On distributive join semilattices

Rodolfo C. Ertola-Biraben, Francesc Esteva, and Lluís Godo

Abstract Motivated by Gentzen's disjunction elimination rule in his Natural Deduction calculus and reading inequalities with meet in a natural way, we conceive a notion of distributivity for join semilattices. We prove that it is equivalent to a notion present in the literature. In the way, we prove that all notions of distributivity for join semilattices we have found in the literature are linearly ordered. We finally consider the notion of distributivity in join semilattices with arrow, that is, the algebraic structure corresponding to the disjunction-conditional fragment of intuitionistic logic.

1 Introduction

Different notions of distributivity for semilattices have been proposed in the literature as a generalization of the usual distributive property for lattices. As far as we know, notions of distributivity for semilattices have been given, in chronological order, by Grätzer and Schmidt [9] in 1962, by Katriňák [12] in 1968, by Balbes [1] in 1969, by Schein [15] in 1972, by Hickman [11] in 1984, and by Larmerová and Rachůnek [14] in 1988. Following the names of its authors, we will use the terminology GS-, K-, B-, S_n-, H-, and LR-distributivity, respectively.

In this paper, motivated by Gentzen's disjunction elimination rule in his Natural Deduction calculus, and reading inequalities with meet in a natural way, we conceive another notion of distributivity for join semilattices, that we call ND-distributivity. We aim to find out whether it is equivalent to any of the notions already present in the literature. In doing so, we also compare the different notions of distributivity for join semilattices we have found. Namely, we see that the given notions imply each other in the following linear order:

$$\mathrm{GS} \Rightarrow \mathrm{K} \Rightarrow (\mathrm{H} \Leftrightarrow \mathrm{LR} \Leftrightarrow \mathrm{ND}) \Rightarrow \mathrm{B} \Rightarrow \cdots\ \mathrm{S}_n \Rightarrow \mathrm{S}_{n-1} \Rightarrow \cdots\ \mathrm{S}_3 \Rightarrow \mathrm{S}_2,$$

and we also provide countermodels for the reciprocals.

Additionally, we show that H-distributivity may be seen as a very natural translation of a way to define distributivity for lattices, fact that will provide more motivation for the use of that notion. Note that Hickman used the term mild distributivity for H-distributivity.

The paper is structured as follows. After this introduction, in Section 2 we provide some notions and notations that will be used in the paper. In Section 3 we show how to arrive to our notion of ND-distributivity for join semilattices. In Section 4 we compare the different notions of distributivity for join semilattices that appear in the literature. We prove

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that one of those is equivalent to the notion of ND-distributivity given in Section 3. Finally, in Section 5 we consider what happens with the different notions of distributivity considered in Section 4 when join semilattices are expanded with a natural version of the relative meet-complement.

2 Preliminaries

In this section we provide the basic notions and notations that will be used in the paper.

Let $\mathbf{J} = (J;\leq)$ be a poset. For any $S \subseteq J$, we will use the notations S^l and S^u to denote the set of lower and upper bounds of S, respectively. That is,

$S^l = \{x \in J : x \leq s$, for all $s \in S\}$ and
$S^u = \{x \in J : s \leq x$, for all $s \in S\}$.

Lemma 1 *Let $\mathbf{J} = (J;\leq)$ be a poset. For all $a,b,c \in J$, the following statements are equivalent:*
(i) for all $x \in J$, if $x \leq a$ and $x \leq b$, then $x \leq c$,
(ii) $\{a,b\}^l \subseteq \{c\}^l$,
(iii) $c \in \{a,b\}^{lu}$.

A poset $\mathbf{J} = (J;\leq)$ is a *join semilattice* (resp. meet semilattice) if $\sup\{a,b\}$ (resp. $\inf\{a,b\}$) exists for every $a,b \in J$. A poset $\mathbf{J} = (J;\leq)$ is a lattice if it is both a join and a meet semilattice. As usual, the notations $a \vee b$ (resp. $a \wedge b$) will stand for $\sup\{a,b\}$ (resp. $\inf\{a,b\}$).

Given a join semilattice $\mathbf{J} = (J;\leq)$, we will use the following notions:

- $\mathbf{J}$ is *downwards directed* iff for any $a,b \in J$, there exists $c \in J$ such that $c \leq a$ and $c \leq b$.
- A non empty subset $I \subseteq J$ is said to be an *ideal* of $\mathbf{J}$ iff
  (1) if $x,y \in I$, then $x \vee y \in I$ and
  (2) if $x \in I$ and $y \leq x$, then $y \in I$.
- The principal ideal generated by an element $a \in J$, denoted by $(a]$, is defined by $(a] = \{x \in J : x \leq a\}$.
- $Id(\mathbf{J})$ will denote the set of ideals of $\mathbf{J}$.
- $Id_{fp}(\mathbf{J})$ will denote the subset of ideals of $\mathbf{J}$ that are intersection of a finite set of principal ideals, that is, $Id_{fp}(\mathbf{J}) = \{(a_1] \cap \cdots \cap (a_k] : a_1,...a_k \in J\}$.

In this paper we are concerned with various notions of distributivity for join semilattices, all of them generalizing the usual notion of distributive lattice, that is, a lattice $\mathbf{J} = (J;\leq)$ is distributive if the following equation holds true for any elements $a,b,c \in J$:

(D) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (equivalently, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$).

There are several equivalent formulations of this property. In particular, we mention the following ones that are relevant for this paper:

- for all $a,b,c \in J$, if $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$, then $b = c$.
- for any two ideals I_1, I_2 of $\mathbf{J}$, the ideal $I_1 \vee I_2$ generated by their union is defined by $I_1 \vee I_2 = \{a \vee b : a \in I_1, b \in I_2\}$.
- the set $Id(\mathbf{J})$ of ideals of $\mathbf{J}$ is a distributive lattice.

In the case of semilattices, several non-equivalent generalizations of these conditions can be found in the literature, already mentioned in the introduction. However, as expected, all of them turn to be equivalent to usual distributivity in the case of lattices.

The class of distributive lattices forms a variety (that is, an equational class). In contrast, in any sense of distributivity for join semilattices that coincides with usual distributivity in the case of a lattice, the class of distributive join semilattices is not even a quasi-variety. Indeed, consider the distributive lattice in Figure 1. Taken as a join semilattice, the set of black-filled nodes is a sub join semilattice that is clearly a non-distributive lattice (a diamond). Thus, it is neither distributive as a join semilattice. This proves that the class of distributive join semilattices (in any sense that coincides with usual distributivity in the case of lattices) is not closed by subalgebras, and hence it is not a quasi-variety.

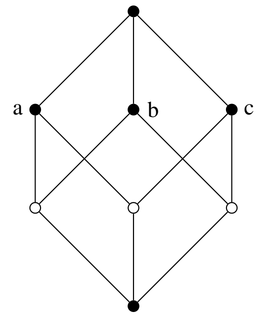


Fig. 1 A distributive lattice with a non-distributive sub join semilattice.

