Degree-preserving Gödel logics with an involution: intermediate logics and (ideal) paraconsistency

Marcelo E. Coniglio[1], Francesc Esteva[2], Joan Gispert[3] and Lluis Godo[2]

[1] Dept. of Philosophy - IFCH and
Centre for Logic, Epistemology and the History of Science,
University of Campinas, Brazil
coniglio@cle.unicamp.br
[2] Artificial Intelligence Research Institute (IIIA) - CSIC, Barcelona, Spain
{esteva,godo}@iiia.csic.es
[3] Dept. of Mathematics and Computer Science, University of Barcelona, Spain
jgispertb@ub.edu

Abstract

In this paper we study intermediate logics between the logic $\mathrm{G}^{\leq}_{\sim}$, the degree preserving companion of Gödel fuzzy logic with involution $G_{\sim}$ and classical propositional logic CPL, as well as the intermediate logics of their finite-valued counterparts $\mathrm{G}^{\leq}_{n\sim}$. Although $\mathrm{G}^{\leq}_{\sim}$ and $\mathrm{G}^{\leq}_{n\sim}$ are explosive w.r.t. Gödel negation $\neg$, they are paraconsistent w.r.t. the involutive negation $\sim$. We introduce the notion of saturated paraconsistency, a weaker notion than ideal paraconsistency, and we fully characterize the ideal and the saturated paraconsistent logics between $\mathrm{G}^{\leq}_{n\sim}$ and CPL. We also identify a large family of saturated paraconsistent logics in the family of intermediate logics for degree-preserving finite-valued Łukasiewicz logics.

1 Introduction

Contradictions frequently arise in scientific theories, as well as in philosophical argumentation. In computer science, techniques for dealing with contradictory information need to be developed, in areas such as logic programming, belief revision, the semantic web and artificial intelligence in general. Since classical logic –as well as many other non-classical logics– trivialize in the presence of inconsistencies, it can be useful to consider logical systems tolerant to contradictions in order to formalize such situations.

A logic L is said to be *paraconsistent* with respect to a negation connective $\neg$ when it contains a $\neg$-contradictory but not trivial theory. Assuming that L is (at least) Tarskian, this is equivalent to say that the $\neg$-explosion rule

$$\frac{\varphi \quad \neg\varphi}{\psi}$$

is not valid in L. The main challenge for paraconsistent logicians is defining logic systems in which not only a contradiction does not necessarily trivialize, but also allowing that useful conclusions can be derived from such inconsistent information.

The first systematic study of paraconsistency from the point of view of formal logic is due to da Costa, which introduces in 1963 a hierarchy of paraconsistent systems called C_n. This is why da Costa is considered one of the founders of the subject of paraconsistency. Under his perspective, propositions in a paraconsistent setting are 'dubious' in the sense that, in general, a sentence and its negation can be hold simultaneously without trivialization. That is, it is possible to consider contradictory but nontrivial theories. Moreover, it is possible to express (in every system C_n) the fact that a given sentence φ has a classical behavior w.r.t. the explosion law. This approach to paraconsistency, in which the explosion law is recovered in a controlled way, was generalized by Carnielli and Marcos in [17] by means of the notion of *Logics of Formal Inconsistency* (**LFI**s, in short). An **LFI** is a paraconsistent logic (w.r.t. a negation $\neg$) having, in addition, an unary connective $\circ$ (a *consistency operator*), primitive or defined, such that any theory of the form $\{\varphi, \neg\varphi, \circ\varphi\}$ is trivial, despite $\{\varphi, \neg\varphi\}$ not being necessarily so. Of course, the main novelty whith respect to da Costa's systems C_n is that the consistency operator (which corresponds to the well-behavior operator) can now be a primitive connective, which allows to consider a more general and expressive theory of paraconsistency. The **LFI**s have been extensively studied since then (for general references, consult [16, 15]). Avron has contributed significantly to the development of **LFI**s, see for instance [7, 8, 9].

According to da Costa, one of the main properties that a paraconsistent logic should have is being as close as possible to classical logic. That is, a paraconsistent logic should retain as much as possible the classical inferences, and still allowing to have non-trivial, contradictory theories. A natural way to formalize this *desideratum* is by means of the notion of maximality of a logic w.r.t. another one. A (Tarskian and structural) logic L_1 is said to be *maximal* w.r.t. another logic L_2 if both are defined over the same signature, the consequence relation of L_1 is contained in that of L_2 (i.e., L_2 is an extension of L_1) and, if φ is a theorem of L_2 which is not derivable in L_1, then the extension of L_1 obtained by adding φ (and all of its instances under uniform substitutions) as a theorem coincides with L_2. Hence, a 'good' paraconsistent logic L should be maximal w.r.t. classical logic CPL (presented over the same signature than L). As observed in [20], the notion of maximality can be vacuously satisfied when both logics (L_1 and L_2) have the same theorems.

In [2], Orieli, Avron and Zamansky propose an interesting notion of maximality w.r.t. paraconsistency: a paraconsistent logic is *maximally paraconsistent* if no proper extension of it is paraconsistent. Thus, they prove that several well-known 3-valued logics such as Sette's P1 and da Costa and Ottaviano's J_3 are maximally paraconsistent. Note that both P1 and J_3 are also maximal w.r.t. CPL.

These strong features satisfied by logics such as P1 and J_3 lead Arieli, Avron and Zamansky to introduce in [4] the notion of ideal paraconsistent logics. Briefly, a logic L is called *ideal paraconsistent* when it is maximally paraconsistent and maximal w.r.t. to classical logic CPL (the formal definition of ideal paraconsistency will we recalled in Section 5). One interesting problem is to find ideal paraconsistent logics, and in this sense [4] provides a vast variety of examples of ideal paraconsistent finite-valued logics, aside from P1 and J_3.

Besides many paraconsistent logicians (including Avron and his collaborators, as we have seen above) agree with da Costa's requirement of maximality w.r.t. CPL for defining reasonable paraconsistent logics, this is not an uncontroversial position. In [30], Wansing and Odintsov extensively criticized that requirement. According to these authors, maximality w.r.t. classical logic is not a good choice: on the one hand, the phenomenon of paraconsistency should be interpreted from an informational perspective instead of considering epistemological or ontological terms. On the other hand, CPL would be inappropriate for reasoning about information:

> "classical logic is not at all a natural reference logic for reasoning about information and information structures. On the other hand, it is reasoning about information that suggests paraconsistent reasoning." [30, p. 181]

> "one may wonder why exactly a *nonclassical* paraconsistent logic, if correct, should have a distinguished status in virtue of being faithful to classical logic "as much as possible"." [30, p. 181]

> "Paraconsistency does deviate from logical orthodoxy, but it is not at all clear that classical logic indeed is the logical orthodoxy from which paraconsistent logics ought to deviate only minimally." [30, p. 183]

Despite it could be argued against this emphatic perspective, it also seems that being maximal w.r.t. CPL should not be a necessary requirement for being an 'ideal' (meaning 'optimal') paraconsistent logic.[1] This is why we propose in this paper the notion of *saturated paraconsistent* logic, which is just a weakening of the concept of ideal paraconsistent logic, by dropping the requirement of maximality w.r.t. CPL. As we shall see along this paper, there are several interesting examples of saturated paraconsistent logics.

[1] It is worth noting that, more recently, the authors have changed the terminology "ideal paraconsistent logic" in [4] to "fully maximal and normal paraconsistent logic" e.g. in [5]. According to them, they choose the latter "to use a more neutral terminology" (see [5, Footnote 9, p. 57]).

While paraconsistency deals with excessive or dubious information, fuzzy logics were designed for reasoning with imprecise information; in particular, for reasoning with propositions containing vague predicates. Given that both paradigms are able to deal with information – unreliable, in the case of paraconsistent logics, and imprecise, in the case of fuzzy logics – it seems reasonable to consider logics which combine both features, namely, paraconsistent fuzzy logic. The first steps along this way were taken in [21], where a consistency operator $\circ$ was defined in terms of the other connectives (for instance, by using the Monteiro-Baaz Δ-operator) in several fuzzy logics. In [18] this approach was generalized to fuzzy **LFI**s extending the logic MTL of pre-linear (integral, commutative, bounded) residuated lattices, in which the consistency operator is primitive.

We have studied in different papers paraconsistent logics arising from the family of mathematical fuzzy logics, see e.g. [21, 18, 19, 20]. In particular, in [21] the authors observe that even though all truth-preserving fuzzy logics L are explosive, their degree-preserving companions $L^{\leq}$ [13] are paraconsistent in many cases. This provides a large family of paraconsistent fuzzy logics. In [19] the authors studied the lattice of logics between the n-valued Łukasiewicz logics Ł_n and their degree-preserving companions $\text{Ł}_n^{\leq}$. Although there are many paraconsistent logics for each n, no one of them is ideal. However, in [20] the authors of this paper consider a wide class of logics between $\text{Ł}_n^{\leq}$ and CPL, and in that case they indeed find and axiomatically characterize a family of ideal paraconsistent logics.

In this paper we study paraconsistent logics arising from Gödel fuzzy logic expanded with an involutive negation $\mathrm{G}_{\sim}$, introduced in [23], as well as from its finite-valued extensions $\mathrm{G}_{n\sim}$. It is well-known that Gödel logic G coincides with its degree-preserving companion (since G has the deduction-detachment theorem), but this is not the case for $\mathrm{G}_{\sim}$. In fact, $\mathrm{G}_{\sim}$ and $\mathrm{G}_{\sim}^{\leq}$ are different logics, and moreover, while $\mathrm{G}_{\sim}^{\leq}$ is explosive w.r.t. Gödel negation $\neg$, it is paraconsistent w.r.t. the involutive negation $\sim$.[2] We also study the logics between $\mathrm{G}_{n\sim}^{\leq}$ (the finite valued Gödel logic with an involutive negation) and CPL, and we find that the ideal paraconsistent logics of this family are only the above mentioned 3-valued logic J_3 and its 4-valued version J_4, introduced in [20]. Moreover, we fully characterize the ideal and the saturated paraconsistent logics between $\mathrm{G}_{n\sim}^{\leq}$ and CPL.

The paper is structured as follows. After this introduction, some basic definitions and known results to be used along the paper will be presented. In Section 3 we show that the logics between $\mathrm{G}_{\sim}^{\leq}$ and CPL are defined by matrices over a $G_{\sim}$-algebra with lattice filters, and in particular we study the logics defined by matrices over $[0,1]_{\sim}$ with order filters. In Section 4 we study the case of finite-valued Gödel logics with involution $G_{n\sim}$, and we observe that $G_{3\sim}$ and

[2]In fact, $\mathrm{G}_{\sim}^{\leq}$ is then a *paradefinite* logic (w.r.t. $\sim$) in the sense of Arieli and Avron [1], as it is both paraconsistent and paracomplete, since the law of excluded middle $\varphi \vee {\sim}\varphi$ fails, as in all fuzzy logics. Logics with a negation which is both paraconsistent and paracomplete were already considered in the literature under different names: *non-alethic* logics (da Costa) and *paranormal* logics (Beziau).

$G_{4\sim}$ coincide with Ł$_3$ and Ł$_4$ (the 3 and 4-valued Łukasiewicz logics) already studied in [20]. We prove that, in the general case, each finite $G_{n\sim}$-algebra is a direct product of subalgebras of $\mathbf{VG_{n\sim}}$, the Gödel chain of n-elements with the unique involution one can define on it, which is given by $\sim x = 1 - x$. This result allow us to characterize the logics between $\mathrm{G}^{\leq}_{n\sim}$ and CPL. In Section 5 the definition of saturated paraconsistent logic is formally introduced, and it is proved that between $\mathrm{G}^{\leq}_{n\sim}$ and CPL there are only three saturated paraconsistent logics: two of them (J_3 and J_4) are already known and are in fact ideal paraconsistent, and there is only one that is saturated but not ideal paraconsistent, that we call $\mathsf{J}_3 \times \mathsf{J}_4$. Finally, in Section 6 we return to the study of finite-valued Łukasiewicz logic and prove that in this framework there is a large family of saturated paraconsistent logics that are not ideal paraconsistent. Some concluding remarks are discussed in the final section.

