Finite Forests. Their Algebras and Logics

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A *forest* is a poset such that the downset of each element is totally ordered. An order-preserving map f between forests is *open* if it carries downsets to downsets, that is, if $y \leq f(x)$ then there exists x' such that $x' \leq x$ and $f(x') = y$. The category $\mathcal{F}_{fin}$ has as objects the finite forests and as morphisms the order-preserving open maps between them.

As is well-known, finite sets and maps between them constitute a category dually equivalent to the category of finitely generated (=finite) Boolean algebras and homomorphisms between them. In this work we show that $\mathcal{F}_{fin}$ is dually equivalent to the finite slice of some varieties of prelinear residuated lattices that constitute the algebraic semantics of some many-valued logics.

In particular we recall the *spectral* equivalence between $\mathcal{F}_{fin}$ and the category $\mathbb{G}_{fin}$ of finite Gödel algebras (see [8],[5]), and we introduce dual equivalences between $\mathcal{F}_{fin}$ and $\mathbb{NM}^{-}_{fin}$, which is the finite slice of the variety of Nilpotent Minimum algebras generated by the NM-chains lacking the negation fixpoint; between $\mathcal{F}_{fin}$ and $\mathbb{NM}^{+}_{fin}$, which are the finite NM-algebras with negation fixpoint; and between $\mathcal{F}_{fin}$ and $\mathbb{IUML}_{fin}$, which is the finite slice of the variety of Involutive Uninorm Mingle algebras (introduced in [10]).

At the interpretation level we can argue that, as a proposition in Boolean propositional logic "selects" a subset of a set, a proposition in one of the logics whose algebraic semantics constitutes a variety dually equivalent to $\mathcal{F}_{fin}$ selects a "part" of a forest. What makes those logics distinct from one another are the different notions of "part" of a forest that they express. This analysis may contribute to a better understanding of the semantics of those logics.

Technically speaking, the dual equivalences we have obtained enable us to derive several results and applications concerning the dual algebras and their logics.

We immediately have that $\mathbb{G}_{fin}$ is categorically equivalent to each one of the categories $\mathbb{NM}^{-}_{fin}$, $\mathbb{NM}^{+}_{fin}$ and $\mathbb{IUML}_{fin}$, and composition provides us with the functors realising those equivalences. Moreover, using the fact that each variety is categorically equivalent to the ind-finite completion of the category of its finitely presented members [9], we automatically extend the aforementioned equivalences to the full varieties, that is, $\mathbb{G} \equiv \mathbb{NM}^{-} \equiv \mathbb{NM}^{+} \equiv \mathbb{IUML}$, and all these varieties are dually equivalent to the pro-finite completion of $\mathcal{F}_{fin}$ (the categorical equivalence between $\mathbb{G}$ and $\mathbb{IUML}$ has been recently proved in [6], using a completely different technique). As a byproduct, we formulate the correct notion that *ideal-determines* IUML-algebras (as defined in [7]).

We study through duality the directly indecomposable members in the above categories and how they are obtained performing various manipulations on Gödel hoops. We characterise the category of finite Gödel hoops $\mathbb{GH}_{fin}$ as the dual of the category $\mathcal{T}_{fin}$ of finite *trees* (see [1] for a similar treatment of Gödel hoops).

A slight enrichment of $\mathcal{F}_{fin}$, consisting in attaching a bit to every tree, and in requiring that maps are non-decreasing on these bits, provides us with a category dually equivalent to the category $\mathbb{NM}_{fin}$ of finite Nilpotent Minimum algebras (see [2],[4]). With another application of duality we further show that $\mathbb{NM}_{fin}$ is equivalent to a category $\mathbb{G}^+_{fin}$ of pointed Gödel algebras. Again we automatically extend the resulting equivalence to the whole varieties, that is $\mathbb{NM} \equiv \mathbb{G}^+$.

We use the equivalences we have obtained to prove that the classes of finitely presented algebras dealt with enjoy a strong form of amalgamation.

We find recursive formulas to compute the exact structure of finitely generated free algebras in the aforementioned varieties as well as formulas to compute their cardinalities (analogous results are in [1], [3]).

For each subvariety of $\mathbb{G}, \mathbb{NM}^-, \mathbb{NM}^+, \mathbb{IUML}, \mathbb{NM}$ we find a dually equivalent category of forests.

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