3 Distributivity and Natural Deduction

Let us consider the disjunction-fragment of intuitionistic logic in the context of Gentzen's Natural Deduction calculus (see [6, p. 186]). It has the following introduction rule for $\vee$ and an analogous one with $\mathfrak{B}$ as only premiss:

$$\textbf{(}\vee\textbf{I):}\ \frac{\mathfrak{A}}{\mathfrak{A}\vee\mathfrak{B}}$$

and the following disjunction elimination rule:

$$\frac{\mathfrak{A}\vee\mathfrak{B} \qquad \begin{array}{c}[\mathfrak{A}]\\ \mathfrak{C}\end{array} \qquad \begin{array}{c}[\mathfrak{B}]\\ \mathfrak{C}\end{array}}{\mathfrak{C}}\ .$$

The last rule may be read as saying that if $\mathfrak{C}$ follows from $\mathfrak{A}$ and $\mathfrak{C}$ follows from $\mathfrak{B}$, then $\mathfrak{C}$ follows from $\mathfrak{A}\vee B$, so reflecting what is usually called "proof by cases". It is possible to give an algebraic translation in the context of a join semilattice $\mathbf{J}=(J;\leq)$:

$$\text{for all } a,b,c\in J,\text{ if } a\leq c \text{ and } b\leq c,\text{ then } a\vee b\leq c,$$

which is easily seen to be one of the conditions stating that $a\vee b$ is the supremum of a and b. Now, the last rule is usually employed in a context with a fourth formula $\mathfrak{H}$:

$$\textbf{(}\vee\textbf{E):}\ \frac{\mathfrak{H},\mathfrak{A}\vee\mathfrak{B} \qquad \begin{array}{c}\mathfrak{H},[\mathfrak{A}]\\ \mathfrak{C}\end{array} \qquad \begin{array}{c}\mathfrak{H},[\mathfrak{B}]\\ \mathfrak{C}\end{array}}{\mathfrak{C}}\ .$$

In the context of a lattice $\mathbf{L}=(L;\leq)$, we would give the following algebraic translation:

($\mathbf{D}_{\wedge\vee}$) for all $h,a,b,c\in L$,
if $h\wedge a\leq c$ and $h\wedge b\leq c$, then $h\wedge(a\vee b)\leq c$.

It is easily seen that **($\mathbf{D}_{\wedge\vee}$)** is equivalent to the usual notion of distributivity for lattices. Now, the natural question arises how to give an algebraic translation of **($\vee$E)** if only $\vee$ is available, for example, if we are in the context of a join semilattice.

Considering that an inequality $u\wedge v\leq w$ in a lattice $\mathbf{L}=(L;\leq)$ is equivalently expressed as the first order statement

$$\text{for all } x\in L,\text{ if } x\leq u \text{ and } x\leq v \text{ then } x\leq w,$$

we may write **($\mathbf{D}_{\wedge\vee}$)** in the context of a join semilattice $\mathbf{J}=(J;\leq)$ as follows:

($D_\vee$) for all $h,a,b,c \in J$,
IF for all $x \in J$ (if $x \leq h$ and $x \leq a$, then $x \leq c$) and for all $x \in J$ (if $x \leq h$ and $x \leq b$, then $x \leq c$),
THEN for all $x \in J$ (if $x \leq h$ and $x \leq a \vee b$, then $x \leq c$).

Alternatively, using the equivalence between parts (i) and (ii) in Lemma 1, we may write

($D_\vee$) for all $h,a,b,c \in J$, if $\{h,a\}^l \cup \{h,b\}^l \subseteq \{c\}^l$, then $\{h,a\vee b\}^l \subseteq \{c\}^l$.

Yet, using the equivalence between parts (ii) and (iii) in Lemma 1, we may also alternatively write

($D_\vee$) for all $h,a,b,c \in J$, $\{h,a\}^{lu} \cap \{h,b\}^{lu} \subseteq \{h,a\vee b\}^{lu}$.

Accordingly, given the above logical motivation, it is natural to consider the following notion of distributivity for join semilattices.

Definition 1 *A join semilattice* $\mathbf{J} = (J;\leq)$ *is called ND-distributive (ND for Natural Deduction) if it satisfies* **($D_\vee$)**.

Now, it happens that there are many different (and non-equivalent) notions of distributivity for semilattices. This is not new:

> The concept of distributivity permits different non-equivalent generalizations from lattices to semilattices. (see [15])

So, it is natural to inquire whether the given notion of ND-distributivity for join semilattices is equivalent to any of the notions already present in the literature and, if so, to which. In what follows we will solve that question. In doing so, we will also compare the different notions of distributivity for join semilattices that we have found.

In this paper, given our logical motivation, we restrict ourselves to study the distributivity property for join semilattices. However, an analogous path could be followed for meet semilattices or even for posets.

Remark 1 *Let us note that the following rule (reflecting proof by three cases) is equivalent to* ($\vee$**E**)*:*

$$\dfrac{\mathfrak{H},\ \mathfrak{A}\vee\mathfrak{B}\vee\mathfrak{C} \qquad \begin{array}{c}\mathfrak{H},\ [\mathfrak{A}]\\ \mathfrak{D}\end{array} \qquad \begin{array}{c}\mathfrak{H},\ [\mathfrak{B}]\\ \mathfrak{D}\end{array} \qquad \begin{array}{c}\mathfrak{H},\ [\mathfrak{C}]\\ \mathfrak{D}\end{array}}{\mathfrak{D}}\ .$$

Indeed, it implies ($\vee$**E**) *taking* $\mathfrak{C} = \mathfrak{B}$*. Also, the following derivation shows that it may be derived using* ($\vee$**E**) *twice:*

$$\dfrac{\mathfrak{H},\ \mathfrak{A}\vee\mathfrak{B}\vee\mathfrak{C} \qquad \begin{array}{c}\mathfrak{H},\ [\mathfrak{A}]\\ \mathfrak{D}\end{array} \qquad \dfrac{\mathfrak{H},\ \mathfrak{B}\vee\mathfrak{C} \qquad \begin{array}{c}\mathfrak{H},\ [\mathfrak{B}]\\ \mathfrak{D}\end{array} \qquad \begin{array}{c}\mathfrak{H},\ [\mathfrak{C}]\\ \mathfrak{D}\end{array}}{\mathfrak{D}}}{\mathfrak{D}}\ .$$

This rule also has the following natural algebraic translation in the case of join semilattices: given a a join semilattice $(J;\leq)$,

($2D_\vee$) for all $a,b,c,h_1,h_2 \in J$,
IF for all $x \in J$ (if $x \leq h_1$, $x \leq h_2$ and $x \leq a$, then $x \leq c$) and
for all $x \in J$ (if $x \leq h_1$, $x \leq h_2$ and $x \leq b$, then $x \leq c$),
THEN for all $x \in J$ (if $x \leq h_1$, $x \leq h_2$ and $x \leq a \vee b$, then $x \leq c$).

