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The Proof by Cases Property and its Variants in Structural Consequence Relations

Abstract. This paper is a contribution to the study of the rôle of disjunction in Abstract Algebraic Logic. Several kinds of (generalized) disjunctions, usually defined using a suitable variant of the proof by cases property, were introduced and extensively studied in the literature mainly in the context of finitary logics. The goals of this paper are to extend these results to all logics, to systematize the multitude of notions of disjunction (both those already considered in the literature and those introduced in this paper), and to show several interesting applications allowed by the presence of a suitable disjunction in a given logic.

Keywords: Abstract Algebraic Logic, generalized disjunction, proof by cases properties, consequence relations, filter-distributive logics, protoalgebraic logics

1. Introduction

Abstract Algebraic Logic (AAL) has developed a corpus of results and techniques for studying logical systems by providing a deep understanding on the nature of the process by which a logic can be endowed with an algebraic semantics. These results allow to study properties of the logical systems, when understood as structural consequence relations over a set of formulae, by means of (equivalent) algebraic properties of their semantics.

In particular, AAL has led to fine analysis on the rôle of the connectives of classical logic, identifying their essential properties, and thus suggesting possible generalizations of these connectives (in non-classical logics) still retaining their essential function(s) in classical logic. Notable examples of this approach are the extensive studies on equivalence connectives (see e.g. [2, 8]). Indeed, the Lindenbaum-Tarski proof of completeness of the classical propositional calculus, based on the fact that the equivalence connective defines a congruence in the algebra of formulae, has been extended to broad classes of logics by considering a suitable generalized notion of equivalence. Namely, equivalence can be taken as a (possibly parameterized, possibly infinite) set of formulae in two variables satisfying certain simple properties. This approach gave rise to the so-called *Leibniz hierarchy*

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of protoalgebraic logics, based on the generalized equivalences that might be present in a logic and their properties. Similarly, there have been many works focusing on implication connectives, studying for instance in what logics they still have some kind of deduction theorem [3, 7, 21], or when they define order in the corresponding algebraic semantics [4]. Other works have provided abstract studies of logics with a suitable notion of conjunction such as [22].

In this paper we focus on disjunction connectives. They have been, of course, already the subject of many important contributions in AAL as well (see e.g. [5, 6, 8, 12, 13, 16, 26, 27]). The *proof by cases property* has been identified as the essential property of classical disjunction[1] and has been formulated in two (non equivalent) ways:

PCP If $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi$.

wPCP If $\varphi \vdash_{\mathrm{L}} \chi$ and $\psi \vdash_{\mathrm{L}} \chi$, then $\varphi \vee \psi \vdash_{\mathrm{L}} \chi$.

The most comprehensive treatment so far, summarizing and generalizing the previous works, was presented by Czelakowski in his book [8] where he considered, in the same fashion as generalized equivalences in the theory of protoalgebraic logics, generalized connectives given by a (possibly parameterized and infinite) set of formulae in two variables satisfying the PCP. This approach allowed to obtain several interesting characterizations in terms of other properties of logics and their semantics, namely: distributivity of the lattice of theories, distributivity of the lattice of filters over any algebraic model of the logic and the behavior of prime filters and substitutions. However, all these results were restricted to finitary logics.

The present paper contributes to the study of the rôle of disjunction in AAL mainly in the following four ways:

- In order to extend the general theory to all infinitary logics we identify a strong form of the proof by cases property (denoted as sPCP):

  $$\text{If } \Gamma, \Phi \vdash_{\mathrm{L}} \chi \text{ and } \Gamma, \Psi \vdash_{\mathrm{L}} \chi, \text{ then } \Gamma, \{\varphi \vee \psi \mid \varphi \in \Phi, \psi \in \Psi\} \vdash_{\mathrm{L}} \chi.$$

  The sPCP is equivalent to the PCP in any finitary logic, but it allows us to formulate for *any* logic characterizations which in the finitary case coincide with those of [8].

[1]For example, Michael Dummett claims in [11] about the weak proof by cases property: "If this law does not hold, the operator $\vee$ could not legitimately be called disjunction operator."

- As a consequence we obtain the known results *not only* for finitary logics but for the strictly larger class of logics with the IPEP (intersection-prime extension property), i.e. logics where the finite meet-irreducible theories form a basis of the closure system of theories. This class includes not only all finitary logics, but also all semilinear logics in the sense of [4].
- We present a hierarchy of logics based on (1) the form of the disjunction they posses (given by a single formula, a set of formulae, or a parameterized set of formulae) and on (2) the kind of proof by cases property this disjunction satisfies (wPCP, PCP, or sPCP). Furthermore, we provide examples showing the separation of all classes in this hierarchy.
- We show several consequences of the presence of a suitable disjunction in a given logic. Namely, we can find an axiomatization of the extension of this logic defined semantically by a positive universal class of its models, and as a particular case we show how to axiomatize the intersection of any finite set of its axiomatic extensions.

Structure of the paper In Section 2 we briefly recall the necessary notions of AAL and introduce several new ones (like (filter)-framality, IPEP, RFSI-completeness, etc.) and prove basic properties needed for the rest of the paper. Section 3 explores the possible definitions for the proof by cases property, uses them to define several notions of disjunction, and gives examples of logics satisfying and separating them. Section 4 collects all of our characterizations for these properties: Subsection 4.1 introduces the syntactical notion of ∇-form of a rule and uses it to characterize proof by cases properties, and gives another syntactical characterization of the wPCP; Subsection 4.2 proves the transfer theorem for the sPCP, and gives a characterization in terms of framality (a strong form of distributivity); Subsection 4.3 gives characterizations in terms of prime filters; and Subsection 4.4 restricts the achieved results to the context of protoalgebraic logics obtaining more characterizations and relations between the involved properties. Finally, Section 5 shows some applications of the theory: Subsection 5.1 studies the preservation of proof by cases properties in expansions, and the identification and axiomatization of the least logic satisfying proof by cases above a given logic; in Subsection 5.2, given a logic L with the PCP, we find an axiomatization of the extension of L defined semantically by a positive universal class of models of L, and as a particular case we show how to axiomatize the intersection of two axiomatic extensions of L; finally, in Subsection 5.3 we consider the analogous problems in the more general case of non-negative universal classes at the price of restricting to finitary logics.

2. Preliminaries

2.1. Basic notions

We use this subsection to fix the notations for the notions of Abstract Algebraic Logic that will be needed in the paper (for comprehensive monographs and survey see [8, 13, 14, 15]). A *propositional language* $\mathcal{L}$ is any type (with no restriction on the cardinality), by $\boldsymbol{Fm}_{\mathcal{L}}$ we denote the free term algebra over an arbitrary (but fixed) infinite set of variables in the language $\mathcal{L}$, by $Fm_{\mathcal{L}}$ we denote its universe. For any sets of formulae Γ, Δ and a formula φ we often write 'Γ, Δ', and 'Γ, φ' for, respectively, '$\Gamma \cup \Delta$', and '$\Gamma \cup \{\varphi\}$'.

An $\mathcal{L}$-*consecution* is a pair $\Gamma \rhd \varphi$. Given a set of $\mathcal{L}$consecutions L, we write $\Gamma \vdash_{\mathrm{L}} \varphi$ rather than $\Gamma \rhd \varphi \in \mathrm{L}$. A *logic* in the language $\mathcal{L}$ is a set of $\mathcal{L}$-consecutions such that $\vdash_{\mathrm{L}}$ is a structural consequence relation. We write $\Gamma \vdash_{\mathrm{L}} \Delta$ when $\Gamma \vdash_{\mathrm{L}} \varphi$ for every $\varphi \in \Delta$. A *theory* of a logic L is a set of formulae closed under the consequence relation. By $\mathrm{Th}(\mathrm{L})$ we denote the set of all theories of L and by $\mathrm{Th}_{\mathrm{L}}(\Gamma)$ the theory generated by Γ.

Logical matrices will be denoted as $\mathbf{A}$, $\mathbf{B}$, …, and their algebraic reducts as $\boldsymbol{A}$, $\boldsymbol{B}$, …. The semantical consequence given by a class $\mathbb{K}$ of $\mathcal{L}$-matrices is denoted as $\models_{\mathbb{K}}$ and it is clearly a logic. Moreover, if $\mathbb{K}$ is a finite set of finite $\mathcal{L}$-matrices, the logic $\models_{\mathbb{K}}$ is finitary. The class of (reduced) matrix models of a logic L is denoted as $\mathbf{MOD}(\mathrm{L})$ (or $\mathbf{MOD}^*(\mathrm{L})$ respectively). A matrix $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ is *relatively finitely subdirectly irreducible in* $\mathbf{MOD}^*(\mathrm{L})$, in symbols $\boldsymbol{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$, if it cannot be decomposed as a non-trivial subdirect product of a finite non-empty family of matrices from $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$; $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is complete lattice and a closure system, and hence it induces a closure operator denoted as $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$ (we write simply Fi when the logic and the algebra are clear from the context).

Definition 2.1. *A logic* L *is* filter-distributive *if for each $\mathcal{L}$-algebra, the lattice $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is distributive. A logic* L *is* filter-framal *if for each $\mathcal{L}$-algebra, the lattice $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a frame, i.e., for each $\mathcal{F} \cup \{G\} \subseteq \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ holds:*

$$G \cap \bigvee_{F \in \mathcal{F}} F = \bigvee_{F \in \mathcal{F}} (G \cap F).$$

We omit the prefix 'filter-' whenever the corresponding property holds for $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$.

It is known that (1) a finitary logic is (filter-)distributive iff it is (filter-) framal [8, Proposition 2.5.1.]; and (2) a finitary protoalgebraic logic is distributive iff it is filter-distributive [8, Proposition 2.5.24.].

2.2. Infinitary axiomatic systems

Since one of the main goals of this paper is to develop a theory of disjunctions coping with infinitary logics, we need to recall the appropriate notion of proof. Note that we assume that axiomatic systems are given by collections of schemata, rather than particular consecution.

DEFINITION 2.2. *An* axiomatic system $\mathcal{AS}$ *in the language* $\mathcal{L}$ *is a set* $\mathcal{AS}$ *of* $\mathcal{L}$*-consecutions closed under arbitrary substitutions. The elements of* $\mathcal{AS}$ *of the form* $\Gamma \rhd \varphi$ *are called* axioms *if* $\Gamma = \emptyset$, finitary deduction rules *if* Γ *is finite, and* infinitary deduction rules *otherwise.*

A proof *of a formula* φ *from a set of formulae* Γ *in* $\mathcal{AS}$ *is a well-founded tree (with no infinitely-long branch) labeled by formulae such that*

- *its root is labeled by* φ *and leaves by axioms of* $\mathcal{AS}$ *or elements of* Γ*,*
- *if a node is labeled by* ψ *and* $\Delta \neq \emptyset$ *is the set of labels of its preceding nodes, then* $\Delta \rhd \psi \in \mathcal{AS}$*.*

We write $\Gamma \vdash_{\mathcal{AS}} \varphi$ *if there is a proof of* φ *from* Γ*. An axiomatic system* $\mathcal{AS}$ *is called a presentation of a logic* L *if* $\vdash_{\mathrm{L}} = \vdash_{\mathcal{AS}}$*.*

DEFINITION 2.3. *Let* L_i *be a logic in language* $\mathcal{L}_i$ *for* $i = 1, 2$ *(such that* $\mathcal{L}_1 \subseteq \mathcal{L}_2$*) and* $\mathcal{C}$ *a set of* $\mathcal{L}_2$*-consecutions. We say that* L_2 *is the* expansion of L_1 by $\mathcal{C}$, *in symbols* $\mathrm{L}_2 = \mathrm{L}_1 + \mathcal{C}$*, if it is the weakest logic in* $\mathcal{L}_2$ *containing* L_1 *and* $\mathcal{C}$*, i.e. the logic axiomatized by all* $\mathcal{L}_2$*-substitutional instances of consecutions from* $\mathcal{C} \cup \mathrm{L}_1$*.*

We say that L_2 *is an* (axiomatic) expansion of L_1 *if* $\mathrm{L}_2 = \mathrm{L}_1 + \mathcal{C}$ *for some set of consecutions (axioms)* $\mathcal{C}$*. Finally, if* $\mathcal{L}_1 = \mathcal{L}_2$*, we use the term* extension *rather than* expansion.

Finally, we generalize [8, Proposition 0.8.1] to all (not necessarily finitary) logics. It allows to generalize the notion of proof (in a given logic L), which can be seen as a way to demonstrate that $\varphi \in \mathrm{Th}_{\mathrm{L}}(\Gamma)$ for a set of $\mathcal{L}$-formulae Γ, to any $\mathcal{L}$-algebra $\boldsymbol{A}$ in order to obtain a way to demonstrate that $a \in \mathrm{Fi}(X)$ for any set $X \cup \{a\} \subseteq A$.

PROPOSITION 2.4 (Proof in algebra). *Let* L *be a logic,* $\mathcal{AS}$ *one of its presentations,* $\boldsymbol{A}$ *an* $\mathcal{L}$*-algebra, and* $X \cup \{a\} \subseteq A$*. Let us define a set* $V_{\mathcal{AS}} \subseteq \mathcal{P}(A) \times A$ *as* $\{\langle e[\Gamma], e(\psi)\rangle \mid e$ *is an* $\boldsymbol{A}$*-evaluation and* $\Gamma \rhd \psi \in \mathcal{AS}\}$*.*[2] *Then* $a \in \mathrm{Fi}(X)$ *iff there is a well-founded tree (called* proof of a from X*) labeled by elements of* A *such that*

- *its root is labeled by* a*, and leaves are labeled by elements* x *such that* $x \in X$ *or* $\langle \emptyset, x\rangle \in V_{\mathcal{AS}}$*,*

[2] Note that if $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$, then $V_{\mathcal{AS}} = \mathcal{AS}$.

