On conditional probabilities and their canonical extensions to Boolean algebras of compound conditionals[1]

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Abstract

In this paper we investigate canonical extensions of conditional probabilities to Boolean algebras of conditionals. Before entering into the probabilistic setting, we first prove that the lattice order relation of every Boolean algebra of conditionals can be characterized in terms of the well-known order relation given by Goodman and Nguyen. Then, as an interesting methodological tool, we show that canonical extensions behave well with respect to conditional subalgebras. As a consequence, we prove that a canonical extension and its original conditional probability agree on basic conditionals. Moreover, we verify that the probability of conjunctions and disjunctions of conditionals in a recently introduced framework of Boolean algebras of conditionals are in full agreement with the corresponding operations of conditionals as defined in the approach developed by two of the authors to conditionals as three-valued objects, with betting-based semantics, and specified as suitable random quantities. Finally we discuss relations of our approach with nonmonotonic reasoning based on an entailment relation among conditionals.

Keywords: Boolean algebras of conditionals, Conditional subalgebras, Conditional probability, Canonical extension, Conjunction and disjunction of conditionals, Nonmonotonic reasoning.

1. Introduction

Conditionals play a key role in different areas of logic, probabilistic reasoning and knowledge representation in AI, and they have been studied from many points of view, see, e.g., [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In particular, a three-valued calculus of conditional objects has been given in [13], where a simple semantics for the preferential entailment studied in [14, 15, 16] has been provided. Other approaches to conditional objects in the realm of Boolean algebras have been studied in [17, 18]. Further results, from the artificial intelligence perspective, have been given, for instance, in [19, 20, 21].

In the recent paper [22], an alternative algebraic setting for Boolean conditionals has been put forward. More precisely, given a finite Boolean algebra $\mathbf{A} = (\mathbb{A}, \wedge, \vee, \bar{\ }, \bot, \top)$ of events, the authors build another (much bigger but still finite) Boolean algebra $\mathcal{C}(\mathbf{A})$ where *basic conditionals*, i.e. objects of the form $(A|B)$ for $A \in \mathbb{A}$ and $B \in \mathbb{A}' = \mathbb{A}\setminus\{\bot\}$, can be freely combined with the usual Boolean operations, yielding

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compound conditional objects, while they are required to satisfy a set of natural properties. Moreover, the set of atoms of $C(\mathbf{A})$ are fully identified and it is shown they are in a one-to-one correspondence with sequences of pairwise different atoms of $\mathbf{A}$ of maximal length. Finally, it is also shown that any positive probability P on the set of events from $\mathbf{A}$ can be *canonically* extended to a probability μ_P on the algebra of conditionals $C(\mathbf{A})$ in such a way that the probability $\mu_P(\text{“}(A|B)\text{”})$ of a basic conditional $(A|B)$ coincides with the conditional probability $P(A|B) = P(A \wedge B)/P(B)$. This is done by suitably defining the probability of each atom of $C(\mathbf{A})$ as a certain product of conditional probabilities.

On the other hand, the recent paper [23] presents results in the setting of conditional random quantities, with values in the unit interval $[0, 1]$, where the numerical approach to conjunctions and disjunctions of conditional events (see, e.g., [24, 25, 26]) is extended in general to cover arbitrarily complex compound conditionals. These objects are conditional random quantities obtained by conjunctions, disjunctions, and negations of conditional events and/or compound conditionals.

In this paper we take the more symbolic algebraic approach to conditionals from [22] a step further and bring it closer to the more numerical approach of [23] and [24, 25, 26]. We do this by first providing new basic results on the algebras $C(\mathbf{A})$ of conditionals themselves, and second by turning operational some of its algebraic and probabilistic definitions. For instance, and in contrast with the above mentioned papers, in [22] precise definitions of conjunction and disjunction of conditionals are not explicitly given. Rather, any compound conditional comes determined by the disjunction of those atoms in $C(\mathbf{A})$ that lie below it. Similarly, the probability of any compound conditional is computed as the sum of the probabilities of the atoms below the conditional. But no operational and systematic procedure to do these computations avoiding a combinatorial explosion is provided in [22]. More precisely, the main novel contributions of the paper are:

- We show that the construction of the algebra of conditionals $C(\mathbf{A})$ from a finite algebra of events $\mathbf{A}$ is compatible with subalgebras. Also we explore the relationship of Goodman and Nguyen's inclusion relation between basic conditionals with the natural order relation $\leqslant$ in $C(\mathbf{A})$.
- We extend the definition from [22] of the canonical extension to $C(\mathbf{A})$ of a positive probability on $\mathbf{A}$ to the case of starting with a general conditional probability on $\mathbb{A} \times \mathbb{A}'$, and we show that this extension is compatible with taking restrictions on subalgebras and with Stalnaker's thesis.
- We derive for the canonical extension the formula to compute the probability of a conjunction and a disjunction of conditionals, and check they coincide with the ones proposed in the literature by McGee and Kaufmann, also in accordance with the random quantities approach.
- Finally, we introduce an entailment relation in terms of the lattice order in $C(\mathbf{A})$ and we characterize probabilistically the entailment relation by canonical extensions. Then, we show that a corresponding nonmonotonic consequence relation on the algebra $\mathbf{A}$ satisfies the well-known rules of the system P.

The paper is structured as follows. After this introduction and some preliminaries in Section 2, we first examine in Section 3 the relation between the lattice order in a conditional algebra $C(\mathbf{A})$ and the inclusion relation defined by Goodman and Nguyen. In Section 4 we show that the positivity assumption for the probability on $\mathbf{A}$, needed for the canonical extension to the algebra of conditionals $C(\mathbf{A})$, can be lifted by starting from a conditional probability (in the axiomatic sense) on $\mathbb{A} \times \mathbb{A}'$. Then in Section 5 we show that, if $\mathbf{B}$ is a subalgebra of events of $\mathbf{A}$ and P a conditional probability on $\mathbb{A} \times \mathbb{A}'$, then the restriction of the canonical extension μ_P on $C(\mathbf{A})$ to $C(\mathbf{B})$ is, in fact, the canonical extension of the restriction of P on $\mathbb{B} \times \mathbb{B}'$. This will allow us to prove that μ_P is such that $\mu_P(\text{“}(A|B)\text{”}) = P(A|B)$ and then in Section 6 that the probability of the conjunction coincides with McGee and Kaufmann's expressions obtained within the approach developed by two of the authors to conditionals as three-valued objects, with betting-based semantics, and specified as suitable random quantities. We also obtain the probability of the disjunction and the probability sum rule, in agreement with the approach given in [24]. In Section 7 we introduce an entailment relation in terms of the lattice order in $\mathcal{C}(\mathbf{A})$; then, we characterize probabilistically the entailment relation by canonical extensions. Then, we examine a nonmonotonic consequence relation on the algebra $\mathbf{A}$, which satisfies the well-known rules of the system P. Moreover, we discuss the Rational Monotony and the disjunctive Weak Rational Monotony rules. We also illustrate an example related to the failure of the transitive property. We conclude in Section 8 with some remarks and prospects for future work.

2. Preliminaries

In this section we recall basic notions and results from [22] where, for any Boolean algebra of events $\mathbf{A} = (\mathbb{A}, \wedge, \vee, \bar{\ }, \bot, \top)$, a Boolean algebra of conditionals, denoted $\mathcal{C}(\mathbf{A})$, is built. We will also denote a conjunction $A \wedge B$ simply by AB. Intuitively, a Boolean algebra of conditionals over $\mathbf{A}$ allows *basic conditionals*, i.e. objects of the form $(A|B)$ for $A \in \mathbb{A}$ and $B \in \mathbb{A}' = \mathbb{A} \backslash \{\bot\}$, to be freely combined with the usual Boolean operations up to certain extent.