The natural question arises about whether **($2D_\vee$)** is equivalent to **($D_\vee$)**. Let us see that to be the case.

Proposition 1 **($2D_\vee$)** *is equivalent to* **($D_\vee$)**.

Proof. Taking $h = h_1 = h_2$, it is immediate to see that **($2D_\vee$)** implies **($D_\vee$)**. For the other direction we prove the contrapositive. Accordingly, let us suppose, given a join semilattice $(J;\leq)$ with join $\vee$, that there exist $a,b,c,h_1,h_2 \in J$ such that

(1) for all $x \in J$ (if $x \leq h_1$, $x \leq h_2$ and $x \leq a$, then $x \leq c$),
(2) for all $x \in J$ (if $x \leq h_1$, $x \leq h_2$ and $x \leq b$, then $x \leq c$), and such that
(3) there exists $x \in J$ such that $x \leq h_1$, $x \leq h_2$, $x \leq a \vee b$, and $x \nleq c$.

Our goal is to prove that there exist $a, b, c, h \in J$ such that

(A) for all $x \in J$ (if $x \leq h$ and $x \leq a$, then $x \leq c$),
(B) for all $x \in J$ (if $x \leq h$ and $x \leq b$, then $x \leq c$), and such that
(C) there exists $y \in J$ such that $y \leq h$, $y \leq a \vee b$, and $y \nleq c$.

Let us now see that our goal is satisfied taking the same a, b, c as in our hypothesis and h to be the element provided by (3), that is, $h \in J$ satisfies

(i) $h \leq h_1$,
(ii) $h \leq h_2$,
(iii) $h \leq a \vee b$, and
(iv) $h \nleq c$.

Let us see that h satisfies (A), (B), and (C). For (A), suppose $x \in J$, $x \leq h$, and $x \leq a$. Then, using (i), (ii) and $\leq$-transitivity, it follows that $x \leq h_1, h_2$. So, using (1), we get $h \leq c$. Item (B) is obtained analogously. Finally, h satisfies (C) using $\leq$-reflexivity, (iii), and (iv). □

Remark 2 *Note that a similar proof may be obtained for any natural number $n \geq 2$, that is, it holds in general that* **(nD$_\vee$)** *is equivalent to* **(D$_\vee$)***, where we define* **(nD$_\vee$)** *taking $h_1, h_2, \ldots, h_n$.*

4 Different notions of distributivity for join semilattices

In the following subsections we consider and compare the notions of distributivity for semilattices we have found in the literature. Some authors have presented their notion for the case of meet semilattices and others for join semilattices. We will make things uniform and, motivated by the logical considerations in the previous section, we will choose to consider join semilattices.

We emphasize that all the distributivity notions for semilattices (and posets) proposed in the literature are generalizations of the distributivity property for lattices, in fact, when restricted to lattices all these notions coincide.

4.1 GS-distributivity

The following seems to be the most popular definition of distributivity for join semilattices.

Definition 2 *A join semilattice $\mathbf{J} = (J; \leq)$ is GS-distributive iff*

(GS) *for all $a, b, x \in J$, if $x \leq a \vee b$, then there exist $a', b' \in J$ such that $a' \leq a$, $b' \leq b$, and $x = a' \vee b'$.*

In order to visualize it, see Figure 2. The given definition seems to have appeared for the first time in [9, p. 180, footnote 4]. It also appears in many other places, e.g., in [8, Sect. II.5.1, pp. 167-168].

Next, note that **(GS)** implies that every pair elements has a lower bound. In fact, we have the following equivalence.

Proposition 2 *Let $\mathbf{J} = (J; \leq)$ be a join semilattice. Then, the following two statements are equivalent:*
(i) Every pair of elements has a lower bound.
(ii) for all $a, b, x \in J$, if $x \leq a \vee b$, then there exist $a', b' \in J$ such that $a' \leq a$, $b' \leq b$, and $a' \vee b' \leq x$.

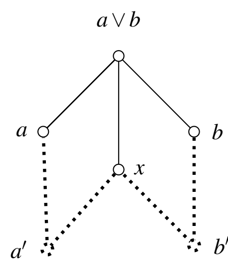


Fig. 2 Diagram for the usual notion of distributivity for join semilattices

Proof. (i) $\Rightarrow$ (ii) Suppose $x \leq a \vee b$. Let a' be a lower bound of $\{a,x\}$ and b' be a lower bound of $\{b,x\}$. Then, $a' \leq a$ and $b' \leq b$. Also, $a' \leq x$ and $b' \leq x$, which implies that $a' \vee b' \leq x$.

(ii) $\Rightarrow$ (i) Let $a,b \in J$. We have $a \leq a \vee b$. Then, by hypothesis, there exist $a' \leq a$, $b' \leq b$ such that $a' \vee b' \leq a$. As $b' \leq a' \vee b'$, it follows that $b' \leq a$. Then, $b' \leq a,b$. That is, b' is a lower bound of $\{a,b\}$. □

Let us now see that there cannot exist finite GS-distributive join semilattices that are not lattices, for which we shall use the following well-known fact.

Lemma 2 *Every finite join semilattice with bottom has meet.*

Proposition 3 *A GS-distributive join semilattice which is not a lattice is infinite.*

Proof. Let us suppose we have a finite GS-distributive join semilattice which is not a lattice. First note that, due to Lemma 2, if a finite join semilattice is not a lattice, then it does not have bottom. Now, a finite join semilattice without bottom must have at least two minimal elements, which contradicts the fact that a GS-distributive join semilattice must be downwards directed due to Proposition 2.

The fact that, as implied by Proposition 2, every GS-distributive join semilattice is downward directed, implies in turn, as shown in [8], that the ideal $I \vee J$, generated by the union of two ideals I, J, is defined as in the case of distributive lattices, namely,

$$I \vee J = \{a \vee b : a \in I, b \in J\}.$$

As a consequence, it follows that the ideals of a GS-distributive join semilattice $\mathbf{J}$ form a lattice that will be denoted by $Id(\mathbf{J})$. Grätzer proved in [8, p. 168] the following characterization result.

Proposition 4 *Let $\mathbf{J}$ be a join semilattice. Then, $\mathbf{J}$ is GS-distributive iff $Id(\mathbf{J})$ is distributive.*

4.2 K-distributivity

The concept given in the following definition is similar to the one in **(GS)**.

Definition 3 *A join semilattice $\mathbf{J} = (J; \leq)$ is K-distributive iff*

(K) *for all $a,b,x \in J$,*
if $x \leq a \vee b$, $x \nleq a$ and $x \nleq b$, then there exist $a',b' \in J$ such that $a' \leq a$, $b' \leq b$, and $x = a' \vee b'$.

