On the non-falsity and threshold preserving variants of MTL logics

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Abstract. In this paper we study the definition and axiomatisation of non-falsity preserving and threshold preserving companions of several extensions of the Monoidal t-norm based fuzzy logic MTL. More in detail, we first extend some recent preliminary results on non-falsity preserving logics, and then we present a new study on threshold-preserving companions of the main three fuzzy logics, Łukasiewicz, Product and Gödel logics.

Keywords: Mathematical fuzzy logic, monoidal t-norm based logic, non-falsity preserving logics, threshold preserving logics

1 Introduction

Fuzzy logics are logics of *graded truth* that have been proposed as a suitable tool for reasoning with imprecise information, in particular for reasoning with propositions containing vague predicates. Their main feature is that they allow to interpret formulas in a linearly ordered scale of truth-values, and this is specially suited for representing the gradual aspects of vagueness. In particular, systems of fuzzy logic have been in-depth developed within the frame of mathematical fuzzy logic (MFL) [15, 8]. In deductive systems in MFL, mostly with semantics in the real unit interval $[0, 1]$, the usual notion of deduction is defined by requiring the preservation of the truth-value 1 (full *truth-preservation*), which is understood as representing the absolute truth. Namely, generalizing the classical notion of consequence, in these systems a formula follows from a set of premises if every algebraic evaluation that interprets the premises as 1-true also interprets the conclusion as 1-true. All the fuzzy logics under the truth-preserving paradigm are explosive in the sense of validating the $\neg$-explosion rule

$$\frac{\varphi \quad \neg\varphi}{\psi}$$

where $\neg$ is the definable negation in systems of MFL. A logic not satisfying this rule is called $\neg$-paraconsistent [9, 17, 4].

In the last years, there have been several works studying paraconsistent variants of fuzzy logics (see e.g. [11, 5, 7]), mainly by moving from the (full) truth-preserving paradigm to the *degree-preserving paradigm*, in which a conclusion follows from a set of premises if, for all evaluations, the truth degree of the conclusion is greater or equal than those of the premises, see e.g. [3]. Still, another way of defining paraconsistent variants of a fuzzy logic is put forward in [1], although for the particular case of Łukasiewicz fuzzy logic. In this approach, the notion of consequence at work is the *non-falsity preservation*, according to which a conclusion follows from a set of premises whenever if the premises are non-false, so must be the conclusion. In other words, assuming a $[0, 1]$-valued semantics, this is the case when, for any evaluation, if truth degrees of the premises are above 0, then the truth-degree of the conclusion is so as well. While this notion of consequence is not weaker than the one in the truth-preserving logics, it is stronger than the one of degree-preserving logics, and has been preliminary studied in [12]. For instance, while all tautologies in a truth-preserving logic keep being obviously valid in the non-falsity preserving variant, usually Modus Ponens is not a valid inference rule any longer, e.g. in the case of Łukasiewicz logic. On the other hand, the excluded-middle axiom $\varphi \vee \neg\varphi$ is a valid formula in the non-falsity variant (it always take a positive truth-value), while this is neither a valid in the truth-preserving and degree-preserving variants.

In this paper, we first further explore non-falsity preserving companions of two classes of extensions of the MTL logic, and second we address the question of defining and syntactically characterising logics that preserve a given truth-value threshold, that can be any real value $a \in (0, 1]$, focusing the study on the three most prominent extensions of MTL, namely Łukasiewicz, Product and Gödel fuzzy logics. In more detail, after this introduction, in Section 2 we gather some preliminaries on various systems of t-norm based fuzzy logics and present basic definitions about variants of these systems corresponding to logical matrices on MTL-chains with lattice filters as sets of designated values. Then in Section 3 we focus on the paraconsistent non-falsity preserving companions of MTL logics, overviewing basic results for the case of Involutive MTL logics from [12] and providing new insights for MTL logics validating a suitable inference rule. Finally in Section 4 we deal with threshold-preserving logics for the above mentioned three particular cases. We conclude with some final remarks in Section 5.

2 Preliminaries

Most well known and studied system of mathematical fuzzy logic are the so-called *t-norm based fuzzy logics*, corresponding to formal many-valued calculi with truth-values in the real unit interval $[0, 1]$ and with a conjunction and an implication interpreted respectively by a (left-) continuous t-norm and its residuum, and thus, including e.g. the well-known Łukasiewicz, Gödel and Product infinitely-valued logics, corresponding to the calculi defined by Łukasiewicz, min and product t-norms respectively. The most general t-norm based fuzzy logic is the logic MTL (monoidal t-norm based logic) introduced in [14], whose theorems correspond to the common tautologies of all many-valued calculi defined by a left-continuous t-norm and its residuum [16].