- *if a node is labeled by x and $Z \neq \emptyset$ is the set of labels of its preceding nodes, then $\langle Z, y\rangle \in V_{\mathcal{AS}}$.*

Proof. Let $D(X)$ be the set of elements of A for which there exists a proof from X. We can easily show that $\mathcal{AS} \subseteq \models_{\langle \boldsymbol{A}, D(X)\rangle}$. Indeed, assume that $\Gamma \rhd \varphi \in \mathcal{AS}$ and $h[\Gamma] \subseteq D(X)$ for some evaluation h. Then for each $x \in h[\Gamma]$ there is a proof from X and, since $\langle h[\Gamma], h[\varphi]\rangle \in V_{\mathcal{AS}}$, we can connect these proofs such that they will form a proof of $h(\varphi)$. Thus $D(X) \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ and, since $X \subseteq D(X)$, we obtain $\mathrm{Fi}(X) \subseteq D(X)$. To prove the converse direction consider $x \in D(X)$ and notice that for each y appearing in its proof we can easily prove that $y \in \mathrm{Fi}(X)$ (because $\mathrm{Fi}(X)$ is closed under all the rules of L, in particular those in $\mathcal{AS}$). □

2.3. Intersection-prime filters

It is well known [8, Proposition 1.3.4.] that $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ iff F is *intersection-prime* in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, i.e. there is no pair of filters F_1, F_2 such that $F = F_1 \cap F_2$ and $F \subsetneq F_1, F_2$.[3] Recall that a family $\mathcal{B} \subseteq \mathcal{C}$ is a *basis* of a closure system $\mathcal{C}$ if for every $X \in \mathcal{C}$ there is a $\mathcal{D} \subseteq \mathcal{C}$ such that $X = \bigcap \mathcal{D}$ (which can be equivalent formulated as an extension property: for every $Y \in \mathcal{C}$ and every $a \in A \setminus Y$ there is $Z \in \mathcal{B}$ such that $Y \subseteq Z$ and $a \notin Z$).

Definition 2.5. *We say that* L *has the* (transferred) intersection-prime extension property, (τ-)IPEP *for short, if the intersection-prime theories form a basis of* Th(L) *(intersection-prime filters form a basis of $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ for each $\mathcal{L}$-algebra $\boldsymbol{A}$, respectively). Finally, we say that a logic* L *is* RFSI*-complete if $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}}$.*

Lemma 2.6. *Any finitary logic has the τ-*IPEP*; any logic with the τ-*IPEP *has the* IPEP*; and any logic with the* IPEP *is* RFSI*-complete.*

Proof. The first claim is well-known (see e.g. [8, Corollary 1.3.3.]); the second is trivial; we prove the last one. Clearly, it is sufficient to show that $T/\Omega T$ is an intersection-prime filter in $\boldsymbol{Fm}_{\mathcal{L}}/\Omega T$ for any intersection-prime theory T. We proceed counterpositively: assume that there are filters $F_i \supsetneq T/\Omega T$ and $T/\Omega T = F_1 \cap F_2$. Thus there must be a pair of formulae φ_1, φ_2 such that $\varphi_i/\Omega T \in F_i \setminus (T/\Omega T)$. Consider the theories T_i generated by $T \cup \{\varphi_i\}$ and observe that clearly $T_i \supsetneq T$ and $T_i/\Omega T \subseteq F_i$ (recall that F_i are filters in $\boldsymbol{Fm}_{\mathcal{L}}/\Omega T$). Thus $T = T_1 \cap T_2$ and the proof is done. □

[3]This property is known in lattice theory as *(finite) meet-irreducibility*. In [8] these filters are called simply *prime*, but because in this paper we introduce prime filters by using a suitable notion of disjunction, we need to separate both notions.

Next we give an example of an infinitary logic with the IPEP; on the other hand, in Example 3.12 we present a logic which (due to Theorem 4.17) does not enjoy the IPEP. Therefore, the class of IPEP logics is a non-trivial proper extension of that of finitary logics.

EXAMPLE 2.7. *The standard infinite-valued Łukasiewicz logic* $Ł_\infty$ *has the* IPEP *but is not finitary.*[4] Recall (see [23]) that $Ł_\infty$ has connectives $\to$, $\neg$ and is given by the matrix $\mathbf{A} = \langle\langle[0,1], \to^{\boldsymbol{A}}, \neg^{\boldsymbol{A}}\rangle, \{1\}\rangle$, where $x \to^{\boldsymbol{A}} y = \min\{1-x+y, 1\}$ and $\neg^{\boldsymbol{A}} x = 1-x$. It is well known that $Ł_\infty$ is not finitary (see e.g. [19]); we show that it enjoys the IPEP.

If $T \nvdash_{Ł_\infty} \chi$, then there is an evaluation e such that $e[T] = \{1\}$ and $e(\chi) \neq 1$. We define $T' = e^{-1}[\{1\}]$. Obviously T' is a theory, $T \subseteq T'$ and $T' \nvdash_{Ł_\infty} \chi$. Assume that T' is not intersection-prime; thus there are formulae $\varphi, \psi \notin T'$ such that $T' = \mathrm{Th}_{Ł_\infty}(T, \varphi) \cap \mathrm{Th}_{Ł_\infty}(T, \psi)$. Assume without loss of generality that $e(\varphi) \leq e(\psi)$, so $e(\varphi \to \psi) = 1$ and so $\varphi \to \psi \in T'$. Thus $\psi \in \mathrm{Th}_{Ł_\infty}(T, \varphi)$ (because $\varphi, \varphi \to \psi \vdash_{Ł} \psi$) and thus $\psi \in T'$—a contradiction.

LEMMA 2.8. *Let* L' *be an axiomatic extension of* L*. If* L *has the* IPEP*, then so has* L'*.*

PROOF. We fix an L'-theory T. There have to be intersection-prime L-theories T_i and $T = \bigcap_i T_i$. From [8, Proposition 0.8.3.] we know that all L-theories containing T (and so in particular T_is) are L'-theories. To complete the proof we just observe that if an L'-theory is intersection-prime in L', then it is also intersection-prime in L. □

3. A hierarchy of disjunctions

Let $\nabla(p, q, \overrightarrow{r})$ be a set of formulae in two variables p, q and possible parameters $\overrightarrow{r}$. We define $\varphi \nabla \psi$ as $\bigcup\{\nabla(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in Fm_{\mathcal{L}}^{\leq\omega}\}$. Given sets $\Phi, \Psi \subseteq Fm_{\mathcal{L}}$, $\Phi \nabla \Psi$ denotes the set $\bigcup\{\varphi \nabla \psi \mid \varphi \in \Phi, \psi \in \Psi\}$. We start with a useful convention:

CONVENTION 3.1. *A parameterized set* $\nabla(p, q, \overrightarrow{r})$ *of formulae is a* p-protodisjunction *(or just* protodisjunction *if* ∇ *has no parameters) in* L *whenever*

(PD) $\quad \varphi \vdash_{\mathrm{L}} \varphi \nabla \psi \quad$ *and* $\quad \psi \vdash_{\mathrm{L}} \varphi \nabla \psi.$

Throughout all the paper the (parameterized) sets ∇ *are assumed to be (p-) protodisjunctions.*

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[4] Actually one can easily find many such examples. Indeed, from [4, Theorem 16] it follows that any (possibly infinitary) weakly implicative semilinear logic (including $Ł_\infty$ and many other well-known fuzzy logics) has the IPEP.

The notion of p-protodisjunction is not interesting on its own because, actually, any theorem (or a set of theorems) in two variables of a given logic would be a protodisjunction in this logic; we only introduce it as a useful means to shorten the formulation of many upcoming definitions and results. A more genuine property of disjunction is the so-called *proof by cases* of classical disjunction, which has been considered for arbitrary logics in the literature in two different versions:

DEFINITION 3.2. *We say that* ∇ *enjoys the* Proof by Cases Property[5] *in* L *if for any set* $\Gamma, \varphi, \psi, \chi$ *of formulae we have:*

PCP *If* $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ *and* $\Gamma, \psi \vdash_{\mathrm{L}} \chi$*, then* $\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$.

We say that ∇ *enjoys the* weak Proof by Cases Property *in* L *if for any formulae* φ, ψ, χ *we have:*

wPCP *If* $\varphi \vdash_{\mathrm{L}} \chi$ *and* $\psi \vdash_{\mathrm{L}} \chi$*, then* $\varphi \nabla \psi \vdash_{\mathrm{L}} \chi$.

Note that the (weak) Proof by Cases Property is defined as a property of a pair: the logic L and the p-protodisjunction ∇. To simplify the formulation, we will write just that '∇ has the PCP' when the logic L is fixed or known from the context. Analogous conventions will be used for all other upcoming properties defined for p-protodisjunctions in given logics.

EXAMPLE 3.3. *Natural examples of protodisjunctions satisfying the* PCP*.* In many logics, the usual lattice connective $\vee$ does indeed behave, as one may expect, as a protodisjunction with proof by cases. This is the case, for instance, in the substructural logic $\mathrm{FL_{ew}}$ (Full Lambek logic with exchange and weakening, which coincides with the multiplicative and additive fragment of affine linear logic aMALL; see [18] for more details) and in all its axiomatic extensions. In weaker logics, the situation can be more complicated. For instance, in $\mathrm{FL_e}$ (Full Lambek logic only with exchange, or equivalently, the multiplicative and additive fragment of linear logic MALL), if t is the 0-ary connective corresponding to the neutral element of the multiplicative conjunction, one can prove that the definable connective $(\varphi \wedge t) \vee (\psi \wedge t)$ satisfies the PCP (it mainly follows from the local deduction theorem: $\Gamma, \alpha \vdash_{\mathrm{L}} \beta$ iff $\Gamma \vdash_{\mathrm{L}} (\alpha \wedge t)^n \to \beta$ for some $n \geq 1$; see e.g. [18]), while $\vee$ does not as we will see in Example 3.7.

[5] We could have introduced the wPCP and the PCP as double direction meta-rules (as it was done and studied e.g. in [13] under the name *(weak) Property of Disjunction*). However reverse directions of these meta-rules could obviously be equivalently replaced by (PD) (one direction is obvious, for the other one observe that from $\varphi \nabla \psi \vdash_{\mathrm{L}} \varphi \nabla \psi$, we would obtain $\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi$ and $\psi \vdash_{\mathrm{L}} \varphi \nabla \psi$). Thus, we prefer our definition because it keeps the interesting implication separated from the trivial one that we can always assume.

The properties of proof by cases are intrinsic for the logic in the sense that all sets satisfying the wPCP are interderivable:

LEMMA 3.4. *Assume that* ∇ *has the* wPCP *and* ∇' *is an arbitrary p-protodisjunction. Then:* ∇' *enjoys the* wPCP *iff* $\varphi \nabla \psi \dashv\vdash_{\mathrm{L}} \varphi \nabla' \psi$.

The weak proof by cases property entails other properties a disjunction is expected to satisfy: commutativity, idempotency and associativity (which, however, are also typically satisfied by conjunction connectives, whereas the PCP and the wPCP are typically satisfied only by disjunction connectives). The following lemma is straightforward:

LEMMA 3.5. *If* ∇ *satisfies the* wPCP*, then it also satisfies:*

(C) $\varphi \nabla \psi \vdash_{\mathrm{L}} \psi \nabla \varphi$
(I) $\varphi \nabla \varphi \vdash_{\mathrm{L}} \varphi$
(A) $\varphi \nabla (\psi \nabla \chi) \dashv\vdash_{\mathrm{L}} (\varphi \nabla \psi) \nabla \chi$

The properties (C), (I), (A) must be properly read: they respectively give commutativity, idempotency and associativity as regards to membership in the filter of matrix models, but they do not imply that these properties hold for disjunctions of arbitrary elements in the matrix. In symbols: if $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ is an L-matrix, (C) means that for every $a, b \in A$, $a \nabla^{\boldsymbol{A}} b \subseteq F$ implies that $b \nabla^{\boldsymbol{A}} a \subseteq F$; but it does not necessarily mean that $a \nabla^{\boldsymbol{A}} b = b \nabla^{\boldsymbol{A}} a$, and analogously for the other two properties.

EXAMPLE 3.6. *An (element-wise) non-commutative protodisjunction satisfying the* PCP. Consider the logic $\mathrm{G}_\triangle$ (in the language $\{\wedge, \vee, \to, \overline{0}, \overline{1}, \triangle\}$ of type $\langle 2,2,2,0,0,1\rangle$) obtained as the expansion of Gödel-Dummett logic (see [10]) with the unary operator $\triangle$ (see [1]). This logic is complete with respect to the matrix given by the filter $\{1\}$ and the algebra $[0,1]_{\mathrm{G}_\triangle} = \langle [0,1], \min, \max, \to, 0, 1, \triangle\rangle$, where $a \to b = b$ if $a > b$ and $a \to b = 1$ otherwise, and $\triangle(1) = 1$ and $\triangle(a) = 0$ for every $a < 1$. Clearly $\{p \vee q\}$ defines a protodisjunction with PCP where the (C), (I), (A) properties are true element-wise. However, we can also consider $\{\triangle p \vee q\}$ which is also a protodisjunction with PCP (observe that $\triangle p \vee q \dashv\vdash_{\mathrm{G}_\triangle} p \vee q$), but provides a counterexample for commutativity when taken element-wise: $\triangle(0.5) \vee 0.3 = 0.3 \neq 0.5 = \triangle(0.3) \vee 0.5$.

We can show that the converse direction of Lemma 3.5 is not valid and also that the wPCP and the PCP are indeed different:

Example 3.7. *A finitary logic with a protodisjunction satisfying the conditions* (C)*,* (I)*,* (A)[6] *but not the* wPCP. Let L be the extension of $\mathrm{FL_e}$ by the axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$. $\vee$ is clearly a protodisjunction satisfying (C), (I), and (A) in L. However, $\vee$ does not even enjoy the wPCP. Indeed, assume that $\vee$ has the wPCP and we show that then L proves the formula χ defined as $(\varphi \vee \psi) \wedge t \to (\varphi \wedge t) \vee (\psi \wedge t)$ which is a contradiction with [20, Example 3.2]. Obviously $\varphi \to \psi \vdash_{\mathrm{L}} \varphi \vee \psi \to \psi$ and so $\varphi \to \psi \vdash_{\mathrm{L}} (\varphi \vee \psi) \wedge t \to \psi \wedge t$. Thus finally $\varphi \to \psi \vdash_{\mathrm{L}} \chi$. Analogously we could prove $\psi \to \varphi \vdash_{\mathrm{L}} \chi$. Using the wPCP and the fact that $\vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ the proof is done.

