y_i\} \leq \inf_{i\in I}\{x_i\} \Rightarrow \inf_{i\in I}\{y_i\},$$
and this is an easy consequence of residuation because for every $i \in I$,
$$\inf_{i\in I}\{x_i \Rightarrow y_i\} * \inf_{i\in I}\{x_i\} \leq (x_i \Rightarrow y_i) * x_i \leq y_i.$$

4. Without loss of generality we can assume that $\sup_{i\in I}\{y_i\} \leq \sup_{i\in I}\{x_i\}$. Thus, $\sup_{i\in I}\{x_i\} \Leftrightarrow \sup_{i\in I}\{y_i\} = \sup_{i\in I}\{x_i\} \Rightarrow \sup_{i\in I}\{y_i\}$. It is obvious that it is enough to prove that
$$\inf_{i\in I}\{x_i \Rightarrow y_i\} \leq \sup_{i\in I}\{x_i\} \Rightarrow \sup_{i\in I}\{y_i\}.$$
This is true because if $a = \inf_{i\in I}\{x_i \Rightarrow y_i\}$, then for every $i \in I$,
$$a * x_i \leq y_i;$$
and hence,
$$a * \sup_{i\in I}\{x_i\} = \sup_{i\in I}\{a * x_i\} \leq \sup_{i\in I}\{y_i\}. \qquad \square$$

The proof we give for Theorem 97 is based on a continuity argument, and resembles the one given in [Hájek, 1998b, Theorem 5.4.30]. The main difference is that while Hájek introduces a *distance* between models on the same domain, in this paper we consider a dual notion, which we call *similarity* and denote by S. In the case of Łukasiewicz, since the duality, there is no essential difference between considering a distance or a similarity, but this is not the case for Product Logic, where it is crucial to consider a similarity.

Definition 94 (Similarity)**.** Let Γ be a predicate language with a finite number of predicate symbols $P_1, \ldots, P_n$, and let $\mathbf{M}, \mathbf{M}'$ be two models over $[0,1]_\Pi$ on the same domain M such that r_{P_i} and r'_{P_i} are the interpretations of the predicate symbols in $\mathbf{M}$ and $\mathbf{M}'$ respectively.

1. For each predicate symbol $P \in \Gamma$ with arity $ar(P)$, we define
$$S(r_P, r'_P) := \inf_{a\in M^{ar(P)}}\{r_P(a) \Leftrightarrow r'_P(a)\}$$
$$= \inf_{a\in M^{ar(P)}}\left\{\frac{\min\{r_P(a), r'_P(a)\}}{\max\{r_P(a), r'_P(a)\}}\right\}$$

2. Moreover, we define

$$S(\mathbf{M}, \mathbf{M}') := S(r_{P_1}, r'_{P_1}) * \ldots * S(r_{P_n}, r'_{P_n}).$$

Definition 95. We define the complexity $\tau(\varphi)$ of a formula φ as follows:

1. $\tau(\varphi) = 0$, if φ is atomic or $\bot$,
2. $\tau(\varphi * \psi) = 1 + max\{\tau(\varphi), \tau(\psi)\}$, if $* \in \{\rightarrow, \otimes\}$,
3. $\tau(Qx\, \varphi) = \tau(\varphi)$, if $Q \in \{\forall, \exists\}$.

This complexity captures the number of nested propositional connectives in the formula.

Lemma 96. *Assume* Γ *is a predicate language with* n *predicate symbols. Let* $\mathbf{M}$ *and* $\mathbf{M}'$ *be two first order structures over* $[0,1]_\Pi$ *on the same domain* M*, and let* φ *be a first order formula. Then, for all* $\varepsilon \in [0, 1)$*,*

$$\textit{if } S(\mathbf{M}, \mathbf{M}') > \sqrt[n\cdot 2^{\tau(\varphi)}]{\varepsilon}, \textit{ then,}$$
$$\textit{for each evaluation } v,\ (\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon.$$

Proof. It is enough to prove that if $\mathbf{M}$ differs from $\mathbf{M}'$ only by the interpretation of one predicate symbol P, then

(C_φ) for all $\varepsilon \in [0, 1)$, if $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\varphi)}]{\varepsilon}$, then, for each evaluation v, $(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon$.

We show that this condition (C_φ) holds by induction on the length of the formula φ.

- If φ is either atomic or $\bot$, then it is obvious.
- Let us suppose $\varphi = \psi \star \chi$ with $\star \in \{\rightarrow, \otimes\}$, and $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\varphi)}]{\varepsilon}$. Then, $S(\mathbf{M}, \mathbf{M}') > \max\{\sqrt[2^{\tau(\psi)}]{\sqrt{\varepsilon}}, \sqrt[2^{\tau(\chi)}]{\sqrt{\varepsilon}}\}$. Using the inductive hypothesis for $\sqrt{\varepsilon}$, we get that
$$(\|\psi\|_{\mathbf{M},v} \Leftrightarrow \|\psi\|_{\mathbf{M}',v}) \geq \sqrt{\varepsilon},$$
$$(\|\chi\|_{\mathbf{M},v} \Leftrightarrow \|\chi\|_{\mathbf{M}',v}) \geq \sqrt{\varepsilon}.$$
  Hence, by the first two items in Lemma 93 we get that
$$(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \sqrt{\varepsilon} * \sqrt{\varepsilon} = \varepsilon.$$
- Let us suppose that $\varphi = Qx\, \psi$, with $Q \in \{\forall, \exists\}$, and $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\varphi)}]{\varepsilon}$. Then, $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\psi)}]{\varepsilon}$. By the inductive hypothesis we get that $(\|\psi\|_{\mathbf{M},v} \Leftrightarrow \|\psi\|_{\mathbf{M}',v}) \geq \varepsilon$ for each evaluation v. Hence,
$$\inf_v\{\|\psi\|_{\mathbf{M},v} \Leftrightarrow \|\psi\|_{\mathbf{M}',v}\} \geq \varepsilon.$$
  By the last two items in Lemma 93 it follows that

$$(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon.$$

Hence, the lemma is proved. □

We are now ready to prove the main result of the present Appendix.

Theorem 97. *A first-order formula* φ *is a* $[0,1]_\Pi$*-tautology if and only if it is a tautology in any one-generated subalgebra of* $[0,1]_\Pi$.

Proof. The result is an obvious consequence of the previous lemma. Suppose that φ is not a $[0,1]_\Pi$-tautology, then there is a structure $\mathbf{M}$ and an evaluation v such that $\|\varphi\|_{\mathbf{M},v} < \varepsilon$ for some $\varepsilon < 1$. Take $s \in (0,1)$ such that $s^n > \sqrt[n \cdot 2^{\tau(\varphi)}]{\varepsilon}$, and denote by $\langle s \rangle$ the subalgebra of $[0,1]$ generated by s. For every predicate symbol P, let $r'_P(a)$ be $\min\{t \in \langle s \rangle \mid t \geq r_P(a)\}$. Now we define the structure $\mathbf{M}' = (M, r'_{P_1}, \ldots, r'_{P_n})$ over the algebra $\langle s \rangle$. An easy computation shows that $S(r_P, r'_P) \geq s$ for every predicate symbol P; hence, $S(\mathbf{M}, \mathbf{M}') \geq s^n > \sqrt[n \cdot 2^{\tau(\varphi)}]{\varepsilon}$. By Lemma 96, $(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon$. This together with the fact that $\|\varphi\|_{\mathbf{M},v} < \varepsilon$ implies that $\|\varphi\|_{\mathbf{M}',v} \neq 1$. This finishes the proof. □

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