The language of MTL consists of denumerably many propositional variables $p_1, p_2, \ldots$, binary connectives $\wedge, \&, \rightarrow$, and the truth constant $\overline{0}$. Formulas, which will be denoted by lower case greek letters $\varphi, \psi, \chi, \ldots$, are recursively defined from propositional variables, connectives and truth-constant as usual. Further connectives and constants are definable, in particular: $\neg\varphi$ stands for $\varphi \rightarrow \overline{0}$ and $\overline{1}$ stands for $\neg\overline{0}$. A Hilbert-style calculus for MTL was introduced in [14] with the following set of axioms:

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $\varphi \,\&\, \psi \rightarrow \varphi$
(A3) $\varphi \,\&\, \psi \rightarrow \psi \,\&\, \varphi$
(A4) $\varphi \wedge \psi \rightarrow \varphi$
(A5) $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$
(A6) $\varphi \,\&\, (\varphi \rightarrow \psi) \rightarrow \varphi \wedge \psi$
(A7a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \,\&\, \psi \rightarrow \chi)$
(A7b) $(\varphi \,\&\, \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A8) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A9) $\overline{0} \rightarrow \varphi$

and whose unique inference rule is *modus ponens*: from φ and $\varphi \rightarrow \psi$ derive ψ.

In Table 1 we gather some of the main axiomatic extensions of MTL together with their additional axioms. Of particular interest in this paper is the Involutive MTL logic (IMTL for short), i.e. the axiomatic extension of MTL with the axiom (INV) which enforces the negation $\neg$ to be involutive [14]. The well-known Łukasiewicz logic is the extension of IMTL with the divisibility axiom (Div), Gödel logic is the extension of MTL with the contraction axiom (Con) while Product logic is the extension of MTL with the (Div) and the cancellation axiom (C) [15]. Both Gödel and Product logics are extensions of SMTL, the extension of MTL with the axiom (PC). In this paper we will also consider in Section 3.2 the extension of non-SMTL logics (i.e. MTL logics that are not SMTL logics) with an additional rule of inference.

MTL is an algebraizable logic in the sense of Blok and Pigozzi [2] and its equivalent algebraic semantics is given by the variety of MTL-algebras. MTL-algebras can be equivalently introduced as commutative, bounded, integral residuated lattices $\langle A, \wedge, \vee, *, \rightarrow, \overline{0}, \overline{1}\rangle$ further satisfying the following prelinearity condition: $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$. Algebras of IMTL are MTL-algebras satisfying the equation $x = \neg\neg x$, algebras of Ł are usually called MV-algebras and are IMTL-algebras further satisfying the equation $x * (x \rightarrow y) = x \wedge y$, Gödel-algebras are MTL-algebras satisfying the equation $x * y = x \wedge y$, while Product algebras are MTL-algebras satifsying $x \wedge y = x * (x \rightarrow y)$ and $\neg x \vee ((x \rightarrow x * y) \rightarrow y) = 1$.

Besides enjoying strong completeness as a consequence of their algebraizability, all the logics in Table 1, enjoy completeness with respect to their corresponding classes of algebras on the real-unit interval $[0, 1]$, as proved e.g. in

Axiom schema	Name
$\neg\neg\varphi \to \varphi$	(Inv)
$\neg\varphi \vee ((\varphi \to \varphi \,\&\, \psi) \to \psi)$	(C)
$\varphi \to \varphi \,\&\, \varphi$	(Con)
$\varphi \wedge \psi \to \varphi \,\&\, (\varphi \to \psi)$	(Div)
$\neg(\varphi \wedge \neg\varphi)$	(PC)
$\neg(\varphi \,\&\, \psi) \vee (\varphi \wedge \psi \to \varphi \,\&\, \psi)$	(WNM)
$\varphi \vee \neg\varphi$	(EM)

Logic	Additional axioms
Strict MTL (SMTL)	(PC)
Involutive MTL (IMTL)	(Inv)
Nilpotent Minimum (NM)	(Inv) and (WNM)
Basic Logic (BL)	(Div)
Strict Basic Logic (SBL)	(Div) and (PC)
Łukasiewicz Logic (Ł)	(Div) and (Inv)
Product Logic (Π)	(Div) and (C)
Gödel Logic (G)	(Con)
Classical Logic (CL)	(EM)

Table 1. Some axiomatic extensions of MTL obtained by adding the corresponding additional axiom schemata.

[16] for MTL and in [13] for IMTL. Furthermore, Łukasiewicz logic, Gödel logic and Product logic are even complete w.r.t. a *single* algebra over $[0, 1]$, the standard MV-algebra, the standard Gödel algebra and the standard Product algebra respectively, see e.g. [15].