Example 3.8. *A finitary logic with a protodisjunction satisfying the* wPCP *but not the* PCP. Consider the non-distributive lattice *diamond*, with the domain $\{\bot, a, b, t, \top\}$ (where $\bot$ is the minimum element and $\top$ is the maximum) and take now the finitary logic given by all the (finitely many) matrices over this algebra with a lattice filter. Observe that for every set $\Gamma \cup \{\varphi\}$ of formulae, $\Gamma \vdash \varphi$ iff $\bigwedge e[\Gamma] \leq e(\varphi)$ for every evaluation e over the diamond. From this it easily follows that $\vee$ is a protodisjunction with wPCP. Assume now, for a contradiction, that it satisfies the PCP too. Then from $\varphi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ and $\chi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ we obtain $\varphi \vee \chi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ and thus also (applying the PCP again) $\varphi \vee \chi, \psi \vee \chi \vdash (\varphi \wedge \psi) \vee \chi$ (a form of distributivity). Then, we reach a contradiction by observing that $a \vee b = t \vee b = \top$ while $(a \wedge t) \vee b = \bot \vee b = b$.

We could also show the independence of the conditions (C), (I), (A) of protodisjunctions by several artificial examples, all of them finitary. We leave it as an exercise for a reader and just mention a natural example: any substructural non-contractive involutive logic (e.g. linear logic or Łukasiewicz infinite-valued logic) has the multiplicative disjunction $\oplus$ which satisfies conditions (PD), (C), and (A) but not (I).

We define a natural intermediate property between the PCP and wPCP:

Definition 3.9. *We say that* ∇ *enjoys the* finitary Proof by Cases Property *in* L *if for any* finite *set* Γ *of formulae and any formulae* φ, ψ, χ *we have:*

fPCP *If* $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ *and* $\Gamma, \psi \vdash_{\mathrm{L}} \chi$*, then* $\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$.

It is straightforward to check that for finitary logics the PCP and fPCP are equivalent. Another natural way of obtaining properties related to the PCP consists in replacing φ and ψ by sets of formulae. If we only allow finite sets, then we only obtain reformulations of the PCP and fPCP respectively:

[6] Logics with a connective satisfying these conditions were studied in [28] under the name of *logics with disjunction*.

LEMMA 3.10. ∇ *has the* (f)PCP *if, and only if, the following meta-rule holds for every (finite) set* $\Gamma \cup \{\chi\}$ *of formulae and every finite sets* Φ, Ψ *of formulae:*

$$\frac{\Gamma, \Phi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \Psi \vdash_{\mathrm{L}} \chi}{\Gamma, \Phi \nabla \Psi \vdash_{\mathrm{L}} \chi}$$

PROOF. We prove it for PCP (the proof for fPCP is analogous). One implication is trivial; we prove the other by induction. Call a pair $\Gamma, \Phi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \Psi \vdash_{\mathrm{L}} \chi$ a *situation*; define the *complexity* of a situation as a pair $\langle n, m\rangle$ where n and m are respectively the cardinals of $\Phi \setminus \Psi$ and $\Psi \setminus \Phi$. We show by the induction over $k = n+m$ that in each situation we obtain $\Gamma, \Phi \nabla \Psi \vdash_{\mathrm{L}} \chi$.

First assume $k \leq 2$. If $n = 0$, i.e. $\Phi \subseteq \Psi$, we obtain $\Phi \nabla \Phi \subseteq \Phi \nabla \Psi$ and since $\Gamma, \Phi \nabla \Phi \vdash_{\mathrm{L}} \Gamma \cup \Phi$ the proof is done. The proof for $m = 0$ is the same. If $n = m = 1$ we use PCP. The induction step: consider a situation with complexity $\langle n, m\rangle$, where $n + m > 2$. We can assume without loss of generality that $n \geq 2$, take a formula $\varphi \in \Phi \setminus \Psi$ and define $\Phi'_1 = \Phi \setminus \{\varphi\}$. We know that $\Gamma, \Phi'_1, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \Psi \vdash_{\mathrm{L}} \chi$. Thus we also know that $\Gamma, \Phi'_1, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \Phi'_1, \Psi \vdash_{\mathrm{L}} \chi$; notice that the complexity of this situation is $\langle 1, m\rangle$ and so we can use the induction assumption to obtain $\Gamma, \Phi'_1, \varphi \nabla \Psi \vdash_{\mathrm{L}} \chi$.

Thus we have the situation $\Gamma, \varphi \nabla \Psi, \Phi'_1 \vdash_{\mathrm{L}} \chi$ and $\Gamma, \varphi \nabla \Psi, \Psi \vdash_{\mathrm{L}} \chi$ (the second claim is trivial); the complexity of this situation is $\langle n', m'\rangle$, where $n' \leq n - 1$ and $m' \leq m$, and so by the induction assumption we obtain $\Gamma, \varphi \nabla \Psi, \Phi'_1 \nabla \Psi \vdash_{\mathrm{L}} \chi$ (which is exactly what we wanted). $\square$

Observe that, if L is finitary, the lemma holds even without requiring that Φ and Ψ are finite. However, for infinitary logics it makes sense to consider it as a stronger property:

DEFINITION 3.11. *We say that* ∇ *enjoys* strong Proof by Cases Property *in* L *if for every sets* Γ, Φ, Ψ *of formulae and every formula* χ *we have:*

sPCP *If* $\Gamma, \Phi \vdash_{\mathrm{L}} \chi$ *and* $\Gamma, \Psi \vdash_{\mathrm{L}} \chi$*, then* $\Gamma, \Phi \nabla \Psi \vdash_{\mathrm{L}} \chi$.

Clearly the sPCP implies the PCP and in finitary logics these properties coincide (due to the remark just before the definition). On the other hand, we can show that even though there are natural infinitary logics with a connective satisfying the sPCP (Example 4.24), this property is *not* in general implied by the PCP, as shown by the next example:

EXAMPLE 3.12. *An infinitary logic with a protodisjunction satisfying the* PCP *but not the* sPCP*.* Let $\boldsymbol{A}$ be a complete distributive lattice such that it is not a dual frame, i.e. there are elements $x_i \in A$ for $i \geq 0$ such that

$$\bigwedge_{i\geq 1}(x_0 \vee x_i) \not\leq x_0 \vee \bigwedge_{i\geq 1} x_i.$$

We expand the lattice language by constants $\{c_i \mid i \geq 0\} \cup \{c\}$ and define an algebra $\boldsymbol{A}'$ in this language by setting $c_i^{\boldsymbol{A}'} = x_i$ and $c = \bigwedge_{i\geq 1} x_i$. Then we define the logic L in this language given semantically by the class of matrices $\{\langle \boldsymbol{A}', F\rangle \mid F$ is a principal lattice filter in $\boldsymbol{A}\}$. Note that $\Gamma \vdash_{\mathrm{L}} \varphi$ iff for each $\boldsymbol{A}$-evaluation e holds: $\bigwedge_{\psi\in\Gamma} e(\psi) \leq e(\varphi)$ (one direction: as $[\bigwedge_{\psi\in\Gamma} e(\psi)]$ clearly contains $e[\Gamma]$ it has to contain $e(\varphi)$; the reverse direction: just observe that any principal filter containing $e[\Gamma]$ has to contain $\bigwedge_{\psi\in\Gamma} e(\psi)$).

First we show that $\vee$ enjoys the PCP: assume that for each e evaluation holds $(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\varphi) \leq e(\chi)$ and $(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\psi) \leq e(\chi)$, thus $[(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\varphi)] \vee [(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\psi)] \leq e(\chi)$, the distributivity of $\boldsymbol{A}$ completes the proof. Finally, by the way of contradiction, assume that $\vee$ enjoys the sPCP. Observe that: $c_0 \vdash_{\mathrm{L}} c_0 \vee c$ and $\{c_i \mid i \geq 1\} \vdash_{\mathrm{L}} c_0 \vee c$. Using the sPCP we obtain $\{c_0 \vee c_i \mid i \geq 1\} \vdash_{\mathrm{L}} c_0 \vee c$—a contradiction.

The strong and finitary proof by cases properties can be written in a more compact way (as a formal generalization of the wPCP):

Proposition 3.13. ∇ *has the* sPCP *(resp.* fPCP*) if, and only if, the following meta-rule holds for any set (resp. any finite set) of formulae* $\Phi \cup \Psi \cup \{\chi\}$*:*

$$\frac{\Phi \vdash_{\mathrm{L}} \chi \qquad \Psi \vdash_{\mathrm{L}} \chi}{\Phi \nabla \Psi \vdash_{\mathrm{L}} \chi}.$$

Proof. The left-to-right direction is easy (trivial in the case of sPCP, or obtained by Lemma 3.10 in the case of fPCP). The reverse direction simply follows using (PD). □

Summing up, by combining restrictions on the cardinality of the context set and on the cardinality of the disjuncts, we have at most the following four properties of proof by cases (in increasing order of strength): wPCP, fPCP, PCP and sPCP. With exception of the pair fPCP and PCP we know that they are different (Example 3.8 provides in fact a finitary logic separating wPCP from fPCP). In addition, we know that the last three are equivalent for finitary logics. The next sections of the paper will be devoted to showing characterizations of these four properties in a general context and to find broad classes of logics (containing the finitary ones) where the properties still collapse.

Since, as we said, we are assuming that all sets ∇ satisfy (PD), we can use the consequence operation to formulate the proof by cases properties in more compact forms as Tarski-style conditions. Namely, ∇ satisfies:

wPCP iff $\mathrm{Th_L}(\varphi) \cap \mathrm{Th_L}(\psi) = \mathrm{Th_L}(\varphi \nabla \psi)$ for each φ, ψ.

fPCP iff $\mathrm{Th_L}(\Phi) \cap \mathrm{Th_L}(\Psi) = \mathrm{Th_L}(\Phi \nabla \Psi)$ for each finite Φ, Ψ
iff $\mathrm{Th_L}(\Gamma, \Phi) \cap \mathrm{Th_L}(\Gamma, \Psi) = \mathrm{Th_L}(\Gamma, \Phi \nabla \Psi)$ for each finite Γ, Φ, Ψ
iff $\mathrm{Th_L}(\Gamma, \varphi) \cap \mathrm{Th_L}(\Gamma, \psi) = \mathrm{Th_L}(\Gamma, \varphi \nabla \psi)$ for each finite $\Gamma \cup \{\varphi, \psi\}$.

PCP iff $\mathrm{Th_L}(\Gamma, \Phi) \cap \mathrm{Th_L}(\Gamma, \Psi) = \mathrm{Th_L}(\Gamma, \Phi \nabla \Psi)$ for each Γ and finite Φ, Ψ
iff $\mathrm{Th_L}(\Gamma, \varphi) \cap \mathrm{Th_L}(\Gamma, \psi) = \mathrm{Th_L}(\Gamma, \varphi \nabla \psi)$ for each $\Gamma \cup \{\varphi, \psi\}$.

sPCP iff $\mathrm{Th_L}(\Phi) \cap \mathrm{Th_L}(\Psi) = \mathrm{Th_L}(\Phi \nabla \Psi)$ for each Φ, Ψ
iff $\mathrm{Th_L}(\Gamma, \Phi) \cap \mathrm{Th_L}(\Gamma, \Psi) = \mathrm{Th_L}(\Gamma, \Phi \nabla \Psi)$ for each Γ, Φ, Ψ.

Since proof by cases is arguably the most characteristic property a disjunction is expected to satisfy, these properties can be used to formally define what a disjunction connective is. Having in general four different properties of proof by cases, we could define four different corresponding notions of disjunction but, taking into account the modest rôle that the fPCP will play in the upcoming characterization results, we decide to dismiss its corresponding definition of disjunction. On the other hand, recalling the fact that (by Lemma 3.4) these properties are intrinsic for a given logic in the sense that all possible sets ∇ satisfying the wPCP (or PCP, or sPCP) are interderivable, it makes sense to define classes of logics according to the presence of such p-protodisjunctions, also distinguishing them based on the structure of the ∇ sets, that is, distinguishing traditional disjunctions defined by a single connective or formula, disjunctions which come from a (possibly infinite) parameter-free ∇, and the most general case where ∇ is allowed to be infinite and parameterized.

DEFINITION 3.14. *We say that* ∇ *is a* strong p-disjunction *(resp.* p-disjunction*, resp.* weak p-disjunction*) if it satisfies the* sPCP *(resp.* PCP*, resp.* wPCP*). If* ∇ *has no parameters, we drop the prefix 'p-'.*

DEFINITION 3.15. *We say that a logic* L *is* strongly (p-)disjunctional *(resp.* (p-)disjunctional*, resp.* weakly (p-)disjunctional*) if it has a strong (p-)disjunction (resp. a (p-)disjunction, resp. a weak (p-)disjunction).*

Furthermore, we say that L *is* strongly disjunctive *(resp.* disjunctive*, resp.* weakly disjunctive*) if it has a strong disjunction (resp. a disjunction, resp. a weak disjunction) given by a single parameter-free formula.*

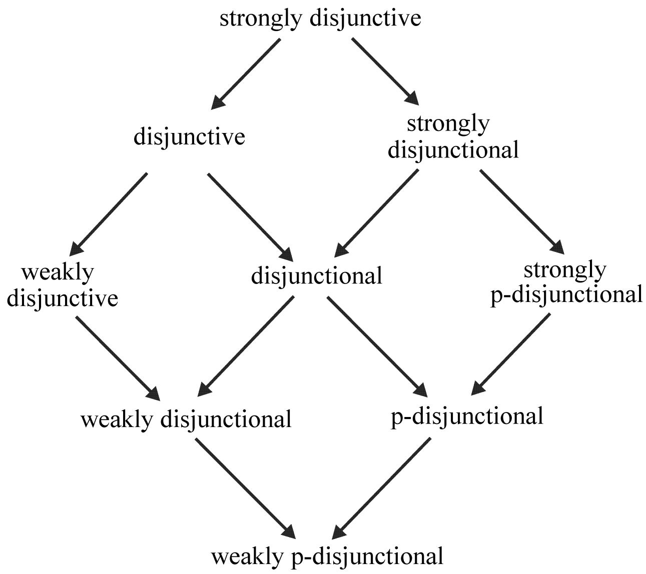

Figure 1. The disjunctional hierarchy of logics.