In order to visualize, see again Figure 2. The given definition seems to have appeared for the first time in [12, Definition 4, p. 122]. It also appears, for example, in [11, p. 167].

From the very definition, it turns out that GS-distributivity implies K-distributivity. In fact, as noted in [12, 1.5, p. 122-123], it is the case that GS-distributivity is equivalent to K-distributivity plus the condition that every pair of elements has a lower bound (that is, downward directed). Therefore, the following proposition makes clear the relationship between GS- and and K-distributivity.

Proposition 5 *GS-distributivity implies K-distributivity, but not conversely.*

The most simple counter-example showing that the reciprocal does not hold is the non-downward directed join semilattice in Figure 3. Indeed, the given join semilattice is K-distributive, as the only way to satisfy the antecedent of **(K)** is to take $1 \leq a \vee b$, but then the consequent is also true. On the other hand, it is not GS-distributive, as we have $a \leq a \vee b$ and, however, there are no $a' \leq a$ and $b' \leq b$ such that $a' \vee b' = a$.

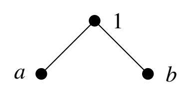


Fig. 3 Join semilattice showing that K- does not imply GS-distributivity

Finally, analogously to Proposition 4, we have the following characterisation of K-distributivity via ideals, a proof of which may be found in [12, p. 123].

Proposition 6 *Let* $\mathbf{J}$ *be a join semilattice. Then,* $\mathbf{J}$ *is K-distributive iff* $Id(\mathbf{J}) \cup \{\emptyset\}$ *is distributive.*

4.3 H-distributivity

In [11] Hickman introduces the concept of *mildly distributive* meet semilattices as those meet semilattices whose strong ideals form a distributive lattice.[1] In [11, Theorem 2.5, p. 290] it is stated that it is equivalent to the following statement: [2]

($\mathbf{H}_\wedge$) for all n and $x, a_1, \cdots, a_n$,
IF for all b (if $a_1 \leq b, \cdots, a_n \leq b$, then $x \leq b$),
THEN there exists $(x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$ and $x \leq (x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$.

The given conditional may be seen as a translation of the following version of distributivity for lattices:

$$\text{IF } x \leq a_1 \vee \cdots \vee a_n, \text{ THEN } x \leq (x \wedge a_1) \vee \cdots \vee (x \wedge a_n).$$

For the case of join semilattices, dualising Hickman's distributivity notion for meet semilattices, we come up with the following definition, using the notion of strong filter: a non-empty set F of a join-semillatice is called *strong filter* if for any finite subset $S \subseteq F$, it holds that $S^{lu} \subseteq F$.

Definition 1. A join semilattice is H-distributive if its strong filters form a distributive lattice.

Similarly to meet semilattices, H-distributivity for join semilattices can be seen to be equivalent to the following condition.

Lemma 1. *A join semilattice* $\mathbf{J} = (J; \leq)$ *is H-distributive iff the following condition holds:*

(H) for all n and $x, a_1, \cdots, a_n \in J$,
IF $x \leq a_1 \vee \cdots \vee a_n$,
THEN there exists $(x \wedge a_1), \ldots, (x \wedge a_n)$ and $x \leq (x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$.

Using quantifiers, **(H)** may be rendered as follows:

[1] A non-empty set I of a meet semillatice is called strong ideal if for any finite subset $S \subseteq I$, it holds that $S^{ul} \subseteq I$.

[2] Note that the original Hickman's statement can be misleading since the condition "there exists $(x \wedge a_1) \vee (x \wedge a_2) \vee \cdots \vee (x \wedge a_n)$" is missing.

for all n and $x, a_1, \ldots, a_n \in J$,
IF $x \leq a_1 \vee \cdots \vee a_n$,
THEN for all y, if for all $i = 1, \ldots, n$ (for all z, IF $z \leq x$ and $z \leq a_i$, THEN $z \leq y$) then $x \leq y$

that, in turn, is equivalent to:

for all n and $x, a_1, \ldots, a_n \in J$,
IF $x \leq a_1 \vee \cdots \vee a_n$,
THEN for all y, if (for all z, IF $z \leq x$ and ($z \leq a_1$ or ... or $z \leq a_n$), THEN $z \leq y$) then $x \leq y$.

Using set-theoretic notation, **(H)** may also be rendered as follows:

(C) for all n and $x, a_1, \ldots, a_n \in J$,
if $x \leq a_1 \vee \cdots \vee a_n$, then $x \in (\{x, a_1\}^l \cup \cdots \cup \{x, a_n\}^l)^{ul}$.

At this point, the reader may wonder whether the number n of arguments is relevant or whether two arguments are enough. Let us settle this question. Firstly, with that in mind, consider

($\mathbf{D}_{\vee_n}$) for all $x, a_1, \ldots, a_n, c$,
if $\{x, a_1\}^l \cup \cdots \cup \{x, a_n\}^l \subseteq \{c\}^l$, then $\{x, a_1 \vee \cdots \vee a_n\}^l \subseteq \{c\}^l$.

Now, let us state the following fact.

Lemma 3 *($\mathbf{D}_{\vee_n}$) is equivalent to* **(C)**.

Proof. $\Rightarrow$) Suppose $x \leq a_1 \vee \cdots \vee a_n$ and $y \in (\{x, a_1\}^l \cup \cdots \cup \{x, a_n\}^l)^u$. Our goal is to see that $x \leq y$. Take $c = y$ and apply ($\mathbf{D}_{\vee_n}$). Then we have $\{x\}^l = \{x, a_1 \vee \cdots \vee a_n\}^l \subseteq \{y\}^l$ and hence $x \leq y$.

$\Leftarrow$) Suppose $\{x, a_1\}^l \cup \{x, a_2\}^l \cup \cdots \cup \{x, a_n\}^l \subseteq \{c\}^l$. We have to prove that, if $y \leq x$ and $y \leq a_1 \vee \cdots \vee a_n$ then $y \leq c$. Now, using **(C)**, and the assumptions $y \leq x$ and $y \leq a_1 \vee \cdots \vee a_n$ it follows that $y \in (\{x, a_1\}^l \cup \cdots \cup \{x, a_n\}^l)^{ul}$. But since $\{x, a_1\}^l \cup \{x, a_2\}^l \cup \cdots \cup \{x, a_n\}^l \subseteq \{c\}^l$, we also have $y \in (\{x, a_1\}^l \cup \cdots \cup \{x, a_n\}^l)^{ul} \subseteq \{c\}^{lul} = \{c\}^l$. Hence $y \leq c$. □

In turn, let us see that ($\mathbf{D}_{\vee_n}$) is equivalent to ($\mathbf{D}_{\vee}$), which proves that having more than two arguments does not make any difference.