In the following, given a left-continuous t-norm $*$, we will denote by $[\mathbf{0}, \mathbf{1}]_*$ the standard MTL-algebra determined by $*$, i.e. $[\mathbf{0}, \mathbf{1}]_* = ([0, 1], \min, \max, *, \to, 0, 1)$, where $\to$ is the residuum of $*$ and the negation $\neg$ is defined as $\neg x = x \to 0$. In the systems of mathematical fuzzy logic considered above, the usual notion of logical consequence has been defined as preservation of the *truth*, represented by the top element of the corresponding algebras. For instance let L be any extension of MTL, which we assume to be complete w.r.t. the family $\mathcal{C}_L = \{[\mathbf{0}, \mathbf{1}]_* \mid [\mathbf{0}, \mathbf{1}]_* \text{ is a L-algebra}\}$ of standard L-algebras. Then the typical notion of logical consequence is the following for every set of formulas $\Gamma \cup \{\varphi\}$:

$$\begin{array}{lll} \Gamma \models_L \varphi & \text{if,} & \text{for any } [0,1]_* \in \mathcal{C}_L \text{ and any } [0,1]_*\text{-evaluation } e, \\ & & \text{if } e(\psi) = 1 \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) = 1 \text{ as well.} \end{array}$$

This can be generalised by considering logics defined by logical matrices $M = \langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is a standard L-chain and F is a non-trivial lattice filter of $\mathbf{A}$ i.e. F is either a closed interval $F_a = [a, 1]$ with $a \in (0, 1]$, or a semi-open interval $F_{(a} = (a, 1]$ with $a \in [0, 1)$. Considering the filters as sets of designated values, then the companions of the logic L given by the classes of matrices $\mathcal{C}_L^a = \{\langle [\mathbf{0}, \mathbf{1}]_*, F_a \rangle \mid [\mathbf{0}, \mathbf{1}]_* \text{ is a L-algebra}\}$ and $\mathcal{C}_L^{(a} = \{\langle [\mathbf{0}, \mathbf{1}]_*, F_{(a} \rangle \mid [\mathbf{0}, \mathbf{1}]_* \text{ is a L-algebra}\}$ are defined respectively as follows:

$$\begin{array}{lll} \Gamma \models_L^a \varphi & \text{if,} & \text{for any } [0,1]_* \in \mathcal{C}_L \text{ and any } [0,1]_*\text{-evaluation } e, \\ & & \text{if } e(\psi) \geq a \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) \geq a \text{ as well.} \end{array}$$

$$\begin{array}{lll} \Gamma \models_L^{(a} \varphi & \text{if,} & \text{for any } [0,1]_* \in \mathcal{C}_L \text{ and any } [0,1]_*\text{-evaluation } e, \\ & & \text{if } e(\psi) > a \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) > a \text{ as well.} \end{array}$$

The extreme cases are the 1-preserving logic $\models^1_L = \models_L$, which is explosive, and the non-falsity preserving logic $\models^{(0}_L$, which is paraconsistent w.r.t. $\neg$. Observe that the finitary versions of both logics are strongly related because, for any L-evaluation e, the condition $e(\neg\varphi) = 1$ if and only if $e(\varphi) = 0$ holds due to the fact the implication in MTL is residuated and thus $e(\varphi \to \psi) = 1$ iff $e(\varphi) \leq e(\psi)$.

Lemma 1. *For every pair of formulas φ, ψ the following relation holds:* $\varphi \models^{(0}_L \psi$ *iff* $\neg\psi \models^1_L \neg\varphi$.

Moreover, if $[\mathbf{0}, \mathbf{1}]_*$ is a standard IMTL-algebra, with c being the fixpoint of the involutive negation $n(x) = x \to 0$, then it is easy to check that

(i) The logic of the matrix $M^a = \langle [\mathbf{0}, \mathbf{1}]_*, F_a \rangle$ is paraconsistent iff $a \leq c$,
(ii) The logic of the matrix $M^{(a} = \langle [\mathbf{0}, \mathbf{1}]_*, F_{(a} \rangle$ is paraconsistent iff $a < c$.

3 Non-falsity preserving companions of two classes of MTL extensions

3.1 The case of extensions of IMTL

In this section we recall from [12] the characterisation of logics defined by (sets of) matrices of the form $\langle [\mathbf{0}, \mathbf{1}]_*, F_{(0} \rangle$, with $[\mathbf{0}, \mathbf{1}]_*$ being a standard IMTL-algebra. We remind that this means that $*$ is a left-continuous t-norm such that the residual negation $\neg$, defined as $\neg x = x \to 0 = \sup\{y \in [0, 1] \mid x * y = 0\}$ satisfies the involutivity condition $\neg(\neg x) = x$. Notable examples of such t-norms are Łukasiewicz t-norm (which is continuous) and Nilpotent Minimum t-norm.