Remark 3.16. *Thanks to Lemma 3.4, in a (strongly) p-disjunctional logic any weak p-disjunction is actually a (strong) p-disjunction.*

Theorem 3.17. *All classes of logics defined in the previous definition are mutually different. Furthermore, the intersection of any two classes is their infimum w.r.t. the subsumption order depicted in Figure 1.*

The intersection property follows from the previous remark and the separation of all the classes is established by an upcoming series of examples. These examples also show that in finitary logics, taking into account the equivalence of sPCP and PCP, there are exactly six mutually distinct classes.

Example 3.18. *A finitary weakly disjunctive but not p-disjunctional logic.* The logic in Example 3.8 based on lattice diamond has a connective $\vee$ satisfying the wPCP but not the PCP. Therefore, it is weakly disjunctive. If it was p-disjunctional with some ∇, then, according to Lemma 3.4, ∇ would be interderivable with $\vee$, so $\vee$ would satisfy the PCP as well—a contradiction.

Example 3.19. *An infinitary disjunctive but not strongly p-disjunctional logic.* The logic in Example 3.12 based on a complete distributive lattice which is not a dual frame has a connective $\vee$ satisfying the PCP but not the sPCP. Therefore, it is disjunctive, but following the same line of reasoning as in the previous example it cannot be strongly p-disjunctional.

EXAMPLE 3.20. *A finitary (strongly) disjunctional but not weakly disjunctive logic.* Consider the implicational fragment of Gödel-Dummett logic; let us call it $G_{\to}$. First we show that the set

$$\varphi \nabla \psi = \{(\varphi \to \psi) \to \psi, (\psi \to \varphi) \to \varphi\}$$

is a protodisjunction satisfying the PCP. Since G is an axiomatic extension of FL_{ew} it satisfies: $\varphi \vdash (\varphi \to \psi) \to \psi$ and $\psi \vdash (\varphi \to \psi) \to \psi$ and so ∇ satisfies (PD). Now observe that $\Gamma, \varphi \to \psi, (\varphi \to \psi) \to \psi, (\psi \to \varphi) \to \varphi \vdash \psi$ and as we assume that $\Gamma, \psi \vdash \chi$ thus $\Gamma, \varphi \to \psi, \varphi \nabla \psi \vdash \chi$ and so by the deduction theorem $\Gamma, \varphi \nabla \psi \vdash (\varphi \to \psi) \to \chi$. Analogously we can prove that $\Gamma, \varphi \nabla \psi \vdash (\psi \to \varphi) \to \chi$ and as the formula $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$ is a theorem of Gödel-Dummett logic we obtain $\Gamma, \varphi \nabla \psi \vdash \chi$ as needed.

Assume that some parameter-free formula $\varphi(p, q)$ has the wPCP. As a consequence of the completeness theorem for G, we know that $G_{\to}$ is complete with respect to the matrix $\mathbf{A}$ whose universe is the real unit interval $[0, 1]$, the filter is $\{1\}$ and the only operation is:

$$a \to^{\mathbf{A}} b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

By Lemma 3.4, the formula $\varphi(p, q)$ and the set $\varphi \nabla \psi$ are mutually derivable in $G_{\to}$. We know that $\varphi \vee \psi \leftrightarrow ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$ holds in Gödel-Dummett logic and so we can use the global deduction theorem we obtain that $\varphi(p, q)$ is interpreted in $\mathbf{A}$ as the function maximum. So, in particular, for every $a, b \in [0, 1)$ we have $\varphi^{\mathbf{A}}(a, b) = \max\{a, b\}$. We show by an infinite descent argument that this is impossible. Since $\to$ is the only connective in the language, we must have $\varphi(p, q) = \alpha(p, q) \to \beta(p, q)$. Take any $a, b \in [0, 1)$. If $a \leq b$, $\varphi^{\mathbf{A}}(a, b) = \alpha^{\mathbf{A}}(a, b) \to^{\mathbf{A}} \beta^{\mathbf{A}}(a, b) = b$, which implies $\beta^{\mathbf{A}}(a, b) = b$. Analogously, if $a > b$, we have $\beta^{\mathbf{A}}(a, b) = a$. Thus, $\beta(p, q)$ would be a strictly shorter formula with the same property. Following this line of reasoning we would derive that for each $a, b \in [0, 1)$ holds either $a \to^{\mathbf{A}} b = \max\{a, b\}$ or $b \to^{\mathbf{A}} a = \max\{a, b\}$; thus obtaining a contradiction.[7]

Finally, we provide an example supporting the necessity of the level of generality of disjunction connectives, defined by possibly infinite sets of formulae with parameters (some claims in the example need to be justified by results in subsequent sections of this paper).

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[7]For the reader's convenience we have included this rather self-contained proof showing that $G_{\to}$ is not weakly disjunctional. However, this fact would also follow from a reasoning analogous to that of the next example.

Example 3.21. *A finitary (strongly) p-disjunctional logic but not weakly disjunctional.* Consider the purely implicational fragment of intuitionistic logic[8] $\mathrm{IPC}_{\rightarrow}$. The fact that $\mathrm{IPC}_{\rightarrow}$ is p-disjunctional follows from Theorem 4.27 (the filter-distributivity of this logic was proved in [9]). Assume, for contradiction, that a set ∇ is a disjunction in $\mathrm{IPC}_{\rightarrow}$. Thus (by Theorem 5.1) it also a disjunction in the full intuitionistic logic IPC. Since, as it is well known, the lattice connective $\vee$ satisfies the Proof by Cases in IPC too, by Lemma 3.4, we have $p \nabla q \dashv\vdash_{\mathrm{L}} p \vee q$. Using finitarity, the presence of the lattice conjunction $\wedge$ in the language of IPC and the deduction theorem we obtain a formula $\vee'$ of two variables p, q built using only implication and lattice conjunction such that $\vdash_{\mathrm{IPC}} p \vee' q \leftrightarrow p \vee q$—which is known to be impossible (see e.g. [24]).

Another example of a finitary logic with a (this time explicit and natural) parameterized infinite disjunction is the Full Lambek logic FL (see e.g. [18]), although in this case we have not succeeded in showing that it is not weakly disjunctive.

Example 3.22. *A finitary logic with a parameterized infinite disjunction.* Consider the logic FL. This logic has a non-commutative conjunction & with right and left residual implications denoted respectively as \ and /. Given formulae α and φ, one defines the *left conjugate* and the *right conjugate* of φ with respect to α respectively as $\lambda_\alpha(\varphi) = (\alpha\backslash\varphi\&\alpha) \wedge t$ and $\rho_\alpha(\varphi) = (\alpha\&\varphi/\alpha) \wedge t$. An *iterated conjugate* of φ with respect to $\alpha_1, \ldots, \alpha_n \in A$ is a composition $\gamma(\varphi) = \gamma_{\alpha_1}(\gamma_{\alpha_2}(\ldots\gamma_{\alpha_n}(\varphi)))$ where $\gamma_{\alpha_i} \in \{\lambda_{\alpha_i}, \rho_{\alpha_i}\}$ for every i. With this notation, one defines the following infinite set with parameters:

$$\varphi \nabla \psi = \{\gamma_1(\varphi \wedge t) \vee \gamma_2(\psi \wedge t) \mid \text{ where } \gamma_1, \gamma_2 \text{ are iterated conjugates}\}$$

which satisfies the sPCP. Sato in [25, Proposition 6.9] showed that there is no finite protodisjunction in FL satisfying the PCP. We conjecture that there is no parameter-free weak disjunction.

4. Characterizations of the proof by cases properties

Let us fix a logic L in the language $\mathcal{L}$ and p-protodisjunction ∇ in L.

4.1. Purely syntactical characterizations

We start with a characterization (inspired by the proofs of Theorems 2.5.8. and 2.5.9. of [8]) of weakly (p-)disjunctional logics by means substitutions.

[8]We thank Ramon Jansana for drawing our attention to this logic.

THEOREM 4.1 (Characterization of weakly (p-)disjunctional logics). *The following are equivalent:*

1. L *is weakly (p-)disjunctional,*
2. *for each (surjective) substitution σ and a pair φ, ψ of formulae holds*
$$\mathrm{Th}_{\mathrm{L}}(\sigma\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\sigma\psi) = \mathrm{Th}_{\mathrm{L}}(\sigma[\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi)]),$$
3. *for each (surjective) substitution σ and a pair of distinct variables p, q holds*
$$\mathrm{Th}_{\mathrm{L}}(\sigma p) \cap \mathrm{Th}_{\mathrm{L}}(\sigma q) = \mathrm{Th}_{\mathrm{L}}(\sigma[\mathrm{Th}_{\mathrm{L}}(p) \cap \mathrm{Th}_{\mathrm{L}}(q)]).$$

PROOF. We first prove that 1 implies 2. Take any weak (p-)disjunction ∇. Notice that if ∇ is parameter-free or σ is surjective we obtain $\sigma\varphi \mathrel{\nabla} \sigma\psi = \sigma[\varphi \mathrel{\nabla} \psi]$ (in the first case it is trivial, in the second case we can write the chain of equations: $\sigma\varphi \mathrel{\nabla} \sigma\psi = \bigcup\{\nabla(\sigma\varphi, \sigma\psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in Fm_{\mathcal{L}}^{\leq\omega}\} = \bigcup\{\nabla(\sigma\varphi, \sigma\psi, \overrightarrow{\sigma\beta}) \mid \overrightarrow{\beta} \in Fm_{\mathcal{L}}^{\leq\omega}\} = \sigma[\varphi \mathrel{\nabla} \psi]$). Thus in both cases we can prove that 1 implies 2 by this chain of equations: $\mathrm{Th}_{\mathrm{L}}(\sigma\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\sigma\psi) = \mathrm{Th}_{\mathrm{L}}(\sigma\varphi \mathrel{\nabla} \sigma\psi) = \mathrm{Th}_{\mathrm{L}}(\sigma[\varphi \mathrel{\nabla} \psi]) = \mathrm{Th}_{\mathrm{L}}(\sigma[\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi)])$.

The implication from 2 to 3 is trivial; we prove that 3 implies 1. Let us assume first that 3 holds only for surjective substitutions. We define $\nabla(p, q, \overrightarrow{\alpha}) = \mathrm{Th}_{\mathrm{L}}(p) \cap \mathrm{Th}_{\mathrm{L}}(q)$. Clearly ∇ is a p-protodisjunction; we show that it satisfies the wPCP. Consider a surjective substitution such that $\sigma p = \varphi$ and $\sigma q = \psi$. Then we can write this chain of equations: $\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi) = \mathrm{Th}_{\mathrm{L}}(\sigma p) \cap \mathrm{Th}_{\mathrm{L}}(\sigma q) = \mathrm{Th}_{\mathrm{L}}(\sigma[\mathrm{Th}_{\mathrm{L}}(p) \cap \mathrm{Th}_{\mathrm{L}}(q)]) = \mathrm{Th}_{\mathrm{L}}(\sigma[p \mathrel{\nabla} q]) \subseteq \mathrm{Th}_{\mathrm{L}}(\varphi \mathrel{\nabla} \psi)$.

Assume now that 3 holds for all substitutions. Take a substitution σ such that $\sigma p = p$ and $\sigma r = q$ for every $r \neq p$. We define $\nabla(p, q) = \sigma[\mathrm{Th}_{\mathrm{L}}(p) \cap \mathrm{Th}_{\mathrm{L}}(q)]$. Then ∇ is clearly a protodisjunction and analogously as in the previous case we show that it enjoys the wPCP. $\square$

REMARK 4.2. *Note that from the proof of this theorem we can infer that if* L *is weakly p-disjunctional logic, then* $\mathrm{Th}_{\mathrm{L}}(p) \cap \mathrm{Th}_{\mathrm{L}}(q)$ *is one of its weak p-disjunctions. In fact, it is the largest weak p-disjunction (written in variables p and q) in the sense of inclusion.*

Next we define the notion of ∇-form of a consecution, inspired by [6]. It will allow us to obtain the upcoming Theorem 4.5 as an extension of Theorem 2.5.3 from [8].

DEFINITION 4.3. *Let $R = \Gamma \rhd \varphi$ be an $\mathcal{L}$-consecution. Then by R^{∇} we denote the set $\{\Gamma \mathrel{\nabla} \chi \rhd \delta \mid \chi \in Fm_{\mathcal{L}}$ and $\delta \in \varphi \mathrel{\nabla} \chi\}$ of consecutions.*

LEMMA 4.4. *Let R be a consecution such that $R^{\nabla} \subseteq$ L.*

1. *If ∇ satisfies* (I)*, then $R \in$ L.*
2. *If ∇ satisfies* (A)*, then $(R^{\nabla})^{\nabla} \subseteq$ L.*

PROOF. The first claim: from the assumption we know $\Gamma \nabla \varphi \vdash_{\mathrm{L}} \varphi \nabla \varphi$, (PD) and (I) complete the proof. To prove the second claim we start with $\Gamma \nabla (\psi_1 \nabla \psi_2) \vdash_{\mathrm{L}} \varphi \nabla (\psi_1 \nabla \psi_2)$; repeated use of (A) completes the proof. □

The first part of this lemma tells us that in the next theorem we could write '$R^{\nabla} \subseteq$ L iff $R \in$ L', instead of '$R^{\nabla} \subseteq$ L for each $R \in$ L'. The second part will be useful later.

THEOREM 4.5 (Syntactical characterizations). ∇ *enjoys the*

1. sPCP *iff* ∇ *satisfies* (C), (I), *and $R^{\nabla} \subseteq$ L for each $R \in$ L.*
2. fPCP *iff* ∇ *satisfies* (C), (I), *and $R^{\nabla} \subseteq$ L for each finitary $R \in$ L.*
3. wPCP *iff* ∇ *satisfies* (C), (I), *and $(\varphi \rhd \psi)^{\nabla} \subseteq$ L whenever $\varphi \vdash_{\mathrm{L}} \psi$.*

PROOF. We prove all left-to-right directions at once. From $\Gamma \vdash_{\mathrm{L}} \varphi$ we obtain and $\Gamma \vdash_{\mathrm{L}} \varphi \nabla \chi$ using (PD). By (PD) we also obtain $\chi \vdash_{\mathrm{L}} \varphi \nabla \chi$. Thus sPCP (for arbitrary Γ), fPCP (for finite Γ), and wPCP (for Γ being a singleton) we get $\Gamma \nabla \chi \vdash_{\mathrm{L}} \varphi \nabla \chi$.