2 Preliminaries

2.1 Truth-preserving Gödel logics

This section is devoted to needed preliminaries on the Gödel fuzzy logic G, its axiomatic extensions, as well as their expansions with an involutive negation. We present their syntax and semantics, their main logical properties and the notation we use throughout the article.

The language of Gödel propositional logic is built as usual from a countable set of propositional variables V, the constant $\bot$ and the binary connectives $\wedge$ and $\rightarrow$. Disjunction and negation are respectively defined as $\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$ and $\neg\varphi := \varphi \rightarrow \bot$, and the constant $\top$ is taken as $\bot \rightarrow \bot$.

The following are the *axioms* of G:

(A1)	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2)	$(\varphi \wedge \psi) \rightarrow \varphi$
(A3)	$(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
(A4a)	$(\varphi \rightarrow (\psi \rightarrow \chi)) \equiv ((\varphi \wedge \psi) \rightarrow \chi)$
(A4b)	$((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A6)	$\varphi \rightarrow (\varphi \wedge \varphi)$
(A7)	$((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A8)	$\bot \rightarrow \varphi$

The *deduction rule* of G is modus ponens.

As a many-valued logic, Gödel logic is the axiomatic extension of Hájek's Basic Fuzzy Logic BL [27] (which is the logic of continuous t-norms and their residua) by means of the contraction axiom (A6).

Since the unique idempotent continuous t-norm is the minimum, this yields that Gödel logic is strongly complete with respect to its standard fuzzy semantics that interprets formulas over the structure $[0,1]_G = ([0,1], \min, \Rightarrow_G, 0, 1)$,[3] i.e. semantics defined by truth-evaluations of formulas e on $[0,1]$ such that $e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$, $e(\varphi \to \psi) = e(\varphi) \Rightarrow_G e(\psi)$ and $e(\bot) = 0$, where $\Rightarrow_G$ is the binary operation on $[0,1]$ defined as

$$x \Rightarrow_G y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise.} \end{cases}$$

As a consequence $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$ and $e(\neg\varphi) = \neg_G e(\varphi) = e(\varphi) \Rightarrow_G 0$.

Gödel logic can also be seen as the axiomatic extension of intuitionistic propositional logic by the prelinearity axiom

$$(\varphi \to \psi) \vee (\psi \to \varphi).$$

Its algebraic semantics is therefore given by the variety of prelinear Heyting algebras, also known as Gödel algebras. A Gödel algebra is thus a (bounded, integral, commutative) residuated lattice $\mathbf{A} = (A, \wedge, \vee, *, \Rightarrow, 0, 1)$ such that the monoidal operation $*$ coincides with the lattice meet $\wedge$, and the pre-linearity equation

$$(x \Rightarrow y) \vee (y \Rightarrow x) = 1$$

is satisfied, where $x \vee y = ((x \Rightarrow y) \Rightarrow y) * ((y \Rightarrow x) \Rightarrow x))$. Gödel algebras are locally finite, i.e. given a Gödel algebra $\mathbf{A}$ and a finite set $F \subseteq A$, the Gödel subalgebra generated by F is finite as well.

It is also well-known that the axiomatic extensions of Gödel logic correspond to its finite-valued counterparts. If we replace the unit interval $[0,1]$ by the truth-value set $VG_n = \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$ in the standard Gödel algebra $[0,1]_G$ then the structure $\mathbf{VG_n} = (VG_n, \min, \Rightarrow_G, 0, 1)$ becomes the "standard" algebra for the n-valued Gödel logic G_n. It turns out that G_n is the axiomatic extension of G with the axiom

$$(A_{G_n}) \quad (\varphi_1 \to \varphi_2) \vee \ldots \vee (\varphi_n \to \varphi_{n+1})$$

In fact the logics G_n are all the axiomatic extensions of G, and for each n, G_n is an axiomatic extension of G_{n+1}, where G_2 coincides with CPL. Thus the set of axiomatic extensions of G form a chain of logics (and of the corresponding varieties of algebras) of strictly increasing strength:

$$G < \ldots \leq G_{n+1} < G_n < \ldots < G_3 < G_2 = \text{CPL}$$

where $L < L'$ denotes L' is an axiomatic extension of L.

Since the negation in Gödel logics is a pseudo-complementation and not an involution, in [23] the authors investigate the residuated fuzzy logics arising from continuous t-norms without non trivial zero divisors and extended with an involutive negation. In particular, they consider the extension of Gödel logic G with an involutive negation $\sim$, denoted as $G_\sim$, and axiomatize it.

[3]Called *standard* Gödel algebra.

The intended semantics of the $\sim$ connective on the real unit interval $[0, 1]$ is an arbitrary order-reversing involution $n : [0, 1] \to [0, 1]$, i.e. satisfying $n(n(x)) = x$ and $n(x) \leq n(y)$ whenever $x \geq y$.

It turns out that in $\mathrm{G}_{\sim}$, with both negations, $\neg$ and $\sim$, the projection Monteiro-Baaz connective Δ is definable as

$$\Delta\varphi := \neg{\sim}\varphi,$$

and whose semantics on $[0, 1]$ is given by the mapping $\delta : [0, 1] \to [0, 1]$ defined as $\delta(1) = 1$ and $\delta(x) = 0$ for $x < 1$.

Axioms of $\mathrm{G}_{\sim}$ are those of G plus

($\sim$1)	${\sim}{\sim}\varphi \leftrightarrow \varphi$	*(Involution)*
($\sim$2)	$\neg\varphi \to {\sim}\varphi$	
($\sim$3)	$\Delta(\varphi \to \psi) \to \Delta({\sim}\psi \to {\sim}\varphi)$	*(Order Reversing)*
(Δ1)	$\Delta\varphi \vee \neg\Delta\varphi$	
(Δ2)	$\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$	
(Δ5)	$\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$	

where $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, and inference rules of $\mathrm{G}_{\sim}$ are *modus ponens* and *necessitation* for Δ: from φ infer $\Delta\varphi$.

$\mathrm{G}_{\sim}$ is an algebraizable logic, whose equivalent algebraic semantics is the quasivariety of $\mathrm{G}_{\sim}$-algebras, defined in the natural way, and generated by the class of its linearly ordered members. Among them, the so-called *standard* $\mathrm{G}_{\sim}$-algebra, denoted $[0, 1]_{G_{\sim}}$, is the algebra on the real interval $[0, 1]$ with Gödel truth functions extended by the involutive negation ${\sim}x = 1 - x$. This $\mathrm{G}_{\sim}$-chain generates the whole quasi-variety of $\mathrm{G}_{\sim}$-algebras. In fact, we have the following strong *standard* completeness result for $\mathrm{G}_{\sim}$ [23, 24]: for any set $\Gamma \cup \{\varphi\}$ of $\mathrm{G}_{\sim}$-formulas, $\Gamma \vdash_{\mathrm{G}_{\sim}} \varphi$ iff $\Gamma \models_{\mathrm{G}_{\sim}} \varphi$.

Finally, remark that, while G enjoys the usual deduction-detachment theorem (i.e. $\Gamma \cup \{\varphi\} \vdash_{\mathrm{G}} \psi$ iff $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$), this is not the case for $\mathrm{G}_{\sim}$, that has only the following form of Δ-deduction theorem: $\Gamma \cup \{\varphi\} \vdash_{\mathrm{G}_{\sim}} \psi$ iff $\Gamma \vdash_{\mathrm{G}_{\sim}} \Delta\varphi \to \psi$.

On the other hand, as in the case of Gödel logic, one can also consider the logics $\mathrm{G}_{n\sim}$ for each $n \geq 2$, the finite-valued counterparts of $\mathrm{G}_{\sim}$. Namely, $\mathrm{G}_{n\sim}$ can be obtained as the axiomatic extension of $\mathrm{G}_{\sim}$ with the axiom ($\mathrm{A}_{\mathrm{G}_n}$),[4] and can be shown to be complete with respect to its intended algebraic semantics, the variety of algebras generated by the linearly ordered algebra $\mathbf{VG_{n}}_{\sim}$ obtained in turn by expanding $\mathbf{VG_n}$ with the involutive negation ${\sim}x = 1 - x$, the only involutive order-reversing mapping that can be defined on VG_n. Clearly, $\mathrm{G}_{2\sim} = \mathrm{CPL}$. The graph of axiomatic extensions of $\mathrm{G}_{n\sim}$ is depicted in Fig. 2.1, where edges denote extensions. It can be observed that, if n is even then $\mathrm{G}_{n\sim}$ is an extension of $\mathrm{G}_{m\sim}$ for any $m > n$, while if n is odd, $\mathrm{G}_{n\sim}$ is an extension of $\mathrm{G}_{m\sim}$ only for those $m > n$ being odd as well. Also, note that, in the figure, $\mathrm{G}^{-}_{\sim}$ denotes the extension of $\mathrm{G}_{\sim}$ with the axiom

[4] Equivalently, as the expansion of G_n with $\sim$ along with the axioms ($\sim$1)-($\sim$3), (Δ1)-(Δ3), and the necessitation rule for Δ.

(NFP) $\quad {\sim}\Delta(\varphi \leftrightarrow {\sim}\varphi)$

that captures the condition that the involutive negation does not have a fixpoint, a condition satisfied by all the logics $G_{n\sim}$ with n even.

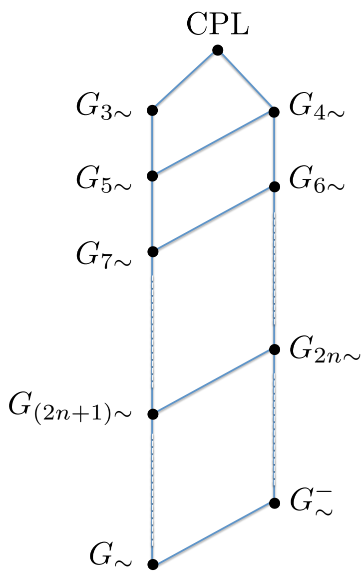

Figure 1: Graph of axiomatic extensions of $G_{\sim}$.

2.2 Degree-preserving Gödel logics with involution

Main logics studied in Mathematical Fuzzy Logic are (full) truth-preserving fuzzy logics, like the Gödel logics introduced in the previous version. But we can also find in the literature companion logics that preserve degrees of truth, see e.g.[25, 13]. Namely, given a fuzzy logic L,[5] one can introduce a variant of L that is usually denoted $L^{\leq}$, whose associated deducibility relation has the following semantics: for every set of formulas $\Gamma \cup \{\varphi\}$,

$\Gamma \vdash_{L^{\leq}} \varphi$ iff for every L-chain $\boldsymbol{A}$, every $a \in A$, and every $\boldsymbol{A}$-evaluation e, if $a \leq e(\psi)$ for every $\psi \in \Gamma$, then $a \leq e(\varphi)$.

[5]For practical purposes, we can assume in this paper that L is an axiomatic extension of Hájek's BL logic.