Assume L is an axiomatic extension of IMTL, complete with respect to a class of standard algebras $\mathcal{C}_L$, and whose corresponding notion of proof is denoted $\vdash_L$. It is immediate to observe that in the case of a IMTL logic L, Lemma 1 can be strengthened in the sense that the 1-preserving logic $\models^1_L$ and the non-falsity preserving logic $\models^{(0}_L$ become interdefinable. Namely, in this case we have both:

(i) $\varphi \models^1_L \psi$ iff $\neg\psi \models^{(0}_L \neg\varphi$, (ii) $\varphi \models^{(0}_L \psi$ iff $\neg\psi \models^1_L \neg\varphi$.

In order to syntactically characterise $\models^{(0}_L$, defined by the class of matrices

$$\mathcal{C}^{(0}_L = \{\langle [0, 1]_*, F_{(0} \rangle \mid \langle [0, 1]_*, F_1 \rangle \in \mathcal{C}_L\},$$

the following system nf-L, called the *non-falsity preserving companion* of L, is defined in [12] as follows.

Definition 1. *The calculus* nf-L *is defined by the axioms of* L *and the following rules:*

- *Rule of Adjunction: (Adj)* $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- *Reverse Modus Ponens: (MP^r)* $\dfrac{\neg\psi \vee \chi}{\neg\varphi \ \vee \ \neg(\varphi \to \psi) \ \vee \ \chi}$

– *Restricted Modus Ponens: (r-MP)* $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}, \qquad \textit{if} \quad \vdash_L \varphi \to \psi$

The corresponding notion of proof will be denoted by $\vdash_{\mathsf{nf}\text{-}L}$.

The above (MP^r) rule captures the following form of reverse of modus ponens: if $\neg\psi$ is non-false then either $\neg\varphi$ is non-false or $\neg(\varphi \to \psi)$ is non-false. The addition of the disjunct χ both in the premise and in the conclusion of the rule is needed for technical reasons. On the other hand, note the usual Modus Ponens rule is not valid in $\models_L^{(0}$ (e.g. we may have $e(\varphi) = e(\neg\varphi) = e(\varphi \to \bar{0}) = a > 0$, with a being the negation fix point in $[0,1]_*$, while $e(\bar{0}) = 0$), thus we need to have the above restricted form.

The following is a syntactic counterpart of part of Lemma 1.

Proposition 1. *[12] If* $\psi \vdash_L \varphi$ *then* $\neg\varphi \vdash_{\mathsf{nf}\text{-}L} \neg\psi$.

Thanks to this relation, the logic nf-L has been shown to be complete in [12] with respect to the intended semantics.

Theorem 1. *Let L be an axiomatic extension of IMTL. The calculus* nf-L *is sound and complete w.r.t. to the class of matrices* $\mathcal{C}_L^{(0}$.

Note that, as a direct corollary, Definition 1 provides us with complete axiomatisations of non-falsity preserving companions of prominent IMTL logics like Łukasiewicz or Nilpotent Minimum logics.

3.2 The non-falsity preserving variant of non-SMTL logics validating the rule ($\mathbf{R}^{\neg\neg}$)

In this section we show that to prove the results in the previous section the requirement of the negation $\neg$ to be involutive, as it happens in IMTL logics, can be significantly weakened. Indeed, let $\mathrm{MTL}^{\neg\neg}$ be the (non-axiomatic) extension of MTL with the rule

$$(\mathrm{R}^{\neg\neg}) \quad \frac{\neg\neg\varphi}{\varphi},$$

introduced in [5]. The algebraic semantics of $\mathrm{MTL}^{\neg\neg}$ consists of the quasi-variety generated by the class of MTL-chains $\mathbf{A}$ whose negation $\neg$ is such that, for any $a \in A$, $\neg a = 0$ iff $a = 1$, or equivalently $\neg a > 0$ iff $a < 1$.

If L is an axiomatic extension of MTL, let us denote by $\mathrm{L}^{\neg\neg}$ the extension of L with the rule ($\mathrm{R}^{\neg\neg}$). If L is complete w.r.t. a class of standard matrices $\mathcal{C}_L$ over the real unit interval $[0,1]$, then $\mathrm{L}^{\neg\neg}$ is complete w.r.t. the class of matrices $\mathcal{C}_{L^{\neg\neg}} = \{\langle [0,1]_*, \{1\}\rangle \mid \langle [0,1]_*, \{1\}\rangle \in \mathcal{C}_L$ s.t. for all $x, \neg x = 0$ iff $x = 1\}$, see [5].

In $\mathrm{L}^{\neg\neg}$ we keep having at the semantical level the equivalence between the 1-preserving logic and the non-falsity preserving logic, in the following sense.