Also the reverse directions will be proven at once: assume that $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$. Based on restrictions on the cardinality of Γ (arbitrary, finite or empty) we can use one of the assumptions to get $\Gamma \nabla \psi, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi \nabla \psi$ and $\Gamma \nabla \chi, \psi \nabla \chi \vdash_{\mathrm{L}} \chi \nabla \chi$. Using (C) and (I) we obtain $\Gamma \nabla \psi, \Gamma \nabla \chi, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$. Since clearly $\Gamma \vdash_{\mathrm{L}} \Gamma \nabla \psi$ and $\Gamma \vdash_{\mathrm{L}} \Gamma \nabla \chi$, the proof is done. □

The next proposition shows that to check the sPCP it is sufficient to show that L is closed under ∇-forms of the elements of any of its presentations.

PROPOSITION 4.6. *Assume $\mathcal{AS}$ is a presentation of* L*. Then ∇ enjoys the* sPCP *iff ∇ satisfies* (C), (I), *and $R^{\nabla} \subseteq$ L for each $R \in \mathcal{AS}$.*

PROOF. Assume that $\Gamma \vdash_{\mathrm{L}} \varphi$ and we show $\Gamma \nabla \chi \vdash_{\mathrm{L}} \delta \nabla \chi$ for each formula χ and each δ appearing in the proof of φ from Γ. If $\delta \in \Gamma$ or δ is an axiom, the proof is trivial. Now assume that $R = \Gamma' \rhd \delta$ is the deduction rule we use to obtain δ (we can assume it because axiomatic systems are closed under substitutions). From the induction assumption we have $\Gamma \nabla \chi \vdash_{\mathrm{L}} \Gamma' \nabla \chi$. Since $R^{\nabla} \in$ L, the proof is done. □

4.2. Proof by cases and properties of the lattice of filters

We start by using the results of the previous section to prove a crucial theorem: the transfer of sPCP. The property proved in this theorem will be called *transferred* sPCP and denoted as τ-sPCP. We use the same denotation for the other three variants of the proof by cases property.

THEOREM 4.7 (Transfer of sPCP). *If* ∇ *enjoys the* sPCP, *then for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and each* $X, Y \subseteq A$ *we have* $\mathrm{Fi}(X) \cap \mathrm{Fi}(Y) = \mathrm{Fi}(X \nabla^{\boldsymbol{A}} Y)$.

PROOF. The inclusion $\mathrm{Fi}(X \nabla^{\boldsymbol{A}} Y) \subseteq \mathrm{Fi}(X) \cap \mathrm{Fi}(Y)$ follows easily from (PD). To prove the converse one, we start by showing that for each $x \in \mathrm{Fi}(X)$ we have $x \nabla^{\boldsymbol{A}} y \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$ for each y. Using Proposition 2.4 we know that if $x \in \mathrm{Fi}(X)$ than there is a proof of x from X in some presentation $\mathcal{AS}$ of L. We show that $z \nabla^{\boldsymbol{A}} y \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$ for each z labeling any node of that proof, i.e. for each $\chi(p, q, r_1, \ldots, r_n) \in \nabla$ and each sequence $u_1, \ldots, u_n$ of elements of A we have $\chi^{\boldsymbol{A}}(z, y, u_1, \ldots, u_n) \in \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$.

If z labels a leaf and $z \in X$, then it is trivial. Otherwise there is a set Z of labels of the preceding nodes (possible empty), a consecution $\Gamma \rhd \varphi \in \mathcal{AS}$, and an evaluation h, such that $h[\Gamma] = Z$ and $h(\varphi) = z$. Without loss of generality we could assume that variables $q, r_1, \ldots, r_n$ do not occur[9] in $\Gamma \cup \{\varphi\}$ and so we can set $h(q) = y$ and $h(r_i) = u_i$ for every $i \in \{1, \ldots, n\}$. Thus $h[\Gamma \nabla q] \subseteq Z \nabla^{\boldsymbol{A}} y \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$ (the last inclusion follows from the induction assumption). From the characterization of the sPCP in Theorem 4.5 we know that $\Gamma \nabla q \vdash_{\mathrm{L}} \chi(\varphi, q, r_1, \ldots, r_n)$ and so $\chi^{\boldsymbol{A}}(z, y, u_1, \ldots, u_n) = h(\chi(\varphi, q, r_1, \ldots, r_n)) \in \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$.

Now we can finally prove that $\mathrm{Fi}(X) \cap \mathrm{Fi}(Y) \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} Y)$. If $z \in \mathrm{Fi}(X)$ then by the just proved claim for each $y \in Y$ holds: $z \nabla^{\boldsymbol{A}} y \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$ and so, by (C), $y \nabla^{\boldsymbol{A}} z \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$. This can be more compactly written as: $Y \nabla^{\boldsymbol{A}} z \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} Y)$. Analogously we obtain $z \nabla^{\boldsymbol{A}} z \subseteq \mathrm{Fi}(Y \nabla^{\boldsymbol{A}} z)$ from $z \in \mathrm{Fi}(Y)$. Thus $z \in \mathrm{Fi}(Y \nabla^{\boldsymbol{A}} z)$ (by (I)) and so $z \in \mathrm{Fi}(X \nabla^{\boldsymbol{A}} Y)$. $\square$

Now we can prove the main theorem of this subsection. Note that it can be seen as a generalization of Theorem 2.5.8. of [8] which was restricted to finitary logics. To do so, we have substituted PCP by sPCP and (filter-) distributivity by (filter-)framality, and we have also identified the exact rôle of the wPCP (thanks to Theorem 4.1).

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[9] We could define a new suitable $\Gamma \rhd \varphi$ with the same properties using a Hilbert-hotel style argument: consider any enumeration of the variables such that $p_0 = q$, $p_i = r_i$, a substitution $\sigma(p_i) = p_{i+n+1}$, and an evaluation h' such that $h'(\sigma p) = h(p)$. Then $\sigma[\Gamma] \rhd \sigma\varphi$ is the needed consecution: indeed $\sigma[\Gamma] \rhd \sigma\varphi \in \mathcal{AS}$, $h'[\sigma[\Gamma]] = Z$, and $h'(\sigma\varphi) = z$. Note that we have used our assumption that axiomatic systems are closed under substitutions.

THEOREM 4.8 (Characterizations of sPCP). *The following are equivalent:*

1. ∇ *enjoys the* sPCP*,*
2. ∇ *enjoys the* wPCP *and the logic* L *is filter-framal,*
3. ∇ *enjoys the* wPCP *and the logic* L *is framal,*
4. ∇ *enjoys the* wPCP *and for any theory* T *and a any set* Γ *of formulae the following holds:*

$$T \cap \bigvee_{\varphi\in\Gamma} \mathrm{Th}_{\mathrm{L}}(\varphi) = \bigvee_{\varphi\in\Gamma} (T \cap \mathrm{Th}_{\mathrm{L}}(\varphi)).$$

PROOF. To prove that 1 implies 2 we use Theorem 4.7 to justify the two non-trivial equations in the following chain:

$$F \cap \bigvee_{G\in\mathcal{F}} G = \mathrm{Fi}(F) \cap \mathrm{Fi}(\bigcup_{G\in\mathcal{F}} G) = \mathrm{Fi}(F \nabla \bigcup_{G\in\mathcal{F}} G) = \mathrm{Fi}(\bigcup_{G\in\mathcal{F}} (F \nabla G)) =$$
$$= \mathrm{Fi}(\bigcup_{G\in\mathcal{F}} \mathrm{Fi}(F \nabla G)) = \mathrm{Fi}(\bigcup_{G\in\mathcal{F}} (F \cap G)) = \bigvee_{G\in\mathcal{F}} (F \cap G).$$

The proofs of the implication from 2 to 3 and the implication from 3 to 4 are trivial. To prove that 4 implies 1 we write a chain of equations. The first equality is trivial, the second is due to the framality of L:

$$\mathrm{Th}_{\mathrm{L}}(\Phi) \cap \mathrm{Th}_{\mathrm{L}}(\Psi) = \mathrm{Th}_{\mathrm{L}}(\Phi) \cap (\bigvee_{\psi\in\Psi} \mathrm{Th}_{\mathrm{L}}(\psi)) = \bigvee_{\psi\in\Psi} (\mathrm{Th}_{\mathrm{L}}(\Phi) \cap \mathrm{Th}_{\mathrm{L}}(\psi)) =$$

we continue by repeating the first step for Φ and wPCP:

$$= \bigvee_{\varphi\in\Phi,\psi\in\Psi} (\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi)) = \bigvee_{\varphi\in\Phi,\psi\in\Psi} \mathrm{Th}_{\mathrm{L}}(\varphi \nabla \psi) =$$

the rest of the proof is simple:

$$= \mathrm{Th}_{\mathrm{L}}(\bigcup_{\varphi\in\Phi,\psi\in\Psi} \mathrm{Th}_{\mathrm{L}}(\varphi \nabla \psi)) = \mathrm{Th}_{\mathrm{L}}(\bigcup_{\varphi\in\Phi,\psi\in\Psi} \varphi \nabla \psi) = \mathrm{Th}_{\mathrm{L}}(\Phi \nabla \Psi). \quad \square$$

COROLLARY 4.9 (Transfer of framality). *Let* L *be a weakly p-disjunctional logic. If* L *is framal, then it is filter-framal.*

Next proposition shows that, also in the general (non-finitary) case, some of the implications of Theorem 4.8 (modulo necessary adjustments) can be proved for the weaker properties of proof by cases.

PROPOSITION 4.10. *If* ∇ *has the (τ-)*wPCP *and* L *is (filter-)distributive, then* ∇ *has the (τ-)*PCP.

Proof. We show the proof for filters (that for theories is analogous). We write a chain of equalities $\mathrm{Fi}(X,x) \cap \mathrm{Fi}(X,y) = (\mathrm{Fi}(X) \vee \mathrm{Fi}(x)) \cap (\mathrm{Fi}(X) \vee \mathrm{Fi}(y)) = \mathrm{Fi}(X) \vee (\mathrm{Fi}(x) \cap \mathrm{Fi}(y)) = \mathrm{Fi}(X) \vee \mathrm{Fi}(x \nabla y) = \mathrm{Fi}(X, x \nabla y)$. $\square$

Remark 4.11. *Note that (as in the case of Theorem 4.8) to obtain (τ-) PCP we do not need full (filter-)distributivity,* $\mathrm{Fi}(X) \vee (\mathrm{Fi}(x) \cap \mathrm{Fi}(y)) = \mathrm{Fi}(X,x) \cap \mathrm{Fi}(X,y)$ *would be sufficient.*

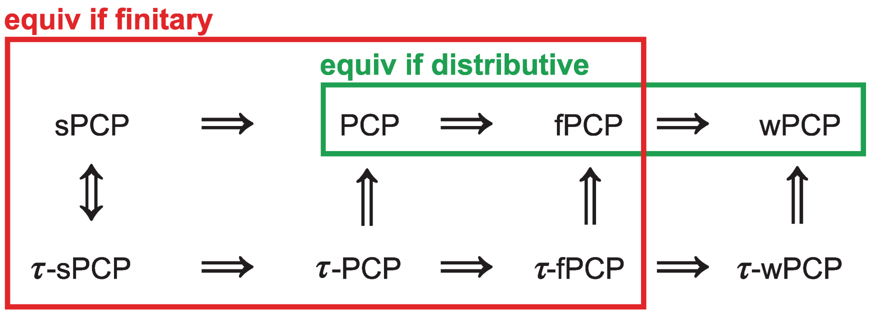

Figure 2. (Transferred) proof by cases properties in finitary and distributive logics.

This proposition together with Theorem 4.8 allows us to depict at Figure 2 the mutual relationship of the four forms of proof by cases property, their transfer variants, and (filter-)distributivity/framality. Note that all the properties depicted in Figure 2 are equivalent in framal logics.

4.3. Proof by cases and properties of prime filters

In this subsection we study semantical characterizations of p-disjunctions in terms of a convenient notion of prime filter and its corresponding extension principle. We show a strong link between this notion of filter and relatively finitely subdirectly irreducible reduced models of the logic.

Definition 4.12. *Let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra and $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. Then, F is called* ∇-prime *if for every $a, b \in A$, $a \nabla^{\boldsymbol{A}} b \subseteq F$ implies $a \in F$ or $b \in F$.*

We say that L *has the* (transferred) prime extension property, (τ-)PEP *for short, if ∇-prime theories form a basis of* Th(L) *(∇-prime filters form a basis of $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ for each $\mathcal{L}$-algebra $\boldsymbol{A}$, respectively).*

Finally, let us by $\mathbf{MOD}^{p}_{\nabla}(\mathrm{L})$ *denote the class of reduced models of* L *whose filter is ∇-prime.*

Notice that our definition naturally extends the usual notion of prime filter. Also note that the fact that ∇ is a p-protodisjunction gives the trivial converse direction: $a \in F$ or $b \in F$ implies $a \nabla^{\boldsymbol{A}} b \subseteq F$.

