Logical approaches to fuzzy similarity-based reasoning: an overview

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1 Introduction: vagueness, uncertainty and truthlikeness

For many years, classical logic has given a formal basis to the study of human reasoning. However, during the last decades, it has become apparent that human practical reasoning demands more than what traditional deductive logic can offer. For instance, classically, the truth of a statement q with respect to a state of knowledge K is determined whenever every model of K is also model of q. But nothing can be said about its truth value if only *most* of the models of K are also models of q or when models of q are *very "close"* to models of our state of knowledge. Moreover, a statement can only be either true or false, but some human expressions, such as "John is bald", often fail to be semantically determined with bivalued precision since they express gradual properties.

Minsky ([Min85]) and McDermott ([McD87]) suggested that classical logic is inappropriate for modeling human reasoning because of the modeling"perfect" nature of the former. They remarked that while classical logical reasoning is both **sound** (all conclusions reached are valid or true) and **complete** (all true facts can be deduced), human reasoning does not possess either one of these qualities. On the contrary, the modeling of human reasoning usually requires "imperfect" knowledge to be taken into account in the form of uncertainty, vagueness, truthlikeness, incompleteness and partial contradictions. These limitations of classical logic in accounting for human reasoning motivated the study of alternative (or perhaps better, complementary) formalisations which already became one of the major research areas in the field of Artificial Intelligence in the recent past.

A variety of *approximate reasoning* models have appeared as possible alternatives and have generated an extensive literature in both Philosophy and Artificial Intelligence. Models of approximate reasoning aim at being more flexible than classical logic and basically work on three "imperfections" information can be pervaded with: vagueness, uncertainty and truthlikeness. Informally, and roughly speaking, we can say that approximate reasoning deals with propositions and "labels" associated to them, which are usually interpreted as degrees of truth, belief or proximity to the truth. Each one of these units of measurement is respectively associated with the notions of vagueness, uncertainty and truthlikeness. Unfortunately, this simplified view does not make clear that each one of these "imperfections" responds to a different semantics. In what follows, we shall try to give an "orthogonal" description of these three "imperfections" which we hope will make clear the distinction among them and then we shall indicate how they may be combined.

The three axes correspond to: crisp vs. many-valued interpretations, complete vs. incomplete information, and error-free vs. mistaken information.

Interpretation Problem:

Whenever we have (or fix) a representation language to describe our information about the world, we should provide a form of interpreting the sentences in such a language, i. e. to establish a correspondence between meaning and truth (or as Carnap says [Car37], between theoretical concepts and observations). There is no consensus about what is the best (or most appropriate) theory to define the truth: there are e. g. pragmatic, coherence, correspondence, redundancy or semantic theories. In our opinion, Tarski's correspondence theory is the most adequate form to establish this relationship. He considers that the truth of sentences, statements, judgments, propositions, beliefs or ideas, consists of their "correspondence" with reality, world, or facts. The fundamental difficulty for this theory is to specify what it means to say that a statement "corresponds" to reality. In Tarski's view ([Tar56]), the truth value of a sentence is determined by an interpretation function e from language $\mathbf{L}$ to models Ω. One familiar objection to Tarski's correspondence theory is that his definition applies only (or at best) to formal languages, but not to natural languages. It is well-known that most of the statements in natural language do not have a precise characterisation of their meanings, i. e. they are not semantically determined (in last sense) because they contain (among other things) different kinds of *vague expressions*, including predicates as 'big', 'short', 'large', etc., modifiers as 'very', 'more or less', 'rather', etc., and quantifiers as 'most', 'some', 'many', 'few', etc.

In brief, some expressions which refer to gradual properties are inherently or semantically vague. In such cases, there are no suitable characterizations of their meaning in terms of true-false interpretations. Hence, if we want to represent knowledge of the type "the mountain is high" we need to increase the interpretation power and hence we have to give up classical logic principles such as the excluded middle principle ($p \vee \neg p$ is always true) or the non-contradiction principle ($p \wedge \neg p$ is always false). In response to this necessity Lotfi Zadeh introduced in 1965 ([Zad65]) *fuzzy set theory*. His fundamental idea consists in understanding lattice-valued maps as generalised characteristic functions of some new kind of objects, the so-called *fuzzy sets*, of a given universe.

In the context of fuzzy set theory, fuzzy logic (FL) was born as a logical system that aims at the formalisation of the reasoning with vague propositions. The term fuzzy logic has been used through the literature with two different meanings (see [Zad94]):

> *In the narrow sense* fuzzy logic, FL_n, is an extension of many-valued logic, where the notion of "degree of membership" of an element x in an universe X with respect to a fuzzy set A over X is regarded as the degree of truth of the statement "$A(x)$" (usually read as "x is A). However, as it is pointed out by Zadeh in [Zad94], the agenda of FL_n is quite different from that of traditional many-valued logical systems, e. g. Łukasiewicz's logic. He states that concepts such as linguistic variables, canonical forms, fuzzy if-then rules, fuzzy quantifiers and such modes of reasoning as interpolative reasoning, syllogistic reasoning, and dispositional reasoning, are not part of traditional many-valued logical systems.
>
> *In the broad sense* fuzzy logic, FL_b, is almost synonymous with fuzzy set theory which is a general theory to represent and to reason over "classes" with unsharp boundaries. Fuzzy set theory includes: fuzzy arithmetic, fuzzy mathematical programming, fuzzy topology, fuzzy graph theory, and fuzzy data analysis.

In the last years, many works have been devoted to the development of the formal background of fuzzy logic in narrow sense (as witnessed by a number of important monographs that have appeared in the literature, e. g. [Haj98, NPM99, Got01, Ger01]), that is, to formal systems of many-valued logics having the real unit interval as set of truth values, and truth functions defined by fuzzy connectives that behave classically on extremal truth values (0 and 1) and satisfy some natural monotonicity conditions. Actually, these connectives originate from the definition and algebraic study of fuzzy set theoretical operations over the real unit interval.

In this paper we consider vagueness as an interpretation problem and it will be formally addressed by means of the use of some kind of many-valued logic. As in classical logic, in many-valued logics sentences of the representation language get a truth-value, possibly an intermediate value between 0 and 1, in every complete description of the world. But, as we discuss next, it may be not always possible to determinate this value because in general we may not have (or/and it is impossible to have) a complete knowledge about the real world.

Incomplete Information Problem:

In a classical logic, there is a clear distinction between a definition of truth (such as it was mentioned above) and a criteria for recognizing the truth. In the latter sense, the truth or validity of a conclusion C is often given in terms of a list of arguments Γ, called premises, knowledge base or theory. Thus, if every interpretation that satisfies Γ (i.e. which is a model of Γ) it satisfies C as well, then we can say that in the context of Γ, C is true or is valid conclusion. On the other hand, the falsehood of C is established when no model of Γ satisfies C. Any other situation (when only some (but not all) models of Γ are also models of C) leaves uncertainty about the truth-value of the conclusion C. Moreover, except for the case in which the theory that describes our knowledge about the state of the world is complete it will not be possible in general to determine the truth or falsehood of every possible conclusion. Therefore, even in classical logic, there are three (not two) different epistemic attitudes on propositions with respect to a given theory modelling the world [Gär88, DP01]. This indetermination caused by the third state (ignorance or indetermination) will be referred to as the incomplete information problem.

In practice, it is usual that those dealing with the task of decision or prediction making do not have complete information about the current state of affairs. This prevents from unequivocal assessments of future states and limits the ability to precisely predict the consequence of the possible choices. In such a situation, a classical logical approach as basic formalization of this kind of reasoning would condemn us to "inaction". Fortunately, it is often the case that the available information, even if incomplete, is useful and sufficient for many purposes. For example, knowledge about the laws of evolution of a physical system may be useful to derive that, given an ideal gas, if its pressure P and temperature *Temp* are known then its volume *Vol* is determined by the expression $Vol = k\frac{Temp}{P}$. Of course, if the gas is not ideal the result provided by this equation will not be accurate. Hence, in a strict sense, it will not be possible to know the exact volume of the gas, at least from that equation. So, this kind of precise laws have limited applicability in real domains, where e. g. statistical mechanics gives an answer to these questions by using probability theory, that aims at capturing the underlying uncertainty. In such a case, a probability distribution over the possible values of the volume would be obtained, which provides a measure (objective or subjective) of confidence on the accuracy in predicting a value of a physical property, in our example the volume. Other popular theories used to formalize and quantify that uncertainty are possibility theory and Dempster-Shafer theory of evidence.

In general, an uncertainty model attaches numbers to logical propositions which do not indicate a degree of truth (as some authors usually point out) but a degree of confidence or belief in the truth-value of these propositions. In this sense, the measure of uncertainty compensates the lack of knowledge at the propositional level with information at a higher level of abstraction. At this point, it is important to differentiate vagueness or imprecision at propositional level (as it was discussed above) from vagueness at the model or interpretation level, as it is the case in possibilistic logic. In the latter case, each (crisp) interpretation is attached a degree that estimates the extent to which it may represent the real state of the world. Such an attachment defines a fuzzy set of interpretations which is in fact a fuzzy set based modeling of our vague knowledge about what the real world is. In this sense, the degrees of possibility and necessity in possibilistic logic may be understood as uncertainty degrees induced by some kind of vague (and hence incomplete) information.

However, although notions of vagueness (at propositional level) and uncertainty are not the same, there are close links between them and in many occasions they need to live together. For example, as mentioned in [DP88], if all we know is that "John is tall" (i.e. a vague knowledge about John's height) then, about the truth of the sentence "John's height is 1.80 m", one can only say that it is more or less possible. More formally, Dubois and Prade in [DP91a] propose to understand each fuzzy assertion of the sort of "X is *Tall*" (where *Tall* is a fuzzy set of an universe of discourse U and X is a variable taking values in U) as a constraint on the unknown possibility of the crisp assertions $X = x$, with $x \in U$, of the form $\Pi(X = x) \leq \mu_{Tall}(x)$. This example makes it clear that vague, incomplete information also produces uncertainty on conclusions.

As it is argued by Resconi, Klir and St. Clair [RKC92], uncertainty is an intensional or metatheoretical notion. For this reason, modal logics provide a unified framework for representing those uncertain theories [RKCH93, Hal03] and are naturally related to various generalisations of the modal system S4.3. The well-known cases are probabilistic [Car50, Hal03], possibilistic [FH91, HK94, LL96] and Dempster-Shafer logics [Rus87, Hal03].

Summarizing, we will associate the term *uncertainty* to a degree of belief regarding the truth of a proposition, usually crisp but not necessarily so. Formally, the uncertainty should correspond to intensional logics which are non-truth functional [DP01].

Mistaken Information Problem:

In classical logic, falsity entails any statement. But, in many occasions, we may want to use "false" theories, for instance, Newton's Gravitational Theory. Although this theory is not true we may accept it is a good approximation to truth. Note that we are not referring to self-contradictory theories, but empirically or factually false ones, i. e. theories that correctly explain most of the observations but have counter-examples. As a first approach, we could measure how close is a theory to the truth according to its amount of counter-examples. According to this measure, it is possible to affirm that there are good reasons to conjecture that Einstein's Gravitational Theory, which is also not true, is a better approximation to the truth than Newton's Theory. In a more general sense, we refer to the notion of "proximity to the truth" of a statement (even though it may not be true or provable) as *truthlikeness* [Odd07].

Popper makes an observation which throws light over the distinction between uncertainty and truthlikeness. He points out that by using the Bayesian inference to establish the strength of belief in a hypothesis h from both a previous knowledge K and an observed evidence e, " if the evidence e contradicts the hypothesis h then the probability $P(h \mid e, K)$ of h given e (in the context of K) is zero; yet, h may be highly truthlike, since false theories (even theories known to be false) may be 'close' to the truth". This point stresses the difference between incompleteness and "falsity" of a theory. The first case indicates its failure to express the whole truth and the second one represents the acceptance of an untrue proposition. For instance, a witness in court who does not lie but conceals some "relevant"[1] of facts, tells an incomplete information. On the contrary, a testimony which is partially true, refers to false information.

In a more pragmatic sense, the concept of truthlikeness appears, for example, when we want to give an answer to a query over a database: if we must match exactly the query against the database, we will possibly need too much time or even we can fail. But, if we allow to match the consult "approximately" enough then a lot of time may be saved. In this case, we also may say that the answer to a query is close to the truth or truthlike. Note that the notion of "approximation" to the truth is in correspondence to the one of error in numerical methods.

If we accept this notion, we will be able to say that a statement is *almost true*, *nearly true* or *approximately true*, indicating thereby that it is false or unprovable but close to being true. For instance, in this sense, the sentence "the height of Mount Everest is 8.800 m" is close to the truth[2]. This concept of "close" to the truth presupposes some metric which allows us to express the degree of approximate truth. Notice that this last concept is different from the two previous: vagueness and uncertainty. The truth-value of our example "the height of Mount Everest is 8.800 m" is certainly false and precisely formulated, therefore it is neither uncertain nor vague.

Summarising, we may say that vagueness, uncertainty and truthlikeness, until few years ago, were not clearly differentiated from each other, possibly because they are usually coded by real numbers from the unit interval $[0, 1]$. In the last years, much effort has been devoted to clarify the conceptual differences between vagueness and uncertainty as it is witnessed in [BDG+99]. However, the distinction between these two notions and truthlikeness is not so clear in the literature. For instance, in [DPB99], similarity logic is classified as a non truth-functional logic dealing with vagueness. We consider that it is useful to clarify the distinctive features of each notion, since they are specially important when we aim to represent knowledge and reason with it.

We think that these three notions, vagueness, uncertainty and truthlikeness, constitute the basic axes of approximate reasoning models. Also, we believe that they may be formalized and combined under a homogeneous framework which should be, we understand, an appropriate extension of fuzzy logic in the narrow sense. Several attempts have been made in this direction. For instance,

- Zadeh in [Zad86] combines fuzziness and probability by suggesting a definition of the probability of a fuzzy proposition.
- In [DP93], the authors extensively survey the literature concerning the relationship between fuzzy sets and probability theories; again, besides pointing out the gaps between them, the authors build bridges between both theories, stressing in this sense the importance of possibility theory.

[1]Relevance is an important notion which we do not consider here, but that it should be taking account in a thinner analysis of truthlikeness

[2]The height of Mount Everest that appears in dictionaries is 8,835m.

- In Hájek et.al.'s paper [HGE95], and in some later elaborations [Haj98, GHE03], there is a further contribution to this bridge building. They propose three different theories in Łukasiewicz-Pavelka's logic to cope with probability, necessity and belief functions respectively. The main idea behind this approach is that uncertainty measures of crisp propositions can be understood as truth-values of some suitable fuzzy propositions associated to crisp propositions (it is worth mentioning that although in this work the propositional variables only take Boolean values it is easy to extend it to the many-valued case).