Lemma 2. *For any logic L extension of MTL, the following conditions hold:*

(i) $\varphi \models^1_{L^{\neg\neg}} \psi$ *iff* $\neg\psi \models^{(0}_{L^{\neg\neg}} \neg\varphi$, $\qquad$ *(ii)* $\varphi \models^{(0}_{L^{\neg\neg}} \psi$ *iff* $\neg\psi \models^1_{L^{\neg\neg}} \neg\varphi$.

Proof. Straighforward: (ii) is Lemma 1, and to prove (i), note that if $[0,1]_*$ is a standard MTL$^{\neg\neg}$-algebra then "$x = 1$ implies $y = 1$" is equivalent to "$y < 1$ implies $x < 1$", and this is in turn equivalent to "$\neg y > 0$ implies $\neg x > 0$". □

Then one can define the non-falsity preserving companion of a MTL$^{\neg\neg}$-logic and prove its completeness as follows. In fact, we can restrict ourselves to extensions of *non-SMTL* logics with the rule (R$^{\neg\neg}$). By a non-SMTL logic we mean a MTL logic that does not satisfy the axiom (PC) $\neg(\varphi \wedge \neg\varphi)$. Indeed, note that if L is a SMTL logic, then L$^{\neg\neg}$ collapses into classical logic. This is so because, using the rule (R$^{\neg\neg}$), from axiom (PC), which can be equivalently expressed in MTL as $\neg\varphi \vee \neg\neg\varphi$, L$^{\neg\neg}$ then derives the Excluded-Middle axiom $\neg\varphi \vee \varphi$.

Theorem 2. *Let* L *be a non-SMTL logic. Then the calculus* nf-$L^{\neg\neg}$*, defined by the axioms of L following rules:*

- *The rule (R$^{\neg\neg}$)*
- *The rule of Adjunction (Adj)*
- *The rule of Reverse Modus Ponens (MP^r)*
- *The rule of Restricted Modus Ponens (r-MP)*

is a sound and complete axiomatisation w.r.t. to the class of matrices $\mathcal{C}^0_{L^{\neg\neg}}$.

The proof is an easy adaptation of the proof of Theorem 1 for IMTL logics in [12].

4 Threshold-preserving logics

In this section we turn our attention to logics preserving lower bounds of truth-values, in other words, logics whose semantic consequence relations are of the form $\models^a_L$ and $\models^{(a}_L$ for some positive value $a \in (0,1]$ (as introduced in Section 2) for some logics L extensions of MTL that are complete with respect to some class of standard L-algebras $\mathcal{C}_L$.

We state two general but sufficient conditions for a logic L to guarantee a finitary axiomatisation $\models^a_L$. Consider the following two conditions on L:

(C1) L satisfies a form of global Deduction Theorem in the sense that there exists a term t such that:

$$\Gamma \cup \{\varphi\} \vdash_L \psi \quad \text{iff} \quad \Gamma \vdash_L t(\varphi) \to \psi$$

(C2) The logic $\models^a_L$ is interpretable in L, that is, there exists a term r such that:

$$\varphi \models_a \psi \quad \text{iff} \quad r(\varphi) \models_L r(\psi)$$

Theorem 3. *Let L be an extension of MTL satisfying conditions (C1) and (C2). Then the calculus L_a defined syntactically by the axioms of L and the following rules:*

- *the rules of L restricted to theorems of L,*
- *the rule of Adjunction, and*
- *the restricted rule:* $(R_{t,r})\quad \dfrac{\varphi, \quad \vdash_L t(r(\varphi)) \to r(\psi)}{\psi}$

is a sound and complete axiomatisation of the finitary $\models^a_L$.

Proof. The following is a sketch of the proof:

(i) $\varphi_1, \ldots, \varphi_n \models^a_L \psi$ iff
(i) $\varphi_1 \wedge \ldots \wedge \varphi_n \models^a_L \psi$ iff
(ii) $r(\varphi) \models_L r(\psi)$ iff –by condition (C2), where $\varphi = \varphi_1 \wedge ... \wedge \varphi_n$
(iii) $\models_L t(r(\varphi)) \to r(\psi)$ iff –by condition (C1)
(iv) $\vdash_L t(r(\varphi)) \to r(\psi)$ iff –by completeness of L
(v) in L there is a proof $\langle \Pi_1, ..., \Pi_n \rangle$ where $\Pi_n = t(r(\varphi) \to r(\psi)$ iff
(vi) in L_a there is a proof $\langle \Pi_0, \Pi_1, ..., \Pi_n, \Pi_{n+1} \rangle$, where the steps $\Pi_1, ..., \Pi_n$ (with applications of the rules restricted to theorems) are as above and where Π_0 is an initial step to obtain φ by the adjunction rule from $\varphi_1, ..., \varphi_n$, and a final step $\Pi_{n+1} = \psi$, where ψ is obtained from Π_0 and Π_n by the application of the rule $(\mathrm{R}_{t,r})$. □

4.1 The case of Łukasiewicz logics

A particular instantiation of the above setting is for finite-valued Łukasiewicz logics[1] $Ł_n$ and for the infinite-valued Łukasiewicz logic Ł expanded with Baaz-Monteiro Δ operator $Ł_\Delta$.