For this reason $L^{\leq}$ is known as a fuzzy logic *preserving degrees of truth*, or the *degree-preserving companion* of L. It is not difficult to show that L and $L^{\leq}$ have the same theorems and also that for every finite set of formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{L^{\leq}} \varphi \text{ iff } \vdash_{L} \Gamma^{\wedge} \to \varphi$$

where $\Gamma^{\wedge}$ means $\gamma_1 \wedge \ldots \wedge \gamma_k$ for $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ (when Γ is empty then $\Gamma^{\wedge}$ is $\top$).

As regards to axiomatization, the logic $L^{\leq}$ admits a Hilbert-style axiomatization having the same axioms as L and the following deduction rules [13]:

(Adj-$\wedge$) from φ and ψ derive $\varphi \wedge \psi$

(MP-r) if $\vdash_{L} \varphi \to \psi$,[6] then from φ and $\varphi \to \psi$, derive ψ

Since Gödel logic G enjoys the deduction-detachment theorem, a key observation is that $G^{\leq} = G$. However, the case is different for the expansion of G with an involutive negation, since $G_{\sim}$ does not satisfy the usual deduction-detachment theorem, and hence $G_{\sim}$ and $G_{\sim}^{\leq}$ are different logics. Moreover, while $G_{\sim}^{\leq}$ keeps being $\neg$-explosive, it is $\sim$-paraconsistent. Indeed, there are φ, ψ such that $\varphi \wedge {\sim}\varphi \nvdash_{G^{\leq}} \psi$.

As for the axiomatization of $G_{\sim}^{\leq}$, we need to consider an extra rule regarding Δ. As shown in [21], a complete Hilbert-style axiomatization for $G_{\sim}^{\leq}$ can be obtained by the axioms of $G_{\sim}$, the previous rules **(Adj-$\wedge$)** and **(MP-r)**,[7] together with the following restricted necessitation rule for Δ:

(Δ-r) if $\vdash_{G_{\sim}} \varphi$, then from φ derive $\Delta\varphi$

Finally, let us consider the logics $G_{n\sim}^{\leq}$, the degree-preserving companions of the finite-valued logics $G_{\sim}^{\leq}$, defined in the obvious way as above for $L = G_{n\sim}$. Similarly to $G_{\sim}^{\leq}$, $G_{n\sim}^{\leq}$ also admits the following Hilbert-style axiomatization: $G_{n\sim}^{\leq}$ has as axioms those of $G_{n\sim}$, and as rules, the rule (Adj-$\wedge$) and the following restricted rules:

(MP-r) if $\vdash_{G_{n\sim}} \varphi \to \psi$, then from φ and $\varphi \to \psi$, derive ψ

(Δ-Nec-r) if $\vdash_{G_{n\sim}} \varphi$, then from φ derive $\Delta\varphi$

3 Logics defined by matrices over $[0,1]_{G\sim}$ by means of order filters

By a logical matrix we understand a pair $\langle \mathbf{A}, F\rangle$ where $\mathbf{A}$ is an algebra and F is a subset of A. The logic $L(M)$ defined by the matrix $M = \langle \mathbf{A}, F\rangle$ is obtained

[6] That is, if $\varphi \to \psi$ is a theorem of L

[7] For $L = G_{\sim}$.

by stipulating, for any set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{L(M)} \varphi \quad \text{if} \quad \begin{array}{l}\text{for every evaluation } e \text{ on } \mathbf{A},\\ \text{if } e(\gamma) \in F \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) \in F.\end{array}$$

On the other hand, the logic $L(\mathcal{M})$ determined by a class of matrices $\mathcal{M}$ is defined as the intersection of the logics defined by all the matrices in the family.

Notation: *In the rest of the paper, without danger of confusion and for the sake of an lighter notation, we will often identify a matrix M or a set of matrices $\mathcal{M}$ with their corresponding logics $L(M)$ and $L(\mathcal{M})$.*

As proved in [13] for logics of residuated lattices, one can show that $\mathrm{G}_{\sim}^{\leq}$, the degree-preserving companion of $\mathrm{G}_{\sim}$, is not algebraizable in the sense of Block and Pigozzi and thus it has no algebraic semantics. But it has a semantics via matrices. Indeed, $\mathrm{G}_{\sim}$ is in fact the logic defined by the set of matrices

$$\mathcal{M}_{\mathrm{G}_{\sim}} = \{\langle \mathbf{A}, F\rangle : \mathbf{A} \text{ is a } \mathrm{G}_{\sim}\text{-algebra and } F \text{ is a lattice filter of } \mathbf{A}\}.$$

Using similar arguments as in the proof of [13, Theorem 2.12], in fact we can also prove that $\mathrm{G}_{\sim}^{\leq}$ is complete with respect to the subset of matrices over the standard $\mathrm{G}_{\sim}$-algebra:

$$\mathcal{M}_{[0,1]} = \{\langle [0,1]_{\mathrm{G}_{\sim}}, F\rangle : F \text{ is an order filter of } [0,1]\}.$$

Next, we study the relationships among all the logics defined by matrices from $\mathcal{M}_{[0,1]}$, i.e. matrices over the algebra $[0,1]_{G_{\sim}}$ by order filters, identifying which ones are paraconsistent. Actually, the order filters on $[0,1]_{G_{\sim}}$ are the following sets: $F_{[a} = \{x \in [0,1] : x \geq a\}$ for all $a \in (0,1]$, and $F_{(a} = \{x \in [0,1] : x > a\}$ for all $a \in [0,1)$. Abusing the notation, we will denote the corresponding logics as

$$\mathrm{G}_{\sim}^{[a} = \langle [0,1]_{G_{\sim}}, F_{[a}\rangle \text{ and } \mathrm{G}_{\sim}^{(a} = \langle [0,1]_{G_{\sim}}, F_{(a}\rangle.$$

The consequence relations corresponding to these logics will be respectively denoted by $\vdash_{[a}$ and $\vdash_{(a}$, while $\vdash^*_{[a}$ and $\vdash^*_{(a}$ will denote the finitary companions of $\vdash_{[a}$ and $\vdash_{(a}$, respectively. Next proposition shows the relationships among all these logics.

Proposition 1. *The logics $\mathrm{G}_{\sim}^{[a} = \langle [0,1]_{G_{\sim}}, F_{[a}\rangle$ for $a \in (0,1]$, and $\mathrm{G}_{\sim}^{(a} = \langle [0,1]_{G_{\sim}}, F_{(a}\rangle$ for $a \in [0,1)$, satisfy the following properties:*

P1. $\vdash_{[p} = \vdash_{[p'}$ *and* $\vdash_{(p} = \vdash_{(p'}$, *for all* $p, p' \in (1/2, 1)$
Moreover, $\vdash^*_{[p} = \vdash^*_{(p}$, *for all* $p \in (1/2, 1)$.

P2. $\vdash_{[n} = \vdash_{[n'}$ *and* $\vdash_{(n} = \vdash_{(n'}$, *for all* $n, n' \in (0, 1/2)$
Moreover, $\vdash^*_{[n} = \vdash^*_{(n}$, *for all* $n \in (0, 1/2)$.

P3. $\vdash_{[p} \;\subsetneq\; \vdash_1$, *for any* $p \in (1/2, 1)$

P4. $\vdash_1$ *and* $\vdash_{[1/2}$ *are not comparable*

P5. $\vdash_{[p}$ *and* $\vdash_{[1/2}$ *are not comparable, for any* $p \in (1/2, 1)$

P6. $\vdash^*_{[p} \subsetneq \vdash_{(1/2}$

P7. $\vdash_{[p}$ *and* $\vdash_n$ *are not comparable, for any* $p \in (1/2, 1)$ *and any* $n \in (0, 1/2)$

P8. $\vdash_{[n} \subsetneq \vdash_{[1/2}$*, for any* $n \in (0, 1/2)$

P9. $\vdash_{[1/2}$ *and* $\vdash_{(1/2}$ *are not comparable*

P10. $\vdash^*_{[n} \subsetneq \vdash_{(0}$*, for any* $n \in (0, 1/2)$*.*

Proof.

P1. We divide the proof in three steps:

(i) That $\vdash_{[p} = \vdash_{[p'}$ and $\vdash_{(p} = \vdash_{(p'}$ is an easy consequence of the fact that for every $p, p' \in (1/2, 1)$ it is possible to define an automorphism f of $[0,1]_{G\sim}$ such that $f(p) = p'$. Let us then show that $\vdash_{[p} = \vdash_{(p}$ for every $p \in (1/2, 1)$.

(ii) Assume $\{\varphi_i : i \in I\} \vdash^*_{[p} \psi$, with I finite, for some $p \in (1/2, 1)$. Let q such that $1/2 < q < p$, and let e be an evaluation such that $e(\varphi_i) > q$ for all $i \in I$. Let $p' = \min_{i \in I} e(\varphi_i)$. Obviously $p' > q$. Then, by (i), we also have $\{\varphi_i : i \in I\} \vdash^*_{[p'} \psi$, and therefore we have $e(\psi) \geq p' > q$, and hence $\{\varphi_i : i \in I\} \vdash^*_{(q} \psi$. Therefore, we have $\vdash^*_{[p} \subseteq \vdash^*_{(q}$ for all $1/2 < q < p$.

(iii) Assume $\{\varphi_i : i \in I\} \vdash^*_{(q} \psi$, with I finite, for some $q \in (1/2, 1)$. Let p such that $q < p < 1$, and let e be an evaluation such that $e(\varphi_i) \geq p$ for all $i \in I$. Let $q' = \min_{i \in I} e(\varphi_i)$. Obviously $q' \geq p$. Then, by (i), we also have $\{\varphi_i : i \in I\} \vdash^*_{(q'} \psi$, and therefore we have $e(\psi) \geq q' \geq p$, and hence $\{\varphi_i : i \in I\} \vdash^*_p \psi$. Therefore, we have $\vdash_{(q} \subseteq \vdash^*_{[p}$ for all $1/2 < q < p$.

P2. The proofs are analogous to those of P1.

P3. Assume $\{\varphi_i : i \in I\} \vdash_{[p} \psi$ for a given $p \in (1/2, 1)$, and let e be an evaluation such that $e(\varphi_i) = 1$ for all $i \in I$. Since it is also true that $e(\varphi_i) \geq p'$ for all $p' \in (1/2, 1)$, by P1 it follows that $\{\varphi_i : i \in I\} \vdash_{[p'} \psi$ for all $p' \in (1/2, 1)$, and hence $e(\psi) \geq p'$ for all $p' \in (1/2, 1)$, and thus $e(\psi) = 1$. Therefore $\{\varphi_i : i \in I\} \vdash_1 \psi$.

The strict inclusion can be easily noticed since, e.g. it holds that $\varphi \vdash_1 \Delta\varphi$ but $\varphi \nvdash_{[p} \Delta\varphi$ for any $p < 1$.

P4. It clearly holds that, on the one hand, $\Delta(\varphi \leftrightarrow {\sim}\varphi) \vdash_{[1/2} \varphi$ but $\Delta(\varphi \leftrightarrow {\sim}\varphi) \nvdash_1 \varphi$, while on the other hand, $\varphi \vdash_1 \Delta\varphi$ but $\varphi \nvdash_{[1/2} \Delta\varphi$

P5. It follows from noticing that $\Delta(\varphi \leftrightarrow {\sim}\varphi) \wedge \varphi \vdash_{[p} \bot$ and $\Delta(\varphi \leftrightarrow {\sim}\varphi) \wedge \varphi \nvdash_{[1/2} \bot$, while $\Delta(\varphi \leftrightarrow {\sim}\varphi) \vdash_{[1/2} \varphi$ and $\Delta(\varphi \leftrightarrow {\sim}\varphi) \nvdash_{[p} \varphi$.