In mathematical terms, the formal construction of the algebra of conditionals $\mathcal{C}(\mathbf{A})$ is done as follows. One first considers the free Boolean algebra $\mathbf{Free}(\mathbb{A}|\mathbb{A}') = (Free(\mathbb{A}|\mathbb{A}'), \sqcap, \sqcup, \bar{\ }, \bot, \top)$[6] generated by the set $\mathbb{A}|\mathbb{A}' = \{(A|B) : A \in \mathbb{A}, B \in \mathbb{A}'\}$. Then, one considers the smallest congruence relation $\equiv_{\mathfrak{C}}$ on $\mathbf{Free}(\mathbb{A}|\mathbb{A}')$ satisfying the following natural properties:

(C1) $(B|B) \equiv_{\mathfrak{C}} \top$, for all $B \in \mathbb{A}'$;

(C2) $(A_1|B) \sqcap (A_2|B) \equiv_{\mathfrak{C}} (A_1 \wedge A_2|B)$, for all $A_1, A_2 \in \mathbb{A}$, $B \in \mathbb{A}'$;

(C3) $\overline{(A|B)} \equiv_{\mathfrak{C}} (\bar{A}|B)$, for all $A \in \mathbb{A}$, $B \in \mathbb{A}'$;

(C4) $(A \wedge B|B) \equiv_{\mathfrak{C}} (A|B)$, for all $A \in \mathbb{A}$, $B \in \mathbb{A}'$;

(C5) $(A|B) \sqcap (B|C) \equiv_{\mathfrak{C}} (A|C)$, for all $A \in \mathbb{A}$, $B, C \in \mathbb{A}'$ such that $A \leqslant B \leqslant C$.

Notice that, if $A = A'$ and $B = B'$ in $\mathbf{A}$, then $(A|B) = (A'|B')$. Then, in a sense, the partial operation "|" is well-defined. Finally, the algebra $\mathcal{C}(\mathbf{A})$ is defined as follows.

Definition 1. For every Boolean algebra $\mathbf{A}$, the *Boolean algebra of conditionals* of $\mathbf{A}$ is the quotient structure $\mathcal{C}(\mathbf{A}) = \mathbf{Free}(\mathbb{A}|\mathbb{A}')/_{\equiv_{\mathfrak{C}}}$.

Since $\mathcal{C}(\mathbf{A})$ is a *quotient* of $\mathbf{Free}(\mathbb{A}|\mathbb{A})$, elements of $\mathcal{C}(\mathbf{A})$ are equivalence classes but, without danger of confusion, one can henceforth identify classes $[t]_{\equiv_{\mathfrak{C}}}$ with one of its representative elements, in particular, by t itself. Conditionals of the form $(A|\top)$ will also be simply denoted as A.

A basic observation is that if $\mathbf{A}$ is finite, $\mathcal{C}(\mathbf{A})$ is finite as well, and hence atomic. Indeed, if $\mathbf{A}$ is a Boolean algebra with n atoms $\mathrm{at}(\mathbf{A}) = \{\alpha_1, \ldots, \alpha_n\}$, i.e. $|\mathrm{at}(\mathbf{A})| = n$, it is shown in [22] that the atoms of $\mathcal{C}(\mathbf{A})$ are in one-to-one correspondence with sequences $\overline{\alpha} = (\alpha_{i_1}, \ldots, \alpha_{i_{n-1}})$ of $n-1$ pairwise different atoms of $\mathbf{A}$, each of these sequences giving rise to an atom $\omega_{\overline{\alpha}}$ of $\mathcal{C}(\mathbf{A})$ defined as the following conjunction of $n-1$ basic conditionals:

$$\omega_{\overline{\alpha}} = (\alpha_{i_1}|\top) \sqcap (\alpha_{i_2}|\bar{\alpha}_{i_1}) \sqcap \ldots \sqcap (\alpha_{i_{n-1}}|\bar{\alpha}_{i_1} \ldots \bar{\alpha}_{i_{n-2}}). \tag{1}$$

[6] We will continue denoting the top and bottom of $\mathbf{Free}(\mathbb{A})$ by $\top$ and $\bot$ respectively without danger of confusion.

In what follows the atom in (1) will be also denoted by $\langle\alpha_{i_1},\ldots,\alpha_{i_{n-1}}\rangle$, or by $\omega_{i_1\cdots i_{n-1}}$. It is then clear that the cardinality of the set of atoms of $\mathcal{C}(\mathbf{A})$ is $|\mathrm{at}(\mathcal{C}(\mathbf{A}))| = n!$. We recall that the lattice order relation $\leqslant$ in $\mathcal{C}(\mathbf{A})$ is defined as

$$t \leqslant s \ \text{ iff } \ t \sqcap s = t \ \text{ iff } \ t \sqcup s = s, \ \text{for every } s,t \in \mathcal{C}(\mathbf{A}). \tag{2}$$

We also observe that, for every $s,t \in \mathcal{C}(\mathbf{A})$,

$$t \leqslant s \quad \text{iff} \quad t \sqcap \bar{s} = \bot, \tag{3}$$

because $t = t \sqcap \top = (t \sqcap s) \sqcup (t \sqcap \bar{s})$. In [22, Proposition 4.7] it is shown that an atom $\omega = (\alpha_{i_1}|\top) \sqcap (\alpha_{i_2}|\bar{\alpha}_{i_1}) \sqcap \cdots \sqcap (\alpha_{i_{n-1}}|\bar{\alpha}_{i_1}\cdots\bar{\alpha}_{i_{n-2}})$ is below a conditional $(A|H)$ w.r.t. the lattice order $\leqslant$ in $\mathcal{C}(\mathbf{A})$, i.e. $\omega \leqslant A|H$, if and only if, letting $\omega = \langle\alpha_{i_1},\alpha_{i_2},\ldots,\alpha_{i_{n-1}}\rangle$, if j is the first index for which $\alpha_{i_j} \leqslant H$, then $\alpha_{i_j} \leqslant A$ as well; in other words, the following condition is satisfied:

either "$\alpha_{i_1} \leqslant AH$", or "$\alpha_{i_1} \leqslant \bar{H}$ and $\alpha_{i_2} \leqslant AH$", or ..., or "$\alpha_{i_1} \leqslant \bar{H}$ and ... and $\alpha_{i_{n_2}} \leqslant \bar{H}$ and $\alpha_{i_{n-1}} \leqslant AH$".