Finite-valued Łukasiewicz logics $Ł_n$ are complete with respect to the matrices $\langle \mathbf{MV}_n, \{1\} \rangle$, where $\mathbf{MV}_n$ is the MV-chain over the n-element set $MV_n = \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$, and they satisfy the above conditions (C1) and (C2). Namely, as is well-known, Baaz-Monteiro operator Δ is definable in $Ł_n$ as $\Delta\varphi := \varphi \& \overset{n}{\ldots} \& \varphi$ and $Ł_n$ enjoys a global deduction theorem: $\varphi \vdash_{Ł_n} \psi$ iff $\vdash_{Ł_n} \Delta\varphi \to \psi$. On the other hand, it is also well-known that, for every $a \in MV_n$, there is a McNaughton term $r_a(x)$ such that $r_a(x) = 1$ iff $x \geq a$. Therefore, it holds that $\varphi \vdash^a_{Ł_n} \psi$ iff $r_a(\varphi) \models_{Ł_n} r_a(\psi)$. As a consequence, according to Theorem 3, the logic $Ł^a_n$ defined there provides a complete axiomatisation of the semantic consequence relation $\models^a_{Ł_n}$. In this case the rule $(\mathrm{R}_{t,r})$ takes this form:

$$\frac{\varphi, \quad \vdash_{Ł_n} \Delta(r_a(\varphi)) \to r_a(\psi)}{\psi}.$$

Note that for $n = 3$ and $a = 1/2$, the resulting logic $Ł^{1/2}_3$ provides an alternative axiomatisation (in the language of $Ł_n$) of the well-known D'Ottaviano and da Costa's paraconsistent logic J_3 [10].

When we move to the case of infinite-valued Łukasiewicz logic Ł, condition (C2) keeps holding at least for every rational a thanks to the McNaughton terms,

[1] This case was partially studied in [6], here we provide more elegant axiomatisations.

but (C1) fails since Ł does not have a global deduction theorem. To overcome this problem we can consider the logic $Ł_{\Delta}$, the expansion of Ł with the Δ operator, already axiomatised by Hájek in [15]. Then in $Ł_{\Delta}$ condition (C2) keeps holding for rational values a, while now condition (C1) is satisfied as well taking $t = \Delta$. Therefore, Theorem 3 can be applied to $Ł_{\Delta}$ to get axiomatisations of $Ł^a_{\Delta}$ for every rational a.

In particular, if we are interested on the logic to reason with *half-true* propositions, it is enough to instantiate Theorem 3 with $a = 1/2$, $t(\varphi) = \Delta\varphi$ and $r(\varphi) = \varphi \otimes \varphi$.

4.2 The cases of Gödel and Product logics

In this final section we consider the cases of Gödel and Product logics. These logics fall outside the scope of Theorem 3, and hence they require a specific consideration.

The case of Gödel logic. The analysis of Gödel logic turns out to be very simple. Gödel logic can be seen as the axiomatic extension of MTL with the axiom (Con), see Table 1. In fact, Gödel logic is standard complete with respect to the single matrix $M_1 = \langle [\mathbf{0}, \mathbf{1}]_{\mathbf{G}}, \{1\} \rangle$, where $[\mathbf{0}, \mathbf{1}]_{\mathbf{G}}$ denotes the standard Gödel algebra $([0, 1], \min, \max, *_G, \to_G, 0, 1)$, with $*_G = \min$ and $\to_G$ is its residuum.

For $a \in (0, 1)$, let us denote by $\models^a_G$ and $\models^{(a}_G$ the logics defined by the logical matrices $M^a = \langle [\mathbf{0}, \mathbf{1}]_{\mathbf{G}}, [a, 1] \rangle$ and $M^{(a} = \langle [\mathbf{0}, \mathbf{1}]_{\mathbf{G}}, (a, 1] \rangle$ respectively. We will also denote the logic $\models^1_G$ simply as $\models_G$.

As is well-known, a distinctive characteristic of Gödel logic is that, for any $a \in [0, 1]$, the mapping $g^a : [0, 1] \to [0, 1]$, defined by $g^a(x) = x$ for $x \in [0, a)$ and $g^a(x) = 1$ for $x \in [a, 1]$, is a morphism of Gödel-algebras, analogously with the mapping $g^{(a} : [0, 1] \to [0, 1]$ defined by $g^a(x) = x$ for $x \in [0, a]$ and $g^a(x) = 1$ for $x \in (a, 1]$. Note that, in particular, $g^{(0}$ maps $[0, 1]$ into $\{0, 1\}$. These well-known facts allow us to prove the following result, see also the left-hand lattice of logics in Fig. 1.