P6. Assume that, for a given $p \in (1/2, 1)$, $\{\varphi_i : i \in I\} \vdash_{[p} \psi$, with I finite, and let e be an evaluation such that $e(\varphi_i) > 1/2$ for all $i \in I$. Let $p' = \min_{i \in I} e(\varphi_i)$. Obviously $p' > 1/2$. Then, from P1 we also have $\{\varphi_i : i \in I\} \vdash_{[p'} \psi$, and therefore we have $e(\psi) \geq p' > 1/2$, and hence $\{\varphi_i : i \in I\} \vdash_{(1/2} \psi$. Therefore, we have $\vdash_{[p} \subseteq \vdash_{(1/2}$.

That the inclusion is strict follows from observing that $\Delta(\sim\varphi \to \varphi) \wedge \sim\Delta(\varphi \leftrightarrow \sim\varphi) \vdash_{(1/2} \varphi$ but $\Delta(\sim\varphi \to \varphi) \wedge \sim\Delta(\varphi \leftrightarrow \sim\varphi) \nvdash_{[p} \varphi$.

P7. It follows from observing (i) $\Delta(\varphi \leftrightarrow \sim\varphi) \vdash_{[n} \varphi$ and $\Delta(\varphi \leftrightarrow \sim\varphi) \nvdash_{[p} \varphi$, and (ii) $\varphi \vdash_{[p} \sim\Delta(\varphi \to \sim\varphi)$ and $\varphi \nvdash_{[n} \sim\Delta(\varphi \to \sim\varphi)$.

P8. The first part is proved in a similar way to P3. The second is a consequence of the following facts:
$\Delta(\sim\varphi \to \varphi) \wedge \sim\Delta(\varphi \to \sim\varphi) \vdash_{[1/2} \bot$ and
$\Delta(\sim\varphi \to \varphi) \wedge \sim\Delta(\varphi \to \sim\varphi) \nvdash_{[n} \bot$ for all $n \in (0, 1/2)$

P9. It results from noticing e.g. (i) $\Delta(\varphi \leftrightarrow \sim\varphi) \vdash_{[1/2} \varphi$ but $\Delta(\varphi \leftrightarrow \sim\varphi) \nvdash_{(1/2} \varphi$, and (ii) $\varphi \vdash_{(1/2} \sim\Delta(\varphi \to \sim\varphi)$ but $\varphi \nvdash_{[1/2} \sim\Delta(\varphi \to \sim\varphi)$.

P10. Assume $\{\varphi_i : i \in I\} \vdash_{[n} \psi$ for a given $n \in (0, 1/2)$ and a finite set I, and let e be an evaluation such that $e(\varphi_i) > 0$ for all $i \in I$. Let $n' = \min_{i \in I} e(\varphi_i)$. Obviously $n' > 0$ and $e(\varphi_i) \geq n'$, for all $i \in I$. Then, from P1, we also have that $\{\varphi_i : i \in I\} \vdash_{[n'} \psi$, and hence we have $e(\psi) \geq n' > 0$. This means $\{\varphi_i : i \in I\} \vdash_{(0} \psi$. Therefore, we have $\vdash^*_{[n} \subseteq \vdash_{(0}$.

On the other hand, $\neg\neg\varphi \vdash_{(0} \varphi$ but $\neg\neg\varphi \nvdash_{[n} \varphi$, hence we have proved that $\vdash^*_{[n} \subsetneq \vdash_{(0}$.

□

It is clear that a matrix logic $\mathrm{G}_\sim^{[a} = \langle [0,1]_{G_\sim}, F_{[a\rangle} \rangle$ (resp. $\mathrm{G}_\sim^{(a} = \langle [0,1]_{G_\sim}, F_{(a)} \rangle$) is paraconsistent only in the case that $a \leq 1/2$ (resp. $a < 1/2$). As a consequence of the above classification, it turns out that there are only three different paraconsistent logics among them.

Corollary 1. *Among the families of logics* $\{\mathrm{G}_\sim^{[a} = \langle [0,1]_{G_\sim}, F_{[a\rangle} \rangle\}_{a \in (0,1]}$ *and* $\{\mathrm{G}_\sim^{(a} = \langle [0,1]_{G_\sim}, F_{(a)} \rangle\}_{a \in [0,1)}$*, there are only three different paraconsistent logics:* $\mathrm{G}_\sim^{[a}$ *for any* $a \in (0, 1/2)$*,* $\mathrm{G}_\sim^{[1/2}$*, and* $\mathrm{G}_\sim^{(0}$*.*

In analogy to [19, Theorem 2], it is easy to show that every intermediate logic L between $\mathrm{G}_\sim^{\leq}$ and CPL is in fact the logic $L(\mathcal{M}^*)$ defined by a subfamily of matrices $\mathcal{M}^* \subseteq \mathcal{M}_{\mathrm{G}_\sim}$. However, note that the set of $\mathrm{G}_\sim$-algebras and their lattice filters is very large. Then, an exhaustive analysis of the set of intermediate logics between $\mathrm{G}_\sim^{\leq}$ and CPL actually seems to be a difficult task. Because of this, in the next section we will restrict ourselves to the case of finite-valued Gödel logics with an involutive negation $\mathrm{G}_{n\sim}$.

4 Logics between $G^{\leq}_{n\sim}$ and CPL

In this section we will study the graph of intermediate logics between $G^{\leq}_{n\sim}$ and CPL, for a natural $n > 2$. The cases $n = 3$ and $n = 4$ are easy to analyze since $G_{3\sim}$ and $G_{4\sim}$ coincide respectively with the 3-valued and 4-valued Łukasiewicz logics $Ł_3$ and $Ł_4$.

Proposition 2. $G_{3\sim}$ *and* $G_{4\sim}$ *are logically equivalent to* $Ł_3$ *and* $Ł_4$ *respectively.*

Proof. First, in the algebra $\mathbf{VG_{3\sim}}$ it is possible to define the binary connective $x \to_{3Ł} y = (x \to y) \vee ({\sim}x \vee y)$, that coincides with the 3-valued Łukasiewicz implication, i.e. $x \to_{3Ł} y = \min(1, 1 - x + y)$ for every $x, y \in VG_3$. Thus in $\mathbf{VG_{3\sim}}$ we can define all the Łukasiewicz connectives.

Second, also in the algebra $\mathbf{VG_{4\sim}}$ we can define the binary connective

$$x \to_{4Ł} y = ({\sim}\Delta x \wedge {\sim}x) \vee [\Delta({\sim}x \to x) \wedge ({\sim}\Delta x) \wedge (\neg\neg y) \wedge x)] \vee (x \to y)$$

which coincides again with the 4-valued Łukasiewicz implication, i.e. $x \to_{4Ł} y = \min(1, 1 - x + y)$ for every $x, y \in VG_4$.

On the other hand, in any finite Łukasiewicz algebra $\mathbf{MV_n}$ we can always define Gödel implication as $x \to_G y = (x \to_Ł y) \vee {\sim}x$ and Gödel negation as $\neg x = \Delta({\sim}x)$. □

Therefore the logics between CPL and $G^{\leq}_{3\sim}$ (resp. $G^{\leq}_{4\sim}$) coincide with the logics between CPL and $Ł^{\leq}_3$ (resp. $Ł^{\leq}_4$) studied in [19] and [20].

Observe, however, that for any $n > 4$, $G_{n\sim}$ is no longer equivalent to $Ł_n$. Thus we need to study the intermediate logics for $G^{\leq}_{n\sim}$ for $n > 4$, and this is the goal of the next subsection, while in Section 4.2 we will have a closer look to the case $n = 5$.

4.1 The intermediate logics of $G^{\leq}_{n\sim}$ for $n > 4$

Throughout this section n will denote a natural number such that $n > 4$.

Following the same arguments as in previous sections, it is easy to check that $G^{\leq}_{n\sim}$ is in fact the logic semantically defined by the class of matrices:

$$\{\langle \mathbf{A}, F\rangle : \mathbf{A} \text{ is a } G_{n\sim}\text{-algebra and } F \text{ is a lattice filter of } \mathbf{A}\}.$$

Therefore, in order to study the intermediate logics between $G^{\leq}_{n\sim}$ and CPL we need to characterize the (finite) $G_{n\sim}$-algebras.

Proposition 3. *Every finite* $G_{n\sim}$*-algebra is a finite direct product of finite* $G_{n\sim}$*-chains.*

Proof. Notice that for every $G_{n\sim}$-chain the term $t(x, y, z) := (\Delta(x \leftrightarrow y) \wedge z) \vee (\neg\Delta(x \leftrightarrow y) \wedge x)$ is a discriminator term,[8] hence every $G_{n\sim}$-variety is a discriminator variety. Then the result is a consequence of a result of universal algebra (see for instance [14, Theorem 9.4, item (d)]). □

[8] In fact, this is a discriminator term in the whole variety of $G_{\sim}$-algebras. For a definition of discriminator term and discriminator variety see [14].

Thus, since every $G_\sim$-algebra is locally finite, every intermediate logic L between $G^{\leq}_{n\sim}$ and CPL is induced by a family of matrices $\langle \mathbf{A}, F\rangle$ where $\mathbf{A}$ is a finite direct product of subalgebras of $\mathbf{VG_{n\sim}}$ and F is a lattice filter of A compatible[9] with L.

First of all we study the matrix logics defined by $\langle \mathbf{VG_n}, F\rangle$ where F is an order filter of $\mathbf{VG_n}$. Observe that in any of these logics it is possible (mainly since we have Δ operator) to build a propositional formula on n variables $\Phi(p_0, p_1, \ldots, p_{n-1})$ such that, for every evaluation e of formulas on $\mathbf{VG_{n\sim}}$, then

$$e(\Phi(p_0, p_1, \ldots, p_n)) = \begin{cases} 1, & \text{if } e(p_i) = \frac{i}{n-1} \text{ for all } i = 0, 1, \ldots, n-1 \\ 0, & \text{otherwise.} \end{cases}$$

In order to simplify the notation, for every nonempty subset $T \subseteq VG_n$ we denote by $L(\mathcal{M}_T)$ the logic defined by the set of matrices $\mathcal{M}_T = \{\langle \mathbf{VG_{n\sim}}, F_t\rangle : t \in T\}$. Let $L(\mathbf{VG_{n\sim}})$ be the set of logics $L(\mathcal{M}_T)$, for $\emptyset \neq T \subseteq VG_n \setminus \{0\}$.

