Corrigendum: Towards a probability theory for product logic: states, integral representation and reasoning

Tommaso Flaminio

Artificial Intelligence Research Institute (IIIA - CSIC), Campus de la Univ. Autonoma de Barcelona s/n, 08193 Bellaterra, Spain. Email: `tommaso@iiia.csic.es`

Lluis Godo

Artificial Intelligence Research Institute (IIIA - CSIC), Campus de la Univ. Autonoma de Barcelona s/n, 08193 Bellaterra, Spain. Email: `godo@iiia.csic.es`

Sara Ugolini

Department of Mathematics, University of Denver, 2390 South York Street, 80208 Denver, Colorado, United States. Email: `sara.ugolini@du.edu`

Abstract

The aim of this short note is to report on a counter-example by Stefano Aguzzoli (private communication) showing that a claim made in a recent paper of ours [2, Proposition 5.2], stating that the class of states of a free product algebra is closed, is in fact not true. That claim was used in turn in the proof of one of the main results of the same paper [2, Theorem 5.4]. However, we also provide in this note an alternative proof for that result, so that it keeps holding true.

1. The state space of a free product algebra is not closed

In our recent paper [2] we introduced states of free, finitely generated, product-algebras. For the sake of completeness, let us recall that, letting $\mathcal{F}_{\mathbb{P}}(n)$ be the free n-generated product algebra, a *state* of $\mathcal{F}_{\mathbb{P}}(n)$ is a map $s : \mathcal{F}_{\mathbb{P}}(n) \to [0, 1]$ satisfying the following conditions [2, Definition 3.1]:

S1. $s(1) = 1$ and $s(0) = 0$,

S2. $s(f \wedge g) + s(f \vee g) = s(f) + s(g)$,

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S3. If $f \leq g$, then $s(f) \leq s(g)$,

S4. If $f \neq 0$, then $s(f) = 0$ implies $s(\neg\neg f) = 0$.

For a better reading of this note, let us recall our main result of [2] which shows an integral representation theorem for states of free, finitely generated, product algebras.

Theorem 1.1 ([2, Corollary 4.10]). *For every $n \in \mathbb{N}$, and for every map $s : \mathcal{F}_{\mathbb{P}}(n) \to [0, 1]$ the following are equivalent:*

(1) s is a state,

(2) there is a unique regular Borel measure μ such that, for every $f \in \mathcal{F}_{\mathbb{P}}(n)$,

$$s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu.$$

In [2, Proposition 5.2] we claimed that, for every $n \in \mathbb{N}$, the class $\mathcal{S}(n)$ of states of $\mathcal{F}_{\mathbb{P}}(n)$ is closed in the product topology of $[0, 1]^{\mathcal{F}_{\mathbb{P}}(n)}$ ([2, Proposition 5.2]). However, the following counter-example, pointed out by Stefano Aguzzoli, shows that such claim is false:

- Let $\{\mu_i\}_{i \in \mathbb{N}}$ be the sequence of Dirac (and hence regular Borel) measures on $[0, 1]$ such that, for each $i \in \mathbb{N}$ and each $A \subseteq [0, 1]$,

  $$\mu_i(A) = 1 \text{ if } 1/(i + 1) \in A, \text{ and } \mu_i(A) = 0 \text{ otherwise.}$$

  Define $s_i : \mathcal{F}_{\mathbb{P}}(1) \to [0, 1]$ as in Theorem 1.1: for all $f \in \mathcal{F}_{\mathbb{P}}(1)$,

  $$s_i(f) = \int_{[0,1]} f \, \mathrm{d}\mu_i.$$

  Then, the sequence $\{s_i\}_{i \in \mathbb{N}}$ converges to a map $s^* : \mathcal{F}_{\mathbb{P}}(1) \to [0, 1]$ which is not a state of $\mathcal{F}_{\mathbb{P}}(1)$. Indeed, take the identity map $f : x \mapsto x$, the single generator of $\mathcal{F}_{\mathbb{P}}(1)$. It is easy to check that, for each $i \in \mathbb{N}$, $s_i(f) = 1/(i + 1)$, $s_i(\neg f) = 0$ and $s_i(\neg\neg f) = 1$. Taking the limit $s^* = \lim_i s_i$ in the cube $[0, 1]^{\mathcal{F}_{\mathbb{P}}(1)}$ endowed with the product topology, one has:

  $$s^*(f) = 0, \; s^*(\neg f) = 0 \text{ and } s^*(\neg\neg f) = 1.$$

But it is clear that s^* does not satisfy the axiom (S4) of [2, Definition 3.1], whence s^* is not a state of $\mathcal{F}_{\mathbb{P}}(1)$.

Therefore, the state space $\mathcal{S}(n)$ of $\mathcal{F}_{\mathbb{P}}(n)$ is convex (see [2, §5]), but not closed. As a direct consequence of this fact, we cannot apply Krein-Milman theorem (see [3]) to show that $\mathcal{S}(n)$ coincides with the topological closure of its extremal points. This observation invalidates an argument in the proof of one of our main results [2, Theorem 5.4], and for which we now propose a valid proof.

First of all, recall that a point c of a convex subset C of an Euclidean space, is *extremal* provided that c cannot be expressed as nontrivial convex combination of elements of C. In other words, if $c = \lambda c_1 + (1-\lambda)c_2$ with $c_1, c_2 \in C$ and $\lambda \in [0,1]$, and c is extremal, then necessarily either $\lambda = 1$ or $\lambda = 0$.

In the next theorem, $\mathcal{H}(\mathcal{F}_{\mathbb{P}}(n))$ denotes the set of homomorphisms of $\mathcal{F}_{\mathbb{P}}(n)$ to $[0,1]_{\mathbb{P}}$, and $\delta : \mathcal{S}(n) \to \mathcal{M}(n)$ is a bijective and affine map which associates, to each state $s \in \mathcal{S}(n)$, the unique regular Borel measure $\mu \in \mathcal{M}(n)$ (provided by Theorem 1.1) such that for every $f \in \mathcal{F}_{\mathbb{P}}(n)$, $s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu$ (see [2, Proposition 5.3]).

Theorem 1.2 ([2, Theorem 5.4])**.** *The following are equivalent for a state* $s : \mathcal{F}_{\mathbb{P}}(n) \to [0,1]$*:*

1. *s is extremal;*
2. *$\mu = \delta(s)$ is a Dirac measure (and hence it is extremal in $\mathcal{M}(n)$);*
3. *$s \in \mathcal{H}(\mathcal{F}_{\mathbb{P}}(n))$.*

New proof. The proof of the equivalence between (2) and (3) is the same as in [2, Theorem 5.4]. Thus, it suffices to prove the implications (2)⇒(1) and (1)⇒(3).

(2)⇒(1). Let $s \in \mathcal{S}(n)$ and $\mu = \delta(s)$, and assume, by way of contradiction, that $s = \lambda e_1 + (1-\lambda)e_2$ with $e_1, e_2 \in \mathcal{S}(n)$ and $0 < \lambda < 1$. Again Theorem 1.1 applied to e_1 and e_2 provides unique regular Borel measures $\mu_1 = \delta(e_1)$ and $\mu_2 = \delta(e_2)$ such that, for all $f \in \mathcal{F}_{\mathbb{P}}(n)$, $e_1 = \int_{[0,1]^n} f \, \mathrm{d}\mu_1$ and

$e_2 = \int_{[0,1]^n} f \, \mathrm{d}\mu_2$. Thus,

$$\begin{aligned} s(f) &= \int_{[0,1]^n} f \, \mathrm{d}\mu \\ &= \lambda \int_{[0,1]^n} f \, \mathrm{d}\mu_1 + (1-\lambda) \int_{[0,1]^n} f \, \mathrm{d}\mu_2 \\ &= \int_{[0,1]^n} f \, \mathrm{d}(\lambda\mu_1 + (1-\lambda)\mu_2). \end{aligned}$$

Therefore $\delta(s) = \mu = \lambda\mu_1 + (1-\lambda)\mu_2$ is not Dirac and this contradicts the hypothesis.