Truthlikeness is probably the least known of the above three notions. The aim of this paper is to survey some logical formalisations of similarity-based reasoning models, where similarity is understood as truthlikeness. To this end, the paper is structured is as follows. In the next section we provide all necessary background about fuzzy similarity relations. In Section 3, we introduce two different logical approaches, one syntactically and another semantically oriented, in order to formalize fuzzy similarity reasoning. In Section 4 we describe the main ideas behind the syntactic model based on the notion of approximate proof, while Section 5 is devoted to the semantical model based on several notions of approximate entailments. In Section 6, we give four different formalisations of these similarity entailments in terms of suitable systems modal and conditional logics, including for each class a system of graded operators with classical semantics and a system with many-valued operators. Finally, Section 7 explores some nonmonotonic issues of similarity-based reasoning, by considering similarity, instead of distance, as a central notion with which to define epistemic orderings and operators of theory revision.

2 Truthlikeness and graded similarity

As it was mentioned above, the dichotomy of the class of propositions into truths and falsehoods should thus be supplemented with a more fine-grained criterion according to their closeness to the truth. The problem of truthlikeness is to give an adequate account of such a concept and to explore its logical properties and its applications to knowledge representation. While a multitude of apparently different solutions to this problem has been proposed, it is now standard to classify them into two main approaches: the *content approach* and the *likeness approach*. The first approach is based on Popper's idea that any theory (or knowledge base) K may be divided in two parts: its truth content K_T, and its falsity content K_F. This partition into true and false propositions is induced by the real world (obviously the epistemological problem is to know which is this world). Following this idea, a knowledge base is closer to the truth than another if it has more truth content (without engendering more falsity content) and less falsity content (without sacrificing truth content). Unfortunately, this account suffers from a fatal flaw, it entails that no false theory is closer to the truth than any other. This was shown independently by Tichý and Miller ([Tic74, Mil74]). After the failure of Popper's idea, the modern definition of truthlikeness follows the likeness approach, and has emerged based on similarity and was proposed independently by Risto Hilpinen within possible worlds semantics (see [Hil76]) and by Pavel Tichý within propositional logic (see [Tic74]). The basic idea of this *similarity approach* is that the degree of truthlikeness of a sentence φ depends on the similarity between the states of affairs that are compatible with φ and the true state of the world (see e. g. [Nii87] p. 198). According to Niiniluoto ([Nii87]), we will consider the *truthlike* value of a sentence as its degree of "proximity to the truth", even though it may not be true or provable. This degree should be given by the "distance" that separates (or dually, by the similarity between) the models of this sentence and the models of the "reality".

Thus, this notion of truthlikeness can be regarded as a special case of the more general concept of similarity and its logical counterparts to some form of similarity-based reasoning, this last concept being often associated with reasoning by analogy which is an important form of non-demonstrative inference. Similarity-based reasoning aims at studying which kinds of logical consequence relations make sense when taking into account that some propositions may be closer to be true than others. A typical kind of inference which is in the scope of similarity-based reasoning responds to the form "if φ is true then ψ is *close* to be true", in the sense that, although ψ may be false (or not provable), knowing that φ is true leads to infer that ψ is semantically close (or similar) to some other proposition which is indeed true. Notice again that the fact of ψ being close to (or approximately) true has nothing to do with a problem of uncertainty, i. e. with a problem of missing information not allowing us to know whether ψ is true or false [DP95]. Essentially, similarity-based reasoning has been investigated from two different perspectives:

- Qualitative or comparative approaches, where the aim is formalizing e. g. expressions like *p is closer to q than r*. The works independently developed by Nicod [Nic70], Williamson [Wil88], and Konikowska [Kon97], belong to the first tradition. At the semantical level, Lewis in [Lew73] uses sphere systems in order to formalize the counterfactual reasoning. Given a possible world w, a sphere system is a set of sets of worlds centered on w, nested, and closed under union and intersection. It is meant to carry information about the comparative overall similarity of worlds. Any particular sphere around world w is to contain just those worlds that resemble w to at least a certain degree. If one world lies within some sphere around w and another world lies outside that sphere, then the first world is more closely similar to w than the second.
- Quantitative approaches, that are based somehow on a numerical definition of degree of truthlikeness or similarity, following the last tradition of truthlikeness as it is pointed out by Niiniluoto in [Nii87] page. 203, and by [Wes87]. This kind of approach, although not always within a formal logical framework, has blossomed after the introduction by Zadeh [Zad71] of fuzzy similarity relations as graded modelings of similarity relations, originally to be used in techniques of categorization and clustering. From then, similarity-based reasoning has taken an important place in the context of fuzzy reasoning. In this second group, we may mention works such as [Rus91], [DP94], [Yin94], [EGG94],[DEG+95], [Kla95],[BJ96], [DEG+97].

In this paper we will be mainly concerned with reviewing the logical formalizations of similarity-based approaches based in one way or another on the notion of fuzzy similarity relations.

A (binary) fuzzy similarity relation S on a given domain D is a mapping $S : D \times D \to [0, 1]$ fulfilling some basic properties trying to capture the notion of similarity.

Reflexivity: $S(u, u) = 1$ for all $u \in D$

Separability: $S(u, v) = 1$ iff $u = v$

Symmetry: $S(u, v) = S(v, u)$, for all $u, v \in D$

$\otimes$-Transivity: $S(u, v) \otimes S(v, w) \leq S(u, w)$, for all $u, v, w \in D$

where $\otimes$ is a t-norm. The reflexivity property establishes that the similarity degree of any world with itself has the highest value. Separability is a bit stronger since it forbids to have $S(u, v) = 1$ for $u \neq v$. Symmetry has a clear meaning, and $\otimes$-Transitivity is a relaxed form of transitivity since it establishes $S(u, v) \otimes S(v, w)$ as a lower bound for $R(u, w)$. Note that $S(u, v) = S(v, w) = 1$ implies $S(u, w) = 1$. Reflexive and symmetric fuzzy relations are often called *closeness* relations, while those further satisfying $\otimes$-transitivity are usually called *$\otimes$-similarity relations*. Sometimes, the name similarity relation is also used to denote in fact min-similarity relations. These relations have the remarkable property that their level cuts $S_\alpha = \{(u, v) \in D \mid S(u, v) \geq \alpha\}$, for any $\alpha \in [0, 1]$, are indeed equivalence relations.

The question which set of the above properties better models the intuitive notion of similarity has led to some interesting discussions in the literature (see e. g. the series of papers [DeCK03a, Bod03, Boi03, Jan03, Kla03, DeCK03b]) related to the Poincaré paradox and the $\otimes$-transitivity property, but such a matter is not in the scope of this paper.

Even though Zadeh introduced both the notions of fuzzy sets and fuzzy similarity relations, only recently it has been remarked the duality between these two notions, which in turn generates another duality between fuzzy reasoning and similarity-based reasoning. Moreover, as it is pointed out by Klawonn and Castro [KC95], even if similarity is not the intended interpretation of fuzzy sets, one can not avoid the effects of similarity which are inherent in fuzzy sets and in fuzzy reasoning.

Indeed, a fuzzy similarity relation $S : D \times D \to [0, 1]$ defines, for each crisp subset $E \subseteq D$, a corresponding fuzzy set *approx_E* of those elements which are *close* to E (in the sense of being close to *some* element of E), just by defining its membership function $\mu_{approx\_E} : D \to [0, 1]$ as

$$\mu_{approx\_E}(u) = \sup\{S(u, v) \mid v \in E\}$$

Note that the membership degree $\mu_{approx\_E}(u)$ is taken as the (highest) similarity degree of u to some element of E, in particular $\mu_{approx\_E}(u) = 1$ if $u \in E$ (and conversely in the case S is separating). Therefore, $E \subseteq approx\_E$, and hence $approx\_E$ can be properly considered an upper (fuzzy) approximation of E. Moreover, if E a set of typical elements satisfying some given property P, $\mu_{approx\_E}(u)$ can also be viewed as a *typicality degree* of u with respect to the property P (in accordance with Niiniluoto's proposal [Nii87]).

Conversely, a fuzzy subset A on a domain D, with membership function $\mu_A : D \to [0,1]$, can be thought of as being defined by

(i) a set of prototypes $E_A = \{u \in D \mid \mu_A(u) = 1\}$, i. e. those elements that fully belong to A, and

(ii) a fuzzy similarity S_A such that the membership degree $\mu_A(u)$ for any $u \in A$ is interpreted as the (highest) similarity degree of u to some prototype of A, that is, $\mu_A(u) = \sup\{S_A(u,v) \mid v \in E_A\}$. Indeed, it suffices to define

$$S_A(u,v) = \min(\mu_A(u) \Rightarrow \mu_A(v), \mu_A(v) \Rightarrow \mu_A(u))$$

for $\Rightarrow$ being the residuum of some (left-continuous) t-norm.

Note, however, that the induced similarity S_A is not unique, it depends on the fuzzy set A.

3 Logical approaches to formalize fuzzy similarity-based reasoning

From a logical point of view, two different paths of research are upheld according to take as primitive notion either a similarity relation between worlds (models), which is then used to define approximate semantical entailments, or a similarity relation between formulas, which is then used to define a notion of approximate (syntactical) proof, by allowing a partial matching mechanism in the inference steps.

In the following, we mention works related to each approach:

- Ruspini presents in [Rus91] "a semantic characterisation of the major concepts and constructs of fuzzy logic in terms of notions of similarity, closeness, and proximity between possible states (**worlds**) of a system that is being reasoned about". Following Ruspini's conception, a family of entailments has been proposed and applied to Case-Based and Interpolative Reasoning ([DEG$^+$95, DEG$^+$97, DEG$^+$98]). In those works, the characterisation of entailments are strictly semantic. Ruspini's perspective is intrinsically modal, although he never produced a full-fledged modal logical framework. However, this gap may be easily overcome by considering a definition of truthlikeness based on similarity measures between worlds and used as accessibility relations in a Kripke's semantics.

- Ming-Sheng Ying presents in [Yin94] "...a propositional calculus in which the truth values of sentences are true or false exactly, but the reasoning may be approximate by allowing the antecedent of a rule to match its premise only approximately". Thus, he wants to give a notion of an approximate proof like one of approximate calculus in, for example, resolution of systems of equations. In [BG98] the authors generalise Ying's proposal and reduce it to a fuzzy logic in the Hilbert style as defined by Pavelka in [Pav79].

Besides these logical-oriented developments, other more fuzzy set based approaches to model patterns of similarity-based reasoning have been developed. For instance, Klawonn *et al.* have developed interpolation methods to obtain fuzzy control functions which are modelled by similarity relations between terms ([KK93, Kla94, Kla95, KC95, KGK95, KN96]). The notion of *extensionality* appears as fundamental in their investigations. Independently, Boixader and Jacas [BJ98] have proposed models of approximate reasoning through the same concept of extensionality with respect to a natural $\otimes$-indistinguishability operator. They consider the degree of indistinguishability between fuzzy sets as a formal measure of its degree of similarity. Although of different nature, it is also worth mentioning Hüllermeier's probabilistic framework for similarity-based inference [Hül01] where he provides a formal model (called similarity profile) of the principle that "similar causes bring about similar effects" which underlies most approaches to similarity-based reasoning and based on a probabilistic characterization of the similarity between observed causes.

In the rest of the paper we overview the above two kinds of logical approaches and related issues.

4 Fuzzy similarity and approximate proofs

Following [Yin94], [BGY00] and [Ses02], the idea is to consider inferences that may be approximated by allowing the antecedent clauses of a rule to match its premises only approximately. In particular, the classical SLD Resolution is modified in order to overcome failure situations in the unification process if the entities involved in the matching have a non-zero similarity degree. Such a procedure allows us to compute numeric values belonging to the interval [0,1], named *approximation degrees*, which provide an approximation measure of the obtained solutions. This framework, which we shall call *Similarity Propositional Logic Programming* (SPLP), is the propositional version of that one proposed by Sessa in [Ses02] which is based upon a first order language. In [GS99b] we find the first proposal to introduce similarity in the frame of the declarative paradigm of Logic Programming. Logic programs on function-free languages are considered and approximate and imprecise information are represented by introducing a similarity relation between constant and predicate symbols. Two transformation techniques of logic programs are defined. In the underlying logic, the inference rule (Resolution rule) as well as the usual crisp representation of the considered universe are not modified. It allows to avoid both the introduction of weights on the clauses, and the use of fuzzy sets as elements of the language. The semantic equivalence between the two inference processes associated to the two kinds of transformed programs has been proved by using an abstract interpretation technique. Moreover, the notion of fuzzy least Herbrand model has been introduced. In [Ses01] the generalization of this approach to the case of programs with function symbols is provided by introducing the general notion of *structural translation* of languages. In [Ses02] the operational counterpart of this extension is faced by introducing a modified SLD Resolution procedure which allows us to perform these kinds of extended computations exploiting the original logic program, without any preprocessing steps in order to transform the given program. Some relations, which allow to state the computational equivalence between these different approaches, has been proved. Finally, for completeness sake, we also cite [FGS00] where a first and different (it takes into account substitutions of variable with sets of symbols) generalized unification algorithm based on similarity has been proposed.

Suppose, as it is the case in [Yin94, BG98, BGY00], that the starting point is a similarity relation $\mathcal{S}$ (reflexive, symmetric and min-transitive relation) defined on the set *Var* of propositional variables. A first problem is how to extend the similarity $\mathcal{S}$ over *Var* to a similarity over a propositional language $\mathbf{L}$ built from *Var*. In Ying and Gerla's papers the extension is done in two steps:

(1) First $\mathcal{S}$ is extended to $\overline{\mathcal{S}}$ on $\mathbf{L}$ by the following recursive definition,

$$\overline{\mathcal{S}}(p,q) = \begin{cases} \mathcal{S}(p,q), & \text{if } p,q \in Var \\ \mathcal{S}(s,s') \wedge \mathcal{S}(t,t'), & \text{if } p = s \to t \text{ and } q = s' \to t' \\ 0, & \text{otherwise} \end{cases}$$

Notice that $\overline{\mathcal{S}}$ is not compatible with the logical equivalence. Take, for example, $F \to p \equiv p \to p$ for every $p \in Var$ and a simple computation shows that $\overline{\mathcal{S}}(F \to p, p \to p) = 0$.