LEMMA 4.13. *Let ∇ be a (p-)protodisjunction and $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ a matrix where F is ∇-prime. Then $h^{-1}[F]$ is ∇-prime for every strict (surjective) homomorphism h.*

PROOF. Let us assume that $h\colon \langle \boldsymbol{B}, G\rangle \to \langle \boldsymbol{A}, F\rangle$ is a strict (surjective) homomorphism and assume that $G = h^{-1}[F]$ is not ∇-prime, i.e. there are $a, b \notin h^{-1}[F]$ and $a \nabla^{\boldsymbol{B}} b \subseteq h^{-1}[F]$. Thus $h(a), h(b) \notin F$ and $h[a \nabla^{\boldsymbol{B}} b] \subseteq F$. Using that h is surjective or that ∇ has no parameters, we get $h(a)\nabla^{\boldsymbol{A}} h(b) = h[a \nabla^{\boldsymbol{B}} b]$ and the proof is done. $\square$

PROPOSITION 4.14. *If ∇ has the* PEP*, then it enjoys the* sPCP*.*

PROOF. Assume that $\Phi \nabla \Psi \nvdash_{\mathrm{L}} \chi$, then using the PEP there has to be a ∇-prime theory $T \supseteq \mathrm{Th}_{\mathrm{L}}(\Phi \nabla \Psi)$ such that $T \nvdash_{\mathrm{L}} \chi$. First assume that $\Phi \subseteq T$. Then $\Phi \nvdash_{\mathrm{L}} \chi$ and the proof is done. Assume otherwise that there is $\varphi \in T \setminus \Phi$. Since $\varphi \nabla \psi \subseteq T$ for each $\psi \in \Psi$ and T is ∇-prime, we obtain that $\Psi \subseteq T$ and so $\Psi \nvdash_{\mathrm{L}} \chi$. $\square$

LEMMA 4.15. *Any ∇-prime filter is intersection-prime. If ∇ has the (τ-)* PCP*, then every intersection-prime theory (every intersection-prime filter in every $\mathcal{L}$-algebra) is ∇-prime.*

PROOF. First assume that F is not intersection-prime; i.e. $F = F_1 \cap F_2$ for some $F_i \supsetneq F$. Let us consider $a_i \in F_i \setminus F$. Thus, by (PD), we know that $a_1 \nabla^{\boldsymbol{A}} a_2 \subseteq F_i$ and so $a_1 \nabla^{\boldsymbol{A}} a_2 \subseteq F$, i.e. F is not ∇-prime.

We show the proof of the second claim for filters (for theories it is the same). Consider any $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ and assume first that F is not ∇-prime, i.e. there are $x \notin F$ and $y \notin F$ such that $x \nabla^{\boldsymbol{A}} y \subseteq F$. By τ-PCP we know that $F = \mathrm{Fi}(F, x \nabla^{\boldsymbol{A}} y) = \mathrm{Fi}(F, x) \cap \mathrm{Fi}(F, y)$, i.e. F is the intersection of two strictly bigger filters. $\square$

The proofs of the next corollary and theorem are simple consequences of the previous proposition and lemma.

COROLLARY 4.16. *A (p-)protodisjunction ∇ has (τ-)*PEP*, if and only if, it has (τ-)*IPEP *and (τ-)*PCP*.*

THEOREM 4.17. *Let* L *be a logic satisfying the* IPEP*. Then the following are equivalent:*

1. *∇ has the* sPCP*,*
2. *∇ has the* PCP*,*
3. *∇ has the* PEP*.*

Note that if in addition L satisfies the τ-IPEP, then we can add one more equivalent condition, namely that ∇ has the τ-PEP.

The next corollary shows that [8, Proposition 2.5.1.] (saying that distributivity implies framality in finitary logics) can be brought to the context of weakly p-disjunctional logics with IPEP (not necessarily finitary). This corollary can be also seen as transfer of distributivity (previously known only for finitary protoalgebraic logics).

COROLLARY 4.18. *Let* L *be a weakly p-disjunctional logic satisfying the* IPEP. *If* L *is distributive, then it is filter-framal.*

PROOF. By Proposition 4.10 we know that if L is distributive, ∇ has the PCP, so by the previous theorem it has the sPCP, and finally by Theorem 4.8 L is filter-framal. □

The next two theorems are both straightforward generalizations of Theorems 2.5.8. and 2.5.9. of [8], from finitary logics to logics with the IPEP. The proof of the first one easily follows from Theorem 4.8 (using Theorem 4.17 and Remark 4.11).

THEOREM 4.19 (Characterizations of PCP). *Let* L *be a logic satisfying the* IPEP. *Then the following are equivalent:*

1. ∇ *has the* PCP,
2. ∇ *has the* wPCP *and* L *is filter-distributive,*
3. ∇ *has the* wPCP *and* L *is distributive,*
4. ∇ *has the* wPCP *and for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae holds:*

$$\mathrm{Th}_{\mathrm{L}}(\Gamma \cup (\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi))) = \mathrm{Th}_{\mathrm{L}}(\Gamma, \varphi) \cap \mathrm{Th}_{\mathrm{L}}(\Gamma, \psi).$$

THEOREM 4.20 (Characterizations of (p-)disjunctional logics). *Let* L *be a logic satisfying the* IPEP. *Then the following are equivalent:*

1. L *is (p-)disjunctional,*
2. L *is filter-distributive and* $h^{-1}[F]$ *is an intersection-prime filter for every matrix* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ *where* F *is intersection-prime and every strict (surjective) homomorphism* h *from any matrix to* $\langle \boldsymbol{A}, F\rangle$,
3. L *is distributive and* $\sigma^{-1}[T]$ *is an intersection-prime theory for every intersection-prime theory* T *and every (surjective) substitution* σ.

PROOF. We show that 1 implies 2. From Theorem 4.17 we know that there is a strong (p-)disjunction ∇. Therefore, by Theorem 4.8 we obtain that L is filter-distributive. On the other hand, from Theorem 4.7, we know that ∇ satisfies the τ-PCP and so Lemmata 4.15 and 4.13 complete the proof.

Clearly 2 implies 3. To prove that 3 implies 1 it is enough (thanks to Propositon 4.10) to show that L is weakly (p-)disjunctional by using Theorem 4.1. Let us fix a (surjective) substitution σ and formulae φ, ψ. Observe that one inclusion is trivial and, since we assume the IPEP, to prove the reverse one it suffices to show that for each intersection-prime theory T,

$$\text{if} \quad \sigma[\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi)] \subseteq T, \quad \text{then} \quad \mathrm{Th}_{\mathrm{L}}(\sigma\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\sigma\psi) \subseteq T.$$

We have this chain of equivalent statements: $\sigma[\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi)] \subseteq T$ iff $\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi) \subseteq \sigma^{-1}[T]$ iff $(\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi)) \vee \sigma^{-1}[T] = \sigma^{-1}[T]$ iff (due to distributivity) $\mathrm{Th}_{\mathrm{L}}(\sigma^{-1}[T], \varphi) \cap \mathrm{Th}_{\mathrm{L}}(\sigma^{-1}[T], \psi) = \sigma^{-1}[T]$ iff (since $\sigma^{-1}[T]$ is intersection-prime) $\mathrm{Th}_{\mathrm{L}}(\sigma^{-1}[T], \varphi) = \sigma^{-1}[T]$ or $\mathrm{Th}_{\mathrm{L}}(\sigma^{-1}[T], \psi) = \sigma^{-1}[T]$ iff $\varphi \in \sigma^{-1}[T]$ or $\psi \in \sigma^{-1}[T]$ iff $\sigma\varphi \in T$ or $\sigma\psi \in T$. The last condition clearly implies $\mathrm{Th}_{\mathrm{L}}(\sigma\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\sigma\psi) \subseteq T$. □

REMARK 4.21. *Note that the following are equivalent:*

- *for every matrix* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ *where* F *is intersection-prime,* $h^{-1}[F]$ *is also intersection-prime for* every *strict homomorphism* h *from any matrix to* $\langle \boldsymbol{A}, F\rangle$.
- *for every matrix* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ *where* F *is intersection-prime,* $h^{-1}[F]$ *and* F' *are also intersection-prime for* every surjective *strict homomorphism* h *from any matrix to* $\langle \boldsymbol{A}, F\rangle$, *and for every submatrix* $\langle \boldsymbol{A}', F'\rangle \subseteq \langle \boldsymbol{A}, F\rangle$.

Thus a p-disjunctional logic with the IPEP *is disjunctional iff for every matrix* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ *and any of its submatrices* $\langle \boldsymbol{A}', F'\rangle$ *holds that if* F *is intersection-prime, then so is* F'.

PROPOSITION 4.22 (∇-prime completeness). *Let* ∇ *be a p-protodisjunction with the* PEP. *Then* $\mathrm{L} = \models_{\mathbf{MOD}^{p}_{\nabla}(\mathrm{L})}$.

PROOF. From Proposition 4.14 we know that the PEP implies the PCP, so (by Lemma 4.15) ∇-prime and intersection-prime theories coincide and hence L enjoys the IPEP. This (by Lemma 2.6) implies RFSI-completeness, which is exactly what we needed (because, by Proposition 4.14 and Theorem 4.7, the PEP implies also the τ-PCP and so ∇-prime and intersection-prime filters coincide as well). □

The converse direction can be easily proved in the parameter-free case (whether it holds in the parameterized case appears to be an open problem).

PROPOSITION 4.23. *A protodisjunction* ∇ *enjoys the* PEP *if, and only if,* $\mathrm{L} = \models_{\mathbf{MOD}^p_\nabla(\mathrm{L})}$.

PROOF. Assume that $T \not\vdash_{\mathrm{L}} \chi$. Thus there is an $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^p_\nabla(\mathrm{L})$ and e such that $e[T] \subseteq F$ and $e(\chi) \notin F$. Define $T' = e^{-1}[F]$. Clearly, T' is a theory, $T' \supseteq T$, and $\chi \notin T'$ and by Lemma 4.13 T' is ∇-prime. □

EXAMPLE 4.24. *The standard infinite-valued Łukasiewicz logic is strongly disjunctive.* Recall the logic Ł introduced in Example 2.7 via the matrix $[0,1]_{\text{Ł}}$. We define $\varphi \vee \psi$ as $(\varphi \to \psi) \to \psi$ and easily compute that $x \vee^{[0,1]_{\text{Ł}}} y = \max\{x, y\}$. Then clearly $\{1\}$ is a $\vee$-prime filter and thus, by the previous proposition, $\vee$ enjoys the PEP and the sPCP.

4.4. Proof by cases properties in protoalgebraic logics

In this subsection we restrict the scope of our study to protoalgebraic logics in order to obtain stronger results. We start with the generalization of one implication in [8, Theorem 2.5.17.] to all (not necessarily finitary) logics.

THEOREM 4.25. *Every protoalgebraic distributive logic is p-disjunctional.*

PROOF. We use Theorem 4.1 to show that there is weak a p-disjunction ∇; Proposition 4.10 then completes the proof.

Let h be a surjective substitution. Let X be the set of theorems of L, $Y = h^{-1}[X]$, $\mathbf{M} = \langle \boldsymbol{Fm}_{\mathcal{L}}, Y\rangle$, $\mathbf{N} = \langle \boldsymbol{Fm}_{\mathcal{L}}, X\rangle$. Clearly $h\colon \mathbf{M} \to \mathbf{N}$ is a strict surjective homomorphism. From [8, Theorem 1.1.8] we know that $\mathbf{h}$ defined as $\mathbf{h}(F) = h[F]$ is an isomorphism from $[Y, Fm_{\mathcal{L}}]$ to $\mathrm{Th}(\mathrm{L})$.[10] Now, we have the following chain of equalities for every Z: $\mathrm{Th}_{\mathrm{L}}(h[Z]) = \mathrm{Th}_{\mathrm{L}}(h[\mathrm{Th}_{\mathrm{L}}(Y, Z)]) = h[\mathrm{Th}_{\mathrm{L}}(Y, Z)] = h[Y \vee \mathrm{Th}_{\mathrm{L}}(Z)]$ (the first equality is [8, Lemma 0.8.4 (v)], the second follows from $\mathrm{Th}_{\mathrm{L}}(Y, Z) \in [Y, Fm_{\mathcal{L}}]$, and the last one is trivial).

Therefore we have: $\mathrm{Th}_{\mathrm{L}}(hp) \cap \mathrm{Th}_{\mathrm{L}}(hq) = h[Y \vee \mathrm{Th}_{\mathrm{L}}(p)] \cap h[Y \vee \mathrm{Th}_{\mathrm{L}}(q)] = h[(Y \vee \mathrm{Th}_{\mathrm{L}}(p)) \cap (Y \vee \mathrm{Th}_{\mathrm{L}}(q))] = h[Y \vee (\mathrm{Th}_{\mathrm{L}}(p) \cap \mathrm{Th}_{\mathrm{L}}(q))] = \mathrm{Th}_{\mathrm{L}}(h[\mathrm{Th}_{\mathrm{L}}(p) \cap \mathrm{Th}_{\mathrm{L}}(q)])$ (we use the preceding observation, isomorphism of h, distributivity, and the observation again). □

This theorem together with Corollary 4.18 give us:

[10] This property (for all matrices) is in fact equivalent to protoalgebraicity of L.

Corollary 4.26. *Let* L *be a protoalgebraic logic satisfying the* IPEP*. If* L *is distributive, then it is filter-framal.*

This corollary allows us to obtain in the following theorem a generalization from finitary to IPEP logics of: (1) the other implication in [8, Theorem 2.5.17.], and (2) the transfer of distributivity [8, Theorem 2.5.24.].

Theorem 4.27 (Transfer of distributivity). *Let* L *be a protoalgebraic logic satisfying the* IPEP*. Then the following are equivalent:*

1. L *is filter-distributive,*
2. L *is distributive,*
3. L *is p-disjunctional.*

Due to the fact that framal logics are distributive we can use Theorem 4.25 together with Theorem 4.8 to obtain another important theorem, which can be seen as a different generalization of the aforementioned theorems from [8]. This time we need not restrict the scope to IPEP logics, but we need to replace the used notions, distributivity and PCP, with the better suited (and in IPEP logics equivalent) notions of framality and sPCP.

Theorem 4.28 (Transfer of framality). *Let* L *be protoalgebraic logic. Then the following are equivalent:*

1. L *is filter-framal,*
2. L *is framal,*
3. L *is strongly p-disjunctional.*

We conclude this subsection with the proof of a transfer theorem for the PCP when restricted to protoalgebraic logics. This interesting fact seems to have surprisingly little consequences for the general theory. At least, it allows us in the parameter-free case (using Proposition 4.23 and Lemma 4.15) to generalize all the previous results in this subsection for IPEP logics to the larger class of RFSI-complete logics (unfortunately we do not know if this class is actually strictly larger). In the same fashion, Theorem 4.17 can also be generalized to RFSI-complete logics.

Proposition 4.29 (Transfer of PCP). *Let* L *be a protoalgebraic logic in a countable language* $\mathcal{L}$*. If* ∇ *has the* PCP*, then it has the* τ-PCP*.*

Proof. This proof assumes some familiarity with properties of protoalgebraic logics; in particular we utilize parameterized equivalence sets of formulae denoted as $\Leftrightarrow$. Let us fix an $\mathcal{L}$-algebra $\boldsymbol{A}$. To prove the non-trivial inclusion we show that for each $t \notin \mathrm{Fi}(X, a \nabla^{\boldsymbol{A}} b)$ we have $t \notin \mathrm{Fi}(X, a)$ or $t \notin \mathrm{Fi}(X, b)$. We distinguish two cases based on the cardinality of A.