Proposition 4. *The logics $L(\mathcal{M}_{\{t\}})$, with $t \in VG_n \setminus \{0\}$, are pairwise incomparable. Moreover, $L(\mathcal{M}_T)$ is not comparable to $L(\mathcal{M}_R)$ if $\emptyset \neq T, R \subseteq VG_n \setminus \{0\}$ such that $T \neq R$ and T and R have the same cardinality. In addition, the set $L(\mathbf{VG_{n\sim}})$ is a meet-semilattice where the logics $L(\mathcal{M}_{\{t\}})$, for $t \in VG_n \setminus \{0\}$, are its maximal elements.*

Proof. Let $\vdash_t$ be the consequence relation of the logic $L(\mathcal{M}_{\{t\}})$ defined by the matrix $\langle \mathbf{VG_{n\sim}}, F_t\rangle$, with $t \in VG_n \setminus \{0\}$. Let $i, j \in \{1, 2, \ldots, n-1\}$ be such that $i < j$. Then,

- $\Phi(p_0, p_1, \ldots, p_n) \wedge p_i \vdash_{\frac{j}{n-1}} \bot$ and $\Phi(p_0, p_1, \ldots, p_n) \wedge p_i \nvdash_{\frac{i}{n-1}} \bot$
- $\Phi(p_0, p_1, \ldots, p_n) \wedge p_j \nvdash_{\frac{j}{n-1}} p_i$ and $\Phi(p_0, p_1, \ldots, p_n) \wedge p_j \vdash_{\frac{i}{n-1}} p_i$

Therefore $\vdash_t$ and $\vdash_{t'}$ are not comparable if $0 < t < t' < 1$. From this, it is easy to prove that for any subsets $\emptyset \neq T, R \subseteq VG_n \setminus \{0\}$ with the same cardinality and such that $T \neq R$, the logic $L(\mathcal{M}_T)$ is not comparable to $L(\mathcal{M}_R)$. Finally, if $\emptyset \neq T, R \subseteq VG_n \setminus \{0\}$ then $L(\mathcal{M}_T) \cap L(\mathcal{M}_R) = L(\mathcal{M}_{T\cup R})$. Hence $L(\mathbf{VG_{n\sim}})$ is a meet-semilattice such that the maximal elements are exactly the logics $L(\mathcal{M}_{\{t\}})$, for $t \in VG_n \setminus \{0\}$. □

On the other hand, as it was done in [19] and [20] for Łukasiewicz finite-valued logics, matrix logics of the form $\langle (\mathbf{VG_{n\sim}})^k, \Pi_{1\leq t_1<t_2<\ldots<t_k} F_{t_i}\rangle$ can be considered, and all the results obtained there are also valid in this case:

Definition 1. Given a nonempty set $T \subseteq VG_n \setminus \{0\}$, $T = \{t_1, \ldots, t_k\}$ (where $t_i < t_j$ if $i < j$), we will denote by $\mathbb{L}(T)$ the matrix logic $\langle (\mathbf{VG_{n\sim}})^k, \Pi_{i\in R} F_{t_i}\rangle$ defined on the direct products of $\mathbf{VG_{n\sim}}$ by means of order filters.

[9] A filter F of an algebra $\mathbf{A}$ is *compatible* with a logic L if, whenever $\Gamma \vdash_L \varphi$, the following holds: for every $\mathbf{A}$-evaluation e, if $e(\gamma) \in F$ for every $\gamma \in \Gamma$ then $e(\varphi) \in F$.

The results become different when studying the matrices logics that involve components over finite subalgebras belonging to the variety generated by $\mathbf{VG_{n\sim}}$ because even though all of them are direct product of subalgebras of $\mathbf{VG_{n\sim}}$, the number of subalgebras of $\mathbf{VG_{n\sim}}$ is significantly larger than in the Łukasiewicz case. Indeed:

- Subalgebras of $\mathbf{VG_{n\sim}}$ are those chains that can be obtained from VG_n by removing either the fix point for $\sim$ (if it exists, i.e. if n is odd) or a set of pairs of elements of the form a_i and $\sim a_i$ (with $a_i \notin \{0, 1/2, 1\}$).
- Therefore the logics between $\mathrm{G}^{\leq}_{n\sim}$ and CPL are those logics defined by matrices over direct products of subalgebras of $\mathbf{VG_{n\sim}}$ and with products of order filters on the corresponding components of the product algebra. Of course, we have to avoid the repetition of components in these products.

Remark 1. Let $L = \langle \Pi_{i\in I}\mathbf{A_i}, \Pi_{i\in I}F_i\rangle$ be a matrix logic over a direct product of subalgebras $\mathbf{A_i}$ of $\mathbf{VG_{n\sim}}$ defined by a product of order filters F_i in $\mathbf{A_i}$. Then, L is the product $\Pi_{i\in I}L_i$ of the matrix logics $L_i = \langle \mathbf{A_i}, F_i\rangle$, where $\Gamma \vdash_L \varphi$ iff, for every evaluation e_i over $\mathbf{A_i}$ (for $i \in I$): if $e_i(\psi) \in F_i$ for every $i \in I$ and every $\psi \in \Gamma$, then $e_i(\varphi) \in F_i$ for every $i \in I$.

Obviously, a matrix logic L as above is paraconsistent iff all the components L_i are paraconsistent. For example, if one component is $\langle \mathbf{VG_{2\sim}}, F_1\rangle$, then the matrix logic is not paraconsistent.

Example 1. Since $\mathbf{VG_3}$ and $\mathbf{VG_4}$ are subalgebras of $\mathbf{VG_5}$, by the characterization of all extensions of $\mathrm{G}^{\leq}_{n\sim}$ we have that $\mathsf{J}_3 \times \mathsf{J}_4 := \langle \mathbf{VG_3} \times \mathbf{VG_4}, F_{\frac{1}{2}} \times F_{\frac{1}{3}}\rangle$ is a paraconsistent extension of $\mathrm{G}^{\leq}_{5\sim}$ that is comparable neither to J_3 nor to J_4. Indeed, it is immediate to see that $\vdash_{\mathsf{J}_3\times\mathsf{J}_4} \varphi$ iff $\vdash_{\mathsf{J}_3} \varphi$ and $\vdash_{\mathsf{J}_4} \varphi$ for every formula φ. Thus, since theorems of J_3 and theorems of J_4 are not comparable, $\mathsf{J}_3 \times \mathsf{J}_4$ is not an extension of J_3 nor J_4. Moreover it is easy to check that $\Delta(\varphi \to \sim\varphi) \vdash_{\mathsf{J}_3\times\mathsf{J}_4} \bot$ and $\Delta(\varphi \to \sim\varphi) \wedge \neg\Delta(\sim\varphi \to \varphi) \vdash_{\mathsf{J}_3\times\mathsf{J}_4} \bot$, while $\Delta(\varphi \to \sim\varphi) \nvdash_{\mathsf{J}_3} \bot$ and $\Delta(\varphi \to \sim\varphi) \wedge \neg\Delta(\sim\varphi \to \varphi) \nvdash_{\mathsf{J}_4} \bot$. Thus, $\mathsf{J}_3 \times \mathsf{J}_4$, J_3 and J_4 are mutually not comparable.

Finally we can characterize the logics satisfying the explosion rule:

- The minimal matrix logic satisfying the explosion rule (like in the Łukasiewicz case [19]) is the logic $L_{exp} = L(\mathcal{M}_n)$ defined by the following set of matrices:

  $$\mathcal{M}_n = \{\langle \mathbf{VG_{n\sim}}, F_1\rangle, \langle \mathbf{VG_{n\sim}}, F_{\frac{n-2}{n-1}}\rangle, \ldots, \langle \mathbf{VG_{n\sim}}, F_{\frac{i}{n-1}}\rangle, \langle (\mathbf{VG_{n\sim}})^{n-i}, \Pi_{r=i}^{n-1} F_{\frac{r}{n-1}}\rangle\}$$

  where i is the first natural such that $\frac{i}{n-1} > 1/2$.
- Therefore, the explosion rule is valid in all the logics extending the logic L_{exp}, while those that do not contain it are paraconsistent.

4.2 Example: the case $n = 5$

As an example we study the case of the set $Int(\mathrm{G}^{\leq}_{5\sim})$ of matrix logics defining intermediate logics between $\mathrm{G}^{\leq}_{5\sim}$ and CPL. Recall that VG_5 denote the ordered set $\{0, 1/4, 1/2, 3/4, 1\}$. We start with some basic facts:

- Consider the subset $L(\mathbf{VG_{5\sim}}) \subset Int(\mathrm{G}^{\leq}_{5\sim})$ of logics defined by the set of matrices $\mathcal{M}_T = \{\langle \mathbf{VG_{5\sim}}, F_t\rangle : t \in T\}$ for $\emptyset \neq T \subseteq VG_5 \setminus \{0\}$, as it was done in Subsection 4.1. According to Proposition 4, the logics of the matrices $\langle \mathbf{VG_{5\sim}}, F_{i/4}\rangle$ for $i \in \{1, 2, 3, 4\}$ are pairwise incomparable, and in fact they are the maximal logics in $L(\mathbf{VG_{5\sim}})$, while $\bigcap_{i\in\{1,2,3,4\}}\langle \mathbf{VG_{5\sim}}, F_{i/4}\rangle = \mathrm{G}^{\leq}_{5\sim}$ is the minimum logic of $L(\mathbf{VG_{5\sim}})$ (and clearly of $Int(\mathrm{G}^{\leq}_{5\sim})$ as well).
- Let $L\Pi(\mathrm{G}_{5\sim}) \subset Int(\mathrm{G}^{\leq}_{5\sim})$ be the set of matrix logics of the form $\mathbb{L}(T)$ defined on direct products of $\mathbf{VG_{5\sim}}$ by means of products of order filters (recall Definition 1). Then, these logics satisfy the following conditions (like in the case of Łukasiewicz logics):
  - If $\emptyset \neq T, S \subseteq VG_5 \setminus \{0\}$ are such that max $T =$ max R, then $\mathbb{L}(T) \cap \mathbb{L}(R) = \mathbb{L}(T \cup R)$.
  - The maximal elements of $L\Pi(\mathrm{G}_{5\sim})$ are the matrix logics of the type $\langle (\mathbf{VG_{5\sim}})^2, F_{i/4} \times F_{j/4}\rangle$ with $i, j \in \{1, 2, 3, 4\}$ and $i < j$.
  - The matrix logic $\langle (\mathbf{VG_{5\sim}})^2, F_i \times F_j\rangle$ for $0 < i < j$ contains $\langle \mathbf{VG_{5\sim}}, F_j\rangle$ and it is not comparable with $\langle \mathbf{VG_{5\sim}}, F_k\rangle$ for $0 < k \neq j$.
- Finally let us consider the subset $L\Pi^*(\mathrm{G}_{5\sim}) \subset Int(\mathrm{G}^{\leq}_{5\sim})$ of matrix logics defined on direct products of $\mathbf{VG_{5\sim}}$ and their subalgebras together with direct products of order filters. The subalgebras of $\mathbf{VG_{5\sim}}$ are (isomorphic to) $\mathbf{VG_{2\sim}}$, $\mathbf{VG_{3\sim}}$ and $\mathbf{VG_{4\sim}}$, and thus the number of matrix logics in $L\Pi^*(\mathrm{G}_{5\sim})$ proliferate in a large number. Namely, to define matrix logics we have the following components to combine: 4 algebras, $\mathbf{VG_{5\sim}}$, $\mathbf{VG_{4\sim}}$, $\mathbf{VG_{3\sim}}$ and $\mathbf{VG_{2\sim}}$, and 10 order filters: 4 over $\mathbf{VG_{5\sim}}$, 3 over $\mathbf{VG_{4\sim}}$, 2 over $\mathbf{VG_{3\sim}}$ and 1 over $\mathbf{VG_{2\sim}}$. Therefore we have all the possible products (without repetitions) of these 10 components.

We can also characterize the minimal extension of $\mathrm{G}^{\leq}_{5\sim}$ with the explosion rule as the logic $L(\mathcal{M}_5)$ of the set of matrices

$$\mathcal{M}_5 = \{\langle \mathbf{VG_5}, F_1\rangle, \langle (\mathbf{VG_5})^3, F_{3/4} \times F_{2/4} \times F_{1/4}\rangle\}.$$

Concerning axiomatization, as in case of Łukasiewicz logics, we can give an axiomatic characterization of the logics of $L\Pi(\mathrm{G}_{5\sim})$. To see this, first of all, observe that in $\mathrm{G}_{5\sim}$, for every value $i/4 \in VG_5 \setminus \{0\}$ there exists a formula in one variable $\varphi(p)$ characterizing the value $i/4$, i.e. such that for any evaluation e, $e(\varphi(p)) = 1$ if $e(p) = i/4$, and 0 otherwise. For example, for the value 1/2 the formula can be $\Delta(p \leftrightarrow \sim p)$. It is also possible to define a formula characterizing the sets of values $\geq i/4, > i/4, \leq i/4$ and $< i/4$.