(2) Second they define what is proved to be the minimal similarity relation S_e over $\mathcal{L}$ compatible with logical equivalence and containing $\overline{S}$, as:

$$\mathcal{S}_e(p,q) = \sup\{\overline{\mathcal{S}}(p_1,p_2) \wedge \ldots \wedge \overline{\mathcal{S}}(p_{2n-1},p_{2n}) \mid p_1 = p,\, p_{2n} = q \text{ and } p_{2k} \equiv p_{2k+1} \text{ for } k = 1, n-1\}.$$

The main problem arising from this definition is that it is not evident how to practically compute the relation S_e. Moreover the following results can be proved:

(i) There does not exist a functional extension of S compatible with logical equivalence.

(ii) Any similarity relation preserving logical equivalence defines a similarity relation between classes of logical equivalent formulas and thus a similarity relation between subsets of interpretations, i. e. subsets of Ω. Take into account that, in the finite case, there exists an isomorphism between propositions and subsets of interpretations. Moreover any similarity relation over the set of subsets of Ω defines a similarity relation over $\mathbf{L}$ compatible with logical equivalence.

(iii) Relations $\mathcal{S}_e$ obtained from a similarity relation $\mathcal{S}$ over the set *Var* by Ying-Gerla's method do not cover all similarity relations compatible with logical equivalence. For example, if $\mathcal{S}_e(p,q) = \alpha \neq 0$, then $\mathcal{S}_e(p, p \wedge q) \geq \min(\overline{S}(p,p), \overline{S}(p \wedge p, p \wedge q)) = \alpha$ and this is not necessarily true in a similarity relation compatible with logical equivalence.

Based on $\mathcal{S}_e$, in [BG98] the authors define a consequence operator, $Con_e : F(\mathbf{L}) \times \mathbf{L} \to [0,1]$, $F(\mathbf{L})$ being the set of fuzzy subsets of $\mathbf{L}$, by

$$Con_e(\Gamma, q) = \bigvee \{\overline{\mathcal{S}}_e(\mathit{Taut} \cup \Gamma, B) \mid B \vdash q\}$$

where *Taut* denotes the set of classical tautologies and $\overline{\mathcal{S}}_e$ is defined as,

$$\overline{\mathcal{S}}_e(\Gamma, B) = \bigwedge_{q \in B} \bigvee_{p \in \mathcal{L}} \Gamma(p) \wedge \mathcal{S}_e(p,q)$$

for all $\Gamma \in F(\mathbf{L})$, $B \subseteq \mathbf{L}$ and $p, q \in \mathbf{L}$. The relation $\overline{\mathcal{S}}_e$ is not symmetrical, it may be interpreted as the degree in which B can be considered included in Γ. In fact, if Γ is a crisp set of formulas, then $\overline{\mathcal{S}}_e(\Gamma, B) = 1$ whenever $B \subseteq \Gamma$. An easy computation shows that a form of generalized Modus ponens is preserved by this consequence operator, since the inequality

$$Con_e(\{p \to q, p'\}, q) \geq \overline{\mathcal{S}}_e(p, p')$$

holds for any propositions p, q and p'.

In the rest of this section we briefly describe an application of these ideas in the framework of logic programming developed in [GS99b, Ses02, FGS00]. For simplicity we only consider below the propositional version. We start by recalling that a logic program P on $\mathbf{L}$ is a conjunction of definite clauses of $\mathbf{L}$, denoted as $q \leftarrow p_1, \ldots, p_n$, $n \geq 0$, and a goal is a negative clause, denoted with $\leftarrow q_1, \ldots, q_n$, $n \geq 1$, where the symbol "," that separates the propositional variables has to be interpreted as conjunction, where $p_1, \ldots, p_n, q, q_1, \ldots, q_n \in \mathit{Var}$. A *SPLP-program* is a pair $(P, \mathcal{S})$, where P is a logic program defined on $\mathbf{L}$ and $\mathcal{S}$ is a similarity on *Var*. Given P, the least Herbrand model of P is given by $M_P = \{p \in \mathit{Var} \mid P \vDash p\}$, where $\vDash$ denotes classical logical entailment. M_P is equivalent to the corresponding procedural semantics of P, defined by considering the SLD Resolution. In the classical case, a mismatch between two propositional constant names causes a failure of the unification process. Then, it is rather natural to admit a more flexible unification in which the syntactical identity is substituted by a Similarity $\mathcal{S}$ defined on *Var*. The modified version of the SLD Resolution, which we shall call *Similarity-based SLD Resolution*, exploits this simple variation in the unification process. The basic idea of this procedure for first order languages has been outlined in [GS99a]. The following definitions formalize these ideas in the case of propositional languages.

Definition 1 *Let $\mathcal{S} : \mathit{Var} \times \mathit{Var} \to [0,1]$ be a similarity and $p, q \in \mathit{Var}$ be two propositional constants in a propositional language $\mathcal{L}$. We define the* unification-degree of p and q with respect to $\mathcal{S}$ *the value $\mathcal{S}(p,q)$. p and q are* λ-unifiable *if $\mathcal{S}(p,q) = \lambda$ with $\lambda > 0$, otherwise we say that they are* not unifiable.

Definition 2 *Given a similarity $\mathcal{S} : \mathit{Var} \times \mathit{Var} \to [0,1]$, a program P and a goal G_0, a* similarity-based SLD derivation of *$P \cup \{G_0\}$, denoted by $G_0 \Rightarrow_{C_1,\alpha_1} G_1 \Rightarrow \cdots \Rightarrow_{C_k,\alpha_k} G_k$, consists of a sequence $G_0, G_1, \ldots, G_k$ of negative clauses, together with a sequence $C_1, C_2, \ldots, C_k$ of clauses from P and a sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ of values in* [0,1]*, such that for all $i \in \{1, \ldots, k\}$, G_i is a resolvent of G_{i-1} and C_i with unification degree α_i. The approximation degree of the derivation is $\alpha = \inf\{\alpha_1, \ldots, \alpha_k\}$. If G_k is the empty clause $\perp$, for some finite k, the derivation is called a* Similarity-based SLD refutation*, otherwise it is called* failed.

It is easy to see that when the similarity $\mathcal{S}$ is the identity, the previous definition provides the classical notion of SLD refutation. The values α_i can be considered as constraints that allow the success of the unification processes. Then, it is natural to consider the best unification degree that allows us to satisfy all these constraints. In general, an answer can be obtained with different SLD refutations and different approximation degrees, then the maximum α of these values characterizes the best refutations of the goal. In particular, a refutation with approximation-degree 1 provides an exact solution. Let us stress that α belongs to the set $\lambda_1, \lambda_2, \ldots$ of the possible similarity values in $\mathcal{S}$.

In the sequel, we assume the leftmost selection rule whenever Similarity-based SLD Resolution is considered. However, all the presented results can be analogously stated for any selection rule that does not depend on the propositional constant names and on the history of the derivation [Apt90]. Similarity-based SLD Resolution provides a characterization of the fuzzy least Herbrand model $M_{P,\mathcal{S}}$ for $(P, \mathcal{S})$ defined in [GS99b], as stated by the following result.

Proposition 1 *Let a similarity $\mathcal{S}$ and a logic program P (on a propositional language $\mathbf{L}$) be given. For any $q \in Var$, $M_{P,\mathcal{S}}(q) = \alpha > 0$ iff α is the maximum value in* (0,1] *for which there exists a Similarity-based SLD refutation for $P \cup \{\leftarrow q\}$ with approximation degree α.*

Intuitively, the degree of membership $M_{P,\mathcal{S}}(q)$ of an atom q is given by the best "tolerance" level $\alpha \in (0,1]$ which allows us to prove q exploiting the Similarity-based SLD Resolution on $P \cup \{\leftarrow q\}$. Moreover, if S is strict and M_P denotes the classical least Herbrand Model of the program P, then $q \in M_P$ iff $M_{P,\mathcal{S}}(q) = 1$.

5 Fuzzy similarity and approximate entailments

The starting point in the semantical approaches is to assume that a possible world or state of a system may resemble more to some worlds than to another ones, and this basic fact may help us to evaluate to what extent a partial description (a proposition) may be close or similar to some other.

Under this perspective, an epistemic (in the sense of similarity) state may be modelled by a set of propositions K, modelling the factual information about the world, together with a similarity relation $S : W \times W \to [0,1]$ on the set of possible worlds W for some classical propositional language, modelling how similar or close are worlds among them. Dually, one can think of $\delta = 1 - S$ as a kind of metric on worlds.

Then, using classical reasoning we may know what are the consequences we can infer from K, i. e. those propositions p which logically follow from K, but we can also be interested in those propositions which are approximate consequences of K, in the sense that they are close to some other proposition which is indeed a classical consequence of K.

Since in classical logic we can identify propositions with sets of worlds (in a finitary setting), the above problem reduces to how do we extend the similarity S between worlds to a measure of similarity between sets of worlds. And as well-known, a metric between points does not univocally extend to a meaningful metric between sets of points.

A first consideration is that such a metric has not to be necessarily symmetric, in fact, the logical consequence relation is related to the subsethood relation on sets of worlds ($K \models p$ iff $[K] \subseteq [p]$), not on the equality relation. So, when trying to evaluate to what extent a proposition p is an approximate consequence of K, one is led to measure to what extent the set of K-worlds are close to be included into the set of p-worlds, and not the other way round. In this direction, Ruspini defined the two measures

$$I_S(p \mid q) = \inf_{\omega \models q} \sup_{\omega' \models p} S(\omega, \omega') \qquad \textit{and} \qquad C_S(p \mid q) = \sup_{\omega \models q} \sup_{\omega' \models p} S(\omega, \omega')$$

which are the lower and upper bounds respectively of the resemblance or proximity degree between p and q. Indeed, I_S is an implication (i.e. inclusion-like) measure, while C_S is a consistency (i.e. intersection-like) measure.

With these measures, he wants to capture inference patterns like so-called generalised modus ponens. The value of $I_S(p \mid q)$ provides the measure of what extent p is close to be true given q for granted and the similarity between worlds represented by S. In particular, when S is separating and the set of worlds is finite then, $I_S(p \mid q) = 1$ iff $q \models p$. Moreover, if S is $\otimes$-transitive, for a t-norm $\otimes$, then I_S is $\otimes$-transitive as well [Rus91], i. e. the inequality

$$I_S(r \mid p) \otimes I_S(p \mid q) \leq I_S(r \mid q)$$

holds for any propositions p, q and r. This property can be seen as a kind of generalized resolution rule

from: $I_S(r \mid p) \geq \alpha$ and $I_S(p \mid q) \geq \beta$
infer: $I_S(r \mid q) \geq \alpha \otimes \beta$.

if one interprets $I_S(\varphi \mid \psi)$ as the truthlike degree of a (non-material) conditional "if ψ then φ". On the other hand, if we keep the conditioning part fixed, I_S fails to cast a generalized pattern of modus ponens of the following kind, given some proposition K:

from: $I_S(p \to q \mid K) \geq \alpha > 0$ and $I_S(p \mid K) \geq \beta > 0$
infer: $I_S(q \mid K) \geq \alpha \otimes \beta > 0$.

Indeed, one can easily produce a counter-example in which we may have $I_S(p \to q \mid K) = I_S(p \mid K) \geq \alpha$, with $0 < \alpha < 1$ and α arbitrarily close to 1, but $I_S(q \mid K) = 0$. For instance consider $\mathcal{L}$ generated by only two propositional variables p and q, hence with only four interpretations $\Omega = \{w_1\ (= p \wedge q), w_2\ (= p \wedge \neg q), w_3\ (= \neg p \wedge q), w_4\ (= \neg p \wedge \neg q)\}$, and let S be such that $S(w_i, w_i) = 1$, $S(w_2, w_4) = S(w_4, w_2) = \alpha$, and $S(w_i, w_j) = 0$ otherwise. If we take $K = \{p \wedge \neg q\}$, then it is easy to check that $I_S(p \to q \mid K) = \alpha$ and $I_S(p \mid K) = 1$, but $I_S(q \mid K) = 0$.

On the other hand, the value of $C_S(p \mid q)$ provides the measure of what extent p can be considered compatible with the available knowledge q. In particular, in the finite case and with S satisfying separation property, $C_S(p \mid q) = 1$ iff $q \not\models \neg p$. Observe that, when the propositional language is finitely generated and q is equivalent to a maximal consistent set of propositions, both measures coincide because there is a unique world w such that $w \models q$[3]. In addition, it is easy to show that, given a fixed r, the measure $C_S(\cdot \mid r)$ is a possibility measure [DLP94] since the following identities hold true:

1. $C_S(\top \mid r) = 1$
2. $C_S(\bot \mid r) = 0$
3. $C_S(p \vee q \mid r) = \max(C_S(p \mid r), C_S(q \mid r))$.

Therefore, we also have $C_S(p \mid r) = \max\{C_S(p \wedge q \mid r),\ C_S(p \wedge \neg q \mid r)\}$. In particular, when $C_S(p \wedge q \mid r) > C_S(p \wedge \neg q \mid r)$, it results that $C_S(p \mid r) = C_S(p \wedge q \mid r)$. This can be interpreted as: the $p \wedge q$-worlds are closer (consistent) to the known r-worlds than the $p \wedge \neg q$-worlds are. In this context, the term "closer" is used in the sense of "more similar". We return to this consideration in Subsection 7.1.

Based on the I_S and C_S measures, a first logical system was introduced in [EGG94] where I_S and C_S were used as lower and upper bounds for the truthlikeness degree with which a proposition can be entailed in a given similarity-based epistemic state (K, S). Namely, formulas in this framework are pairs of the form $(p, [\alpha, \beta])$, with $\alpha \leq \beta$ are from the unit interval [0, 1]. Then we define

$$(K, S) \models (p, [\alpha, \beta]) \text{ iff } I_S(p \mid K) \geq \alpha \text{ and } C_S(p \mid K) \leq \beta\ .$$

Here we shall go a bit further in this framework along this notion of logical entailment. If we fix the similarity S, the above satisfaction relation can be extended to a consequence relation in the usual way. Let $\Gamma = \{(q_i, [\alpha_i, \beta_i])\}_{i \in I}$ be a set of graded formulas, and say that (K, S) satisfies Γ, written $(K, S) \models \Gamma$, when $(K, S) \models (q_i, [\alpha_i, \beta_i])$ for each $i \in I$. Then we define

$$\Gamma \models_S (p, [\alpha, \beta]) \text{ iff for each } K,\ (K, S) \models (p, [\alpha, \beta]) \text{ whenever } (K, S) \models \Gamma\ .$$

Analogously to classical logic, this notion of logical consequence can be reduced to involving only worlds. Indeed, if for each proposition p and each world w we define $I(p \mid w) = \sup\{S(w', w) \mid w' \models p\}$, then it can be shown that

[3]By an abuse of notation, in this case we will also write $I_S(p \mid w)$ or $C_S(p \mid w)$.

$$\Gamma \models_S (p, [\alpha, \beta]) \text{ iff for each } w,\ w \models_S (p, [\alpha, \beta]) \text{ whenever } w \models_S \Gamma,$$

where $w \models_S (p, [\alpha, \beta])$ iff $\alpha \le I_S(p \mid w) \le \beta$, and $w \models_S \Gamma$ iff $w \models_S (q_i, [\alpha_i, \beta_i])$ for each $(q_i, [\alpha_i, \beta_i]) \in \Gamma$.