1) Firstly assume that $|A|$ is countable.[11] We can assume that the set **VAR** of propositional variables contains (or is equal to) the set $\{v_z \mid z \in A\}$. Consider the following set of formulae:

$$\Gamma = \{v_z \mid z \in \mathrm{Fi}(X, a \nabla^{\boldsymbol{A}} b)\} \cup \bigcup_{\langle c,n\rangle \in \mathcal{L}} \{c(v_{z_1}, \dots, v_{z_n}) \Leftrightarrow v_{c^{\boldsymbol{A}}(z_1,\dots,z_n)} \mid z_i \in A\}.$$

Clearly, $\Gamma, v_a \nabla v_b \nvdash_{\mathrm{L}} v_t$ (because $\langle \boldsymbol{A}, \mathrm{Fi}(X, a \nabla^{\boldsymbol{A}} b)\rangle \in \mathbf{MOD}(\mathrm{L})$ and for the $\boldsymbol{A}$-evaluation $e(v_z) = z$ we obtain $e[\Gamma, v_a \nabla v_b] \subseteq \mathrm{Fi}(X, a \nabla^{\boldsymbol{A}} b)$ and $e(v_t) \notin \mathrm{Fi}(X, a \nabla^{\boldsymbol{A}} b)$). Thus by the PCP we have $\Gamma, v_a \nvdash_{\mathrm{L}} v_t$ or $\Gamma, v_b \nvdash_{\mathrm{L}} v_t$. Assume (without loss of generality) the former case and denote $T' = \mathrm{Th}_{\mathrm{L}}(\Gamma, v_a)$.

We show that the mapping $h\colon \boldsymbol{A} \to \boldsymbol{Fm}_{\mathcal{L}}/\Omega T'$ defined as $h(z) = [v_z]_{T'}$ is a homomorphism by a simple chain of equalities: $h(c^{\boldsymbol{A}}(z_1, \dots, z_n)) = [v_{c^{\boldsymbol{A}}(z_1,\dots,z_n)}]_{T'} = [c(v_{z_1}, \dots, v_{z_n})]_{T'} = c^{\boldsymbol{Fm}_{\mathcal{L}}/\Omega T'}([v_{z_1}]_{T'}, \dots, [v_{z_n}]_{T'}) = c^{\boldsymbol{Fm}_{\mathcal{L}}/\Omega T'}(h(z_1), \dots, h(z_n))$. Thus $F = h^{-1}([T']) \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ and, since clearly $X \cup \{a\} \subseteq F$ and $t \notin F$, we have established that $t \notin \mathrm{Fi}(X, a)$.

2) *Secondly assume that A is uncountable.* We introduce a new set of propositional variables $\mathbf{VAR}' = \{v_z \mid z \in A\}$ which can be safely assumed to contain the original set **VAR**. We define a new logic L' in the language $\mathcal{L}'$ which has the same connectives as $\mathcal{L}$ and atoms from $\mathbf{VAR}'$. If we show that this logic has the PCP, then we can repeat the constructions from the first part of this proof. From our assumption we know that there is a presentation $\mathcal{AS}$ of L such that each of its rules has countably many premises.

We define $\mathcal{AS}' = \{\sigma[\Sigma] \rhd \sigma(\varphi) \mid \Sigma \rhd \sigma \text{ is an } \mathcal{L}'\text{-substitution}, \varphi \in \mathcal{AS}\}$ and $\mathrm{L}' = \vdash_{\mathcal{AS}'}$. Observe that $\Gamma \vdash_{\mathrm{L}'} \varphi$ iff there is a countable set $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}'} \varphi$ (clearly any proof in $\mathcal{AS}'$ has countably many leaves, because all of its rules have countably many premises). Next observe that L' is a conservative expansion of L (consider the substitution σ sending all atoms from **VAR** to themselves and the rest to a fixed $p \in \mathbf{VAR}$, take any proof of φ from Γ in $\mathcal{AS}'$ and observe that the same tree with labels ψ replaced by $\sigma\psi$ is a proof of φ from Γ in L).

We show that L' has the PCP: assume $\Gamma, \varphi \vdash_{\mathrm{L}'} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}'} \chi$. There is a countable subset $\Gamma' \subseteq \Gamma$ such that $\Gamma', \varphi \vdash_{\mathrm{L}'} \chi$ and $\Gamma', \psi \vdash_{\mathrm{L}'} \chi$. Consider the set $\mathbf{VAR}_0$ of variables occurring in $\Gamma' \cup \{\varphi, \psi, \chi\}$ and a bijection g on the set $\mathbf{VAR}'$ such that the image of $\mathbf{VAR}_0$ is a subset of **VAR** (such bijection clearly exists). Thus for the $\mathcal{L}'$-substitution σ induced by g there exists an inverse substitution σ^{-1} and $\sigma[\Gamma'] \cup \{\sigma\varphi, \sigma\psi, \sigma\chi\} \subseteq Fm_{\mathcal{L}}$. Clearly also $\sigma[\Gamma'], \sigma\varphi \vdash_{\mathrm{L}'} \sigma\chi$ and $\sigma[\Gamma'], \sigma\psi \vdash_{\mathrm{L}'} \sigma\chi$. Using the fact that L'

[11] In this proof we will, for simplicity, assume that the set **VAR** of propositional variables is denumerable. The proofs for arbitrary infinite cardinalities would be analogous.

expands L conservatively, we obtain $\sigma[\Gamma'], \sigma\varphi \vdash_{\mathrm{L}} \sigma\chi$ and $\sigma[\Gamma'], \sigma\psi \vdash_{\mathrm{L}} \sigma\chi$. From the PCP of L we know that $\sigma[\Gamma'], \sigma\varphi \nabla_{\mathcal{L}} \sigma\psi \vdash_{\mathrm{L}} \sigma\chi$ and thus also $\sigma[\Gamma'], \sigma\varphi \nabla_{\mathcal{L}} \sigma\psi \vdash_{\mathrm{L}'} \sigma\chi$. Thus by structurality for the inverse substitution σ^{-1} also $\Gamma', \sigma^{-1}[\sigma\varphi \nabla_{\mathcal{L}} \sigma\psi] \vdash_{\mathrm{L}'} \chi$. Observe that $\sigma^{-1}[\sigma\varphi \nabla_{\mathcal{L}} \sigma\psi] \subseteq \varphi \nabla'_{\mathcal{L}} \psi$ completes the proof (indeed $\chi \in \sigma^{-1}[\sigma\varphi \nabla_{\mathcal{L}} \sigma\psi]$ if there is $\delta(p, q, \vec{r}) \in \nabla$ and a sequence $\alpha_1, \dots, \alpha_n$ of $\mathcal{L}$-formulae and $\chi = \sigma^{-1}\delta(\sigma\varphi, \sigma\psi, \alpha_1, \dots, \alpha_n)$, thus $\chi = \delta(\varphi, \psi, \sigma^{-1}\alpha_1, \dots, \sigma^{-1}\alpha_n)$ and so clearly $\chi \in \varphi \nabla'_{\mathcal{L}} \psi$). $\square$

5. Applications

5.1. Proof by cases properties in expansions of a given logic

We start by characterizing under which conditions the sPCP is preserved in expansions of a given strongly p-disjunctional logic.

THEOREM 5.1 (Preservation of sPCP). *Let* L_1 *be a logic in a language* $\mathcal{L}_1$ *with the* sPCP*, and* L_2 *an expansion of* L_1 *in a language* $\mathcal{L}_2 \supseteq \mathcal{L}_1$ *by a set* $\mathcal{C}$ *of consecutions closed under* $\mathcal{L}_2$*-substitutions. Then* L_2 *enjoys the* sPCP *iff* $R^{\nabla} \subseteq \mathrm{L}_2$ *for each* $R \in \mathcal{C}$*. In particular, the* sPCP *is preserved in axiomatic expansions.*

PROOF. The left-to-right direction is a straightforward application of Theorem 4.5. For the reverse direction take a presentation $\mathcal{AS}$ of L_1. We know that L_2 has a presentation $\mathcal{AS}' = \{\sigma[\Gamma] \rhd \sigma\varphi \mid \sigma \text{ is an } \mathcal{L}_2\text{-substitution}, \Gamma \rhd \varphi \in \mathcal{AS} \cup \mathcal{C}\}$. Thus we need to prove that for each $\Gamma \rhd \varphi \in \mathcal{AS} \cup \mathcal{C}$ and for each $\mathcal{L}_2$-substitution σ we have $(\sigma[\Gamma] \rhd \sigma\varphi)^{\nabla} \subseteq \mathrm{L}_2$, i.e. for each $\mathcal{L}_2$-formula χ, each $\delta(p, q, r_1, \dots, r_n) \in \nabla$ and each sequence $\alpha_1, \dots, \alpha_n$ of $\mathcal{L}_2$-formulae we have $\sigma[\Gamma] \nabla \chi \vdash_{\mathrm{L}_2} \delta(\sigma\varphi, \chi, \alpha_1, \dots, \alpha_n)$. If $\Gamma \rhd \varphi \in \mathcal{C}$, this is the assumption; we solve the remaining case.

Consider any enumeration of the propositional variables such that $p_0 = q$, $p_i = r_i$, and $\mathcal{L}_1$-substitutions ρ, ρ^{-1} and $\mathcal{L}_2$-substitution $\bar{\sigma}$ defined as:

- $\rho p_i = p_{i+n+1}$,
- $\rho^{-1} p_i = p_{i-n-1}$ for $i > n$ and p_i otherwise,
- $\bar{\sigma} p_i = \sigma(p_{i-n-1})$ for $i > n$, $\bar{\sigma} p_i = \alpha_i$ for $1 \leq i \leq n$ and $\bar{\sigma} p_0 = \chi$.

Observe that $\rho^{-1}\rho\psi = \psi$ and $\bar{\sigma}\rho\psi = \sigma\psi$. From $\Gamma \rhd \varphi \in \mathcal{AS}$ we know that $\rho[\Gamma] \rhd \rho\varphi \in \mathcal{AS}$ and because clearly $(\rho[\Gamma] \rhd \rho\varphi)^{\nabla} \subseteq \mathrm{L}_1 \subseteq \mathrm{L}_2$ we obtain: $\rho[\Gamma] \nabla q \vdash_{\mathrm{L}_2} \delta(\rho\varphi, q, r_1, \dots, r_n)$. Thus $\bar{\sigma}[\rho[\Gamma] \nabla q] \vdash_{\mathrm{L}_2} \bar{\sigma}\delta(\rho\varphi, q, r_1, \dots, r_n)$. Obviously, $\bar{\sigma}\delta(\rho\varphi, q, r_1, \dots, r_n) = \delta(\sigma\varphi, \chi, \alpha_1, \dots, \alpha_n)$, if we prove $\bar{\sigma}[\rho[\Gamma] \nabla q] \subseteq \sigma[\Gamma] \nabla \chi$ the proof is done. To show this it is enough to observe that the formulae in $\bar{\sigma}[\rho[\Gamma] \nabla q]$ are of the form $\delta'(\sigma\psi, \chi, \bar{\sigma}\alpha_1, \dots, \bar{\sigma}\alpha_k) \in \nabla$ for some $\psi \in \Gamma$, $\delta'(p, q, r_1, \dots, r_k) \in \nabla$ and a sequence of $\mathcal{L}_2$-formulae $\alpha_1, \dots, \alpha_k$. $\square$

Analogous results can be shown for the preservation of fPCP and wPCP in expansions by restricting the condition to finitary consecutions or consecutions with only one premise, respectively. This theorem together with Lemma 2.8 and Corollary 4.16 give us:

THEOREM 5.2 (Preservation of PEP). *Let* L' *be an axiomatic extension of* L. *If* L *has the* PEP *then so has* L'.

Observe that if we take a system of logics where ∇ has the sPCP, ∇ will retain the sPCP in the intersection of the system. Also observe that, trivially, any set ∇ has the sPCP in the inconsistent logic. Thus, the following definition is sound:

DEFINITION 5.3. *Let* L *be a logic and* ∇ *a p-protodisjunction. We denote by* L^{∇} *the least logic extending* L *where* ∇ *has the* sPCP.

We sometimes refer to L^{∇} as the ∇-*extension of* L. The next proposition shows that the ∇-extension of a finitary logic is finitary, and then we characterize this logic both syntactically and semantically. Unfortunately in both cases we need to restrict to parameter-free protodisjunctions; the question whether this restriction can be omitted is left open.

PROPOSITION 5.4. *Let* L *be a finitary logic. Then* L^{∇} *is finitary and* L^{∇} *is the intersection of all finitary extensions of* L *where* ∇ *has the* sPCP.

PROOF. Recall the notion of finitary companion of a logic S, denoted as $\mathcal{FC}(\mathrm{S})$, which is the largest finitary logic contained in S. Thus, since L is finitary, we know that $\mathrm{L} \subseteq \mathcal{FC}(\mathrm{L}^{\nabla}) \subseteq \mathrm{L}^{\nabla}$. If we show that ∇ has the sPCP in $\mathcal{FC}(\mathrm{L}^{\nabla})$, we obtain $\mathcal{FC}(\mathrm{L}^{\nabla}) = \mathrm{L}^{\nabla}$ and hence L^{∇} is finitary. Actually, one can easily show in general that if ∇ has the sPCP in S, then it has the sPCP in $\mathcal{FC}(\mathrm{S})$ as well. $\square$

PROPOSITION 5.5. *Let* L *be a logic and* ∇ *a protodisjunction such that* L^{∇} *has the* IPEP.[12] *Then:*

$$\mathrm{L}^{\nabla} = \models_{\mathbf{MOD}^{p}_{\nabla}(\mathrm{L})}.$$

PROOF. First observe that, since the notion of ∇-primality does not depend on the logic, we have: $\mathbf{MOD}^{p}_{\nabla}(\mathrm{L}) = \mathbf{MOD}^{p}_{\nabla}(\models_{\mathbf{MOD}^{p}_{\nabla}(\mathrm{L})})$. Then, due to Propositions 4.23 and 4.14, ∇ has the sPCP in $\models_{\mathbf{MOD}^{p}_{\nabla}(\mathrm{L})}$. From the assumption that L^{∇} has the IPEP, Theorem 4.17, and Proposition 4.23 we know that $\mathrm{L}^{\nabla} = \models_{\mathbf{MOD}^{p}_{\nabla}(\mathrm{L}^{\nabla})}$. Since clearly $\mathbf{MOD}^{p}_{\nabla}(\mathrm{L}^{\nabla}) \subseteq \mathbf{MOD}^{p}_{\nabla}(\mathrm{L})$, we have that $\models_{\mathbf{MOD}^{p}_{\nabla}(\mathrm{L})} \subseteq \mathrm{L}^{\nabla}$ and the proof is done. $\square$

[12] Note that, thanks to the previous proposition, L^{∇} has the IPEP whenever L is finitary.