Next we will recall some properties holding in $\mathcal{C}(\mathbf{A})$ that will be useful for next sections. For each subvector $(i_1,\ldots,i_k)$ of $(1,\ldots,n)$ we set

$$\omega_{i_1\cdots i_k} = \alpha_{i_1} \sqcap (\alpha_{i_2}|\bar{\alpha}_{i_1}) \sqcap \cdots \sqcap (\alpha_{i_k}|\bar{\alpha}_{i_1}\cdots\bar{\alpha}_{i_{k-1}}), \tag{4}$$

that is, the conjunction $\omega_{i_1\cdots i_k}$, which we also denote by $\langle\alpha_{i_1},\ldots,\alpha_{i_k}\rangle$, stands for the initial conjunction of k components of the atom $\omega_{i_1\cdots i_{n-1}}$. Indeed, as $(\alpha_{i_n}|\bar{\alpha}_{i_1}\cdots\bar{\alpha}_{i_{n-1}}) = (\alpha_{i_n}|\alpha_{i_n}) = \top$, for each permutation $(i_1,\ldots,i_n)$ of $(1,\ldots,n)$, we obtain the following atom of $\mathcal{C}(\mathbf{A})$:

$$\omega_{i_1\cdots i_n} = \omega_{i_1\cdots i_{n-1}} = \alpha_{i_1} \sqcap (\alpha_{i_2}|\bar{\alpha}_{i_1}) \sqcap \cdots \sqcap (\alpha_{i_{n-1}}|\bar{\alpha}_{i_1}\cdots\bar{\alpha}_{i_{n-2}}).$$

We hence recall that, from [22, Proposition 4.3], for each k, the conjunctions $\omega_{i_1\cdots i_k}$'s constitute a partition of the algebra $\mathcal{C}(\mathbf{A})$. In particular this implies that $\bigsqcup_{(i_1,\ldots,i_k)\in\Pi_{\{j_1,\ldots,j_k\}}} \omega_{i_1\cdots i_k} = \top$, where $\Pi_{\{j_1,\ldots,j_k\}}$ is the set of all permutations $(i_1,\ldots,i_k)$ of the set $\{j_1,\ldots,j_k\}$.

Example 1. An example of an algebra $\mathbf{A}$, with 3 atoms, is obtained by considering the partition $\{\alpha_1,\alpha_2,\alpha_3\} = \{EH,\bar{E}H,\bar{H}\}$, where E,H are two uncertain logically independent events. In this case

$$\mathbb{A} = \{\bot,\top,EH,\bar{E}H,\bar{H},H,EH \vee \bar{H},\bar{E}H \vee \bar{H}\}.$$

The basic conditionals of $\mathcal{C}(\mathbf{A})$ are the elements of the set $\mathbb{A}|\mathbb{A}' = \{(A|B) : A \in \mathbb{A}, B \in \mathbb{A}'\}$, where $\mathbb{A}' = \mathbb{A}\backslash\{\bot\}$. Moreover, the atoms of $\mathcal{C}(\mathbf{A})$ are the 3! elements of the form $\omega_{ij} = \alpha_i \sqcap (\alpha_j|\bar{\alpha}_i)$, with $i \neq j$, that is,

$$\mathrm{at}(\mathcal{C}(\mathbf{A})) = \{\omega_{12},\omega_{13},\omega_{21},\omega_{23},\omega_{31},\omega_{32}\},$$

where

$$\begin{array}{ll}
\omega_{12} = \alpha_1 \sqcap \alpha_2|\bar{\alpha}_1 = EH \sqcap (\bar{E}H|\bar{E} \vee \bar{H}), & \omega_{13} = \alpha_1 \sqcap \alpha_3|\bar{\alpha}_1 = EH \sqcap (\bar{H}|\bar{E} \vee \bar{H}), \\
\omega_{21} = \alpha_2 \sqcap \alpha_1|\bar{\alpha}_2 = \bar{E}H \sqcap (EH|E \vee \bar{H}), & \omega_{23} = \alpha_2 \sqcap \alpha_3|\bar{\alpha}_2 = \bar{E}H \sqcap (\bar{H}|E \vee \bar{H}), \\
\omega_{31} = \alpha_3 \sqcap \alpha_1|\bar{\alpha}_3 = \bar{H} \sqcap (EH|H) = \bar{H} \sqcap (E|H), & \omega_{32} = \alpha_3 \sqcap \alpha_2|\bar{\alpha}_3 = \bar{H} \sqcap (\bar{E}H|H) = \bar{H} \sqcap (\bar{E}|H).
\end{array}$$

A pictorial representation of the algebra $\mathcal{C}(\mathbf{A})$, with $2^{3!} = 2^6 = 64$ elements, can be found in Figure 1. We observe that

$$\begin{array}{l}
\omega_{12} \sqcup \omega_{13} = (EH \sqcap (\bar{E}H|\bar{E} \vee \bar{H})) \sqcup (EH \sqcap (\bar{H}|\bar{E} \vee \bar{H})) = EH = \omega_1 = \alpha_1, \\
\omega_{21} \sqcup \omega_{23} = (\bar{E}H \sqcap (EH|E \vee \bar{H})) \sqcup (\bar{E}H \sqcap (\bar{H}|E \vee \bar{H})) = \bar{E}H = \omega_2 = \alpha_2, \\
\omega_{31} \sqcup \omega_{32} = (\bar{H} \sqcap (E|H)) \sqcup (\bar{H} \sqcap (\bar{E}|H)) = \bar{H} = \omega_3 = \alpha_3, \\
\omega_{12} \sqcup \omega_{13} \sqcup \omega_{21} \sqcup \omega_{23} \sqcup \omega_{31} \sqcup \omega_{32} = \omega_1 \sqcup \omega_2 \sqcup \omega_3 = \top.
\end{array}$$

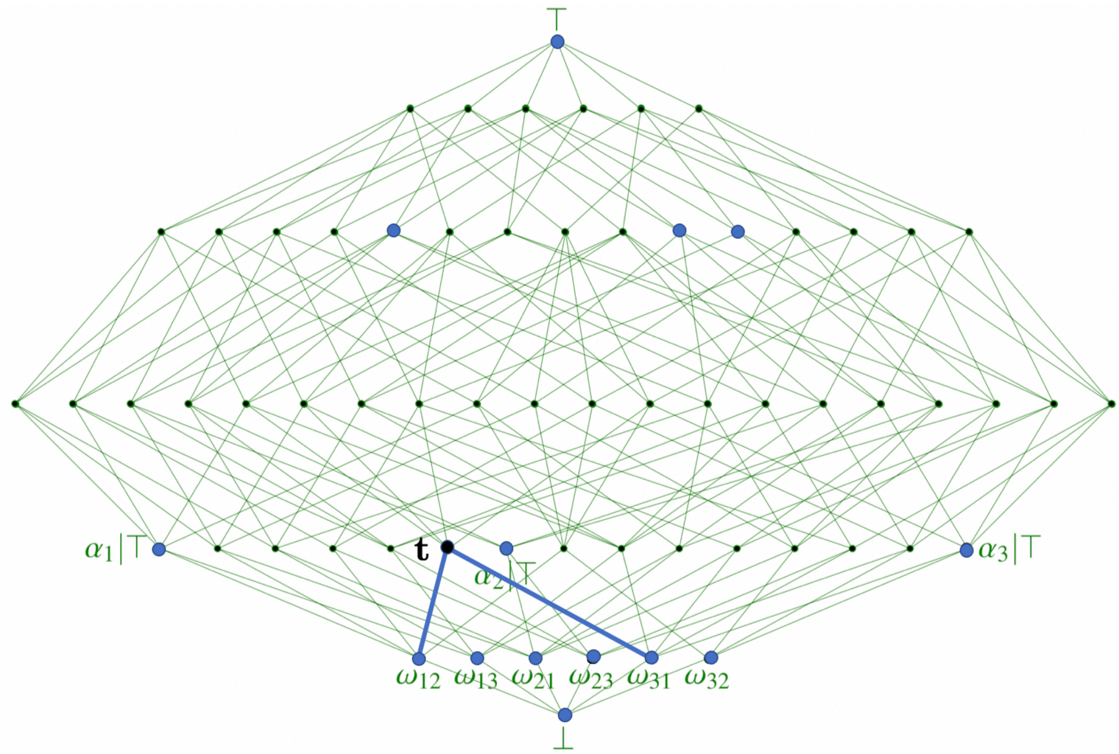

Figure 1: The algebra of conditionals $\mathcal{C}(\mathbf{A})$ of Example 1, where $\text{at}(\mathbf{A}) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\text{at}(\mathcal{C}(\mathbf{A})) = \{\omega_{12}, \omega_{13}, \omega_{21}, \omega_{23}, \omega_{31}, \omega_{32}\}$. The element t is obtained as $\omega_{12} \sqcup \omega_{31}$. The atoms of $\mathcal{C}(\mathbf{A})$, the elements of the original algebra $\mathbf{A}$, and the element t are identified with big dots.