Using this observation, it is easy to see that every matrix logic of type $\langle \mathbf{VG_5}, F_i \rangle$ or $\mathbb{L}(T) \in L\Pi(\mathrm{G}_{5\sim})$ can be axiomatized. For instance, here we give the following example:

- The matrix logic $(\mathbf{VG_5}, F_{i/4})$ is axiomatized by adding to the axioms and rules of $\mathrm{G}_{5\sim}^{\leq}$ the following restricted inference rule:

  if $\vdash_{\mathrm{G}_{5\sim}} (\varphi < i/4) \vee ((\varphi \geq i/4) \wedge (\psi \geq i/4)$, from φ derive ψ

Other matrix logics of $L\Pi(\mathrm{G}_{5\sim})$ can be axiomatized in an analogous way. Notice that these axiomatizations are possible since, in $\mathrm{G}_{5\sim}$, for every element $a \in VG_5$ there exists a characterizing formula in one variable. This is not true in $\mathrm{G}_{n\sim}$ for $n > 5$, and thus the previous axiomatization results are not generalizable to $\mathrm{G}_{n\sim}$ for $n > 5$.

5 Ideal and saturated paraconsistent extensions of $\mathrm{G}_{n\sim}^{\leq}$

As already noticed, matrix logics over direct products of subalgebras of $\mathbf{VG_{n\sim}}$ with products of order filters are $\sim$-paraconsistent iff all the components are $\sim$-paraconsistent. In this section, using the results of the previous section, we study the status of the logics between $\mathrm{G}_{n\sim}^{\leq}$ and CPL in relation to ideal $\sim$-paraconsistency. Namely, we show that there are only two extensions of $\mathrm{G}_{n\sim}^{\leq}$ which are ideal $\sim$-paraconsistent. Moreover we show that there is another $\sim$-paraconsistent extension of $\mathrm{G}_{n\sim}^{\leq}$ which, although not being ideal $\sim$-paraconsistent, it has the remarkable property of not having any proper $\sim$-paraconsistent extension.

We have already briefly discussed in the Introduction the concept of *ideal paraconsistent logics*, introduced by Arieli et al. in [4].[10] We recall here this notion.

Definition 2 (c.f. [4])**.** Let L be a propositional logic defined over a signature Θ (with consequence relation $\vdash_L$) containing at least a unary connective $\urcorner$ and a binary connective $\rightarrow$ such that:

(i) L is paraconsistent w.r.t. $\urcorner$ (or simply $\urcorner$-paraconsistent), that is, there are formulas $\varphi, \psi \in \mathcal{L}(\Theta)$ such that $\varphi, \urcorner\varphi, \nvdash_L \psi$;

[10]The authors, as it was mentioned in Section 1, have changed the terminology "ideal paraconsistent logic" to "fully maximal and normal paraconsistent logic". However, it should be noticed that being normal, according to [5, Definition 1.32], means that the logic L has, besides a deductive implication, a conjunction and a disjunction satisfying the usual properties in terms of consequence relations. Here we decide to keep the original definition of ideal paraconsistency. It is worth noting that all the ideal (and saturated) logics considered in this paper and in [20] are normal in the sense of [5].

(ii) $\to$ is an implication for which the deduction-detachment theorem holds in L, that is, $\Gamma \cup \{\varphi\} \vdash_L \psi$ iff $\Gamma \vdash_L \varphi \to \psi$, for every set for formulas $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}(\Theta)$.

(iii) There is a presentation of CPL as a matrix logic $L' = \langle \mathbf{A}, \{1\}\rangle$ over the signature Θ such that the domain of $\mathbf{A}$ is $\{0, 1\}$, and $\urcorner$ and $\to$ are interpreted as the usual 2-valued negation and implication of CPL, respectively.

(iv) L is a sublogic of CPL in the sense that $\vdash_L \subseteq \vdash_{L'}$, that is, $\Gamma \vdash_L \varphi$ implies $\Gamma \vdash_{L'} \varphi$, for every set for formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(\Theta)$.

Then, L is said to be an *ideal* $\urcorner$*-paraconsistent logic* if it is maximal w.r.t. CPL, and every proper extension of L over Θ is not $\urcorner$-paraconsistent.

An implication connective satisfying the above condition (ii) is usually called *deductive implication*.

Remark 2. In Proposition2 it was stated that J_3 is equivalent to $\langle \mathbf{VG}_{3\sim}, F_{\frac{1}{2}}\rangle$ and therefore for every odd number $n \geq 3$, J_3 is an extension of any $\mathrm{G}^{\leq}_{n\sim}$. Similarly, J_4 is equivalent to $\langle \mathbf{VG}_{4\sim}, F_{\frac{1}{3}}\rangle$. Thus J_4 is an extension of $\mathrm{G}^{\leq}_{n\sim}$ for every $n \geq 4$. In [20, Proposition 6.3] it is shown that J_3 and J_4 are ideal $\sim$-paraconsistent logics where the deductive implication in the signature of $G_\sim$ is the term-defined implication $x \Rightarrow y := \neg x \vee y$.[11]

As discussed in Section 1, requiring to a paraconsistent logic to be maximal w.r.t. CPL in order to be 'ideal' (in the sense of being 'optimal') is a debatable issue (see [30]). On the other hand, the other requirements of Definition 2 seem interesting, and a system enjoying such features could be considered as a remarkable paraconsistent logic. This motivates the following definition.

Definition 3. A logic L is *saturated* $\urcorner$-paraconsistent if it satisfies all the conditions (i) to (iv) of the previous definition, and every proper extension of L over Θ is not $\urcorner$-paraconsistent.[12]

Remark 3. In [29, p. 273] it was introduced the notion of maximality of a logic L w.r.t. an inference rule r. Namely, given a Tarskian and structural propositional logic L over a signature Θ, and given an inference rule r over Θ, L is r*-maximal* if r is not derivable in L, but any proper extension of L over Θ derives r.[13] Clearly ideal and saturated paraconsistent logics are special cases of r-maximal logics, where r is the explosion rule.[14]

[11] Observe that in [20], $\neg$ denotes the Łukasiewicz negation, while Gödel negation for J_3 and J_4 is respectively denoted by $\sim^{1}_{2}$ and $\sim^{1}_{3}$.

[12] Using the terminology of [5], a saturated paraconsistent logic is a logic such that: (i) it has an implication, (ii) it is **F**-contained en CPL, and (iii) it is strongly maximal.

[13] This was denoted by $L \in \mathbf{Triv}_\Theta \bot \{r\}$ in [29], where $\mathbf{Triv}_\Theta$ denotes the trivial logic over the signature Θ.

[14] Indeed, by means of the notion of *reminder set* $L \bot R$ of a logic L w.r.t. a set of rules R introduced in [29, Definition 7], several concepts relative to maximality proposed in the literature can be easily represented, see [29, p. 273].

Proposition 5. $\mathsf{J}_3 \times \mathsf{J}_4 := \langle \mathbf{VG_{3\sim}} \times \mathbf{VG_{4\sim}}, F_{\frac{1}{2}} \times F_{\frac{1}{3}} \rangle$ *is saturated* $\sim$*-paraconsistent, but not ideal* $\sim$*-paraconsistent.*

Proof. Since $\mathbf{VG_{3\sim}}$ and $V\mathbf{G_{4\sim}}$ are subalgebras of $\mathbf{VG_{5\sim}}$, by the characterization of all extensions of $\mathrm{G}^{\leq}_{n\sim}$ given in subsection 4.1, $\langle \mathbf{VG_{3\sim}} \times \mathbf{VG_{4\sim}}, F_{\frac{1}{2}} \times F_{\frac{1}{3}} \rangle$ is an extension of $\mathrm{G}^{\leq}_{5\sim}$ satisfying conditions (i) to (iv) because every component is $\sim$-paraconsistent and $x \Rightarrow y := \neg x \vee y$ is a term-defined deductive implication. We prove by contradiction that $\mathsf{J}_3 \times \mathsf{J}_4$ has no proper $\sim$-paraconsistent extensions. Assume there is a proper $\sim$-paraconsistent extension L of $\mathsf{J}_3 \times \mathsf{J}_4$. In this case there is a matrix $\langle \mathbf{A_1} \times \cdots \times \mathbf{A_k}, F_{i_1} \times \cdots \times F_{i_k} \rangle$ which is an extension of L such that each $\langle \mathbf{A_j}, F_{i_j} \rangle$ is either J_3, J_4, $\langle \mathbf{VG_{5\sim}}, F_{\frac{1}{2}} \rangle$ or $\langle \mathbf{VG_{5\sim}}, F_{\frac{1}{4}} \rangle$. Since J_3 is not comparable with $\mathsf{J}_3 \times \mathsf{J}_4$ and J_3 is a submatrix of $\langle \mathbf{VG_5}, F_{\frac{1}{2}} \rangle$ and also a submatrix of $\langle \mathbf{VG_{5\sim}}, F_{\frac{1}{4}} \rangle$, there is a component $\langle \mathbf{A_{j0}}, F_{j0} \rangle = \mathsf{J}_4$. Similarly, there should be a different component $\langle \mathbf{A_{j1}}, F_{j1} \rangle \neq \mathsf{J}_4$, otherwise J_4 would be an extension of $\mathsf{J}_3 \times \mathsf{J}_4$. Finally, in the case $\langle \mathbf{A_1} \times \cdots \times \mathbf{A_k}, F_{i_1} \times \cdots \times F_{i_k} \rangle$ has a component equal to J_4 and another which is different to J_4, then $\mathsf{J}_3 \times \mathsf{J}_4$ is a submatrix of $\langle \mathbf{A_1} \times \cdots \times \mathbf{A_k}, F_{i_1} \times \cdots \times F_{i_k} \rangle$, which contradicts the fact that L is a proper extension of $\mathsf{J}_3 \times \mathsf{J}_4$.

Let φ a theorem of J_3 which is not a theorem of J_4. Then, the matrix logic $J_2 \times \mathsf{J}_3 := \langle \mathbf{VG_{2\sim}} \times \mathbf{VG_{3\sim}}, F_1 \times F_{\frac{1}{2}} \rangle$ is an extension of $\mathsf{J}_3 \times \mathsf{J}_4$ different of CPL such that $\vdash_{J_2 \times \mathsf{J}_3} \varphi$. Thus $\mathsf{J}_3 \times \mathsf{J}_4$ is not maximal w.r.t. CPL. □

Theorem 1. *Let n be an integer number such that $n > 4$ and let L be an extension of $\mathrm{G}^{\leq}_{n\sim}$.*

1. *If n is an even number, then L is saturated $\sim$-paraconsistent iff L is ideal $\sim$-paraconsistent iff $L = \mathsf{J}_4$*
2. *If n is an odd number, then L is saturated $\sim$-paraconsistent iff $L = \mathsf{J}_3$, $L = \mathsf{J}_4$ or $L = \mathsf{J}_3 \times \mathsf{J}_4$.*
3. *If n is an odd number, then L is ideal $\sim$-paraconsistent iff $L = \mathsf{J}_3$ or $L = \mathsf{J}_4$.*

