Modalities and many-valued: modelling uncertainty measures and similarity-based reasoning and application to Fuzzy description Logics

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1 Abstract

The talk will began remembering results on papers [22, 15, 16] where the authors model uncertainty measures as fuzzy modalities on a many-valued residuated logics. the basic idea is that uncertainty measures is not a truth degree but could be interpreted as a truth degree of the fuzzy sentence ”the probability of φ”. The main properties being that the language does not allow nested modalities and their semantic is not defined as generalized Kripke models in strict sense. We will give the example of probabilities (possibilities, belief functions) over classical propositions and its generalizations to uncertainty measures on many-valued (fuzzy) propositions.

But the main goal of the talk is modal many-valued fuzzy logics defined by Kripke models and some applications. Modal many-valued logics is a topic that has deserved few attention until the nineties. The first known papers are the papers [12, 13] of Fitting where the author defines a modal system over a logic of a finite Heyting algebra and give a complete axiomatization of them. In its late paper Fitting justify it in a very nice an elegant semantics based on a multi-agent system each one using classical logic and having a preference relation. He defined the modal operators based on semantics and his definitions is the ones used in later papers on the topic as generalizations modal operators on many-valued systems based on Kripke semantics (with many-valued worlds and many-valued accessibility relations). In these approaches Modal many-valued language is built taking the language of the many-valued logic $(\wedge, \vee, \&, \rightarrow_{\&}, \neg_{\&}, 0, 1)$ plus at least one the usual modal operators (necessity $\Box$ and possibility $\Diamond$) and its semantic is defined by generalized Kripke models taking many-valued evaluation in each

world $w \in W$ and many-valued accessibility relations $S : W \times W \to [0, 1]$. The evaluation of modal operators are given (following Fitting [12, 13]) by

- $V(\Box\varphi, w) = \bigwedge\{R(w, w') \to V(\varphi, w') : w' \in W\}$.
- $V(\Diamond\varphi, w) = \bigvee\{R(w, w')\&V(\varphi, w') : w' \in W\}$.

There are different attempts and approaches to motivate and study modal logic formalisms based on Kripke semantics to the many-valued setting. Roughly speaking we can classify the approaches in three groups depending how the corresponding Kripke frames look like, in the sense of how many-valuedness affects the worlds and the accesibility relations. Next we describe these three groups and comment about our work in each of them.

A first group (see e.g. [5, 10, 28]) is formed by those logical systems whose class of Kripke frames are such that their *worlds are classical* (i.e. they follow the rules of classical logic) but their *accessibility relations are many-valued*, with values in some suitable lattice A. In such a case, the usual approach to capture the many-valuedness of an accessibility relation $R : W \times W \to A$ is by considering the induced set of classical accessibility relations $\{R_a \mid a \in A\}$ defined by the different level-cuts of R, i.e. $\langle w, w' \rangle \in R_a$ iff $R(w, w') \geq a$. At the syntactical level, one then introduces as many (classical) necessity operators $\Box_a$ (or possibility operators $\Diamond_a$) as elements a of the lattice A, interpreted by (classical) relations R_a. Therefore, in this kind of approach, one is led to a multi-modal language but where (both modal and non modal) formulas are Boolean in each world.

In this setting we will present as example the similarity-based reasoning developed in [8, 9, 10, 6, 11]. The starting point is the paper by Ruspini [27] about a possible semantics for fuzzy set theory. He develops the idea that we could represent a fuzzy concept by its set of prototypical elements (which will have fully membership to the corresponding fuzzy set) together with a similarity relation giving the degree of similarity of each element of the universe to the closest prototype. This degree is taken then as the membership to the fuzzy set. From this basic idea, we will see three graded entailments (approximate, proximity and strong) that can be represented in a multi-modal systems with frames where the (graded) accessibility is given by a fuzzy similarity relation on pairs of worlds, and for which we have proved completeness in several cases [7].

A second group of approaches are the ones whose corresponding Kripke frames have *many-valued worlds*, evaluating propositional variables in a suitable lattice of truth-values A, but with *classical accessibility relations* (see e.g. [23, 20, 24]). In this case, we have languages with only one necessity and/or possibility operator $(\Box, \Diamond)$, but whose truth-evaluation rules in the worlds is many-valued, so modal (and non-modal formulas) are many-valued.

Finally, a third group of approaches are *fully many-valued*, in the sense that in their Kripke frames, both worlds and accessibility relations are many-valued, again over a suitable lattice A. In that case, some approaches (like [12, 13, 25, 4]) have a language with a single necessity/possibility operator $(\Box, \Diamond)$, and some (like [26, 2]) consider a multi-modal language with a family of indexed operators $\Box_a$ and $\Diamond_a$ for each $a \in A$, interpreted in the Kripke models via the level-cuts R_a of a many-valued accessibility relations R. Actually, these two kinds of approaches are not always equivalent, in the sense that the operator $\Box$ and the set of operators $\{\Box_a \mid a \in A\}$ are not always interdefinable (or analogously with the possibility operators).

In this setting we will present recent results a summary [2, 4] on minimum modal many-valued logic over the logic of a finite residuated lattice and modal Gödel logic respectively.

Next we will sketch what Fuzzy description logic could be following the proposal of Hájek in [18, 19] and developed in [14]. Finally we propose a research proposal which main goal to be the study of n-graded Description Logics (depending of the underlying logic and the expressiveness of the description language we want), a topic for which we have at hand many results: canonical completeness of first order finite-valued residuated logic, modal many-valued results, decidability results and many possible reasoning algorithms.

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