Moreover, we have the following properties:

$$\omega_i \sqcap \omega_{ij} = \omega_{ij}, \text{ if } i \neq j; \quad \omega_{ij} \sqcap \omega_{rk} = \bot, \text{ if } (i,j) \neq (r,k); \quad \omega_i \sqcap \omega_{rk} = \alpha_i \sqcap \alpha_r \sqcap \alpha_k | \bar{\alpha}_r = \bot, \text{ if } i \neq r.$$

Finally any compound conditional $t \in \mathcal{C}(\mathbf{A})$ is a disjunction of the atoms below t. For instance, let

$$t = (\alpha_1|\alpha_1 \vee \alpha_2) \sqcap ((\alpha_3|\alpha_1 \vee \alpha_3) \sqcup (\alpha_2|\alpha_2 \vee \alpha_3)) = E|H \sqcap ((\bar{H}|EH \vee \bar{H}) \sqcup (\bar{E}H|\bar{E} \vee \bar{H})).$$

Based on [22, Proposition 4.7], we observe that $E|H = \omega_{12} \sqcup \omega_{13} \sqcup \omega_{31}$, $(\bar{H}|EH \vee \bar{H}) = \omega_{31} \sqcup \omega_{32} \sqcup \omega_{23}$ and $(\bar{E}H|\bar{E} \vee \bar{H}) = \omega_{21} \sqcup \omega_{23} \sqcup \omega_{12}$. Then we have:

$$t = (\omega_{12} \sqcup \omega_{13} \sqcup \omega_{31}) \sqcap (\omega_{31} \sqcup \omega_{32} \sqcup \omega_{23} \sqcup \omega_{21} \sqcup \omega_{23} \sqcup \omega_{12}) = \omega_{12} \sqcup \omega_{31}.$$

Proposition 1. Consider two sequences of indices $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_t)$, with $k \leqslant t$. Then:

(i) $\omega_{i_1 \cdots i_k} \sqcap \omega_{j_1 \cdots j_t} = \omega_{j_1 \cdots j_t}$, if $(i_1, \ldots, i_k) = (j_1, \ldots, j_k)$,

(ii) $\omega_{i_1 \cdots i_k} \sqcap \omega_{j_1 \cdots j_t} = \bot$, if $i_h \neq j_h$ for some index $h \in \{1, \ldots, k\}$.

Moreover,

(*iii*) For every sequence $(i_1, \dots, i_k)$, it holds that

$$\omega_{i_1\cdots i_k} = \bigsqcup_{\{(j_1,\dots,j_{n-1})\in\Pi_{(1,\dots,n-1)}:j_r=i_r,\,1\leqslant r\leqslant k\}} \omega_{j_1\cdots j_{n-1}}\,.$$

Proof. (*i*) Given two sequences $(i_1, \dots, i_k)$ and $(j_1, \dots, j_t)$, with $k \leqslant t$ and $(j_1, \dots, j_k) = (i_1, \dots, i_k)$, it holds that

$$\begin{aligned}&\omega_{i_1\cdots i_k} \sqcap \omega_{j_1\cdots j_t} = \omega_{i_1\cdots i_k} \sqcap \omega_{i_1\cdots i_k} \sqcap (\alpha_{j_{k+1}}|\bar\alpha_{j_1}\cdots\bar\alpha_{j_k}) \sqcap \cdots \sqcap (\alpha_{j_t}|\bar\alpha_{j_1}\cdots\bar\alpha_{j_{t-1}}) =\\ &= \omega_{i_1\cdots i_k} \sqcap (\alpha_{j_{k+1}}|\bar\alpha_{j_1}\cdots\bar\alpha_{j_k}) \sqcap \cdots \sqcap (\alpha_{j_t}|\bar\alpha_{j_1}\cdots\bar\alpha_{j_{t-1}}) = \omega_{j_1\cdots j_t}.\end{aligned}$$

(*ii*) Denote by h the first index in the set $\{i_1, \dots, i_k\}$ such that $i_h \neq j_h$. Then

$$\omega_{i_1\cdots i_h} = \omega_{i_1\cdots i_{h-1}} \sqcap (\alpha_{i_h}|\bar\alpha_{i_1}\cdots\bar\alpha_{i_{h-1}}), \quad \omega_{j_1\cdots j_h} = \omega_{i_1\cdots i_{h-1}} \sqcap (\alpha_{j_h}|\bar\alpha_{i_1}\cdots\bar\alpha_{i_{h-1}}),$$

with $(\alpha_{i_h}|\bar\alpha_{i_1}\cdots\bar\alpha_{i_{h-1}}) \sqcap (\alpha_{j_h}|\bar\alpha_{i_1}\cdots\bar\alpha_{i_{h-1}}) = \bot$, so that

$$\omega_{i_1\cdots i_h} \sqcap \omega_{j_1\cdots j_h} = \omega_{i_1\cdots i_{h-1}} \sqcap (\alpha_{i_h}|\bar\alpha_{i_1}\cdots\bar\alpha_{i_{h-1}}) \sqcap (\alpha_{j_h}|\bar\alpha_{i_1}\cdots\bar\alpha_{i_{h-1}}) = \bot\,.$$

Therefore,

$$\begin{aligned}&\omega_{i_1\cdots i_k} \sqcap \omega_{j_1\cdots j_t} =\\ &= \omega_{i_1\cdots i_{h-1}} \sqcap (\alpha_{i_h}|\bar\alpha_{i_1}\cdots\bar\alpha_{i_{h-1}}) \sqcap \cdots \sqcap (\alpha_{i_k}|\bar\alpha_{i_1}\cdots\bar\alpha_{i_{k-1}}) \sqcap (\alpha_{j_h}|\bar\alpha_{i_1}\cdots\bar\alpha_{i_{h-1}}) \sqcap \cdots \sqcap (\alpha_{j_t}|\bar\alpha_{i_1}\cdots\bar\alpha_{i_{t-1}}) = \bot.\end{aligned}$$

(*iii*) From (*i*), $\omega_{i_1\cdots i_k} \sqcap \omega_{j_1\cdots j_t} = \omega_{j_1\cdots j_t}$, when $i_r = j_r$, $1 \leqslant r \leqslant k$. In particular for $t = n-1$ it holds that $\omega_{i_1\cdots i_k} \sqcap \omega_{j_1\cdots j_{n-1}} = \omega_{j_1\cdots j_{n-1}} \in \mathrm{at}(C(\mathbf{A}))$. Then, as $\top = \bigsqcup_{\{(j_1,\dots,j_{n-1})\in\Pi_{(1,\dots,n-1)}\}} \omega_{j_1\cdots j_{n-1}}$, from (*ii*) it follows that

$$\omega_{i_1\cdots i_k} = \omega_{i_1\cdots i_k} \sqcap \top = \bigsqcup_{\{(j_1,\dots,j_{n-1})\in\Pi_{(1,\dots,n-1)}:j_r=i_r,\,1\leqslant r\leqslant k\}} \omega_{j_1\cdots j_{n-1}}\,.$$

□

Let us notice that the construction of the algebra $C(\mathbf{A})$ presented above can be seen as a map that, for every finite Boolean algebra $\mathbf{A}$ gives its associated Boolean algebra of conditionals $C(\mathbf{A})$. For a later use, it is convenient to observe that such construction preserves subalgebras in the sense made clear by the next easy result.