Proof.
1. Assume that n is even. After Remark 2 and Proposition 5 we only need to prove that if L is saturated $\sim$-paraconsistent then $L = \mathsf{J}_4$. Since n is even then, as observed in Subsection 4.1, every extension L' of $\mathrm{G}^{\leq}_{n\sim}$ is induced by a family of matrices of the form $\langle \mathbf{A}, F \rangle = \langle \mathbf{VG_{n_1\sim}} \times \cdots \times \mathbf{VG_{n_k\sim}}, F_{\frac{i_1}{n_1-1}} \times \cdots \times F_{\frac{i_k}{n_k-1}} \rangle$ where each n_j is also an even number. If L' is $\sim$-paraconsistent then there is a member in that family, say $\langle \mathbf{VG_{n_1\sim}} \times \cdots \times \mathbf{VG_{n_k\sim}}, F_{\frac{i_1}{n_1-1}} \times \cdots \times F_{\frac{i_k}{n_k-1}} \rangle$, such that $2 < n_j \leq n$ and $0 < \frac{i_j}{n_j-1} \leq \frac{1}{2}$ for every j such that $1 \leq j \leq k$. Then, J_4 is an extension of every $\sim$-paraconsistent extension of $\mathrm{G}^{\leq}_{n\sim}$. In particular, J_4 extends L. Thus $L = \mathsf{J}_4$, since L is maximal paraconsistent.
2. Right to left implicacion follows from Remark 2 and Proposition 5. To prove the converse, let L be an saturated $\sim$-paraconsistent extension of

$G^{\leq}_{n\sim}$. Since L is $\sim$–paraconsistent and it has no proper $\sim$–paraconsistent extension, L is induced by a single $\sim$–paraconsistent matrix $\langle \mathbf{A}, F\rangle$ such that $\langle \mathbf{A}, F\rangle = \langle \mathbf{VG_{n_1\sim}} \times \cdots \times \mathbf{VG_{n_k\sim}}, F_{\frac{i_1}{n_1-1}} \times \cdots \times F_{\frac{i_k}{n_k-1}}\rangle$ where $2 < n_j \leq n$ and $0 < \frac{i_j}{n_j-1} \leq \frac{1}{2}$ for every j such that $1 \leq j \leq k$. If all n_j's are even, as in previous item $L = \mathsf{J}_4$. If all n_j's are odd, then J_3 is a $\sim$–paraconsistent extension of L, thus $L = \mathsf{J}_3$. Assume n is odd and some n_j's are even and some are odd, all of them bigger than 2. Then in that case $\mathsf{J}_3 \times \mathsf{J}_4 := \langle \mathbf{VG_{3\sim}} \times \mathbf{VG_{4\sim}}, F_{\frac{1}{2}} \times F_{\frac{1}{3}}\rangle$ is a $\sim$–paraconsistent extension of L, thus $L = \mathsf{J}_3 \times \mathsf{J}_4$.

3. Immediate after Proposition 5 and item 2.

□

6 Saturated paraconsistency and finite-valued Łukasiewicz logics

In [20] the authors study maximality conditions for intermediate logics between CPL and the degree-preserving finite-valued Łukasiewicz logics. In particular we have characterized the ideal paraconsistent logics in this family. Since in the last section we have introduced the weaker notion of saturated paraconsistency in the setting of degree-preserving Gödel logics with involution, we deem interesting to also explore this notion for the above mentioned setting of finite-valued Łukasiewicz logics. This is done in this section, after briefly recalling some basic notions about (degree-preserving) finite-valued Łukasiewicz logics.

The $(n + 1)$-valued Łukasiewicz logic can be semantically defined as the matrix logic

$$Ł_{n+1} = \langle \mathbf{ŁV}_{n+1}, \{1\}\rangle,$$

where $\mathbf{ŁV}_{n+1} = (ŁV_{n+1}, \neg, \rightarrow)$ is the $n + 1$-elements MV-chain with $ŁV_{n+1} = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$, and operations defined as follows: for every $x, y \in ŁV_{n+1}$,

$\neg x = 1 - x$ (Łukasiewicz negation)

$x \rightarrow y = \min\{1, 1 - x + y\}$ (Łukasiewicz implication)

In fact $Ł_{n+1}$ is algebraizable and its generated quasi-variety $MV_{n+1} := \mathcal{Q}(\mathbf{ŁV}_{n+1})$ is its equivalent algebraic semantics.

The (finitary) degree preserving $(n + 1)$-valued Łukasiewicz logic, denoted $Ł^{\leq}_{n+1}$, can be semantically defined the following way: For every finite set of formulas $\Gamma \cup \{\varphi\}$

$\Gamma \models_{Ł^{\leq}_{n+1}} \varphi$ if for every evaluation e over $\mathbf{ŁV}_{n+1}$ and every $a \in ŁV_{n+1}$, if $e(\gamma) \geq a$ for every $\gamma \in \Gamma$, then $e(\gamma) \geq a$.

Following [20] we denote by L_n^i the logic obtained by the matrix $\langle \mathbf{ŁV}_{n+1}, F_{\frac{i}{n}} \rangle$, where $F_{\frac{i}{n}}$ is the order filter $\{x \in ŁV_{n+1} : x \geq i/n\}$. Notice that with this notation the $n+1$-valued Łukasiewicz logic $Ł_{n+1}$ is also denoted by L_n^n.

As proved in [20, Theorem 5.2], for every $1 \leq i \leq n$, L_n^i is equivalent, as a deductive system, to $Ł_{n+1}$ (see [11] and also [12]). Since algebraizability is preserved by equivalence, L_n^i is algebraizable and MV_{n+1} is also its equivalent algebraic semantics. Thus, the lattice of finitary extensions of L_n^i is isomorphic to the lattice of subquasivarieties of $MV_{n+1} = \mathcal{Q}(\mathbf{ŁV}_{n+1})$.

Using this correspondence and some results on quasivarieties of MV_{n+1} (see [26] and [20]) we obtain, in analogy to [19, Theorem 3], that every extension L_n^i is induced by a finite family of matrices $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is a critical[15] MV_{n+1}-algebra and F is a lattice filter of $\mathbf{A}$ compatible with L. In fact, $\mathbf{A}$ is isomorphic to a direct product of MV_{n+1}-chains $\mathbf{ŁV}_{n_0+1} \times \cdots \times \mathbf{ŁV}_{n_{l-1}+1}$, where

1. for every $j < l$, $n_j | n$
2. For every $j, k < l$, $k \neq j$ implies $n_k \neq n_j$.
3. If there exists n_j, $j < l$ such that $n_k | n_j$ for some $k \neq j$, then n_j is unique.

and $F = (F_{\frac{i}{n}})^l \cap (ŁV_{n_0+1} \times \cdots \times ŁV_{n_{l-1}+1})$.

As mentioned in Subsection 2, $\mathbf{ŁV}_3$ is polinomially equivalent to $\mathbf{VG}_{3\sim}$ and $\mathbf{ŁV}_4$ is polinomially equivalent to $\mathbf{VG}_{4\sim}$, where the involutive negation $\sim$ in $\mathbf{VG}_{3\sim}$ and $\mathbf{VG}_{4\sim}$ is in fact the MV-negation $\neg$. Then, as mentioned in Remark 2, the matrix logics $\mathsf{J}_3 = \langle \mathbf{ŁV}_3, F_{\frac{1}{2}} \rangle$ and $\mathsf{J}_4 = \langle \mathbf{ŁV}_4, F_{\frac{1}{3}} \rangle$, expressed in the signature of Łukasiewicz logic, are ideal $\neg$-paraconsistent. We recall here that this can be generalized in the following way.

Proposition 6. [20, Proposition 6.2] *Let q be a prime number, and let $1 \leq i < q$ such that $i/q \leq 1/2$. Then, L_q^i is a $(q+1)$-valued ideal $\neg$-paraconsistent logic.*

In fact, we can also prove that the converse implication also holds under some circumstances. This result is not present in [20].

Theorem 2. *Let $0 < i < n$ such that $\frac{i}{n} \leq \frac{1}{2}$. If L is an extension of L_n^i, then, L is ideal $\neg$-paraconsistent iff $L = \mathsf{L}_q^j$ for some prime number q such that $q|n$ and some $1 \leq j$ such that $j/q \leq 1/2$*

Proof. Let L be an ideal $\neg$-paraconsistent extension of L_n^i. Since L is maximal, it is induced by a single matrix $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is critical and F is compatible with L. In fact, as mentioned above, $\langle \mathbf{A}, F \rangle$ is of type $\langle \mathbf{ŁV}_{n_1+1} \times \cdots \times \mathbf{ŁV}_{n_k+1}, (F_{\frac{i}{n}})^k \cap (ŁV_{n_1+1} \times \cdots \times ŁV_{n_k+1}) \rangle$ where

1. for every $1 \leq i \leq k$, $n_i | n$
2. For every $1 \leq i, j \leq k$, $i \neq j$ implies $n_i \neq n_j$.

[15] An algebra is said to be *critical* if it is a finite algebra not belonging to the quasivariety generated by all its proper subalgebras.

3. If there exists n_j, $1 \leq j \leq k$ such that $n_i | n_j$ for some $1 \leq i \neq j$, then n_j is unique.

Since L is $\neg$-paraconsistent, none of the components $\mathbf{ŁV}_{n_i+1}$ can be $\mathbf{ŁV}_2$ (otherwise L would be explosive), and hence $1 < n_i$ for all $1 \leq i \leq k$. If $k > 1$, then

- If there is n_j, with $1 \leq j \leq k$, such that $n_i | n_j$ for some $1 \leq i \neq j$, then without loss of generality assume that $j = k$. In that case $\langle \mathbf{ŁV}_{n_1+1} \times \cdots \times \mathbf{ŁV}_{n_{k-1}+1}, (F_{\frac{i}{n}})^{k-1} \cap (ŁV_{n_1+1} \times \cdots \times ŁV_{n_{k-1}+1}) \rangle$ is a $\neg$-paraconsistent extension of L which contradicts the assumption of L being ideal $\neg$-paraconsistent.
- If there is no n_j, with $1 \leq j \leq k$, such that $n_i | n_j$ for some $1 \leq i \neq j$, then $n_k \neq n$ and L is not maximal because $\langle \mathbf{ŁV}_2 \times \mathbf{ŁV}_{n_k+1}, (F_{\frac{i}{n}})^2 \cap (ŁV_2 \times ŁV_{n_k+1}) \rangle$ is an extension of L different from CPL and there is a formula φ valid in $\langle \mathbf{ŁV}_2 \times \mathbf{ŁV}_{n_k+1}, (F_{\frac{i}{n}})^2 \cap (ŁV_2 \times ŁV_{n_k+1}) \rangle$ and not valid in L. A contradiction again.

Thus $k = 1$. In that case n should be prime because otherwise for any prime number p such that $p|n$, $\langle \mathbf{ŁV}_{p+1}, F_{\frac{i}{n}} \cap ŁV_{p+1} \rangle$ would be an extension of L different from CPL and there is a formula φ valid in $\langle \mathbf{ŁV}_{p+1}, F_{\frac{i}{n}} \cap ŁV_{p+1} \rangle$ and not valid in L. □

As regards saturated paraconsistency we have the following results:

Theorem 3. *Let $0 < i < n$ such that $\frac{i}{n} \leq \frac{1}{2}$. Let*

$$X = \big\{ p \ : \ p \text{ prime}, \ p|n, \ F_{\frac{i}{n}} \cap ŁV_{p+1} = \big\{ \frac{m}{p} \ : \ m \geq k \big\} \text{ and } \frac{k}{p} \leq \frac{1}{2} \big\}.$$

For every finite subset $\{p_1, \ldots, p_j\} \subseteq X$, the logic defined by the matrix

$$\langle \mathbf{ŁV}_{p_1+1} \times \cdots \times \mathbf{ŁV}_{p_j+1}, (F_{\frac{i}{n}})^j \cap (ŁV_{p_1+1} \times \cdots \times ŁV_{p_j+1}) \rangle$$

is saturated $\neg$-paraconsistent.