Lemma 4 *($\mathbf{D}_{\vee_n}$) is equivalent to ($\mathbf{D}_{\vee}$).*

Proof. We just prove that ($\mathbf{D}_{\vee}$) implies ($\mathbf{D}_{\vee_3}$), the reciprocal being immediate. Suppose $\{h, a_1\}^l \cup \{h, a_2\}^l \cup \{h, a_3\}^l \subseteq \{c\}^l$. Then, we get both $\{h, a_1\}^l \subseteq \{c\}^l$ and $\{h, a_2\}^l \cup \{h, a_3\}^l \subseteq \{c\}^l$, the last of which, using ($\mathbf{D}_{\vee}$), implies that $\{h, a_2 \vee a_3\}^l \subseteq \{c\}^l$, which, together with the first, using ($\mathbf{D}_{\vee}$) again, finally implies that $\{h, a_1 \vee a_2 \vee a_3\}^l \subseteq \{c\}^l$. □

As a consequence, H-distributivity coincides with the notion of ND-distributivity for join semilattices introduced in Section 3. Accordingly, we have the following proposition.

Proposition 7 *A join semilattice is H-distributive iff it is ND-distributive.*

Analogously to Propositions 4 and 6, we also have a characterization of H-distributivity for join semilattices in terms of distributivity of the sublattice of some of their ideals. This appears as Corollary 2.4 in [11, p. 290]), where $Id_{fp}(\mathbf{J})$ denotes the set $\{(a_1] \cap \cdots \cap (a_k] : a_1, \ldots a_k \in J\}$, that is, the set of ideals that are intersection of a finite set of principal ideals of the join semilattice $\mathbf{J} = (J; \leq)$.

Proposition 8 *Let $\mathbf{J}$ be a join semilattice. Then, $\mathbf{J}$ is H-distributive iff $Id_{fp}(\mathbf{J})$ is distributive.*

Let us now compare H- with K-distributivity.

Proposition 9 *Let $\mathbf{J} = (J; \leq)$ be a join semilattice. Then, K-distributivity implies H-distributivity.*

Proof. Suppose

(i) for all $x \in J$, if $x \leq h$ and $x \leq a$, then $x \leq c$, and
(ii) for all $x \in J$, if $x \leq h$ and $x \leq b$, then $x \leq c$.

Further, suppose both (S1) $x \leq h$ and (S2) $x \leq a \vee b$. The goal is to prove $x \leq c$. Let us suppose that $x \leq a$. Then, using (i) and (S1), it follows that $x \leq c$. The case $x \leq b$ is analogous using (ii). Finally, suppose both $x \nleq a$ and $x \nleq b$. Using (K) and (S2), it follows that there exist a', $b' \in J$ such that $a' \leq a$, $b' \leq b$ and (F) $x = a' \vee b'$, which implies $a' \leq x$, which using (S1) gives $a' \leq h$. As we also have $a' \leq a$, using (i) we get $a' \leq c$. Reasoning analogously, we get $b' \leq c$. So, using (F) it follows that $x \leq c$. □

The reciprocal of Proposition 9 does not hold considering the model in Figure 4 (with the understanding that there is no element in the white node). The given model appears as a poset in [7, Figure 2.7, p. 37].[3] We provide a proof using the characterization of K- and H-distributivity by their ideals (Propositions 6 and 8).

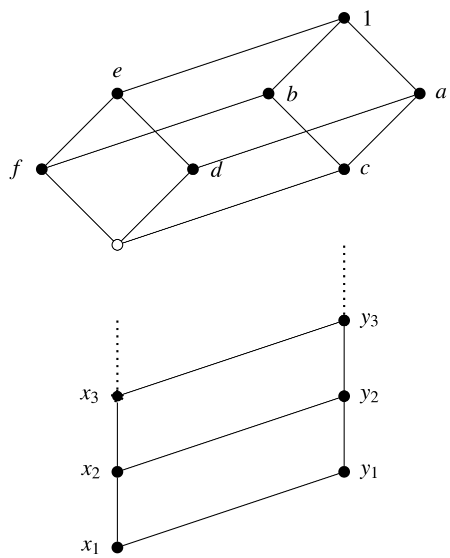


Fig. 4 H-distributive, but not K-distributive join semilattice

Proposition 10 *H-distributivity does not imply K-distributivity.*

Proof. Let us characterize the sets $Id_{fp}(\mathbf{J})$ and $Id(\mathbf{J})$, where $\mathbf{J} = (J, \leq)$ is the join semilattice of Figure 4. An easy computation proves, on the one hand, that $Id_{fp}(J)$ is isomorphic to the ordered set of Figure 4 plus the ideal $I_{\overline{x}} = (f] \wedge (d]$, whose elements are $\{x_i : i \in \omega\}$, that does not exist in the original join semilattice. On the other hand, $Id(\mathbf{J})$ is the set of ideals in $Id_{fp}(\mathbf{J})$ plus the ideal $I_{\overline{y}}$ generated by the set $\{y_i : i \in \omega\}$, that is, the ideal with elements $I_{\overline{y}} = \{y_i : i \in \omega\} \cup \{x_i : i \in \omega\}$. Clearly, this ideal is not a finite intersection of principal ideals. Both $Id_{fp}(\mathbf{J})$ and $Id(\mathbf{J})$ are lattices. Moreover, it is obvious that $Id_{fp}(\mathbf{J})$ is a distributive lattice and thus the join semilattice of the example is H-distributive. But this is not the case for $Id(\mathbf{J})$, since it has a sublattice isomorphic to the pentagon formed by the elements $(a]$, $(d]$, $(c]$, $I_{\overline{y}}$, and $I_{\overline{x}}$. Thus, the join semilattice of the example is not K-distributive. □

It is natural to ask whether it is possible to find a finite example in order to prove the reciprocal of Proposition 9. Let us see that the answer is negative.

Proposition 11 *For finite join semilattices, H-distributivity and K-distributivity coincide.*

[3] We thank the author of this PhD thesis for communicating this example.

Proof. Consider a finite H-distributive join semilattice. We want to see that it is K-distributive. Accordingly, suppose $x \leq a \vee b$, $x \nleq a$, and $x \nleq b$. It is natural to consider $\bigvee\{a,x\}^l$ and $\bigvee\{b,x\}^l$ as candidates for a' and b' in the definition of K-distributivity. Now, in order to do that, we first need to prove that the sets $\{a,x\}^l$ and $\{b,x\}^l$ are not empty. Suppose, say, $\{a,x\}^l = \emptyset$. Then, we have :

- for all y, if $y \leq x$ and $y \leq a$, then $y \leq b$ (as $\{a,x\}^l = \emptyset$),
- for all y, if $y \leq x$ and $y \leq b$, then $y \leq b$,
- $x \leq x$, and
- $x \leq a \vee b$.