(1)⇒(3). Let $s \in \mathcal{S}(n)$ be extremal. Then, by Theorem 1.1, there exists a regular Borel measure $\mu = \delta(s)$ such that $s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu$. Assume now that s is not a product-homomorphism. Then, from (2)⇒(3), μ is not extremal (and hence not Dirac). Therefore, let μ_1, μ_2 and $0 < \lambda < 1$ such that $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$. The same argument as above shows that, letting e_1 and e_2 be the integrals with respect to μ_1 and μ_2 (respectively), one has $s = \lambda e_1 + (1-\lambda)e_2$, proving that s is not extremal, a contradiction. □

2. Kolmogorov maps and the closure of the state space

According to the notation used in [1], let us say that a map $\kappa : \mathcal{F}_{\mathbb{P}}(n) \to [0,1]$ is a Kolmogorov map, if κ satisfies $(S1)-(S3)$ of [2, Definition 3.1] and let $\mathcal{K}(n)$ be the class of Komogorov maps of $\mathcal{F}_{\mathbb{P}}(n)$.

In [1] the authors proved that, for the case of free n-generated Gödel algebras $\mathcal{F}_{\mathbb{G}}(n)$, $\mathcal{K}(n)$ coincides with the closure of the convex hull of the set of homomorphisms of $\mathcal{F}_{\mathbb{G}}(n)$ to the standard Gödel algebra $[0,1]_{\mathbb{G}}$ (we invite the reader to consult [1, Section 3] for further details). Here we analyze the same problem in the realm of free product-algebras.

First of all notice that $\mathcal{S}(n) \subseteq \mathcal{K}(n)$ and indeed the inclusion is proper since, e.g., the map s^* of the above example by Aguzzoli is a Kolmogorov map but not a state. Further, we can prove the following.

Theorem 2.1. *For every n, $\mathcal{K}(n)$ is closed in the product topology of $[0,1]^{\mathcal{F}_{\mathbb{P}}(n)}$.*

Proof. Let us start showing that $\mathcal{K}(n)$ is closed.[1] Let $\{s_i\}_{i \geq 0}$ be a sequence of states of $\mathcal{F}_{\mathbb{P}}(n)$ such that $\lim_{i \in \mathbb{N}} s_i$ exists, and let us prove that such $s =$

[1]The proof of this claim is the first part of the proof of [2, Proposition 5.2]. We report it here for the sake of completeness.

$\lim_{i\in\mathbb{N}} s_i$ is a state. Condition $S1$ is clearly verified. Let us show that s respects condition $S2$. We need to prove that $s(f\vee g) = s(f)+s(g)-s(f\wedge g)$. Since each s_n is a state, we have that:

$$\lim_{i\in\mathbb{N}} s_n(f \vee g) = \lim_{i\in\mathbb{N}}(s_n(f) + s_n(g) - s_n(f \wedge g)),$$

and also, it clearly holds that:

$$\lim_{i\in\mathbb{N}}(s_n(f) + s_n(g) - s_n(f \wedge g)) = \lim_{i\in\mathbb{N}} s_n(f) + \lim_{i\in\mathbb{N}} s_n(g) - \lim_{i\in\mathbb{N}} s_n(f \wedge g),$$

thus the claim directly follows. It is easy to prove condition $S3$, since given $f, g \in \mathcal{F}_{\mathbb{P}}(n)$, if $f \leq g$ then $s_n(f) \leq s_n(g)$ for every $n \in \mathbb{N}$. Thus,

$$s(f) = \lim_{i\in\mathbb{N}} s_n(f) \leq \lim_{i\in\mathbb{N}} s_n(g) = s(g).$$

□

Therefore, since $\mathcal{S}(n) \subseteq \mathcal{K}(n)$, it is immediate that the closure of the states space of $\mathcal{F}_{\mathbb{P}}(n)$ is contained in $\mathcal{K}(n)$. Whether the other inclusion also holds is an open problem that we shall consider in our future work.

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