Proof. By the previous result,$\langle \mathbf{ŁV}_{p_1+1} \times \cdots \times \mathbf{ŁV}_{p_j+1}, (F_{\frac{i}{n}})^j \cap (ŁV_{p_1+1} \times \cdots \times ŁV_{p_j+1}) \rangle$ is an extension of L^i_n. Moreover, it is $\neg$-paraconsistent, because every component is $\neg$-paraconsistent. Let $\Rightarrow^i_n$ defined as $\varphi \Rightarrow^i_n \psi := \sim^i_n \varphi \vee \psi$ where $\sim^i_n(x)$ is the single variable McNaughton term such that for every $a \in ŁV_{n+1}$,

$$\sim^i_n(a) = \begin{cases} 0, & \text{if } a \geq \frac{i}{n} \\ 1, & \text{otherwise.} \end{cases}$$

Similarly to the proof of [20, Proposition 6.2], the logic $\langle \mathbf{ŁV}_{p_1+1} \times \cdots \times \mathbf{ŁV}_{p_j+1}, (F_{\frac{i}{n}})^j \cap (ŁV_{p_1+1} \times \cdots \times ŁV_{p_j+1}) \rangle$ satisfies conditions (i) to (iv) in Definition 2, the definition of ideal $\neg$-paraconsistency. Let L be a $\neg$-paraconsistent extension of $\langle \mathbf{ŁV}_{p_1+1} \times \cdots \times \mathbf{ŁV}_{p_j+1}, (F_{\frac{i}{n}})^j \cap (ŁV_{p_1+1} \times \cdots \times ŁV_{p_j+1}) \rangle$, then L is induced by a finite family of matrices $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is critical and F is compatible with L. Since L is $\neg$-paraconsistent, there is at least one matrix $\langle \mathbf{ŁV}_{n_0+1} \times \cdots \times \mathbf{ŁV}_{n_{l-1}+1}, (F_{\frac{i}{n}})^l \cap (ŁV_{n_0+1} \times \cdots \times ŁV_{n_{l-1}+1}) \rangle$ where

1. for every $m < l$, $n_m | n$
2. for every $m, k < l$, $k \neq m$ implies $n_k \neq n_m$
3. if there exists n_m with $m < l$ such that $n_m | n_k$ for some $k \neq m$, then n_k is unique,

which is $\neg$-paraconsistent. Thus for every $m < l$, it is the case that $2 \leq n_m$. Since $\langle \mathbf{ŁV}_{n_0+1} \times \cdots \times \mathbf{ŁV}_{n_{l-1}+1}, (F_{\frac{i}{n}})^l \cap (ŁV_{n_0+1} \times \cdots \times ŁV_{n_{l-1}+1}) \rangle$ is an extension of $\langle \mathbf{ŁV}_{p_1+1} \times \cdots \times \mathbf{ŁV}_{p_j+1}, (F_{\frac{i}{n}})^j \cap (ŁV_{p_1+1} \times \cdots \times ŁV_{p_j+1}) \rangle$, then $\langle \mathbf{ŁV}_{n_0+1} \times \cdots \times \mathbf{ŁV}_{n_{l-1}+1}, (F_{\frac{i}{n}})^l \cap (ŁV_{n_0+1} \times \cdots \times ŁV_{n_{l-1}+1}) \rangle$ is a submatrix of $\langle \mathbf{ŁV}_{p_1+1} \times \cdots \times \mathbf{ŁV}_{p_j+1}, (F_{\frac{i}{n}})^j \cap (ŁV_{p_1+1} \times \cdots \times ŁV_{p_j+1}) \rangle$. Therefore, by [20, Lemma 5.6], for every $m < l$ there is a $0 < k \leq j$ such that $n_m | p_k$, since $2 \leq n_m$ and p_k is prime, then $n_m = p_k$. Moreover for every $0 < k \leq j$, there is $m < l$ such that $n_m | p_k$. Thus $\mathbf{ŁV}_{p_1+1} \times \cdots \times \mathbf{ŁV}_{p_j+1} \cong \mathbf{ŁV}_{n_0+1} \times \cdots \times \mathbf{ŁV}_{n_{l-1}+1}$ and $L = \langle \mathbf{ŁV}_{p_1+1} \times \cdots \times \mathbf{ŁV}_{p_j+1}, (F_{\frac{i}{n}})^j \cap (ŁV_{p_1+1} \times \cdots \times ŁV_{p_j+1}) \rangle$, proving that any proper extension of $\langle \mathbf{ŁV}_{p_1+1} \times \cdots \times \mathbf{ŁV}_{p_j+1}, (F_{\frac{i}{n}})^j \cap (ŁV_{p_1+1} \times \cdots \times ŁV_{p_j+1}) \rangle$ is not $\neg$-paraconsistent. □

Remark 4. One may wonder whether the saturated $\neg$-paraconsistent logics identified in the above theorem are in fact ideal paraconsistent. However, it is easy to see that this is not the case unless they are of the type describe in Theorem 2. Indeed, this is a consequence of the fact that the logics considered in Theorem 3 (and in Corollary 2 below) are extensions of logics of the type L_n^i, and in Theorem 2 we have exactly characterized which of these extensions are ideal paraconsistent.

Corollary 2. *Let $\{p_1, \ldots, p_j\}$ be any finite set of prime numbers, then $\langle \mathbf{ŁV}_{p_1+1} \times \cdots \times \mathbf{ŁV}_{p_j+1}, F_{\frac{1}{p_1}} \times \cdots \times F_{\frac{1}{p_j}} \rangle$ is saturated $\neg$-paraconsistent.*

Contrary to the case of $G^{\leq}_{n\sim}$ in Theorem 1, not every saturated $\neg$-paraconsistent extension of L_n^i is of the type of the above corollary. For instance L_{15}^7 is saturated $\neg$-paraconsistent. Indeed, it is a $\neg$-paraconsistent logic where $\Rightarrow_{15}^7$ is a deductive implication and $\mathsf{L}_1^1 = \text{CPL}$ is a submatrix logic of L_{15}^7. Moreover, every proper extension L of L_{15}^7 is induced by a family of proper submatrices of L_{15}^7, of type $\langle \mathbf{ŁV}_{n_0+1} \times \cdots \times \mathbf{ŁV}_{n_{l-1}+1}, (F_{\frac{7}{15}})^l \cap (ŁV_{n_0+1} \times \cdots \times ŁV_{n_{l-1}+1}) \rangle$ where at least there is some $j < l$ such that $n_j | 15$ and $n_j \neq 15$. Hence, n_j is either 1, 3 or 5, in which case the j-th component $\langle \mathbf{ŁV}_{n_j+1}, F_{\frac{7}{15}} \cap ŁV_{n_j+1} \rangle$ is not $\neg$-paraconsistent. Thus L is not $\neg$-paraconsistent and, therefore, L_{15}^7 is saturated $\neg$-paraconsistent.

To finalize, an additional analysis – from the point of view of paraconsistency – of the logics discussed in this paper will be done. Recall from Section 1 the class of paraconsistent logics known as *logics of formal inconsistency* (**LFI**s). It is easy to see that all the paraconsistent logics considered in the present paper are, in particular, **LFI**s. Indeed, in [21] it was shown that, if L is a (Δ-)core fuzzy logic which is not pseudo-complemented, then $L^{\leq}$ is an **LFI** such that the consistency operator is given by $\circ\varphi = \Delta(\neg\varphi \vee \varphi)$. This shows that all the paraconsistent logics based on Gödel fuzzy logic presented here are **LFI**s. With respect to the paraconsistent logics based on finite-valued Łukasiewicz logics analyzed in this section, they are also **LFI**s, as the following result states:

Proposition 7. *Let L be one of the matrix logics in Proposition 6, or one the products of matrix logics in Theorem 3. Then, L is an* **LFI** *w.r.t. $\neg$.*

Proof. Concerning the logics of Proposition 6, by [20, Proposition 6.3] we know that each logic L^i_n for $i/n \leq 1/2$ is an **LFI** w.r.t. $\neg$, where the consistency operator is given by $\circ^i_n\alpha := \sim^i_n(\alpha \wedge \neg\alpha)$. Here, $\sim^i_n$ is the unary connective defined as in the proof of Theorem 3. Now, let

$$L = \langle \mathbf{ŁV}_{p_1+1} \times \cdots \times \mathbf{ŁV}_{p_j+1}, (F_{\frac{i}{n}})^j \cap (ŁV_{p_1+1} \times \cdots \times ŁV_{p_j+1})\rangle$$

be one of the logics in Theorem 3 given by a product of matrix logics, for some $\{p_1, \ldots, p_j\} \subseteq X$. By definition of X, for every $1 \leq s \leq j$ there exists $1 \leq k_s < p_s$ such that $k_s/p_s \leq 1/2$ and $\langle \mathbf{ŁV}_{p_s+1}, F_{\frac{i}{n}} \cap ŁV_{p_s+1}\rangle = \mathsf{L}^{k_s}_{p_s}$. This means that $L = \mathsf{L}^{k_1}_{p_1} \times \cdots \times \mathsf{L}^{k_j}_{p_j}$. Using again [20, Proposition 6.3] it follows that each $\mathsf{L}^{k_s}_{p_s}$ is an **LFI** w.r.t. $\neg$, with consistency operator $\circ^{k_s}_{p_s}$ defined as above. It is immediate to see that $\sim^i_n$ restricted to $ŁV_{p_s+1}$ coincides with $\sim^{k_s}_{p_s}$, and so $\circ^i_n$ restricted to $ŁV_{p_s+1}$ coincides with $\circ^{k_s}_{p_s}$, for every $1 \leq s \leq j$. Therefore L is an **LFI** w.r.t. $\neg$, with consistency operator given by $\circ\alpha := \circ^i_n\alpha$.

Indeed, it is clear that $\varphi, \neg\varphi, \circ\varphi \vdash_L \psi$ for every formulas φ, ψ. Let q and r be two different propositional variables, and let e be an evaluation over L such that $e(q) = 1$ and $e(r) = 0$. This ensures that $q, \circ q \nvdash_L r$. On the other hand, any evaluation e' over L such that $e'(q) = e'(r) = 0$ guarantees that $\neg q, \circ q \nvdash_L r$. Hence, L is an **LFI** w.r.t. $\neg$ and $\circ$ (recall the definition of **LFI**s in [17, 16, 15]). □

7 Conclusions

In this paper the Gödel fuzzy logic G expanded with an involutive negation $\sim$, denoted $G_\sim$, together with its finite-valued extensions $G_{n\sim}$, were studied from the point of view of paraconsistency. In order to do this, the respective degree-preserving companions $G^\leq_\sim$ and $G^\leq_{n\sim}$ where analyzed given that, in contrast to $G_\sim$ and $G_{n\sim}$, these logics are $\sim$-paraconsistent. Observe that G coincides with $G^\leq$, since G satisfies the deduction-detachment theorem; hence, the addition of an involutive negation $\sim$ to G allows to develop such kind of study. The question of determining the lattice of intermediate logics between $G^\leq_\sim$ and CPL, as well as the logics between $G^\leq_{n\sim}$ and CPL, was addressed. After introducing the concept of saturated paraconsistent logic, which is weaker than the notion of ideal paraconsistency, it was shown that there are only three saturated paraconsistent logics, two of them (J_3 and J_4) being in fact ideal paraconsistent and the other (namely, $\mathsf{J}_3 \times \mathsf{J}_4$) being saturated but not ideal. Finally, the study of finite valued Łukasiewicz logic we started in [20] has been taken up again, in order to find additional interesting examples of saturated but not ideal paraconsistent logics.

As for future work we aim at performing similar studies for other locally finite fuzzy logics, in particular for the Nilpotent Minimum logic (NM) [22], that combines and shares many features of both Gödel and Łukasiewicz logics. It is indeed logically equivalent to Gödel logic with involution when NM is expanded with the Baaz-Monteiro operator Δ.

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