So, using H-distributivity, it follows that $x \leq b$, a contradiction.

Having proved that both $\{a,x\}^l \neq \emptyset$ and $\{b,x\}^l \neq \emptyset$, let us note that both $\bigvee\{a,x\}^l$ and $\bigvee\{b,x\}^l$ exist, due to having a finite structure. Next, it is clear that $\bigvee\{a,x\}^l = \inf\{a,x\}$ (analogously, $\bigvee\{b,x\}^l = \inf\{b,x\}$).

It remains to be seen that 1) $\inf\{a,x\} \leq a$, 2) $\inf\{b,x\} \leq b$, and 3) $x = \inf\{a,x\} \vee \inf\{b,x\}$. Now, 1) and 2) are easy to see. Regarding 3), as we have both that $\inf\{a,x\} \leq x$ and $\inf\{b,x\} \leq x$, it follows that $\inf\{a,x\} \vee \inf\{b,x\} \leq x$. Finally, observe that the inequality $x \leq \inf\{a,x\} \vee \inf\{b,x\}$ follows from

- for all y, if $y \leq x$ and $y \leq a$, then $y \leq \inf\{a,x\} \vee \inf\{b,x\}$,
- for all y, if $y \leq x$ and $y \leq b$, then $y \leq \inf\{a,x\} \vee \inf\{b,x\}$,
- $x \leq x$, and
- $x \leq a \vee b$

(use H-distributivity). □

In fact, it is easy to observe that in the case of a finite join semilattice $\mathbf{J}$, the sets of ideals $Id(\mathbf{J})$ and $Id_{fp}(\mathbf{J})$ coincide since, for any two elements $a,\ b$, either there is no lower bound, that is, $\{a,b\}^l = \emptyset$, or there exists their meet $a \wedge b = \bigvee\{a,b\}^l$.

4.4 LR-distributivity

Larmerová-Rachůnek version of distributivity (see [14]) was given for posets, as we next see.

Definition 4 *A poset* $\mathbf{P} = (P;\leq)$ *is* LR-distributive *iff*

(LRP) *for all* $a,b,c \in P$, $(\{c,a\}^l \cup \{c,b\}^l)^{ul} = (\{c\} \cup \{a,b\}^u)^l$.

Remark 3 *In the given definition, it is enough to take one inclusion. Indeed, given a poset* $\mathbf{P} = (P;\leq)$ *and* $a,b,c \in P$, *it is always the case that* $(\{c,a\}^l \cup \{c,b\}^l)^{ul} \subseteq (\{c\} \cup \{a,b\}^u)^l$.

It is natural to ask for LR-distributivity in the case of a join semilattice. The following definition follows from the fact that in a join semilattice $\mathbf{J} = (J;\leq)$ it holds that $(\{c\} \cup \{a,b\}^u)^l = \{c, a \vee b\}^l$.

Definition 5 *A join semilattice* $\mathbf{J} = (J;\leq)$ *is* LR-distributive *iff*

(LR) *for all* $a,b,c \in J$, $\{c, a \vee b\}^l \subseteq (\{c,a\}^l \cup \{c,b\}^l)^{ul}$.

Now, it can be seen that LR-distributivity is equivalent to H-distributivity, and hence to the condition $(\mathbf{D}_\vee)$ as well.

Proposition 12 *Let* $\mathbf{J} = (J;\leq)$ *be a join semilattice. Then the following conditions are equivalent:*
(i) $\mathbf{J}$ *satisfies* **(LR)**,
(ii) $\mathbf{J}$ *satisfies* **(H)**,
(iii) $\mathbf{J}$ *satifies* $(\mathbf{D}_\vee)$.

Proof. The equivalence between (ii) and (iii) is Prop. 7. Let us prove that **(LR)** is equivalent to $(\mathbf{D}_\vee)$. If $(\mathbf{D}_\vee)$ is written in the form $\{h,a\}^{lu} \cap \{h,b\}^{lu} \subseteq \{h, a \vee b\}^{lu}$, then it is equivalent to $\{h, a \vee b\}^l \subseteq (\{h,a\}^{lu} \cap \{h,b\}^{lu})^l = (\{h,a\}^l \cup \{h,b\}^l)^{ul}$, that is to **(LR)**.[4] □

[4] We thank the referee for pointing out this short proof.

4.5 B-distributivity

The following definition seems to have appeared for the first time in [1, Theorem 2.2. (i), p. 261].

Definition 6 *A join semilattice* $\mathbf{J} = (J;\leq)$ *is* B-distributive *iff*

(B) *for all n and* $a_1, a_2, \ldots, a_n,\ x \in J$,
if $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ *exists, then also* $(x \vee a_1) \wedge (x \vee a_2) \wedge \cdots \wedge (x \vee a_n)$ *exists and equals* $x \vee (a_1 \wedge a_2 \wedge \cdots \wedge a_n)$.

We have the following fact.

Proposition 13 *H-distributivity implies B-distributivity.*

Proof. Given an H-distributive join semilattice $\mathbf{J} = (J;\leq)$, let us take $a,b,x \in J$ (the general case follows by induction). Let us suppose that $a \wedge b$ exists in $\mathbf{J}$. Then, also $x \vee (a \wedge b)$ exists in $\mathbf{J}$. Our goal is to see that $x \vee (a \wedge b) = \inf\{x \vee a, x \vee b\}$. It is clear that $x \vee (a \wedge b) \leq x \vee a, x \vee b$. Now, suppose both (F1) $y \leq x \vee b$ and (F2) $y \leq x \vee a$. We have to see that $y \leq x \vee (a \wedge b)$. It immediately follows that

(i) for all $w \in J$, if $w \leq x \vee b$ and $w \leq x$, then $w \leq x \vee (a \wedge b)$.

Now, suppose (F3) $w \leq x \vee b$ and (F4) $w \leq a$. Then, we have both

(i') for all $y \in J$, if $y \leq a$ and $y \leq x$, then $y \leq x \vee (a \wedge b)$, and
(ii') for all $y \in J$, if $y \leq a$ and $y \leq b$, then $y \leq x \vee (a \wedge b)$.

So, applying H-distributivity to (F3), (F4), (i'), and (ii'), we have $w \leq x \vee (a \wedge b)$. That is, we have proved

(ii) for all $w \in J$, if $w \leq x \vee b$ and $w \leq a$, then $w \leq x \vee (a \wedge b)$.

Using H-distributivity, (F1), (F2), (i) and (ii), it finally follows that $y \leq x \vee (a \wedge b)$, as desired. □

The reciprocal of Proposition 13 does not hold as may be seen in Figure 5.

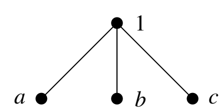


Fig. 5 Join semilattice showing that B- does not imply H-distributivity

Observe also that the lattice $Id_{fp}(\mathbf{J})$, for $\mathbf{J}$ being the join semilattice of Figure 5, is not distributive since it is a diamond.