Proposition 2. Let $\mathbf{A}$ be a finite Boolean algebra and let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$. Then $C(\mathbf{B})$ is a subalgebra of $C(\mathbf{A})$.

Proof. Since $\mathbf{A}$ and $\mathbf{B}$ are finite algebras, so are $C(\mathbf{A})$ and $C(\mathbf{B})$. Moreover, by the way atoms are characterized in every boolean algebra of conditionals, it is clear that $|\mathrm{at}(C(\mathbf{B}))| \leqslant |\mathrm{at}(C(\mathbf{A}))|$. Thus, by an easy cardinality argument, it immediately follows that $C(\mathbf{B})$ is isomorphic to a subalgebra of $C(\mathbf{A})$. More concretely, if $\beta_1, \dots, \beta_t$ are the atoms of $\mathbf{B}$, $C(\mathbf{B})$ is the subalgebra of $C(\mathbf{A})$ whose atoms are of the form $\omega_i^{\mathbf{B}} = (\beta_{i_1}|\top) \sqcap (\beta_{i_2}|\bar\beta_{i_1}) \sqcap \ldots \sqcap (\beta_{i_{t-1}}|\bar\beta_{i_1}\ldots\bar\beta_{i_{t-2}})$. Notice that each $\omega_i^{\mathbf{B}}$ clearly is an element of $C(\mathbf{A})$, whence $C(\mathbf{B})$ is indeed the concrete subalgebra of $C(\mathbf{A})$ having the $\omega_i^{\mathbf{B}}$'s as atoms. □

Subalgebras of $C(\mathbf{A})$ of the form $C(\mathbf{B})$, with $\mathbf{B}$ being a subalgebra of $\mathbf{A}$, will be called *conditional subalgebras* of $C(\mathbf{A})$.

A particularly useful class of subalgebras of a given algebra $\mathbf{A}$ are those generated by partitions of $\mathbf{A}$ standing for the different truth conditions of a set $\mathcal{F}$ of conditionals of $C(\mathbf{A})$ taken as three-valued objects.

In e.g. [25, 26], the elements of such a partition are called the *constituents* generated by $\mathcal{F}$. The simplest example is the case of a single conditional $\mathcal{F}_1 = \{(A|H)\}$, with A and H (uncertain) logically independent events, that generates the partition of $\mathbf{A}$ $\pi_1 = \{AH, \bar{A}H, \bar{H}\}$, where the event AH makes the conditional $(A|H)$ true, the event $\bar{A}H$ makes the conditional $(A|H)$ false, while the event $\bar{H}$ makes the conditional $(A|H)$ void. In the case of two conditionals $\mathcal{F}_2 = \{(A|H), (B|K)\}$, we therefore have in principle $3^2 = 9$ different combined truth conditions, leading to the following 9-element partition

$$\pi_2 = \{AHBK, \bar{A}HBK, AH\bar{B}K, \bar{A}H\bar{B}K, AH\bar{K}, \bar{A}H\bar{K}, \bar{H}BK, \bar{H}\bar{B}K, \bar{H}\bar{K}\}, \tag{5}$$

assuming all these events are different from $\perp$ (i.e. assuming the uncertain events A, H, B, K logically independent). When one deals with sets of more conditionals, the corresponding partitions can be defined by an easy generalisation of the previous procedure.

Now, consider a *positive* probability $P : \mathbf{A} \to [0, 1]$ on the algebra of plain events $\mathbf{A}$. It is shown in [22] that P can be extended to a probability $\mu_P : C(\mathbf{A}) \to [0, 1]$ on the Boolean algebra of conditionals $C(\mathbf{A})$, called *canonical extension*, in such a way that $\mu_P(\text{“}(A|B)\text{”})$, the probability of a basic conditional $(A|B)$, coincides with the conditional probability of A given B, i.e.

$$\mu_P(\text{“}(A|B)\text{”}) = P(A|B) = P(A \wedge B)/P(B),$$

in accordance with the so-called *Stalnaker's thesis*, stating that the probability of a conditional is the conditional probability, whenever the antecedent has non-zero probability [27]. In particular, $\mu_P(\text{“}(A|\top)\text{”}) = P(A|\top) = P(A)$ for any $A \in \mathbb{A}$. Actually, the probability μ_P is first defined on the atoms of $C(\mathbf{A})$ as follows: for any atom $\omega_{i_1 \cdots i_{n-1}} = \alpha_{i_1} \sqcap (\alpha_{i_2}|\bar{\alpha}_{i_1}) \sqcap \cdots \sqcap (\alpha_{i_{n-1}}|\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_{n-2}})$, its probability is defined as the following product of conditional probabilities:

$$\mu_P(\omega_{i_1 \cdots i_{n-1}}) = P(\alpha_{i_1}) \cdot P(\alpha_{i_2}|\bar{\alpha}_{i_1}) \cdots P(\alpha_{i_{n-1}}|\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_{n-2}}) = P(\alpha_{i_1}) \cdot \frac{P(\alpha_{i_2})}{P(\bar{\alpha}_{i_1})} \cdots \frac{P(\alpha_{i_{n-1}})}{P(\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_{n-2}})}.$$

This is well defined because of the assumption that P is positive. Then μ_P is extended to the whole algebra $C(\mathbf{A})$ of conditionals by additivity: for any element t of $C(\mathbf{A})$,

$$\mu_P(t) = \sum_{\omega_{i_1 \cdots i_{n-1}} \leqslant t} \mu_P(\omega_{i_1 \cdots i_{n-1}}),$$

where the lattice order $\leqslant$ in $C(\mathbf{A})$ is as defined in (2). Moreover, it is shown in [22] that for any k, the following factorization holds:

$$\mu_P(\omega_{i_1 \cdots i_k}) = \sum_{(i_{k+1},\dots,i_n) \in \Pi_{\{1,\dots,n\} \setminus \{i_1,\dots,i_k\}}} \mu_P(\omega_{i_1 \cdots i_{n-1}}) = P(\alpha_{i_1}) \cdot P(\alpha_{i_2}|\bar{\alpha}_{i_1}) \cdots P(\alpha_{i_k}|\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_{k-1}}). \tag{6}$$

We finally notice that, as observed above, since for each k the conjunctions $\omega_{i_1 \cdots i_k}$'s constitute a partition of $C(\mathbf{A})$, the sum of the probabilities over all of them is 1, that is:

$$1 = \sum_i P(\alpha_i) = \sum_i \mu_P(\omega_i) = \sum_{i \neq j} \mu_P(\omega_{ij}) = \cdots = \sum_{(i_1,\dots,i_n) \in \Pi_{\{1,\dots,n\}}} \mu_P(\omega_{i_1 \cdots i_{n-1}}).$$

3. On the relation between the lattice order in $C(\mathbf{A})$ and Goodman-Nguyen's inclusion relation

In [28] Goodman and Nguyen introduced an *inclusion relation* between conditional objects in the context of measure-free conditionals. Adapted to the setting of conditionals in an algebra of conditionals it amounts to the following definition.

Definition 2. The Goodman-Nguyen inclusion relation between basic conditionals in an algebra $C(\mathbf{A})$ is defined as follows: for any $A|H, B|K \in C(\mathbf{A})$,

$$A|H \subseteq B|K \text{ iff } AH \leqslant BK \text{ and } \bar{B}K \leqslant \bar{A}H,$$

where $\leqslant$ is the lattice order relation in $\mathbf{A}$.