Of particular interest are formulas of the kind $(p, [\alpha, 1])$ referring only to lower bounds for I_S, which seem to be more relevant for our purposes. In such a case we can just write (p, α). For this subset of formulas one can define a consequence operator similar to the one defined by Biacino and Gerla for fuzzy sets of formulas. Indeed, a set of graded formulas $\Gamma = \{(q_i, \alpha_i)\}_{i \in I}$ can be seen as a fuzzy set of classical formulas with membership function

$$\Gamma(q) = \begin{cases} \alpha_i, & \text{if } q = q_i \\ 0, & \text{otherwise} \end{cases}.$$

Then one can define a consequence operator $\mathcal{C}_S$ based on S such that, for every fuzzy set of formulas Γ, $\mathcal{C}_S(\Gamma)$ is the fuzzy set of approximate consequences of Γ with the following membership function:

$$\mathcal{C}_S(\Gamma)(p) = \sup\{\alpha \mid \Gamma \models_S (p, \alpha)\},$$

for any proposition p.

Lemma 1 $\mathcal{C}_S(\Gamma)(p) = \min\{I_S(p \mid w) \mid w \models_S \Gamma\}$.

In fact, one can show that, for any S, $\mathcal{C}_S$ is a fuzzy consequence operator since it verifies:

(i) $\Gamma \le \mathcal{C}_S(\Gamma)$

(ii) if $\Gamma \le \Gamma'$ then $\mathcal{C}_S(\Gamma) \le \mathcal{C}_S(\Gamma')$

(iii) $\mathcal{C}_S(\mathcal{C}_S(\Gamma)) = \mathcal{C}_S(\Gamma)$

The closure property (iii) is a direct consequence from the above lemma and of the fact that, for any world w, $w \models_S \Gamma$ iff $w \models_S \mathcal{C}(\Gamma)$. When Γ is not a fuzzy but a crisp set of formulas, then it is easy to check that one has

$$\mathcal{C}_S(\Gamma)(p) = I_S(p \mid \wedge\{q \mid q \in \Gamma\}).$$

Another way of looking at the above similarity-based consequence operator is by means of a notion of approximate entailment. Given a $*$-similarity relation S on the set W of classical interpretations of a propositional language, one starts by defining a (graded) approximate satisfaction relation $\models_S^\alpha$, for each $\alpha \in [0, 1]$ by

$$\omega \models_S^\alpha p \quad \text{iff} \quad \begin{array}{l} \text{there exists a model } \omega' \text{ of } p \\ \text{which is } \alpha\text{-similar to } \omega, \text{ i.e. } S(\omega, \omega') \ge \alpha \end{array}$$

If $\omega \models_S^\alpha p$ we say that w is an *approximate model* (at level α) of p. The approximate satisfaction relation can be extended over to an approximate entailment relation in the following way: a proposition p entails a proposition q at degree α, written $p \models^\alpha q$, if each model of p is an approximate model of q at level α, that is,

$$p \models_S^\alpha q \text{ holds iff } w \models_S^\alpha q \text{ for all model } w \text{ of } p, \text{ i.e. iff } I(q \mid p) \ge \alpha$$

$p \models_S^\alpha q$ means "p entails q, approximately" and α is a level of strength. The properties of this graded entailment relation are:

(1)	**Nestedness:**	if $p \models^\alpha q$ and $\beta \le \alpha$ then $p \models^\beta q$;
(2)	**$\otimes$-Transitivity:**	if $p \models^\alpha r$ and $r \models^\beta q$ then $p \models^{\alpha \otimes \beta} q$;
(3)	**Reflexivity:**	$p \models^1 p$;
(4)	**Right weakening:**	if $p \models^\alpha q$ and $q \models r$ then $p \models^\alpha r$;
(5)	**Left strengthening:**	if $p \models r$ and $r \models^\alpha q$ then $p \models^\alpha q$;
(6)	**Left OR:**	$p \vee r \models^\alpha q$ iff $p \models^\alpha q$ and $r \models^\alpha q$;
(7)	**Right OR:**	if r has a single model, $r \models^\alpha p \vee q$ iff $r \models^\alpha p$ or $r \models^\alpha q$.

The fourth and fifth properties are consequences of the transitivity property (since $q \models r$ entails $q \models^1 r$) and express a form of monotonicity. The transitivity property is weaker than usual and the graceful degradation of the strength of entailment it expresses, when $\otimes \neq \min$, is rather natural. It must be noticed that $\models^\alpha$ does not satisfy the Right And property, i. e. from $p \models^\alpha q$ and $p \models^\alpha r$ it does not follow in general that $p \models^\alpha q \wedge r$. Hence the set of approximate consequences of p in the sense of $\models^\alpha$ will not be deductively closed. The left OR shows how disjunctive information is handled, while the right OR reflects the decomposability of the approximate satisfaction relation with respect to the $\vee$ connective.

In the case where some (imprecise) knowledge about the world is known and described under the form of some proposition K (i.e. the actual world is in the set of worlds satisfying K), then an approximate entailment relative to K can be straightforwardly defined as

$$p \models^\alpha_{S,K} q \text{ iff } p \wedge K \models^\alpha_S q \text{ iff } I_S(q \mid p \wedge K) \geq \alpha$$

See [DEG+97] for more details and properties of this derived notion of relative entailment.

The above approximate satisfaction relation $w \models^\alpha_S p$ can be also extended over another entailment relation $\mathrel{|\!\equiv}_S$ among propositions as follows: $p \mathrel{|\!\equiv}^\alpha_S q$ holds whenever each approximate model of p at a given level β is also an approximate model of q but at a possibly lower level $\alpha \otimes \beta$. Formally:

$$p \mathrel{|\!\equiv}^\alpha_S q \text{ holds iff for each } w,\ w \models^\beta_S p \text{ implies } w \models^{\alpha\otimes\beta}_S q$$

Now, $p \mathrel{|\!\equiv}^\alpha_S q$ means "approximately-p entails approximately-q" and α is a level of strength, or in other words, when worlds in the vicinity of p-worlds are also in the vicinity (but possibly a bit farther) of q-worlds. This notion of entailment, called *proximity entailment* in [DEG+97], also admits a characterization in terms of another similarity-based measure

$$J_S(q \mid q) = \inf_w I_S(p \mid w) \Rightarrow I_S(q \mid w)$$

where $\Rightarrow$ is the residuum of the (left-continuous) t-norm $\otimes$ and $I_S(p \mid w) = \sup_{w' \models p} S(w, w')$. Indeed, one can easily check that $p \mathrel{|\!\equiv}^\alpha_S q$ holds iff $J_S(q \mid p) \geq \alpha$. This notion of approximate entailment relation can be easily made relative to a context, described by a set of propositions K we know for sure to hold, sometimes called *background knowledge*, by defining

$$p \mathrel{|\!\equiv}^\alpha_{S,K} q \text{ holds iff for each } w \text{ model of } K,\ w \models^\beta_S p \text{ implies } w \models^{\alpha\otimes\beta}_S q$$

One can analogously characterize this entailment by a generalized measure $J_{S,K}$, namely it holds that $p \mathrel{|\!\equiv}^\alpha_{K_S} q$ iff $J_{K,S}(q \mid p) \geq \alpha$, where $J_{K,S}(q \mid q) = \inf_{w: w \models K} I_S(p \mid w) \Rightarrow I_S(q \mid w)$.

The entailment $\mathrel{|\!\equiv}^\alpha_K$ satisfies similar properties to those satisfied by $\models^\alpha$. Characterizations of both similarity-based graded entailments in terms of these properties are given in [DEG+97]. It is also shown there that $\mathrel{|\!\equiv}^\alpha$ and $\models^\alpha$ actually coincide, i. e. when there is no background knowledge K, or equivalently when K is a tautology. However, when K is not a tautology, $\models^\alpha$ is generally stronger than $\mathrel{|\!\equiv}^\alpha_K$.

6 Modal and conditional logic accounts of the similarity-based entailments

In the notions of approximate entailments described in the previous section, the key presence of a similarity relation on the set of interpretations strongly suggests a modal logic setting for similarity-based reasoning. Indeed, modal logic has always received a lot of attention from logicians and after the publication of Kripke's semantics ([Kri59a, Kri59b]), the notion of possible worlds and of accessibility relation has been inseparably associated with modal logic. For instance, taking classical propositional logic interpretations as possible worlds, each level cut S_α of the (fuzzy) similarity relation S defines an accessibility relation: $(w, w') \in S_\alpha$ if $S(w, w') \geq \alpha$. Therefore it makes sense to consider a modal approach to similarity-based reasoning based on Kripke structures of the form

$$M = (W, S, e),$$

where W is a set of possible worlds, $S : W \times W \to [0, 1]$ a similarity relation between worlds, and e a classical two-valued truth assignment of propositional variables in each world $e : W \times Var \to \{0, 1\}$. Then, for each $\alpha \in [0, 1]$ one can consider the accessibility relation S_α on W, which gives meaning to a pair of dual possibility and necessity modal operators $\Diamond_\alpha$ and $\Box_\alpha$:

$$(M,w) \models \Diamond_\alpha \varphi \text{ if there is } w' \in W \text{ such that } (w,w') \in S_\alpha \text{ and } (M,w') \models \varphi.$$

This defines in fact, a multi-modal logical framework (with as many modalities as level cuts in the similarity relations). Such a multimodal logic setting is systematically developed by Esteva *et al.* [EGGR97] and will be reviewed in Section 6.1.

Note that, if W is the set of classical interpretations of a propositional language $\mathcal{L}$, then the above notion of modal satisfiability for the possibility operators $\Diamond_\alpha$ captures precisely the notion of approximate satisfiability considered in Section 5, in the sense that, for any $p \in \mathcal{L}$, $(M,w) \models \Diamond_\alpha p$ holds iff $w \models^\alpha_S p$ holds. Moreover, the *approximate entailments* $p \models^\alpha_S q$ can also be captured by the formula

$$p \to \Diamond_\alpha q,$$

in the sense that $p \models^\alpha_S q$ holds iff $M \models p \to \Diamond_\alpha q$, i.e. iff $p \to \Diamond_\alpha q$ is valid in $M = (W,S,e)$. As for the *proximity entailments* $\models\!\!\!\equiv^\alpha$, recall that $p \models\!\!\!\equiv^\alpha_K q$ holds iff for all w model of K and for all β, $w \models^\beta p$ implies $w \models^{\alpha\otimes\beta} q$. Therefore, it cannot be represented in the multi-modal framework unless the similarity relations are forced to have a fixed, predefined set of finitely-many different levels, say $G \subset [0,1]$. In that case, the validity of the formula

$$K \to (\bigwedge_{\beta \in G} \Diamond_\alpha p \to \Diamond_{\alpha\otimes\beta} q)$$

in the model (W,S,e) is equivalent to the entailment $p \models\!\!\!\equiv^\alpha_{S,K} q$. Obviously, when C is not finite, this representation is not suitable any longer.

Partly due to these difficulties, an alternative approach developed in [Rod02] is to consider a graded conditional logic, where each (approximate and proximity) entailment is directly represented in the object language by a family of binary operators indexed by degrees. Indeed, the idea is to introduce in the language graded binary modalities $>_\alpha$ and $\gg_\alpha$, for each $\alpha \in G$, with the following semantics: given a similarity Kripke model $M = (W,S,e)$, the following satisfiability conditions are defined:

$$
\begin{array}{lll}
(M,w) \models \varphi >_\alpha \psi & \text{iff} & \text{for all } w' \in W,\ (W,w') \models \varphi \text{ implies } (W,w'') \models \psi \\
 & & \text{for some } w'' \text{ s.t. } S(w',w'') \geq \alpha \\[1em]
(M,w) \models \varphi \gg_\alpha \psi & \text{iff} & \text{for each } \beta,\ (W,w') \models \varphi \text{ for some } w' \text{ s.t. } S(w,w) \geq \beta \text{ implies} \\
 & & (W,w'') \models \psi \text{ for some } w'' \text{ s.t. } S(w,w'') \geq \alpha \otimes \beta
\end{array}
$$

Note that the first condition is actually independent from the world w, it is thus a global condition which is indeed equivalent to the validity in M of $\varphi \to \Diamond_\alpha \psi$ in the previous multi-modal framework, and hence to the validity of the approximate entailment $\varphi \models^\alpha_S \psi$ (when φ and ψ are non modal). The second condition is indeed local, and it is easy to check that the condition of $\varphi \gg_\alpha \psi$ being valid in M is indeed captures the proximity entailment $\varphi \models\!\!\!\equiv^\alpha \psi$. The technical details of this graded conditional approach will be described in Section 6.2.

In both the graded modal and conditional logical frameworks, the following classes of models will be considered:

$$
\begin{array}{lll}
\Sigma_0 & = & \{M = (W,S,e) \mid S \text{ is a fuzzy relation }\}, \\
\Sigma_1 & = & \{M = (W,S,e) \mid S \text{ is a serial fuzzy relation }\}, \\
\Sigma_2 & = & \{M = (W,S,e) \mid S \text{ is a reflexive fuzzy relation }\}, \\
\Sigma_3 & = & \{M = (W,S,e) \mid S \text{ is a reflexive and symmetric fuzzy relation }\}, \\
\Sigma_4 & = & \{M = (W,S,e) \mid S \text{ is a reflexive and } \otimes\text{-transitivity fuzzy relation }\}, \\
\Sigma_\otimes & = & \{M = (W,S,e) \mid S \text{ is a } \otimes\text{-similarity relation }\}.
\end{array}
$$

where we assume the fuzzy relations to take values on some given countable $C \subset [0,1]$, i.e. $S : W \times W \to C$, and in the class $\Sigma_\otimes$ we are also assuming that the t-norm $\otimes$ is closed on C. Furthermore, the notations Σ^*_i and Σ_{if} will be used to denote the subclasses of Σ_i ($i \in \{1,2,3,4,\otimes\}$) where the fuzzy relation is separating as well, and where the set of worlds is finite, respectively. As it is obvious, we have that $\Sigma_0 \supseteq \Sigma_1 \supseteq \Sigma_2 \supseteq \Sigma_3, \Sigma_4 \supseteq \Sigma_\otimes$ and therefore, their corresponding sets of valid formulas satisfy the inverse inclusion.