Proposition 2. $\models^{(0}_G$ *coincides with classical logic, while for any* $a \in (0, 1]$, $\models^a_G = \models^{(a}_G = \models_G$.

The case of Product logic. Product logic is defined as the axiomatic extension of MTL with axioms (Div) and (C), see Table 1. Product logic is standard complete with respect to the single matrix $M_1 = \langle [\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}, \{1\} \rangle$, where $[\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}$ denotes the standard product algebra $([0, 1], \min, \max, *_\Pi, \to_\Pi, 0, 1)$, with $*_\Pi$ being the product t-norm and $\to_\Pi$ is its residuum.

For $a \in (0, 1)$, let us denote by $\models^a_\Pi$ and $\models^{(a}_\Pi$ the logics defined by the logical matrices $M^a = \langle [\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}, [a, 1] \rangle$ and $M^{(a} = \langle [\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}, (a, 1] \rangle$ respectively. We will also denote the logic $\models^1_\Pi$ simply as $\models_\Pi$.

In the following we will make use of a known result about automorphisms of the standard product algebra $[\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}$. Namely, let $\alpha \in \mathbb{R}^+$ and define the mapping

$h^\alpha : [0,1] \to [0,1]$ by $h(x) = x^\alpha$. Then h^α is an automorphism of $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$.[2] This means that, for $\otimes \in \{\min, \max, *_\Pi, \to_\Pi\}$ and every α, $h^\alpha(e(\varphi \otimes \psi)) = h^\alpha(e(\varphi)) \otimes h^\alpha(e(\psi))$, for any formulas φ, ψ, every $[0,1]_\Pi$-evaluation e.

Now we can prove a series of results that we will allow to completely characterise all the $\models_\Pi^a$ and $\models_\Pi^{(a}$ logics.

Proposition 3. *The following conditions hold:*

(1) For any $a \in (0,1)$, $\models_\Pi \varphi$ iff $\models_\Pi^a \varphi$.
(2) For any $a, b \in (0,1)$, $\models_\Pi^a = \models_\Pi^b$.
(3) $\models_\Pi^{(0}$ coincides with classical logic, and for $a \in (0,1]$, $\models_\Pi^a = \models_\Pi^{(a}$
(4) For any $a \in (0,1)$, $\varphi \models_\Pi^a \psi$ iff $\models_\Pi^a \varphi \to \psi$.

Proof. We prove the above conditions:

(1) It is clear that if $\models_\Pi \varphi$ then $\models_\Pi^a \varphi$. Conversely, assume $\not\models_\Pi \varphi$. Then there is e such that $e(\varphi) < 1$. Let $b = e(\varphi)$. Then $\not\models_\Pi^a \varphi$ for all a such that $b < a < 1$. Hence, by (i), $\not\models_\Pi^a \varphi$ for any $a \in (0,1)$ as well.

(2) Indeed, assume $\varphi \not\models_\Pi^a \psi$. Then there exists an evaluation e such that $e(\varphi) \geq a$ and $e(\psi) < a$. We know there exists $\alpha \in \mathbb{R}^+$ such that $a^\alpha = b$. Then, if we let $e' = h^\alpha \circ e$, we have that $e'(\varphi) \geq a^\alpha = b$ and $e'(\psi) < a^\alpha = b$, hence $\varphi \not\models_\Pi^b \psi$.

(3) That $\models_\Pi^{(0}$ coincides with classical logic is a direct consequence of the fact that the mapping $k : [0,1] \to \{0,1\}$ such that $h(0) = 0$ and $h(x) = 1$ for $x \in (0,1]$ is a morphism of product algebras.

To show that $\models_\Pi^a = \models_\Pi^{(a}$, first assume $\varphi \not\models_\Pi^a \psi$, and hence there is e such that $e(\varphi) \geq a > e(\psi)$. Let $\alpha < 1$ such that $(e(\psi))^\alpha = a$, then $(e(\varphi))^\alpha \geq a^\alpha > a = (e(\psi))^\alpha$. Therefore, for $e' = h^\alpha \circ e$, we have $e'(\varphi) > a \geq e'(\psi)$, and hence $\varphi \not\models_\Pi^{(a} \psi$. Conversely, assume $\varphi \not\models_\Pi^{(a} \psi$. Then there is e such that $e(\varphi) > a \geq e(\psi)$. Let $\alpha > 1$ be such that $(e(\varphi))^\alpha = a$ and hence we have $a = (e(\varphi))^\alpha > (e(\psi))^\alpha$. This means that $e'(\varphi) \geq a > e'(\psi)$, where $e' = h^\alpha \circ e$, that is, $\varphi \not\models_\Pi^a \psi$. □