In this section we explore the relationship of this inclusion relation in $C(\mathbf{A})$ with the natural order relation $\leqslant$ in $C(\mathbf{A})$, and we provide a full characterisation of $\leqslant$ in terms of $\subseteq$, extending partial results in [22].

Theorem 1. Given any conditional events $A|H$ and $B|K$ of an algebra $C(\mathbf{A})$, it holds that

$$A|H \leqslant B|K \iff AH = \bot\,, \text{ or } \bar{B}K = \bot\,, \text{ or } A|H \subseteq B|K. \tag{7}$$

Proof. ($\Longrightarrow$) If $AH = \bot$, or $\bar{B}K = \bot$, then the statement holds. Assume that $AH \neq \bot$, $\bar{B}K \neq \bot$, and by absurd that $A|H \nsubseteq B|K$, that is $AH \nleqslant BK$ or $\bar{B}K \nleqslant \bar{A}H$. If it were $AH \nleqslant BK$, as $AH \neq \bot$ there would exist an atom $\alpha \in \mathrm{at}(\mathbf{A})$ such that $\alpha \leqslant AH$ and $\alpha \leqslant \bar{B}K \vee \bar{K}$. On the other hand, since $\bar{B}K \neq \bot$ there would exist an atom $\beta \in \mathrm{at}(\mathbf{A})$ such that $\beta \leqslant \bar{B}K$. Now let ω be an atom of $C(\mathbf{A})$ of the form $\omega = \langle \alpha, \beta, \ldots \rangle$. Then, it would be $\omega \leqslant A|H$ and $\omega \leqslant \bar{B}|K$, hence $(A|H) \sqcap (\bar{B}|K) \neq \bot$ because $\omega \leqslant (A|H) \sqcap (\bar{B}|K)$. This leads to a contradiction because by hypothesis $A|H \leqslant B|K$, that is, by recalling (3), $(A|H) \sqcap (\bar{B}|K) = \bot$.

If it were $\bar{B}K \nleqslant \bar{A}H$, as $\bar{B}K \neq \bot$ there would exist an atom $\alpha \in \mathrm{at}(\mathbf{A})$ such that $\alpha \leqslant \bar{B}K$ and $\alpha \leqslant AH \vee \bar{H}$. On the other hand, since $AH \neq \bot$, there would exist an atom $\beta \in \mathrm{at}(\mathbf{A})$ such that $\beta \leqslant AH$. Now let ω be an atom of $C(\mathbf{A})$ of the form $\omega = \langle \alpha, \beta, \ldots \rangle$. Then, it would be $\omega \leqslant A|H$ and $\omega \leqslant \bar{B}|K$, which is absurd because, it contradicts the hypothesis $A|H \leqslant B|K$.

($\Longleftarrow$) Observe first that if $AH = \bot$ or $\bar{B}K = \bot$, then $A|H = \bot$ or $B|K = \top$, respectively. Thus, in both cases it holds that $A|H = (A|H) \sqcap (B|K)$ and hence, by recalling (2), the condition $A|H \leqslant B|K$ is satisfied.

Assume now that $AH \neq \bot$ and $\bar{B}K \neq \bot$. Moreover, assume that $A|H \subseteq B|K$, that is $AH \leqslant BK$ and $\bar{B}K \leqslant \bar{A}H$. As $AH \leqslant BK$ and $\bar{B}K \leqslant \bar{A}H$, it holds that $AH\bar{B}K = AH\bar{K} = \bar{H}\bar{B}K = \bot$. Taking into account these last logical relationships, it follows that the partition π_2 of $\mathbf{A}$ generated by $A|H$ and $B|K$ is contained in the set $\{\alpha_1, \ldots, \alpha_6\}$,[7] where

$$\alpha_1 = AHBK = AH,\ \alpha_2 = \overline{A}HBK,\ \alpha_3 = \overline{A}H\overline{B}K = \bar{B}K,\ \alpha_4 = \overline{A}H\,\overline{K},\ \alpha_5 = \overline{H}BK,\ \alpha_6 = \overline{H}\,\overline{K}.$$

For the sake of simplicity we first assume all these α_i's are different from $\bot$, i.e. that $\pi_2 = \{\alpha_1, \ldots, \alpha_6\}$, and let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by π_2. In other words we assume $\mathrm{at}(\mathbf{B}) = \pi_2$. Let us consider any atom $\omega_{i_1 \cdots i_5} = (\alpha_{i_1}|\top) \sqcap (\alpha_{i_2}|\bar{\alpha}_{i_1}) \sqcap \cdots \sqcap (\alpha_{i_5}|\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_4})$ of the conditional algebra $C(\mathbf{B})$, that is a subalgebra of $C(\mathbf{A})$. According to Section 2, in order the relation $\omega_{i_1 \cdots i_5} \leqslant A|H$ be satisfied, in the sequence $(i_1 \cdots i_5)$ the number 1 must appear before the numbers 2, 3, and 4. Then, the sequence must be such that $i_1 = 1$, or $(i_1, i_2) = (5, 1)$, or $(i_1, i_2) = (6, 1)$, $(i_1, i_2, i_3) = (5, 6, 1)$, or $(i_1, i_2, i_3) = (6, 5, 1)$. From Proposition 1, we

[7] Note that, in principle, depending on the logical relations among the events A, H, B and K, some of these α_i's might be $\bot$.

observe that, for instance, the disjunction of all the $\omega_{1i_2i_3i_4i_5}$'s coincides with ω_1 and the same for the other cases. Then,

$$A|H = \omega_1 \sqcup \omega_{51} \sqcup \omega_{61} \sqcup \omega_{561} \sqcup \omega_{651}. \tag{8}$$

In order for the relation $\omega_{i_1\cdots i_5} \leqslant \bar{B}|K$ to be satisfied, in the sequence $(i_1 \cdots i_5)$ the number 3 must appear before the numbers 1, 2, and 5. Then, the sequence must be such that $i_1 = 3$, or $(i_1, i_2) = (4, 3)$, or $(i_1, i_2) = (6, 3)$, or $(i_1, i_2, i_3) = (4, 6, 3)$, or $(i_1, i_2, i_3) = (6, 4, 3)$. Then

$$\bar{B}|K = \omega_3 \sqcup \omega_{43} \sqcup \omega_{63} \sqcup \omega_{463} \sqcup \omega_{643}. \tag{9}$$

By Proposition 1, $v \sqcap w = \perp$, for each $v \in \{\omega_1, \omega_{51}, \omega_{61}, \omega_{561}, \omega_{651}\}$ and $w \in \{\omega_3, \omega_{43}, \omega_{63}, \omega_{463}, \omega_{643}\}$. Then, from (8) and (9) it follows that $(A|H) \sqcap (\bar{B}|K) = \perp$. Therefore

$$(A|H) = ((A|H) \sqcap (B|K)) \sqcup ((A|H) \sqcap (\bar{B}|K)) = (A|H) \sqcap (B|K),$$

that is $A|H \leqslant B|K$. Finally, notice that in the case where $\pi_2 \subset \{\alpha_1, \ldots, \alpha_6\}$, by a similar reasoning we would still obtain that $A|H \leqslant B|K$. $\square$

Remark 1. Notice that formula (7) is also valid in terms of a numerical inequality where the conditional events are replaced by their indicators ([29, Equation (15)]). We also observe that in [30, Theorem 6] it has been proved that the condition

$$AH = \perp\,, \text{ or } \; \bar{B}K = \perp\,, \text{ or } A|H \subseteq B|K \tag{10}$$

on the right side of formula (7) is equivalent to the property that, denoting by Π the set of coherent probability assessments (x, y) on $\{A|H, B|K\}$, it holds that $x \leqslant y$, for every $(x, y) \in \Pi$. Therefore, since every coherent probability assessment (x, y) can be extended to a conditional probability P (see Remark 2), by Theorem 1 it follows that

$$A|H \leqslant B|K \Longleftrightarrow P(A|H) \leqslant P(B|K), \forall\, P. \tag{11}$$

The next result directly follows from Theorem 1 and specifies under which conditions the inequality $\leqslant$ and the inclusion relation $\subseteq$ between two conditional events are equivalent.