Yet another line of modeling, alternative to the two above multi-modal frameworks, has been proposed in the literature. It consists in understanding the grades of the modal and conditional operators as truth-values of some related syntactic many-valued objects. For instance, if p is a proposition, one can consider another (fuzzy) proposition $\Diamond p$, read as "*approximately p*" and, given a similarity Kripke frame (W, S), define the truth-value of $\Diamond p$ in a world w as the value $e(w, \Diamond p) = I_S(p \mid w) \in [0,1]$, i.e. the greatest α for which $(M, w) \models \Diamond_\alpha p$. Then one can use a suitable t-norm based fuzzy logic [Haj98, GH05], like Gödel or Łukasiewicz logics, expanded with truth-constants [EGGN07] as base logic to reason about the modalities. In such a framework, the evaluation $e(w, \overline{\alpha} \to \Diamond p)$ of a formula of the form $\overline{\alpha} \to \Diamond p$, where $\overline{\alpha}$ is a truth-constant representing the value α and $\to$ is interpreted as the residuum of a t-norm, takes value 1 iff $e(w, \Diamond p) \geq \alpha$. Hence, the 1-validity of $\overline{\alpha} \to \Diamond p$ in (W, S) is again equivalent to the validity of $\Diamond_\alpha p$.

Analogously, one may introduce (fuzzy) modalities $>$ and $\gg$ in such a way that the truth values of $p > q$ and $p \gg q$ in a world $w \in W$ be $e(w, p > q) = I_S(q \mid p)$ and $e(w, p \gg q) = J_{S,w}(q \mid q)$. These approaches, fully developed in [Rod02], are recalled in Sections 6.3 and 6.4 respectively.

In what follows we will use the special symbol $\mathcal{M}_S$ to denote the similarity Kripke model (Ω, S, e) where Ω is the set of all boolean interpretations of $\mathbf{L}$, $S : \Omega \times \Omega \to C \subset [0,1]$ is a similarity relation (of some of the above types), and $e : \Omega \times \mathit{Var} \to \{0,1\}$ is the truth-evaluations of variables naturally induced by the elements of Ω, i.e. $e(w, p) = w(p)$ for any $w \in \Omega$ and any propositional variable p.

6.1 Multi-modal logic approach

The use of *graded modalities* is a very well known tool in Philosophy and Computer Science. Several authors, for instance Goble [Glo70], Fine [Fin72], Fattorosi-Barnaba and De Caro [FD85], provide graded modal operators $\Box_n$ (with $n \in N$) interpreted as "*there are more than (or at least) n accessible worlds such that...*". Graded languages with this interpretation were applied to the areas of epistemic logic [HM92] and of generalised quantifiers [HR91]. Here, the conceptual framework and technical features are very different.

A general formalization of the similarity-based graded modal logic, as proposed in [EGGR97], can be summarized as follows:

- Modal Language: The new language $\mathcal{L}$ is built over $\mathbf{L}$ by adding modal operators $\Diamond^c_\alpha$ and $\Diamond^o_\alpha$ for every rational $\alpha \in C$, where $\{0,1\} \subseteq C \subseteq [0,1]$.
- Formulae: They are built from a set V (not necessarily finite) of propositional variables using the classical binary connectives $\wedge$, $\vee$ and $\to$, and the unary operators $\neg$, $\Diamond^c_\alpha$ and $\Diamond^o_\alpha$ for every rational $\alpha \in C$, in the usual way.
- Satisfiability: Let $M = (W, S, e)$, $\omega \in W$ and φ be a formula of $\mathcal{L}$. Then, we define:

  $(M, \omega) \models \Diamond^c_\alpha \varphi$ if $I_S(\varphi \mid \omega) \geq \alpha$,
  $(M, \omega) \models \Diamond^o_\alpha \varphi$ if $I_S(\varphi \mid \omega) > \alpha$.

  The rest of the conditions are the usual ones. Note that this notion of satisfiability needs a definition of implication measure for modal formulas since the definition given above is only valid for non modal formulas. Nevertheless, the implication measure for modal formulas φ is defined as a natural extension in the following way,

  $$I_S(\varphi \mid \omega) = \sup\{S(\omega, \omega') \mid (\mathcal{M}, \omega') \models \varphi\}.$$

  we shall also introduce the corresponding family of dual modal operators $\Box^c_\alpha$ and $\Box^o_\alpha$ as $\neg\Diamond^c_\alpha\neg$ and $\neg\Diamond^o_\alpha\neg$ respectively, and whose satisfiability conditions are:

  $(M, \omega) \models \Box^c_\alpha \varphi$ if $I_S(\neg\varphi \mid \omega) < \alpha$,
  $(M, \omega) \models \Box^o_\alpha \varphi$ if $I_S(\neg\varphi \mid \omega) \leq \alpha$.

  It is easy to see that whenever W is finite, $\Diamond^c_\alpha$ and $\Box^c_\alpha$ have the usual Kripke semantics with respect to the accessibility relation S^c_α defined as

  $$\omega S^c_\alpha \omega' \text{ iff } S(\omega, \omega') \geq \alpha.$$

  In contrast, the *strict* cuts S^o_α of S, i.e. $\omega S^o_\alpha \omega'$ iff $S(\omega, \omega') > \alpha$, always provide the modal operators $\Diamond^o_\alpha$ and $\Box^o_\alpha$ with the usual Kripke semantics, even when W is not finite.

- Axioms: For the axiomatic characterization of the different multi-modal systems, let us consider the following schemes, where, as usual, C denotes the range of the fuzzy relations and it is assumed to be of the form $\{0,1\} \subseteq C \subseteq [0,1]$ and closed with respect to the operation $\otimes$:

  K^c: $\Box^c_\alpha(\varphi \to \psi) \to (\Box^c_\alpha \varphi \to \Box^c_\alpha \psi)$, $\forall \alpha \in C$
  K^o: $\Box^o_\alpha(\varphi \to \psi) \to (\Box^o_\alpha \varphi \to \Box^o_\alpha \psi)$, $\forall \alpha \in C$
  D: $\Box^c_1 \varphi \to \Diamond^c_1 \varphi$
  T^c: $\Box^c_\alpha \varphi \to \varphi$, $\forall \alpha \in C$
  T^o: $\Box^o_\alpha \varphi \to \varphi$, for $\alpha < 1$
  C^c: $\varphi \to \Box^c_1 \varphi$
  B^c: $\varphi \to \Box^c_\alpha \Diamond^c_\alpha \varphi$, for $\alpha > 0$
  B^o: $\varphi \to \Box^o_\alpha \Diamond^o_\alpha \varphi$, $\forall \alpha \in C$
  4^c: $\Box^c_{\alpha \otimes \beta} \varphi \to \Box^c_\beta \Box^c_\alpha \varphi$, $\forall \alpha, \beta \in C$
  4^o: $\Box^o_{\alpha \otimes \beta} \varphi \to \Box^o_\beta \Box^o_\alpha \varphi$, $\forall \alpha, \beta \in C$
  N^c: $\Box^c_\alpha \varphi \to \Box^c_\beta \varphi$, for $\beta \geq \alpha$,
  N^o: $\Box^o_\alpha \varphi \to \Box^o_\beta \varphi$, for $\beta \geq \alpha$
  EX^c: $\Diamond^c_0 \varphi$,
  EX^o: $\neg \Diamond^o_1 \varphi$,
  CO: $\Box^c_\alpha \varphi \to \Box^o_\alpha \varphi$, $\forall \alpha \in C$
  OC: $\Box^o_\alpha \varphi \to \Box^c_\beta \varphi$, for $\alpha < \beta$,

  and the following inference rules

  MP: From φ and $\varphi \to \psi$ infer ψ.
  RN^c: From φ infer $\Box^c_\alpha \varphi$, for $\alpha > 0$.
  RN^o: From φ infer $\Box^o_\alpha \varphi$, $\forall \alpha \in C$.

- Completeness: The following completeness results, where PL stands for propositional tautologies, have been proved in [EGGR97]:

  - The axiom system **MS5(C, $\otimes$)$^{++}$** $= PL + K^c + K^o + CO + OC + EX^c + EX^o + T^c + B^o + B^c + 4^o + 4^c + C^c$ is complete with respect to the subclass of finite models of $\Sigma^*_\otimes$ when C is a dense and denumerable and $\otimes = \min$.
  - If C is finite, then the axiom system **MS5(C, $\otimes$)$^{+}$** consisting of PL, K^c, B^c, 4^c, C^c, N^c, EX^c, plus MP and RN^c is complete with respect to the class of models $\Sigma^*_\otimes$, for any t-norm $\otimes$. In this case we shall see that the open and closed modalities are interdefinable, and the resulting modal system can be simplified.
  - If we remove axiom C^c from the system **MS5(C, $\otimes$)$^{++}$** we get a complete system with respect to the subclass of models $\Sigma_{\otimes f}$ when C is dense and denumerable and $\otimes = \min$.
  - If C is finite and we remove axiom C^c from the system **MS5(C, $\otimes$)$^{+}$** we get a complete system with respect to $\Sigma_\otimes$.
  - If C is finite and we remove axiom 4^c (+ C^c resp.) from the system **MS5(C, $\otimes$)$^{+}$** we get a complete system with respect to Σ^*_2 (with respect to Σ_2 resp.).

Once the presentation of the logics is done, we are able to formally claim that the basic similarity-based graded consequence relation proposed in [DEG+95] is fully captured inside these multi-modal systems. Namely, given a similarity relation S on the set of interpretations Ω of a propositional language $\mathbf{L}$, if p and q are non-modal formulas, then we have that the *approximate entailment* corresponds to

$$p \models^\alpha_S q \text{ iff } \mathcal{M}_S \models p \to \Diamond^c_\alpha q.$$

If C is finite, then proximity entailments can be captured as well:

$$p \models\!\!\!\models^\alpha_{S,K} q \text{ iff } \mathcal{M}_S \models K \to \bigwedge_{\beta \in C} (\Diamond^c_\beta p \to \Diamond^c_{\beta \otimes \alpha} q).$$

Finally, we briefly describe some modal systems in the literature which are close to the above mentioned ones:

- In [LL92, LL95] Liau and Lin define a multi-modal system like the one presented here. One goal of that paper is the relationship of their modal system with *possibilistic logic* and therefore they consider models such that the relation R only satisfies the so-called *serial* property, i. e. for all $\omega \in W$, $\sup_{\omega' \in W} R(\omega, \omega') = 1$. Obviously this property is weaker than reflexivity, but to model truthlikeness does not seem meaningful to consider serial relations which are not reflexive, since in that case the corresponding mapping might be such that the approximation p^* of a proposition p would not contain the set $[p]$ of interpretations of p. In their works, Liau and Lin propose a Quantitative modal logic (QML) with $C = [0, 1]$ and prove the following completeness results:
  - The axiom system **SK** consisting of PL, K^c, K^o, CO, OC, EX^c, EX^o, together with the MP and RN^o inference rules is complete with respect to the class of models Σ_0.
  - The axiom system **SKD** = SK + D is complete with respect to the class of models Σ_1.
  - The axiom system **SKT** = SK + T^c is complete with respect to the class of models Σ_2.
- In a very interesting work, Suzuki [Suz97] proposes a more general semantics by considering a partial fuzzy accessibility function instead of a total fuzzy function, as it is the case with our fuzzy similarity relations. He also describes almost all families of modal systems that we considered above. But, he only gives a logic of similarity relations when they are min-transitive. Besides, he only establishes completeness results for the cases that the range of the partial fuzzy function is an arbitrary *finite* subset of $[0, 1]$. Moreover, strong completeness is not available in the general case. However, we think this work is important because some other general results which are natural extensions of well-known ones in classical modal logic, are presented in his work, as for instance, the definition of F-filtration, Craig's interpolation theorem, etc.
- Finally, another similar logic is proposed in [CF92]. The logic is called lattice-based graded logic and contains modal operators $\Box_\alpha$ which are formed for all α from a finite lattice structure instead of the countable set C considered above where $\{0, 1\} \subseteq C \subseteq [0, 1]$. They adopt a semantics which involves a family of accessibility relations R_α for each α in the lattice (also called multi-relational model in [FH91]). In the finite case, when R_α's are nested equivalence relations, their semantics is equivalent to the one with min-transitive similarity relations.

6.2 Multi-conditional logic approach

The idea of the graded conditional logic approach is to encode in the language syntactical objects representing both approximate and proximity entailments $p \models^\alpha q$ and $p \models\!\!\!\equiv^\alpha q$. To do so, binary (graded) modal operators are introduced (under some restrictions, e. g. nested modal formulas are not allowed, and the language is finitely generated) and given appropriate semantics in terms of similarity Kripke structures. Following [Rod02], the main notions involved in the graded conditional logical framework to model similarity-based reasoning can be summarized as follows:

- <u>Conditional Language</u>: The propositional language $\mathbf{L}_f$ generated from a finite set *Var* of propositional variables is extended by two families $\{>_\alpha\}_{\alpha \in C}$ and $\{\gg_\alpha\}_{\alpha \in C}$ of binary operators, where $\{0, 1\} \subseteq C \subseteq [0, 1]$.
- <u>Conditional Formulae</u>:
  - If p is a propositional formula then it is also a conditional formula.
  - If p and q are propositional formulas in $\mathbf{L}_f$ then for every $\alpha \in C : p >_\alpha q$ and $p \gg_\alpha q$ are conditional formulas.
  - If φ and ϕ are conditional formulas then $\varphi \circ \phi$ is a conditional formula, where $\circ \in \{\wedge, \vee, \rightarrow\}$
  - If φ is a conditional formula then $\neg\varphi$ is a conditional formula.

  Note that in this language, nested modal formulas are not allowed.

- Satisfiability: Given a model $\mathcal{M} = (W, S, \models)$, a world $\omega \in W$ and formulas p and q of $\mathbf{L}_f$, we define:

$$(\mathcal{M}, \omega) \models p >_\alpha q \quad \text{if} \quad I_S(q \mid p) \geq \alpha,$$
$$(\mathcal{M}, \omega) \models p \gg_\alpha q \quad \text{if} \quad I_{S,\bar{\omega}}(q \mid p) \geq \alpha,$$

where $\bar{\omega}$ is the maximal elementary conjunction[4] corresponding to ω[5]. The rest of the conditions are the usual ones. Note that the notion of satisfiability for $>_\alpha$ is independent of any particular world, i. e. it is a global notion. The last conditions of satisfiability make clear that in the object language $p >_\alpha q$ and $p \gg_\alpha q$ represent lower bounds of $I_S(q \mid p)$ and $I_{S,\bar{\omega}}(q \mid p)$ respectively.