(4) Assume $\varphi \not\models_\Pi^a \psi$. Then there is e such that $e(\varphi) \geq a > e(\psi)$. It follows that $1 > e(\varphi \to \psi) = e(\psi)/e(\varphi)$, and thus there is α such that $(e(\psi)/e(\varphi))^\alpha < a$, that is, $e'(\varphi \to \psi) < a$, where $e' = h^\alpha \circ e$. Thus, $\not\models_\Pi^a \varphi \to \psi$. Conversely, assume $\not\models_\Pi^a \varphi \to \psi$. Then there is e such that $e(\varphi \to \psi) < a$, hence $e(\psi)/e(\varphi) < a$, that is, $e(\psi) < a \cdot e(\varphi)$. Let α such that $(e(\varphi))^\alpha = a$. Then we have $(e(\psi))^\alpha < a^\alpha \cdot (e(\varphi))^\alpha = a^\alpha \cdot a < a$. Hence, we have $e'(\varphi) = a$ while $e'(\psi) < a$, where again $e' = h^\alpha \circ e$. Therefore, $\varphi \not\models_\Pi^a \psi$. □

The intersection of all the logics $\models_\Pi^b$ for all $b \in (0,1]$ is what is known as the degree-preserving companion of $\models_\Pi$, and is denoted as $\models_\Pi^\leq$, see [3]. Then as a consequence of (1) and (2) of Proposition 3 we have that $\models_\Pi^\leq = \models_\Pi \cap \models_\Pi^a$ for any $a \in (0,1)$. Next is the final summary result, which is also graphically shown in the right-hand lattice of logics in Figure 1.

Theorem 4. – *For every $a \in (0,1)$, $\models_\Pi^a = \models_\Pi^\leq$.*

[2] In fact, all the automorphisms of $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ are of form h^α [18].

– $\models_{\Pi}^{(0}$ *is classical logic, while for any* $a \in (0,1]$, $\models_{\Pi}^{a} = \models_{\Pi}^{(a} \subsetneq \models_{\Pi}$.

Proof. The inclusion $\models_{\Pi}^{\leq} \subseteq \models_{\Pi}^{a}$ is clear. Assume $\varphi \not\models_{\Pi}^{\leq} \psi$. Then there exists e such that $e(\varphi) > e(\psi)$. Let α such that $(e(\varphi))^{\alpha} = a$, hence $(e(\psi))^{\alpha} < a$. Therefore, if $e' = h^{\alpha} \circ e$, then $e'(\varphi) = a > e'(\psi)$, and thus $\varphi \not\models_{\Pi}^{a} \psi$. Therefore $\models_{\Pi}^{a} \subseteq \models_{\Pi}^{\leq}$. $\square$

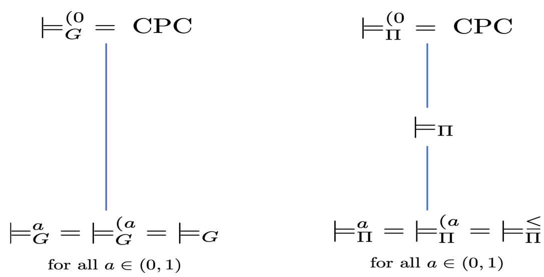

Fig. 1. Lattice of threshold-preserving Gödel and Product logics

5 Conclusions

In this paper we have been concerned with the definition and axiomatisation of both non-falsity preserving and threshold preserving companions of several extensions of the Monoidal t-norm based fuzzy logic MTL, extending some preliminary results in the recent paper [12]. All the axiomatisarions provided make use of restricted inference rules. The question whether we could find axiomatisations with "pure" inference rules is currently an open problem. On the other hand, while the study and characterisation of non-falsity preserving companions of MTL logics is already quite exhaustive, the general study of threshold-preserving companions is only partial, with the exception of the main three fuzzy logics, Łukasiewicz, Product and Gödel logics. In future work we aim at filling this gap, and consider possible applications of these logics to reasoning under uncertainty or with preferences.

Acknowledgements

The authors are indebted to the anonymous reviewers for their comments and suggestions. They also acknowledge support by the MOSAIC project (EU H2020-MSCA-RISE Project 101007627). Esteva and Godo acknowledge partial support by the Spanish project LINEXSYS (PID2022-139835NB-C21), and Gispert by the Spanish project SHORE (PID2022-141529NB-C21), both funded by MCIU/AEI/10.13039/501100011033. Gispert also acknowledges the project 2021 SGR 00348 funded by AGAUR.

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