In the parameterized case we could only prove a weaker statement:

$$\mathrm{L}^{\nabla} = (\models_{\mathbf{MOD}^p_{\nabla}(\mathrm{L})})^{\nabla}.$$

THEOREM 5.6. *Let* L *be a logic with a presentation* $\mathcal{AS}$ *and* ∇ *a protodisjunction satisfying* (C), (I), *and* (A). *Then the logic* L^{∇} *is axiomatized by* $\mathcal{AS} \cup \bigcup\{R^{\nabla} \mid R \in \mathcal{AS}\}$.

PROOF. Let $\hat{\mathrm{L}}$ denote the logic axiomatized by $\mathcal{AS} \cup \bigcup\{R^{\nabla} \mid R \in \mathcal{AS}\}$ (this set is closed under all substitutions because we assume ∇ to be parameter-free). Clearly, for each R from this axiomatic system we have $R^{\nabla} \subseteq \hat{\mathrm{L}}$ (due to Lemma 4.4 part 2), hence we can use Theorem 4.6 to obtain that $\hat{\mathrm{L}}$ has the sPCP.

Let L' be any logic with the sPCP extending L. Notice that for any $R \in \mathcal{AS}$ we have both $R \in \mathrm{L}'$ and $R^{\nabla} \subseteq \mathrm{L}'$ (due to Theorem 4.5). Thus clearly $\hat{\mathrm{L}} \subseteq \mathrm{L}'$. □

Observe that we could relax some of the (C), (I), (A) conditions if we would add them and their ∇-forms to obtain the axiomatization of L^{∇}.

5.2. Axiomatization of positive universal classes

Let us recall that matrices can be regarded as first-order structures where the filter corresponds to a unary predicate F, i.e. all atomic formulae in the corresponding classical first-order language are of the form $F(\varphi)$ where φ is a formula. Recall that a positive clause C is a disjunction $\bigvee_{\varphi\in\Sigma_C} F(\varphi)$ of finitely many atomic formulae. A set of positive clauses $\mathcal{C}$ is said to be valid in a matrix $\mathbf{M} = \langle \boldsymbol{A}, F\rangle$, written as $\mathbf{M} \models \mathcal{C}$, if for each $C \in \mathcal{C}$ and each $\mathbf{M}$-evaluation e there is a $\varphi \in \Sigma_C$ such that $e(\varphi) \in F$. A positive universal class of matrices is the collection of all models of a set of universal closures of positive clauses.[13] The next theorem presents an axiomatization, by means of a p-disjunction, of any logic given by a positive universal class of (RFSI) matrices.

THEOREM 5.7. *Let* L *be a logic with the* IPEP, ∇ *a p-disjunction, and* $\mathcal{C}$ *a set of positive clauses. Then:*

$$\models_{\{\mathbf{A}\in\mathbf{MOD}^*(\mathrm{L}) \;\mid\; \mathbf{A}\models\mathcal{C}\}} = \mathrm{L} + \bigcup\{\nabla_{\psi\in\Sigma_C}\psi \mid C \in \mathcal{C}\}.$$

[13]Positive universal classes are usually defined as the collection of all models of a set of *positive universal formulae*, i.e. the universal closure of formulae build from atoms using conjunction and disjunction. Clearly each formula of this kind can be written as the universal closure of a conjunction of positive clauses and so its generated positive universal class is just the positive universal class generated by the collection of these positive clauses.

PROOF. Let us first denote the set of formulae[14] $\nabla_{\psi\in\Sigma_C}\psi$ as C_∇ and observe that for each matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ we have: if $\mathbf{A} \models C$, then $\models_{\mathbf{A}} C_\nabla$. Moreover, if F is ∇-prime, the reverse implication holds as well.

This observation, together with the fact that $\mathrm{L} + \bigcup\{\nabla_{\psi\in\Sigma_C}\psi \mid C \in \mathcal{C}\}$ is complete w.r.t. ∇-prime matrices (this follows from Theorems 5.1 and 4.17, Proposition 4.22 and Corollary 5.2) completes the proof. □

Note that, in fact, we could have proved the following:

$$\models_{\{\mathbf{B}\in\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}\ \mid\ \mathbf{B}\models\mathcal{C}\}} = \mathrm{L} + \bigcup\{\nabla_{\psi\in\Sigma_C}\psi \mid C \in \mathcal{C}\}.$$

This theorem generalizes and explicates [17, Corollary 3.4.] which was restricted to the framework of extensions of the logic FL.

COROLLARY 5.8. *Let* L *be a logic with the* IPEP, ∇ *a p-disjunction, and let* L_1, L_2 *be axiomatic extensions of* L *by sets of axioms* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*, respectively. Without loss of generality we can assume that* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are written in disjoint sets of variables.*[15] *Then:*

$$\mathrm{L}_1 \cap \mathrm{L}_2 = \mathrm{L} + \bigcup\{\varphi \nabla \psi \mid \varphi \in \mathcal{A}_1, \psi \in \mathcal{A}_2\}.$$

PROOF. Recall that $\mathrm{L}_1 \cap \mathrm{L}_2 = \models_{\mathbf{MOD}^*(\mathrm{L}_1)\cup\mathbf{MOD}^*(\mathrm{L}_2)}$ and denote: $\mathcal{A} = \{F(\varphi)\vee F(\psi) \mid \varphi \in \mathcal{A}_1, \psi \in \mathcal{A}_2\}$. If we show that $\mathbf{MOD}^*(\mathrm{L}_1)\cup\mathbf{MOD}^*(\mathrm{L}_2) = \{\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L}) \mid \mathbf{A} \models \mathcal{A}\}$, the proof is done by Theorem 5.7.

One inclusion is trivial. We prove the converse one counterpositively: consider $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ such that $\mathbf{A} \notin \mathbf{MOD}^*(\mathrm{L}_1) \cup \mathbf{MOD}^*(\mathrm{L}_2)$, i.e. there is $\varphi_i \in \mathcal{A}_i$ such that $\not\models_{\boldsymbol{A}} \varphi_i$. Consider evaluations e_i witnessing these facts. Since φ_1 and φ_2 do not share any propositional variable, there is an evaluation e witnessing both facts. This evaluation also shows that $\mathbf{A} \not\models F(\varphi_1) \vee F(\varphi_2)$. □

5.3. Axiomatization of non-negative universal classes

In this subsection we extend the results of the previous one at the price of restricting to finitary logics. A non-negative clause H is a formula (of

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[14] The extension of ∇ from a binary constructor to an operator applied to finite sets is well defined (as long as provability concerns) thanks to its associativity.

[15] To show this, let us fix an enumeration of propositional variables $\{p_i \mid i \leq |\mathrm{VAR}|\}$ and define substitutions $\sigma_1 p_i = p_{2i}$ and $\sigma_2 p_i = p_{2i+1}$. Clearly there are substitutions σ'_i such that $\sigma'_i\sigma_i\varphi = \varphi$. Thus for each set of axioms $\mathcal{A}$ holds: $\{\sigma[\mathcal{A}] \mid \sigma \text{ is a substitution}\} = \{\sigma[\sigma_i[\mathcal{A}]] \mid \sigma \text{ is a substitution}\}$ and so $\mathrm{L} + \mathcal{A}_i = \mathrm{L} + \sigma_i[\mathcal{A}_i]$. Clearly the set of variables occurring in $\sigma_1[\mathcal{A}_1]$ and the set of those occurring in $\sigma_2[\mathcal{A}_2]$ are disjoint.

classical predicate logic) of the form

$$\bigwedge_{\varphi\in\Gamma_H} F(\varphi) \to \bigvee_{\psi\in\Sigma_H} F(\psi)$$

where Σ_H, Γ_H are finite sets of atomic formulae and Σ_H is non-empty.

A set of non-negative clauses $\mathcal{H}$ is said to be valid in a matrix $\mathbf{M} = \langle \boldsymbol{A}, F\rangle$, written as $\mathbf{M} \models \mathcal{H}$, if for $H \in \mathcal{H}$ and each $\mathbf{M}$-evaluation e such that $e[\Gamma_H] \subseteq F$ there is some $\varphi \in \Sigma_H$ such that $e(\varphi) \in F$.

THEOREM 5.9. *Let* L *be a finitary logic,* ∇ *a p-protodisjunction, and* $\mathcal{H}$ *a set of non-negative clauses. Then:*

$$(\models_{\{\mathbf{B}\in\mathbf{MOD}^*(\mathrm{L})\ \mid\ \mathbf{B}\models\mathcal{H}\}})^{\nabla} = (\mathrm{L} + \bigcup\{\Gamma_H \rhd \nabla_{\psi\in\Sigma_H}\psi \mid H \in \mathcal{H}\})^{\nabla}.$$

PROOF. Let us first denote the consecution $\Gamma_H \rhd \nabla_{\psi\in\Sigma_H}\psi$ as H_∇ and observe that for each matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ we have: if $\mathbf{A} \models H$, then $\Gamma_H \models_{\mathbf{A}} \nabla_{\psi\in\Sigma_H}\psi$. Moreover, if F is ∇-prime, the reverse implication holds as well.

The first part of this observation tells us that

$$\models_{\{\mathbf{A}\in\mathbf{MOD}^*(\mathrm{L})\ \mid\ \mathbf{A}\models\mathcal{H}\}} \supseteq \mathrm{L} + \bigcup\{\Gamma_H \rhd \nabla_{\psi\in\Sigma_H}\psi \mid H \in \mathcal{H}\}.$$

Therefore, the same holds for their ∇-extensions. Next we use Proposition 5.4 to observe that the logic on the right-hand side is finitary and so it has the PEP (Corollary 4.16). Thus by Proposition 4.22, this logic is complete w.r.t. the class $\mathbb{K}$ of *its* reduced ∇-prime matrices. Clearly $\mathbb{K} \subseteq \mathbf{MOD}^p_\nabla(\mathrm{L})$ and thus we can use the second part of the previous observation, to show that $\mathbb{K} \subseteq \{\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L}) \mid \mathbf{A} \models \mathcal{H}\}$ and so

$$\models_{\{\mathbf{B}\in\mathbf{MOD}^*(\mathrm{L})\ \mid\ \mathbf{B}\models\mathcal{H}\}} \subseteq \models_{\mathbb{K}} = (\mathrm{L} + \bigcup\{\Gamma_H \rhd \nabla_{\psi\in\Sigma_H}\psi \mid H \in \mathcal{H}\})^{\nabla}.$$

The rest of the proof is trivial. □

Let $R = \Gamma \rhd \varphi$ and $S = \Delta \rhd \psi$ be consecutions. By $R \nabla S$ we denote the set of consecutions $\{\Gamma, \Delta \rhd \chi \mid \chi \in \varphi \nabla \psi\}$.

THEOREM 5.10. *Let* L *be a finitary logic,* ∇ *a p-protodisjunction, and let* L_1 *and* L_2 *be finitary extensions of* L *respectively obtained by adding sets of finitary consecutions* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*. Without loss of generality we can again assume that* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are written in disjoint sets of variables. Then:*

$$(\mathrm{L}_1 \cap \mathrm{L}_2)^{\nabla} = (\mathrm{L} + \bigcup\{R \nabla S \mid R \in \mathcal{C}_1, S \in \mathcal{C}_2\})^{\nabla}.$$

Proof. Recall that $\mathrm{L}_1 \cap \mathrm{L}_2 = \models_{\mathbf{MOD}^*(\mathrm{L}_1)\cup\mathbf{MOD}^*(\mathrm{L}_2)}$. If $R = \Gamma \rhd \varphi \in \mathcal{C}_1$ and $S = \Delta \rhd \psi \in \mathcal{C}_2$, we denote by $R \vee S$ denote the following non-negative universal clause:

$$\bigwedge_{\chi\in\Gamma\cup\Delta} F(\chi) \to F(\varphi) \vee F(\psi).$$

Finally, we define $\mathcal{H} = \{R \vee S \mid R \in \mathcal{C}_1, S \in \mathcal{C}_2\}$. If we show that $\mathbf{MOD}^*(\mathrm{L}_1) \cup \mathbf{MOD}^*(\mathrm{L}_2) = \{\mathbf{B} \in \mathbf{MOD}^*(\mathrm{L}) \mid \mathbf{B} \models \mathcal{H}\}$, the proof is done by Theorem 5.9.

One inclusion is trivial. We prove the converse one counterpositively: consider $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ such that $\mathbf{A} \notin \mathbf{MOD}^*(\mathrm{L}_1) \cup \mathbf{MOD}^*(\mathrm{L}_2)$, i.e. there is $R_i = \Gamma_i \rhd \varphi_i \in \mathcal{C}_i$ such that $\Gamma_i \not\models_{\boldsymbol{A}} \varphi_i$. Consider evaluations e_1 and e_2 witnessing these facts. Since R_1 and R_2 do not share any propositional variable, there is an evaluation e witnessing both facts. This evaluation also shows that $\mathbf{A} \not\models R_1 \vee R_2$. □

Of course, if the intersection $\mathrm{L}_1 \cap \mathrm{L}_2$ is p-disjunctional, what we obtain is an axiomatization for this logic. If, in addition, L shares the p-disjunction ∇ with $\mathrm{L}_1 \cap \mathrm{L}_2$, we obtain:

$$\mathrm{L}_1 \cap \mathrm{L}_2 = \mathrm{L} + \bigcup\{(R \nabla S)^{\nabla} \mid R \in \mathcal{C}_1, S \in \mathcal{C}_2\}.$$

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