- Axioms: The following schemes will be used to characterise the different classes of models (Σ_i) above mentioned, where p and q are propositional formulas in $\mathbf{L}_f$, and α and β represent any element in the range C of fuzzy relations.

N:	$p >_\alpha q \to p >_\beta q$ if $\beta \leq \alpha$.
	$p \gg_\alpha q \to p \gg_\beta q$ if $\beta \leq \alpha$
CS:	$p >_1 q \to (p \to q)$.
	$p \gg_1 q \to (p \to q)$
EX:	$p >_0 q$.
	$p \gg_0 q$
B:	$r >_\alpha r' \to r' >_\alpha r$, if r and r' are m.e.c.'s
4:	$(p >_\alpha q) \wedge (q >_\beta r) \to p >_{\alpha\otimes\beta} r$
	$(p \gg_\alpha q) \wedge (q \gg_\beta r) \to p \gg_{\alpha\otimes\beta} r$
LO:	$(p \vee q >_\alpha r) \leftrightarrow (p >_\alpha r) \wedge (q >_\alpha r)$
	$(p \vee q \gg_\alpha r) \leftrightarrow (p \gg_\alpha r) \wedge (q \gg_\alpha r)$
RO:	$(r >_\alpha p \vee q) \leftrightarrow (r >_\alpha p) \vee (r >_\alpha q)$, if r is a m.e.c.
	$(s \gg_\alpha p \vee q) \leftrightarrow (s \gg_\alpha p) \vee (s \gg_\alpha q)$

and the following inference rules:

MP:	From φ and $\varphi \to \psi$ infer ψ
RK:	From $p_1 \wedge \cdots \wedge p_n \to q$ infer $p_1 \wedge \cdots \wedge p_n >_\alpha q$
	From $p_1 \wedge \cdots \wedge p_n \to q$ infer $p_1 \wedge \cdots \wedge p_n \gg_\alpha q$

- Completeness: The following completeness results have been proved [Rod02] for different classes of models Σ in which the set W is fixed to the set of all boolean interpretations of $\mathbf{L}_f$, for any t-norm $\otimes$ and range C. From now on, PL will stand for propositional tautologies. Here, two kinds of logical systems: **CSI** and **CSJ**. In the first, the operators $\gg_\alpha$ do not appear and in the second, the operators $>_\alpha$ are not used. We split the completeness results for a **CSI** logic and for a **CSJ** logic.
  - The approximate conditional system $\mathbf{CSI(C, \otimes)} = PL + N + EX + LO + RO$ and closed under MP and RK is complete with respect to Σ_{2f}. Furthermore, it is possible to prove completeness with respect to the subclasses of models Σ_{3f}, Σ_{4f} and $\Sigma_{\otimes f}$ if we add to $\mathbf{CSI(C, \otimes)}$ the axioms B, 4 and both B and 4, respectively.
  - The system $\mathbf{CSI(C, \otimes)}^+$, the extension of $\mathbf{CSI(C, \otimes)}$ with axiom CS, is complete with respect to the subclass of models Σ^*_{2f}. Again, it is possible to extend this result of completeness for the subclasses of models Σ^*_{3f}, Σ^*_{4f} and $\Sigma^*_{\otimes f}$ by adding to $\mathbf{CSI(C, \otimes)}^+$ the axioms B, 4 and both B and 4, respectively.
  - The proximity conditional system $\mathbf{CSJ(C, \otimes)} = PL + N + EX + LO + RO + 4$ and closed under MP and RK is complete with respect to Σ_{4f}.
  - The system $\mathbf{CSJ(C, \otimes)}^+$, the extension of $\mathbf{CSJ(C, \otimes)}$ with axiom CS, is complete with respect to the subclass of models Σ^*_{4f}.

[4] A maximal elementary conjunction, m.e.c. for short, is a conjunction where each propositional variable in *Var* appears either in positive or negative form (remember that we are assuming *Var* be finite).

[5] That is, the conjunction $\bar{\omega} = \bigwedge_{p_i \in Var : \omega(p_i)=1} p_i \wedge \bigwedge_{p_i \in Var : \omega(p_i)=0} \neg p_i$.

Graded approximate and proximity entailments are captured in $\mathbf{CSI}(\mathbf{C}, \otimes)$ and $\mathbf{CSJ}(\mathbf{C}, \otimes)$ as follows:

$$\textbf{approximate entailment:} \quad p \models^{\alpha}_{S,K} q \quad \text{iff} \quad \mathcal{M}_S \models K \to p >_{\alpha} q$$

$$\textbf{proximity entailment:} \quad p \models\!\!\!\equiv^{\alpha}_{S,K} q \quad \text{iff} \quad \mathcal{M}_S \models K \to p \gg_{\alpha} q$$

where $\mathcal{M}_S$ is the similarity Kripke model over the set of all Boolean interpretations of the finitely generated language $\mathbf{L}_f$.

Regarding related work, let us mention that Liau [Lia98] defines what he calls *residuated implication operators* $\stackrel{\alpha}{\Longrightarrow}$ and $\stackrel{\alpha^+}{\Longrightarrow}$ corresponding to $\gg_\alpha$ and its strict counterpart, respectively. He shows how to capture the approximate and proximity entailments proposed in [DEG+95] with these implication operators. However, his considerations are purely semantical. Besides, his motivation is very different because he aims at defining a logical system where quantitative and qualitative uncertainty may be combined. According to this author, probabilistic, Dempster-Shafer and possibilistic theories are included in the first kind of uncertainty, and rough sets and nonmonotonic theories belong to the second class. In fact, his residuated implication operators may be seen as graded generalisations of qualitative possibility relations [Dub86]. We also mention that Liau and Lin [LL96] define a logic for conditional possibility (LCP) based on Dempster's conditional rule, but LCP is able to model similarity-based entailment only when the similarity relation is min-transitive. Besides, although an axiomatic system for LCP is exhibited in its appendix, completeness results are not established. They mention that their main difficulty in order to obtain a completeness result lies in the infiniteness of the language. This problem is however different from the above conditional logics: the need of considering a finite language was due to properly cope with the Symmetry and Right-Or properties.

6.3 Many-valued modal logic approach

As already mentioned in the introduction of this section, a possibly more elegant way of formalising similarity-based reasoning in a modal framework is to shift from a family of graded classical modalities $\Diamond_\alpha$ (one for each $\alpha \in G$) to a single many-valued modality $\Diamond$. The idea is that, even if p is a two-valued formula, $\Diamond p$, to be read as *approximately* p, is a many-valued formula which takes $I_S(p \mid w) = \sup\{S(w, w') \mid (M, w') \models p\}$ as truth-value in a world w from a model $M = (W, S, e)$, i. e. such that $e(w, \Diamond p) = I_S(p \mid w)$. Therefore one needs to choose a suitable (t-norm based) fuzzy logic as base logic to reason about the modal formulas. If one also wants to reason explicitly with truthlikeness degrees, then the base fuzzy logic has to be expanded with truth constants, for instance by adding a constant $\overline{\alpha}$ for each $\alpha \in C$. These expansions have been studied for instance in [EGGN07] for the case of logics of continuous t-norms (thus including Łukasiewicz, Gödel and Product fuzzy logics).

However, with the above semantics, one would be led to define a language without nested modal operators. If one wants to be as general as possible, one has to generalize the above semantics and allow to deal with modal formulas of the form $\Diamond\varphi$, where φ may be in turn a many-valued formula. Indeed, there have been many attempts in the literature to mix many-valued (or fuzzy) and modal logic semantics, obeying to very different motivations. These logics consider the fuzzification of either the valuation function or the accessibility relation in the Kripke model (or both). A complete analysis of all alternative semantics is provided by Thiele in [Thi93]. As it is reported there, there are different ways to face with the problem of how the truth values of a formula p in two worlds ω and ω' can be combined with the "degree of accessibility" of ω' from ω, expressed as the value of a fuzzy relation $R(\omega, \omega')$.

Most of the alternatives have appeared as a direct generalisation of the classical definitions of possibility and necessity. It is easy to see that the classical definitions are equivalent to the following ones:

$$\begin{aligned} e(\omega, \Diamond p) &= \sup_{\omega' \in W} R(\omega, \omega') \wedge e(\omega', p) \\ e(\omega, \Box p) &= \inf_{\omega' \in W} R(\omega, \omega') \Rightarrow e(\omega', p) \end{aligned}$$

where the existential and universal quantifiers are interpreted by supremum and infimum operators, respectively. When we shift from $\{0, 1\}$-valued to $[0, 1]$-valued evaluations e and accessibility relations R, following the tradition of fuzzy logics the *and* connective $\wedge$ is usually associated to a (left-continuous) t-norm (e. g. Gödel, Łukasiewicz or Product), and the *implication* connective $\Rightarrow$ has is associated to its residuum (although other interpretations exist, see e. g. [Yin88]).

This is the alternative taken by Fitting (see [Fit91, Fit92, Fit95]) in his many-valued modal logic. His many-valued modal logic includes truth constants that are the syntactical counterpart of truth values, but it is confined to the case where the set of truth values is finite and the t-norm is min. As another example of this alternative, although somehow particular, we mention Hájek and Harmancová's work [HH96] (see also [Haj98] for a further elaboration) where they study a modal logic over a Pavelka-like extension of the infinitely valued Łukasiewicz's logic. Their logic is a many-valued counterpart of the well-known classical system **S5** and proved to complete with respect to Kripke models with the universal accessibility relation (i.e. $\forall \omega, \omega' : R(\omega, \omega') = 1$).

But probably the most interesting work for the purposes of modelling similarity-based reasoning is [CR07] where Caicedo and Rodríguez define a very general many-valued modal logic over Gödel fuzzy logic by introducing independently a possibility modal operator $\Diamond$ (with the intended meaning of $\Diamond p$ as *approximately-p*) and a necessity modal operator $\Box$ (since they are are not dual). Moreover, to explicitly deal with similarity degrees in the language they take as base logic the expansion of Gödel logic with rational truth-constants, called RG in [EGN06], but with only a finite set of truth-constants. In the following we summarize the most interesting features of the $\Diamond$-fragment of that related modal systems

- Language: propositional variables, truths constant $\bar{\alpha}$ for each rational $\alpha \in C$ (where $\{0,1\} \subseteq C \subset [0,1]$ is finite) logical connectives of Gödel fuzzy logic $\wedge, \rightarrow$ (other connectives are definable, e. g. $\neg\varphi$ is $\varphi \rightarrow \bar{0}$) and one modality $\Diamond$.

- Formulae: they are built in the usual way from a set *Var* of propositional variables using the binary connectives $\wedge$, $\rightarrow$, truth-constants and the unary operator $\Diamond$

- Satisfiability: models are Gödel similarity Kripke models $M = \langle W, S, e_G \rangle$, in which $W \neq \emptyset$ is a set of possible worlds, S is a similarity relation on $W \times W$ and e represents an evaluation assigning to each atomic formula p_i and each interpretation $w \in W$ a truth value $e(p_i, w) \in [0,1]$ of p_i in w. e is extended to formulas by means of Gödel logic truth functions by defining

  $e(\varphi \wedge \psi, w) = \min(e(\varphi, w), e(\psi, w))$,
  $e(\varphi \rightarrow \psi, w) = e(\varphi, w) \Rightarrow_G e(\psi, w)$,

  where $\Rightarrow_G$ is the well-known Gödel implication function[6], and

  $e(\overline{r}, w) = r$, for all $r \in C$,
  $e(\Diamond\varphi, w) = \sup_{w' \in W} \min\{S(w, w'), e(\varphi, w')\}$.

- Axioms: we include here below the axioms of Rational Gödel logic and a list of modal axioms

  Axioms of Rational Gödel logic (RG):
  $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
  $\varphi \rightarrow (\psi \rightarrow \varphi)$
  $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
  $(\varphi \wedge (\psi \wedge \chi)) \rightarrow ((\psi \wedge \varphi) \wedge \chi)$
  $(\varphi \rightarrow (\psi \rightarrow \chi)) \equiv ((\varphi \wedge \psi) \rightarrow \chi)$
  $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
  $\bar{0} \rightarrow \varphi$
  $\varphi \rightarrow (\varphi \wedge \varphi)$
  $\neg\varphi \equiv \varphi \rightarrow \bar{0}$
  $\overline{r} \wedge \overline{s} \equiv \overline{\min\{r, s\}}$, for $r, s \in C$
  $\overline{r} \rightarrow \overline{s} \equiv \overline{r \Rightarrow s}$, for $r, s \in C$

  Modal Axioms:

[6] $\Rightarrow_G$ is defined as $x \Rightarrow_G y = 1$ if $x \leq y$ and $x \Rightarrow_G y = y$, otherwise

$D_\Diamond$: $\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$
$Z^+_\Diamond$: $\Diamond\neg\neg\varphi \to \neg\neg\Diamond\varphi$
$T_\Diamond$: $\varphi \to \Diamond\varphi$
$B_\Diamond$: $\varphi \to \neg\Diamond\neg\Diamond\varphi$
$4_\Diamond$: $\Diamond\Diamond\varphi \to \Diamond\varphi$
$R1$: $\Diamond\overline{r} \to \overline{r}$
$R2$: $\Diamond(\overline{r} \to \varphi) \to (\overline{r} \to \Diamond\varphi)$
$R3$: $\Diamond((\varphi \to \overline{r}) \to \overline{r}) \to ((\Diamond\varphi \to \overline{r}) \to \overline{r})$

and the following inference rules:

$RN^+_\Diamond$: From $\varphi \to \psi$ infer $\Diamond\varphi \to \Diamond\psi$
MP: From φ and $\varphi \to \psi$, infer ψ

- Completeness:

  According to [CR07], the system $FMT_{G,\Diamond}(C, \min) = RG + D_\Diamond + Z^+_\Diamond + T_\Diamond + R1 + R2 + R3$ is complete with respect to the class of Gödel similarity Kripke models with reflexive similarity relations; the system $FMTB_{G,\Diamond}(C, \min) = FMT_{G,\Diamond}(C, \min) + B_\Diamond$ is complete with respect to the class of Gödel similarity Kripke models with reflexive and symmetric similarity relations, and the system $FMS5_{G,\Diamond}(C, \min) = FMTB_{G,\Diamond}(C, \min) + 4_\Diamond$ is complete with respect to the class of Gödel similarity Kripke models with min-transitive similarity relations.