Corollary 1. Given any conditional events $A|H$ and $B|K$, with either $AH \neq \perp$ and $\bar{B}K \neq \perp$, or $AH = \bar{B}K = \perp$, then

$$A|H \leqslant B|K \iff A|H \subseteq B|K.$$

It is interesting to remark that, regarding the above Corollary 1, when either $AH = \perp$ and $\bar{B}K \neq \perp$, or $AH \neq \perp$ and $\bar{B}K = \perp$, it holds that $A|H \leqslant B|K$, but it could be that $A|H \nsubseteq B|K$. For instance, if $A = \bar{H}$ and $\bar{H}\bar{B}K \neq \perp$, then $\perp = A|H \leqslant B|K$, but $\bar{B}K \nleqslant \bar{A}H = H$ and hence $A|H \nsubseteq B|K$.

A slightly different (but still equivalent) characterization of the lattice order relation $\leqslant$ among conditional events can be given as follows.

Theorem 2. Given any conditional events $A|H$ and $B|K$, it holds that

$$A|H \leqslant B|K \qquad \iff \qquad AH\bar{B}K = (A|H) \sqcap \bar{H}\bar{B}K = (\bar{B}|K) \sqcap AH\bar{K} = \perp.$$

Proof. ($\Longrightarrow$). The condition $A|H \leqslant B|K$ amounts to $(A|H) \sqcap (\bar{B}|K) = \bot$. Then, by observing that $(A|H) \sqcap H = AH$ and $(\bar{B}|K) \sqcap K = \bar{B}K$, it follows that

$$\begin{aligned}&(A|H) \sqcap (\bar{B}|K) \sqcap HK = AH\bar{B}K = \bot,\\ &(A|H) \sqcap (\bar{B}|K) \sqcap H\bar{K} = AH\bar{K} \sqcap (\bar{B}|K) = \bot, \text{ and}\\ &(A|H) \sqcap (\bar{B}|K) \sqcap \bar{H}K = \bar{H}\,\bar{B}K \sqcap (A|H) = \bot.\end{aligned}$$

($\Longleftarrow$). Assume that $AH\bar{B}K = (A|H) \sqcap \bar{H}\,\bar{B}K = (\bar{B}|K) \sqcap AH\bar{K} = \bot$. Then, the atoms are $\alpha_1, \ldots, \alpha_{k+1}$, with $k \leqslant 7$. Moreover, by recalling that $H \vee K = HK \vee \bar{H}K \vee H\bar{K}$, we obtain

$$(A|H) \sqcap (\bar{B}|K) \sqcap (H \vee K) = (A|H) \sqcap (\bar{B}|K) \sqcap HK \sqcup (A|H) \sqcap (\bar{B}|K) \sqcap H\bar{K} \sqcup (A|H) \sqcap (\bar{B}|K) \sqcap \bar{H}K = \bot.$$

Then,

$$(A|H) \sqcap (\bar{B}|K) = [(A|H) \sqcap (\bar{B}|K) \sqcap (H \vee K)] \sqcup [(A|H) \sqcap (\bar{B}|K) \sqcap \bar{H}\,\bar{K}] = (A|H) \sqcap (\bar{B}|K) \sqcap \bar{H}\,\bar{K}$$

and it can be proved that $(A|H) \sqcap (\bar{B}|K) \sqcap \bar{H}\,\bar{K} = \bot$. Indeed, if there would exist $\omega_{i_1 i_2 \cdots i_k} \leqslant (A|H) \sqcap (\bar{B}|K) \sqcap \bar{H}\,\bar{K}$, then it would be $\alpha_{i_1} = \bar{H}\,\bar{K}$ and $\alpha_{i_j} \neq \bot$, for $j = 2, \ldots, k$. Then $\bot \neq \omega_{i_2 \cdots i_k i_1} \leqslant (A|H) \sqcap (\bar{B}|K) \sqcap (H \vee K)$, which is absurd. Therefore $(A|H) \sqcap (\bar{B}|K) \sqcap (\bar{H}\bar{K}) = \bot$ and hence $(A|H) \sqcap (\bar{B}|K) = \bot$, that is $A|H \leqslant B|K$. □

Notice that Theorem 1 also follows as a corollary from Theorem 2. Indeed, if $A|H \subseteq B|K$, then $AH\bar{B}K = AH\bar{K} = \bar{H}\,\bar{B}K = \bot$, and hence $AH\bar{B}K = (A|H) \sqcap \bar{H}\,\bar{B}K = (\bar{B}|K) \sqcap AH\bar{K} = \bot$.

4. Canonical extension of a conditional probability

In the definition of the canonical extension μ_P on $\mathcal{C}(\mathbf{A})$ in [22] that we recalled in Section 2, a crucial assumption is that P is positive, i.e. that $P(\alpha) > 0$ for every $\alpha \in \mathrm{at}(\mathbf{A})$, otherwise $\mu_P(\omega)$ will be undefined for some $\omega \in \mathrm{at}(\mathcal{C}(\mathbf{A}))$ (it would be of the form 0/0). A way to overcome this limitation is, instead of starting with a positive (unconditional) probability on $\mathbf{A}$, to directly start with a *conditional probability* on $\mathbb{A} \times \mathbb{A}'$ in the axiomatic sense, that is to say, with a binary map $P : \mathbb{A} \times \mathbb{A}' \to [0, 1]$, where $\mathbb{A}' = \mathbb{A} \backslash \{\bot\}$, such that

(CP1) For all $B \in \mathbb{A}'$, $P(\cdot|B) : \mathbb{A} \to [0, 1]$ is a finitely additive probability on $\mathbf{A}$;

(CP2) For all $A \in \mathbb{A}$ and $B \in \mathbb{A}'$, $P(A|B) = P(A \wedge B|B)$;

(CP3) For all $A \in \mathbb{A}$, $B, C \in \mathbb{A}'$, if $A \leqslant B \leqslant C$, then $P(A|C) = P(A|B) \cdot P(B|C)$.

As usual, we will also denote $P(A|\top)$ simply by $P(A)$, for every $A \in \mathbb{A}$.

Remark 2. As has already been mentioned above, differently from the approach in [22], we do not assume here the positivity of the (conditional) probability P. Then, the function P may be such that $P(A|B) = 0$ and/or $P(B) = 0$ for some $A \in \mathbb{A}$ and $B \in \mathbb{A}'$. Moreover, we recall that, requesting $P : \mathbb{A} \times \mathbb{A}' \to [0, 1]$ to satisfy the above three postulates, assures that P is a coherent conditional probability assessment in the sense of de Finetti to all the conditional objects $(A|B)$, with $A, B \in \mathbb{A}$ and $B \neq \bot$. In fact, a conditional probability assessment on an arbitrary family of (basic) conditional events $P(A_1|B_1) = x_1, \ldots, P(A_n|B_n) = x_n$, is coherent iff it can be extended to a full conditional probability (in the above sense) on $\mathbb{A} \times \mathbb{A}'$ (see, e.g., [31]). In this paper, instead of starting from a (coherent) probability assessment on an arbitrary family of conditional events, we directly start with a full conditional probability P defined on $\mathbb{A} \times \mathbb{A}'$.