4.6 S_n-distributivity

The following definition seems to have appeared for the first time in [15].

Definition 7 *A join semilattice* $(J;\leq)$ *is said to be* S_n-distributive *for n a natural number,* $2 \leq n$*, iff*

($\mathbf{S}_n$) *for all* $a_1, a_2, \ldots, a_n, x \in J$,
if $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ *exists, then also* $(x \vee a_1) \wedge (x \vee a_2) \wedge \cdots \wedge (x \vee a_n)$ *exists and equals* $x \vee (a_1 \wedge a_2 \wedge \cdots \wedge a_n)$.

It is easy to see that B-distributivity implies S_n-distributivity, for any $n \geq 2$. It is also clear that for any $n \geq 2$, S_{n+1} implies S_n. On the other hand, we have that for no natural $n \geq 2$ it holds that S_n-distributivity implies B-distributivity. In fact, it was proved that for any $n \geq 2$, S_n does not imply S_{n+1} (see [13]), where infinite models using the real numbers were provided. As in the case of GS- and H-distributivity, it is natural to ask whether, for example, finite models are possible. As in the cases just mentioned, the answer is negative as already proved in [17, Theorem 7.1, p. 1071]. In [16, Theorem, p. 26] it is also proved that it is not possible to find infinite wellfounded models.

Therefore, so far we have seen that, in the case of a join semilattice, we have the following chain of implications among the different distributivity conditions:

$$\mathbf{(GS)} \Rightarrow \mathbf{(K)} \Rightarrow (\mathbf{(H)} \Leftrightarrow \mathbf{(LR)} \Leftrightarrow \mathbf{(ND)}) \Rightarrow \mathbf{(B)} \Rightarrow \cdots\ \mathbf{(S}_n\mathbf{)} \Rightarrow \mathbf{(S}_{n-1}\mathbf{)} \Rightarrow \cdots\ \mathbf{(S}_2\mathbf{)}.$$

5 Join semilattices with arrow

The expansion of semilattices with an arrow operation has been well studied in the literature in the case of meet semilattices under the name of relatively pseudocomplemented semilattices (see, for example, [8, Section I. 6. 2]). However, as far as we know, the expansion of join semilattices with an arrow has not received much attention, excepting, for instance, [4, 5]. In this section we deal with distributivity of join semilattices expanded with an arrow operation.

Definition 2. A join semilattice with arrow is a structure $(J;\leq,\rightarrow)$ where $(J;\leq)$ is a join semilattice and the arrow $\rightarrow$ is a binary operation such that for all $a,b \in J$:

$$a \rightarrow b = \max\{c \in J : \text{ for all } x \in J, \text{ if } x \leq a \text{ and } x \leq c, \text{then } x \leq b\}.$$

The existence of the $\rightarrow$ operation is clearly equivalent to the requirement that $\rightarrow$ satisfies the following two conditions:

($\rightarrow$E) for all $x \in J$, if $x \leq a$ and $x \leq a \rightarrow b$, then $x \leq b$,
($\rightarrow$I) for all $c \in J$, IF for all $x \in J$, if $x \leq a$ and $x \leq c$, then $x \leq b$, THEN $c \leq a \rightarrow b$.

Note that ($\rightarrow$I) and ($\rightarrow$E) imply the following two usual properties:

$b \leq a \rightarrow b$, for all $a,b \in J$,
if $c \leq a$ and $c \leq a \rightarrow b$, then, $c \leq b$, for all $a,b,c \in J$.

Remark 4 *The idea of defining an arrow in a poset was already present in [10] (see Definition 4, where the author uses the terminology of Brouwer poset and also proves that a poset with arrow is LR-distributive, where the arrow $a \rightarrow b$ is defined as the max $\{c \in J : \{a,c\}^l \subseteq \{b\}^l\}$).*

Remark 5 *In a lattice, or even in a meet semilattice, the arrow (if there exists) coincides with the usual relative pseudocomplement. This follows from the fact that, as previously mentioned, the inequality $a \wedge x \leq b$ is equivalent to the following universal quantification: for all y, if $y \leq a$ and $y \leq x$, then $y \leq b$. By the way, we prefer to use "arrow" instead of "relative pseudocomplement", because meet is not necessarily present.*

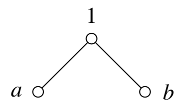
As is well known, a lattice with relative pseudocomplement is distributive (see [18] or [19]). The natural question arises whether a join semilattice with arrow is distributive in any of the senses considered in Section 4. The answer is negative in the case of GS-distributivity, as the join semilattice in Figure 6 has arrow and is not GS-distributive.

Remark 6 *However, note that a meet semilattice with arrow is always GS-distributive, see [2, Proposition 2.1].*

A similar question in the case of K-distributivity has also a negative answer, as the the join semilattice in Figure 7, already given in Figure 4, has arrow and is not K-distributive.

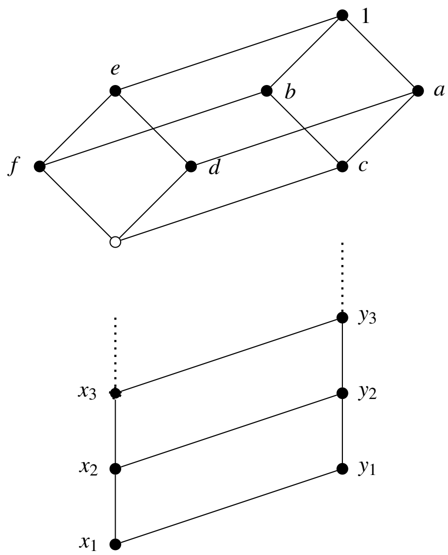
The case of H-distributivity is different, as we see next.

Proposition 14 *Every join semilattice expanded with arrow is H-distributive.*



$\to$	a	b	1
a	1	b	1
b	a	1	1
1	a	b	1

Fig. 6 A non-GS-distributive join semilattice with arrow



$\to$	x_1	x_2	x_n	y_1	y_2	y_n	f	d	e	c	b	a	1
x_1	1	1	1	1	1	1	1	1	1	1	1	1	1
x_2	y_1	1	1	y_1	1	1	1	1	1	1	1	1	1
x_n	y_1	y_2	1	y_1	y_2	1	1	1	1	1	1	1	1
y_1	e	e	e	1	1	1	e	e	e	1	1	1	1
y_2	e	e	e	y_1	1	1	e	e	e	1	1	1	1
y_n	x_1	x_2	e	y_1	y_2	1	1	1	1	1	1	1	1
f	y_1	y_2	y_n	y_1	y_2	y_n	1	a	1	c	1	a	1
d	y_1	y_2	y_n	y_1	y_2	y_n	b	1	1	b	b	1	1
e	y_1	y_2	y_n	y_1	y_2	y_n	b	a	1	c	b	a	1
c	x_1	x_2	x_n	y_1	y_2	y_n	e	e	e	1	1	1	1
b	x_1	x_2	x_n	y_1	y_2	y_n	e	d	e	a	1	a	1
a	x_1	x_2	x_n	y_1	y_2	y_n	f	e	e	b	b	1	1
1	x_1	x_2	x_n	y_1	y_2	y_n	f	d	e	c	b	a	1

Fig. 7 A non-K-distributive join semilattice with arrow

Proof. Let $\mathbf{J} = (J; \leq)$ be a join semilattice with arrow. Take $a, b, c, h \in J$. Suppose

(x1) for all $x \in J$, if $x \leq h$ and $x \leq a$, then $x \leq c$ and
(x2) for all $x \in J$, if $x \leq h$ and $x \leq b$, then $x \leq c$.