  It is worth mentioning that if one adds the axiom

  Bool: $\varphi \vee \neg\varphi$, if φ is not modal.

  to the above systems, one gets completeness with respect to the corresponding class of models restricted to those where all the propositional (non-modal) formulas are classical (e. g. two-valued), and hence, only the modal formulas can get intermediate truth-values.

As for the question of capturing the approximate and proximity entailments in this many-valued modal framework, it is easy to check that the following statements hold for any Boolean propositions p and q:

approximate entailment: $p \models^\alpha_S q$ iff $\bar{\alpha} \to (p \to \Diamond q)$ is valid in $\mathcal{M}_S$

proximity entailment: $p \mathrel{|\!\!\models}^\alpha_S q$ iff $\bar{\alpha} \to (\Diamond p \to \Diamond q)$ is valid in $\mathcal{M}_S$

where $\mathcal{M}_S = (\Omega, S, e_G)$ is the Gödel similarity Kripke model with Ω being the set of *Boolean* interpretations of the propositional variables *Var*. In other words, $p \models^\alpha_S q$ iff $e(w, \bar{\alpha} \to (p \to \Diamond q)) = 1$ for all $w \in \Omega$, and $p \mathrel{|\!\!\models}^\alpha_S q$ iff $e(w, \bar{\alpha} \to (\Diamond p \to \Diamond q)) = 1$ for all $w \in \Omega$.

It turns out that the main difficulty for defining similar many-valued modal logics over a t-norm-based fuzzy logic different from Gödel logic is the fact that the resulting logics are generally not normal (they do not satisfy axiom K). In particular, this is the case with Łukasiewicz logic with general Kripke semantics (see [GR98] for an attempt), even though in such a case the modal operators $\Diamond$ and $\Box$ are dual ($\Box$ can be defined as $\neg\Diamond\neg$), in contrast to Gödel-based many-valued modal logics . One possibility to avoid this difficulty is to introduce graded modalities $\Box_t$ (where $t \in C$) corresponding to the cuts of the many-valued accessibility relation, i.e. using a semantics of the form

$$e(w, \Box_t\varphi) = \inf\{e(\varphi, w') : R(w, w') \geq t\}$$

to extend the valuation. Then, it is easy to see that all modalities $\Box_t$ are normal. We notice that in some particular cases, axiomatizations for these graded modalities can be found in the literature (see for instance [EGGR97, Suz97, BEGR07]). The case considered in [BEGR07] corresponds to considering the n-valued Łukasiewicz logic $Ł_n$ as base logic and having constants in the language for every element in the standard n-valued Łukasiewicz algebra $AŁ_n = \{0, 1/n, ..., (n-1)/n, n\}$. An interesting fact about this case is that $\Box$ is definable in the new language as

$$\Box\varphi := (\overline{1/n} \to \Box_{1/n}\varphi) \wedge \ldots \wedge (\overline{(n-1)/n} \to \Box_{(n-1)/n}\varphi) \wedge \Box_1\varphi$$

Finally, let us mention a recent paper by Hansoul and Teheux [HT06] where they axiomatize a modal system over the infinitely-valued Łukasiewicz logic. The proof is based on the construction of a classical canonical model. Surprisingly this proof does not need the presence in the language of constants for every truth value. The trick to avoid the introduction of constants is based on a result of [Ost88] (see [HT06, Definition 5.3]).

6.4 Many-valued conditional logic approach

Combining the approaches of Sections 6.2 and 6.3, the main idea behind a many-valued conditional logic approach is that the truthlikeness degree with which a (classical) proposition q is an approximate or proximity consequence of another (classical) proposition p is understood as the truth-value of a (many-valued) conditional formula that has p as its antecedent and q as its consequent. Therefore one needs to make use of a suitable fuzzy logic to reason about the conditional formulas. The choice of the particular fuzzy logic is determined by the class of similarity Kripke models defining the intended semantics. Namely, if the intended semantics is the class of $*$-transitive similarity Kripke models, for some (left-continuous) t-norm $*$, then the fuzzy logic to be chosen will be the t-norm fuzzy logic L_*, extension of MTL or BL, which is complete with respect to the standard MTL-algebra over $[0,1]_* = ([0,1], *, \Rightarrow, \min, \max, 0, 1)$ defined by t-norm $*$ and its residuum $\Rightarrow$. Such logics have been axiomatized for the whole family of continuous t-norms [EGM03] as well as for other left-continuous t-norms [GH05]. Moreover if one needs to explicitly deal with degrees then such logics have to expanded by a countable set of truth-constants [EGGN07]. We will denote by $\mathrm{L}_*(\mathcal{C})$ the logic of the t-norm $*$ with truth-constants from a suitable countable set $C \subset [0,1]$.

In the following we summarize the main characteristics of the many-valued conditional logic built over the logic $\mathrm{L}_*(\mathcal{C})$.

- Language: the language is built from a finite set of propositional variables *Var* plus two binary operators $>, \gg$ and a constant $\bar{r}$ for each element r in the range C. The set of propositional formulas built from *Var* is denoted as usual by $\mathbf{L}_f$
- Conditional Formulas: The set of conditional formulas $\mathcal{L}$ is built as follows:
  - Every propositional formula is also a conditional formula.
  - If p and q are propositional formulas in $\mathbf{L}_f$ then $p > q$ and $p \gg q$ are (atomic) conditional formulas.
  - Truth-constants $\bar{r}$, with $r \in C$, are conditional formulas.
  - If φ and ϕ are conditional formulas then $\varphi \circ \phi$ is a conditional formula where $\circ \in \{\wedge, , \&, \rightarrow\}$ (connectives $\neg$, $\vee$ and $\leftrightarrow$ are definable).
- Satisfiability: A $\mathrm{L}_*(\mathcal{C})$-similarity Kripke model is just a usual similarity Kripke model $M = (W, S, e)$, where now $e(w, \cdot)$ is a $\{0,1\}$-valued interpretation of propositional variables for each $w \in W$, and is extended to atomic conditional formulas as follows:

  $$\begin{array}{lcl} e(\omega, p > q) & = & I_S(q \mid p) \\ e(\omega, p \gg q) & = & I_{S,\bar{\omega}}(q \mid p) \end{array}$$

  and to compound conditional formulas as usual ones in t-norm-based logics, i.e.

  $$\begin{array}{lcl} e(\omega, \varphi \wedge \psi) & = & \min(e(\omega, \varphi), e(\omega, \psi)) \\ e(\omega, \varphi \& \psi) & = & e(\omega, \varphi) * e(\omega, \psi) \\ e(\omega, \varphi \rightarrow \psi) & = & e(\omega, \varphi) \Rightarrow e(\omega, \psi) \end{array}$$

  Again, notice that the notion of satisfiability for a conditional formula of the kind $p > q$ is independent of the particular world, i.e. it is global notion, but is not the case with the conditional formulas of the form $p \gg q$. In this case it is also clear that a formula $p > q$ directly represents $I_S(q \mid p)$ whilst $p \gg q$ represents $I_{S,\bar{\omega}}(q \mid p)$.
- Axioms:

  Besides the axioms of $\mathrm{L}_*(\mathcal{C})$, the following axioms will be used to characterise our many-valued conditional logic:

Bool:	$p \vee \neg p$, for p propositional (non conditional).
B:	$p > q \to q > p$, if p and q are m.e.c.
4:	$(p > q)\&(q > s) \to p > s$.
	$(p \gg q)\&(q \gg s) \to p \gg s$.
LO:	$(p \vee q) > r \leftrightarrow (p > r) \wedge (q > r)$.
	$(p \vee q) \gg r \leftrightarrow (p \gg r) \wedge (q \gg r)$.
RO:	$(r > p \vee q) \leftrightarrow (r > p) \vee (r > q)$, if r is a m.e.c.
	$(s \gg p \vee q) \leftrightarrow (s \gg p) \vee (s \gg q)$

and the following inference rules:

MP:	From φ and $\varphi \to \psi$ infer ψ
RK:	From $p \to q$ infer $p > q$
	From $p \to q$ infer $p \gg q$

- Completeness: Again, two kinds of systems are considered: approximate fuzzy conditional systems **FCSI** and proximity fuzzy conditional systems **FCSJ**. In the first case only the operator $>$ is used and in the second case, the operator $\gg$ is the only used.

  The completeness results may be summarised as follows:

  – The system $\mathbf{FCSI}(\mathbf{C}, \otimes)$, which has as axioms: $\mathrm{L}_\otimes(\mathcal{C}) + \mathit{Bool} + \mathit{LO} + \mathit{RO}$ and is closed under MP and RK, is complete with respect to the class of models Σ_{2f}. Moreover, we obtain completeness with respect to the subclasses of models Σ_{3f}, Σ_{4f} and $\Sigma_{\otimes f}$ if we add to $\mathbf{FCSI}(\mathbf{C}, \otimes)$ the axioms B, 4 and both B and 4, respectively.

  – The system $\mathbf{FCSJ}(\mathbf{C}, \otimes) = \mathrm{L}_\otimes(\mathcal{C}) + \mathit{Bool} + \mathit{LO} + \mathit{RO} + 4$ and closed under MP and RK is complete with respect to Σ_{4f}.

  Note that these conditional logics do not have the axiom CS as in the multi-conditional framework and hence they cannot be complete for the classes of models Σ_i^* (with $i \in \{1, 2, 3, 4, \otimes\}$). To be so, one would need to introduce in the logics Baaz's projection connective Δ.

If S is $\otimes$-transitive similarity relation on the set Ω of all Boolean interpretations of the propositional language $\mathbf{L}_f$, the corresponding approximate and proximity entailments are captured by the $\mathbf{FCSI}(\mathbf{C}, \otimes)$ and $\mathbf{FCSJ}(\mathbf{C}, \otimes)$ systems in the following sense:

approximate entailment:	$p \models_S^\alpha q$	iff	$\overline{\alpha} \to (p > q)$ is valid in $\mathcal{M}_S$
proximity entailment:	$p \models_{S,K}^\alpha q$	iff	$\overline{\alpha} \to (K \to (p \gg q))$ is valid in $\mathcal{M}_S$

where $\mathcal{M}_S = (\Omega, S, e)$ is defined as in the previous section.

7 Other issues in similarity-based reasoning

Traditional entailments are always monotonic: adding new premises never invalidate old conclusions, i. e. the set of conclusions increases monotonically with the set of premises. In this sense, the approximate and proximity entailments are also monotonic, because due to their definitions of satisfiability, for any similarity relation S, the following occurs:

$$\begin{array}{rcl} p \models_S^\alpha q & \text{implies} & p \wedge r \models_S^\alpha q \\ p \models_{S,K}^\alpha q & \text{implies} & p \wedge r \models_{S,K}^\alpha q \end{array}$$

However, in some kinds of reasoning like approximate, case-based or interpolative where the notion of similarity between situations plays a central role, sometimes it is necessary to have nonmonotonic entailments based on similarity like Lehmann's Stereotypical reasoning [Leh98], or a most recent proposal to provide a logical interpretation (in terms of nonmonotonic inferences) of dilation and erosion operators used in mathematical morphology techniques [BL02]. Essentially, this kind of reasoning tries to "jump" to conclusions without having complete information about the state of the world, i. e. since the descriptions of complex domains are naturally incomplete it is necessary to resort to assumptions, "defaults" , etc. in order to "fill up" holes of ignorance with assumptions which are taken as valid while there is not any evidence against them. They are nonmonotonic in the sense that the increase of the amount of available information as premises may sometimes lead to the loss of some of previously drawn conclusions. This is in contrast with the situation for purely deductive reasoning.

In this section we describe some forms of nonmonotonic inference based on similarity measures between situations as discussed in [GR02], in particular, those that can be interpreted in terms of consistency and implication measures. Finally, we also consider the relation between similarity reasoning and a very close topic to nonmonotonic reasoning which is belief revision.

7.1 Similarity-based nonmonotonic reasoning

In the recent past, a lot of efforts have been devoted for developing various approaches to combine uncertain and nonmonotonic reasoning. For instance, probabilistic semantics for defaults have been developed by Geffner [Gef88] and Pearl [Pea88] on the basis of Adam's logic of conditionals, and the relation between possibilistic logic and nonmonotonicity was early established by Dubois and Prade [DP91c].

In [GR02], the authors consider the issue of combining both similarity-based and nonmonotonic reasonings. Namely, they study which kinds of nonmonotonic inference relations naturally arise when using implication and consistency measures to rank propositions à la Gärdenfors and Makinson [GM94]. These measures generate two different types of nonmonotonic inferences, namely pessimistic and optimistic inferences. The approach based on *consistency* measures is indeed very close to Possibility theory, and we refer to it as optimistic because it takes into account the "closest" or "best" situations. On the contrary, the approach based on *implication* measures is based on two new ideas: a new kind of orderings between sentences called *inclusion orderings* and a new implication-like measure, which is called *counter-implication* measure, where $L_S(p \mid K)$ indicates the degree to how close is $\neg p$ to imply $\neg K$. This approach may be called pessimistic because it considers the worst situation in order to make an assumption. These two notions are combined to obtain a new form to define comparative entailments.

In both cases, the starting point is to use an ordering between formulas to determinate when a proposition p nonmonotonically implies another proposition q meaning that q follows from p together with all the propositions that are expected in the light of p. In order to formalise this notion of expectation, Makinson and Gärdenfors in [GM94] assume that there is an ordering $\leq^E$ of the sentences in a given language $\mathcal{L}$. Thus, given two sentences p and q, $p \leq^E q$ should be interpreted as "q is at least as expected as p" or "p is at least as surprising as q" (we shall write"$p <^E q$" as an abbreviation for "not $q \leq^E p$"). Makinson and Gärdenfors propose three properties which, they argue, must be satisfied by any reasonable ordering. They are:

transitivity: If $p \leq^E q$ and $q \leq^E r$, then $p \leq^E r$.

dominance: If $p \models q$, then $p \leq^E q$.

conjunctiveness: $p \leq^E (p \wedge q)$ or $q \leq^E (p \wedge q)$.

The authors point out that the first postulate on *expectation ordering* is very natural for an ordering relation, the second postulate says that a logically stronger sentence is always less expected and the third constraint concerns the relation between the degrees of expectation of a conjunction $p \wedge q$ and the corresponding degrees of p and q respectively. Note that the three conditions imply reflexivity ($p \leq^E p$) and connectivity (either $p \leq^E q$ or $q \leq^E p$). By way of comparison, it may be mentioned that these axioms are three of the five conditions used in [Gär88] and [GM94] to define the notion of *epistemical entrenchment* for the logic of theory change (see next section).

Now this ordering can be used to determine when p nonmonotonically implies q in the case q follows from p together with all the propositions that are expected in the light of p. The natural idea, according to Makinson and Gärdenfors, is to require that the added sentences must be those which are strictly more expected than $\neg p$. This motivates the following definition of comparative entailment $\mid\!\sim$:

$$p \mid\!\sim q \quad \text{iff} \quad p \models q \quad \text{or there is a proposition } r \text{ such that } p \wedge r \models q \text{ and } \neg p <^E r, \tag{1}$$

where $\leq^E$ is an ordering satisfying transitivity, dominance and conjunctiveness.

It has been proven in [GM94] that comparative entailments satisfy the desirable properties of Supraclassicality (SC), Left Logical Equivalence (LLE), And, Consistency Preservation (CP), Cut, Or and Rational Monotony (RM) (see [GM94] for their definitions) and vice-versa. So these properties characterize comparative entailments.

An alternative form to define an ordering $\leq^F$ between sentences is proposed by [FHL94], it is called *possibility ordering* and it is required to satisfy the axioms of transitivity, dominance together with:

disjunctiveness $p \vee q \leq^F p$ or $p \vee q \leq^F q$

In this case, $p \leq^F q$ denotes that q is at least as possible as p.

As pointed out in [FHL94], the dual of a possibility ordering, defined as $p \leq^F q$ iff $\neg q \leq^E \neg p$, is an expectation ordering in the above sense of Gärdenfors and Makinson. So, if the condition $\neg p <^E r$ is changed by $\neg r <^F p$ in (1), we shall obtain an equivalent comparative entailment. Furthermore, in [FHL94] it is shown that the following three clauses are equivalent for a possibility ordering $\leq^F$:

1. there is a proposition r such that $p \wedge r \models q$ and $\neg r <^F p$.
2. $p \wedge \neg q <^F p$.
3. $p \wedge \neg q <^F p \wedge q$.

These conditions allow us to give different (but equivalent) versions of (1) in terms of possibility orderings, e. g. $p \mid\!\sim_F q$ iff either $p \models q$ or $p \wedge \neg q <^F p \wedge q$.

Now, we are in condition to focus on the orderings on propositions that a body of evidence K and a similarity measure S on worlds induce when the corresponding consistency measure $C_S(\cdot \mid K)$ is used to rank propositions. As it was mentioned in Section 5, since the consistency measure $C_S(\cdot \mid K)$ is also a possibility measure, the ordering induced on formulas defined by:

$$p \leq_C q \text{ iff } C_S(p \mid K) \leq C_S(q \mid K), \tag{2}$$

read as "q is at least as consistent (with K) as p", is is a *qualitative possibility* relation in the sense of Dubois [Dub86], i. e. the qualitative counterpart of a possibility measure. A qualitative possibility relation is a possibility ordering (as defined above) together with this further axiom

non-triviality: $\perp <_C \top$

where $<_C$ is the strict part of the ordering $\leq_C$. According to the previous section, the corresponding comparative entailment $\mid\!\sim_C$ is then defined as

$$p \mid\!\sim_C q \text{ iff either } p \models q \text{ or } p \wedge \neg q <_C p \wedge q.$$

Although some are stronger than the others, in [FHL94] it is shown that qualitative possibility relations and possibility orderings generate the same family of nonmonotonic entailments [7]. Consequently, a consequence relation $\mid\!\sim$ satisfies SC, LLE, And, CP, Cut, Or and RM iff there exists a proposition K and a similarity S on possible worlds such that $p \mid\!\sim q$ iff $p \models q$ or $C_S(p \wedge q \mid K) > C_S(p \wedge \neg q \mid K)$. Indeed, although the orderings of sentences defined by consistency measures $\leq_C$ are qualitative possibility orderings, they have a different meaning because $p \leq_C q$ means q is at least as consistent with K as p, where the level of consistency is understood as a degree of closeness to K. This way of interpreting the ordering is different to the ones based on preference or possibility. Next, we consider another way to define nonmonotonic inference relations from a more interesting perspective because the ordering is induced taking into account the most distant worlds instead of the closest ones.

As it is pointed out by Makinson in [Mak94, pag. 46], if we want to abandon monotony then we will also have to abandon contraposition. However, in many occasions, the information we get

[7] However, the non-triviality property will become relevant when we will analyse the relationship between similarity logic and belief revision

is in a different way from the one we need it. For instance, we know for sure that if the battery is discharged the car will not start, thus a very common trouble shooting rule is the following: "if a car engine does not start up then it is possible that its battery is discharged". And from this rule of thumb, one can derive, by contraposition, another rule, in this case a predictive rule: "If the car battery is charged up then probably the engine will start up". Note that this last rule (like the first one) is nonmonotonic because it is not intended to assert that the antecedent alone is a sufficient condition of the consequent, but jointly with a set of assumptions commonly accepted in the context of this rule. In order to capture this intuition, we now consider a notion of nonmonotonic consequence $p \mid\!\sim q$ that it is based in the degree of implication of $\neg p$ by $\neg q$.

As we have already mentioned, an implication measure $I_S(\cdot \mid K)$ does not verify any interesting decomposability property and this makes it quite difficult to grasp which properties may satisfy an ordering on propositions defined as

$$p \leq_I q \text{ iff } I_S(p \mid K) \leq I_S(q \mid K)$$

However, in [GR02] they find a way out by contrapositive reasoning. Namely, if $I_S(p \mid K)$ measures to what extent p is implied by K, one can also consider another implication-like index $L_S(p \mid K)$ measuring to what extent $\neg p$ implies $\neg K$ defined by:

$$L_S(p \mid K) = I_S(\neg K \mid \neg p).$$

L_S is called a *counter-implication* measure. It is easy to show that, given a fixed consistent K, the measure $L_S(\cdot \mid K)$ fulfills the following properties:

1. $L_S(\top \mid K) = 1$
2. $L_S(p \wedge q \mid K) = \min(L_S(p \mid K), L_S(q \mid K))$

but fails to satisfy $L_S(\perp \mid K) = 0$. This means that $L_S(\cdot \mid K)$ is very close to a necessity measure[8]. So close, that the ordering induced by it,

$$p \leq_L q \text{ iff } L_S(p \mid K) \leq L_S(q \mid K), \tag{3}$$

is a genuine expectation ordering, that is, it satisfies transitivity, dominance and conjunctiveness. Indeed, the ordering $\leq_L$ will be a qualitative necessity relation (i.e. the dual of a qualitative possibility relation) and the condition $\perp <_L \top$ will hold, if S is separating . Therefore we can also prove that an inference relation $\mid\!\sim$ satisfies SC, LLE, And, Or, RM, CP, Cut, Or and RM iff there exists a proposition K and a similarity S on possible worlds such that $p \mid\!\sim q$ iff either $p \models q$ or $L_S(p \to q \mid K) > L_S(p \to \neg q \mid K)$.

Finally, it is interesting to also express $\mid\!\sim_L$ in terms of the graded approximate entailment $\models^\alpha$ introduced in Section 2. Just by applying the definitions, it turns out that the following condition holds:

$$\begin{array}{lll} p \mid\!\sim_L q & \text{iff} & \text{either } p \models q \\ & & \text{or there exists } \alpha \in [0,1] \text{ such that } p \wedge \neg q \models^\alpha \neg K \text{ and } p \wedge q \not\models^\alpha \neg K. \end{array}$$

In other words, q nonmonotonically follows from p, in a context of K and S, when $\neg K$ is approximately entailed by $\neg(p \to q)$ to a higher degree than by $\neg(p \to \neg q)$, or roughly speaking, when falsifying $p \to q$ falsifies K *more* than when falsifying $p \to \neg q$.

7.2 Belief revision and similarity logic

Theory change formalisms deal with mechanisms for adding (or retracting) a proposition to (from) an existing knowledge base. The natural question addressed by these formalisms is what should the resulting theory be. In particular, one of the basic problems is whether the new information to be added is inconsistent with the given knowledge base. Concerning this problem, most relevant works take as a departure point the postulates proposed by Alchourrón, Gärdenfors and Makinson ([AGM85]) for the so-called belief revision operators. More specifically, in [AGM85], the authors

[8]A formal study of a weaker notion of necessity which it is not required to satisfy that the measure of $\perp$ should be 0 is given in [BG92]

proposed eight postulates which, they argued, must be satisfied by any reasonable revision operator $\star$. In what follows, given a knowledge base K and a formula φ, $K \star \varphi$ denotes the result of adding φ to K. The postulates consist of [9]:

- six basic postulates

  Closure: $K \star \varphi = Cn(K \star \varphi)$

  Success: $\varphi \in K \star \varphi$

  Inclusion: $K \star \varphi \subseteq Cn(K \cup \{\varphi\})$

  Vacuity: If $\neg\varphi \notin Cn(K)$, then $Cn(K \cup \{\varphi\}) \subseteq K \star \varphi$

  Consistency: If $\neg\varphi \notin Cn(\emptyset)$ then $\perp \notin Cn(K \star \varphi)$

  Extensionality: If $\varphi \leftrightarrow \psi \in Cn(\emptyset)$, then $K \star \varphi = K \star \psi$

- and two supplementary postulates.

  Superexpansion: $K \star (\varphi \wedge \psi) \subseteq Cn((K \star \varphi) \cup \{\psi\})$

  Subexpansion: If $\neg\psi \notin Cn(K \star \varphi)$, then $Cn((K \star \varphi) \cup \{\psi\}) \subseteq K \star (\varphi \wedge \psi)$

Gärdenfors [Gär90] has suggested that nonmonotic reasoning and belief revision are two sides of a same coin. This is specially true for nonmonotonic logic based on *expectation orderings*. In fact, while the above postulates leave the choice of the revision operator quite open, Gärdenfors proves [Gär88] that any a such operator underlines an ordering $\leq_{EE}$ on the formulas of a knowledge base that guides the revision procedure. He calls this ordering *epistemic entrenchment*, and it is an expectation ordering (i.e. it satisfies the *transitivity, dominance* and *conjunctiveness* properties) which additionally satisfies two further properties:

Minimality: If $\perp \notin Cn(K)$, then $\varphi \notin Cn(K)$ iff $\varphi \leq_{EE} \psi$ for all ψ

Maximality: If $\psi \leq_{EE} \varphi$ for all ψ, then $\varphi \in Cn(\emptyset)$

The connections between epistemic entrenchment orderings and revision operators is witnessed by the following relationships [Gär88, LR91, Rot91]:

- given an epistemic entrenchment ordering $\leq$ on a consistent belief set K the operator $\star$ defined by

  (EBR) $\qquad \psi \in K \star \varphi$ iff either $(\varphi \rightarrow \neg\psi) < (\varphi \rightarrow \psi)$ or $\varphi \vdash \perp$.

  is a belief revision that satisfies the eight AGM postulates.

- conversely, if $\star$ is an operation on a consistent belief set K that satisfies the eight AGM postulates, then the relation $\leq$ defined from $\star$ by

  $(C \leq)$ $\qquad \varphi \leq \psi$ iff: If $\varphi \in K \star \neg(\varphi \wedge \psi)$ then $\psi \in K \star \neg(\varphi \wedge \psi)$.

  is an epistemic entrechment ordering.

According to these relationships between orderings and belief revision, and taking account the previous subsection, it is not surprising then that there is also a connection between similarity-based logical formalism and belief revision. Indeed, in [DP92], Dubois and Prade have pointed out that the relation $\leq_{EE}$ has exactly the same properties as a qualitative necessity relation. Hence, the only numerical counterpart of epistemic entrenchment orderings are exactly those induced by necessity measures. Taking this into account, the above two relationships also hold when the ordering is an $\leq_C$- (or $\leq_L$)-ordering, as defined in (2) and in (3) respectively, induced by a separating similarity relation S. As a final remark, let us notice that, actually, the symmetry and transitivity properties of the similarity relations are not needed to generate the orders for defining revision operators. Therefore it is possible to consider models with a fuzzy binary relation, representing some more general notion of similarity or "closeness", for which only the reflexivity and separating (discriminant) properties would be required.

[9] where Cn is any consequence operator which includes classical propositional logic, is compact and satisfies the deduction theorem.

8 Summary and conclusions

In this paper we have surveyed different approaches to formalize similarity-based reasoning, in the sense of logical systems that provide a formal account of the graded notion of truthlikeness. For this, we have first clarified the differences between the notion of truthlikeness and the better known notions of uncertainty and vagueness and we have introduced fuzzy similarity relations as the main tool used to model a graded notion of truthlikeness. In fact, fuzzy similarity relations can be used, either syntactically or semantically, to define notions of approximate proofs or approximate entailments respectively. Although both have been addressed in the paper, we have put more emphasis on the semantical approach where the starting point is to assume there are possible worlds or situations that resemble more than others, and this is reflected by a given fuzzy similarity relation between worlds. Indeed, we have shown how similarity relations on possible worlds can be used to extend the classical notion of logical consequence leading to new notions of graded entailments, basically the so-called approximate and proximity entailments. These ideas go back to Ruspini [Rus91] and are captured by similarity-based Kripke structures. Based on these semantics, we have described in detail four formalisations, based on different modal and conditional logical frameworks, capturing different aspects of these similarity entailments. Finally, we also have addressed the issue of exploring nonmonotonic aspects in similarity-based reasoning. By following ideas of Gärdenfors and Makinson [GM94], where they reduce the notion of nonmonotonic reasoning to the notion of ordering between formulas, we have described some approaches that consider different kinds of similarity-based orderings to define nonmonotonic consequence relations and operators of theory revision.

As concluding remarks we may point out that similarity-based reasoning is a research topic that has many different and interesting facets. In this paper we have addressed only some issues in the task of logical formalisation of different notions of approximate consequence that make sense in this framework. Therefore we have not covered many other reasoning models where the notion of similarity or truthlikeness plays a key role, like case-based reasoning or case-based decision. Finally, regarding open problems, we note a couple of questions. In subsection 6.3, we have described a many-valued modal system, based on Gödel logic semantics, only for a $\Diamond$ operator. This logic is very important because it allows us to formalize other related notions, like interpolative reasoning ([DEG$^+$97]), similarity-based SLD resolution ([BGR05]), and fuzzy description logic ([Haj05]). However, Gödel logic is only one of prominent fuzzy logics. One of the main open problems in this field is the search for axiomatization for similar many-valued modal systems based on other fuzzy logics. The difficulty is essentially due to their lack of normality, i. e. they do not satisfy the K axiom. In Section 7 we have considered two kinds of nonmonotonic inference based on similarity orderings. Another approach would be to follow Schlechta's ideas in [Sch97] where he reduces the notion of nonmonotonic reasoning to the notion of distance. The use of similarity relations instead of distances seems an interesting line for future research.

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