Take $y \in J$ and suppose

(F1) $y \leq h$ and
(F2) $y \leq a \vee b$.

Now, using ($\to$I), (x1) implies $a \leq h \to c$ and (x2) implies $b \leq h \to c$. These inequalities together with (F2) imply $y \leq h \to c$, which, using (F1) and ($\to$E), gives $y \leq c$. ☐

Analogously to what happens when considering lattices, in the finite case we have the following fact.

Proposition 15 *Every finite H-distributive join semilattice has arrow.*

Proof. Let $\mathbf{J} = (J; \leq)$ be a finite H-distributive join semilattice. Due to finiteness, $c_1 \vee c_2 \vee \cdots \vee c_n = \bigvee\{c \in J :$ for all $x \in J$, if $x \leq a$ and $x \leq c$, then $x \leq b\}$ exists, for any $a,\ b \in J$. It is clear that for any $c_i, 1 \leq i \leq n$, it holds that

(F) for all x, if $x \leq a$ and $x \leq c_i$, then $x \leq b$.

Now, let us see that $c_1 \vee c_2 \vee \cdots \vee c_n$ is in fact $a \to b$.

First, let us see that $c_1 \vee c_2 \vee \cdots \vee c_n \in \{c \in J :$ for all $x \in J$, if $x \leq a$ and $x \leq c$, then $x \leq b\}$. That is, we have to see that

(T) for all $x \in J$, if $x \leq a$ and $x \leq c_1 \vee c_2 \vee \cdots \vee c_n$, then $x \leq b$.

Now, (T) clearly follows from (F) by H-distributivity.

Secondly, let us take $c \in J$ such that for all $x \in J$, if $x \leq a$ and $x \leq c$, then $x \leq b$. Then, obviously, $c \in \{c \in J :$ for all $x \in J$, if $x \leq a$ and $x \leq c$, then $x \leq b\}$. Then, $c \leq c_1 \vee c_2 \cdots \vee c_n$, as $c_1 \vee c_2 \vee \cdots \vee c_n = \bigvee\{c \in J :$ for all $x \in J$, if $x \leq a$ and $x \leq c$, then $x \leq b\}$. □

Finally, the natural question arises whether the class of join semilattices expanded with arrow forms a variety or at least a quasi variety. The following example proves that the answer is negative. Indeed, consider the distributive lattice in Figure 8, which is the direct product $\mathbf{J} = (L \times L; \leq)$ where $L = \{0, \frac{1}{2}, 1\}$. It is clear that the arrow operation $\to$ exists in $\mathbf{J}$, in fact, $\mathbf{J}^* = (L \times L; \leq, \to)$ becomes a Heyting algebra. Now, consider $\mathbf{J}^*$ as a join semilattice with arrow, and observe that the set B of elements represented by black nodes in the figure is the domain of a subalgebra $(B; \leq, \to)$ of $\mathbf{J}^*$, since both $\vee$ and $\to$ are closed on B. However, $(B; \leq, \to)$ is not a join semilattice with arrow since $\to$ is not an arrow in the sense of Def. 2. In particular, $(\frac{1}{2}, \frac{1}{2}) \to (0,0)$ does not coincide with the maximum (in B) of the set

$$\{(c,d) \in B : \forall (x,y) \in B, \text{if } (x,y) \leq (c,d) \text{ and } (x,y) \leq (\tfrac{1}{2}, \tfrac{1}{2}), \text{then } (x,y) \leq (0,0)\}$$

that does not exist. This shows that the class of join semilattices with arrow is not a quasi variety.

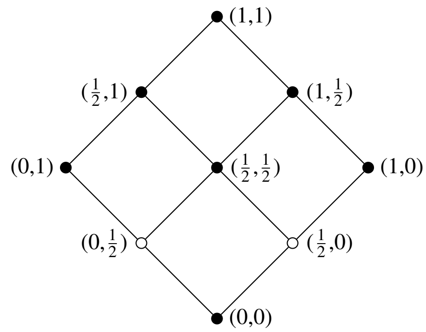


Fig. 8 A distributive join semilattice with a definable arrow.

Moreover, since $\mathbf{J}$ is a distributive lattice while $(B; \leq)$ is not a distributive lattice (it contains a pentagon), the class of distributive join semilattices with arrow is not a quasi variety either. Note that, since $\mathbf{J}$ and $(B; \leq)$ are lattices, this claim is valid for any notion of distributivity for join semilattices.

6 Conclusions

In this paper we have proposed a notion of distributivity for join semilattices with logical motivations related to Gentzen's disjunction elimination rule in the $\{\vee, \to\}$-fragment of intuitionistic logic, and we have compared it to other notions of distributivity for join semilattices proposed in the literature.

There are a number of open problems that we plan to address as future research. In particular, we can mention the following ones:

- As for the logical motivation, similar to the $(\vee\mathbf{E})$ rule in Section 3, one can consider the following rule with two contexts:

$$\frac{\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{A} \vee \mathfrak{B} \qquad \begin{array}{c}\mathfrak{H}_1, \mathfrak{H}_2, [\mathfrak{A}] \\ \mathfrak{C}\end{array} \qquad \begin{array}{c}\mathfrak{H}_1, \mathfrak{H}_2, [\mathfrak{B}] \\ \mathfrak{C}\end{array}}{\mathfrak{C}} .$$

This rule also has a natural algebraic translation in the case of join semilattices. The question arises whether it is equivalent to the condition $(\mathbf{D}_\vee)$ or if it leads to a different one.

- Distributive lattices are characterized by their lattice of ideals. In the case of join semilattices, there are similar characterizations for GS-, K- and H-distributivity, but not for B- and S_n-distributivity. The question is whether B- and S_n-distributive join semilattices can be characterized by means of their ideals.
- In [3] the authors generalize the well-known characterization of distributive lattices in terms of forbidden sublattices (diamond and pentagon) of distributive posets, also identifying the set of forbidden subposets. A similar study for distributive join semilattices is an open question.

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