Then, given a conditional probability $P : \mathbb{A} \times \mathbb{A}' \to [0, 1]$, we can proceed as in the previous section to define a (unconditional) probability μ_P in $C(\mathbf{A})$.

Definition 3. For any conditional probability $P : \mathbb{A} \times \mathbb{A}' \to [0, 1]$, we define a mapping $\mu_P : \mathrm{at}(C(\mathbf{A})) \to [0, 1]$ as follows: for any atom $\omega = (\alpha_1|\top) \sqcap (\alpha_2|\bar{\alpha}_1) \sqcap \ldots \sqcap (\alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2})$,

$$\mu_P(\omega) = P(\alpha_1|\top) \cdot P(\alpha_2|\bar{\alpha}_1) \cdot \ldots \cdot P(\alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2}). \tag{12}$$

Of course, $\mu_P(\omega) = 0$ if either $P(\alpha_1|\top) = 0$, or $P(\alpha_2|\bar{\alpha}_1) = 0$, ..., or $P(\alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2}) = 0$. That is, differently from Section 2, as the positivity property has been lifted, it may be that $\mu_P(\omega) = 0$ for some $\omega \in \mathrm{at}(C(\mathbf{A}))$.

One can check that μ_P so defined is a probability distribution on $\mathrm{at}(C(\mathbf{A}))$.

Proposition 3. $\sum_{\omega \in \mathrm{at}(C(\mathbf{A}))} \mu_P(\omega) = 1$.

Proof. Although one could adapt here the proof of [22, Lemma 6.8], we provide below a direct proof. Let $\mathrm{at}(\mathbf{A}) = \{\alpha_1, \ldots, \alpha_n\}$. Clearly, $\sum_{\alpha \in \mathrm{at}(\mathbf{A})} P(\alpha|\top) = 1$; moreover, we have

$$P(\alpha|\top) = P(\alpha|\top) \cdot P(\bar{\alpha}|\bar{\alpha}) = P(\alpha|\top) \cdot \sum_{\beta \neq \alpha} P(\beta|\bar{\alpha}) = \sum_{\beta \neq \alpha} P(\alpha|\top) \cdot P(\beta|\bar{\alpha}),$$

and thus

$$1 = \sum_{\alpha} P(\alpha|\top) = \sum_{\alpha} \sum_{\beta \neq \alpha} P(\alpha|\top) \cdot P(\beta|\bar{\alpha}).$$

Now, if we consider sets of three atoms of $\mathbf{A}$ we get

$$\begin{aligned} 1 &= \sum_{\alpha} P(\alpha|\top) \cdot \sum_{\beta \neq \alpha} P(\beta|\bar{\alpha}) = \sum_{\alpha} P(\alpha|\top) \cdot \sum_{\beta \neq \alpha} P(\beta|\bar{\alpha}) \cdot P(\bar{\alpha}\bar{\beta}|\bar{\alpha}\bar{\beta}) = \\ &= \sum_{\alpha} P(\alpha|\top) \cdot \sum_{\beta \neq \alpha} P(\beta|\bar{\alpha}) \cdot \sum_{\gamma \notin \{\alpha,\beta\}} P(\gamma|\bar{\alpha}\bar{\beta}) = \sum_{\alpha} \sum_{\beta \notin \{\alpha\}} \sum_{\gamma \notin \{\alpha,\beta\}} P(\alpha|\top) \cdot P(\beta|\bar{\alpha}) \cdot P(\gamma|\bar{\alpha}\bar{\beta}). \end{aligned}$$

Iterating this procedure for sets of $n-1$ atoms of $\mathbf{A}$ we finally get:

$$\begin{aligned} 1 &= \sum_{\beta_1} \sum_{\beta_2 \notin \{\beta_1\}} \ldots \sum_{\beta_{n-1} \notin \{\beta_1, \ldots, \beta_{n-2}\}} P(\beta_1|\top) \cdot P(\beta_2|\bar{\beta}_1) \cdot \ldots \cdot P(\beta_{n-1}|\bar{\beta}_1 \ldots \bar{\beta}_{n-2}) = \\ &= \sum_{\langle \beta_1, \ldots, \beta_{n-1} \rangle \in \mathrm{at}(C(\mathbf{A}))} P(\beta_1|\top) \cdot P(\beta_2|\bar{\beta}_1) \cdot \ldots \cdot P(\beta_{n-1}|\bar{\beta}_1 \ldots \bar{\beta}_{n-2})) = \sum_{\omega \in \mathrm{at}(C(\mathbf{A}))} \mu_P(\omega). \end{aligned}$$

□

Then, we can extend μ_P to a probability on the whole algebra $C(\mathbf{A})$ in the usual way by additivity, as in the previous case: for any $t \in C(\mathbf{A})$, $\mu_P(t) = \sum_{\omega \leqslant t} \mu_P(\omega)$. We will keep referring to μ_P as the *canonical extension* of P.

We now check that Equation (6) keeps holding in this more general setting. Indeed, concerning the canonical extension on the conjunctions $\omega_{i_1 \cdots i_k}$'s, we first observe that, as $\omega_{1 \cdots n-2\, n-1} \sqcup \omega_{1 \cdots n-2\, n} = \omega_{1 \cdots n-2}$, from (12) it holds that:

$$\begin{aligned} \mu_P(\omega_{1 \cdots n-2}) &= \mu_P(\omega_{1 \cdots n-2\, n-1}) + \mu_P(\omega_{1 \cdots n-2\, n}) = \\ &= P(\alpha_1)P(\alpha_2|\bar{\alpha}_1) \cdots P(\alpha_{n-2}|\bar{\alpha}_1 \ldots \bar{\alpha}_{n-3})[P(\alpha_{n-1}|(\alpha_{n-1} \vee \alpha_n)) + P(\alpha_n|(\alpha_{n-1} \vee \alpha_n))] = \\ &= P(\alpha_1)P(\alpha_2|\bar{\alpha}_1) \cdots P(\alpha_{n-2}|\bar{\alpha}_1 \ldots \bar{\alpha}_{n-3}). \end{aligned}$$

Likewise $\mu_P(\omega_{i_1\cdots i_{n-2}}) = P(\alpha_{i_1})P(\alpha_{i_2}|\bar{\alpha}_{i_1})\cdots P(\alpha_{i_{n-2}}|\bar{\alpha}_{i_1}\cdots\bar{\alpha}_{i_{n-3}})$. Then, by backward iteration, for each $k \leqslant n-1$, it holds that

$$\mu_P(\omega_{i_1\cdots i_k}) = P(\alpha_{i_1})P(\alpha_{i_2}|\bar{\alpha}_{i_1})\cdots P(\alpha_{i_k}|\bar{\alpha}_{i_1}\ldots\bar{\alpha}_{i_{k-1}}). \tag{13}$$

In particular

$$\mu_P(\omega_i) = P(\alpha_i), \quad i = 1, \ldots, n. \tag{14}$$

The question of whether μ_P actually extends P, in the sense that, for any basic conditional $(A|B) \in C(\mathbf{A})$, it holds $\mu_P(\text{“}(A|B)\text{”}) = P(A|B)$ is deferred to Theorem 5 in the next section. From now on, we will simply use the notation $\mu_P(A|B)$ instead of $\mu_P(\text{“}(A|B)\text{”})$ without danger of confusion.

5. The canonical extension and subalgebras, Stalnaker's thesis, coherence and convexity

In this section we first prove two basic properties of the canonical extensions of conditional probabilities on an algebra of events $\mathbf{A}$ to the algebra of conditionals $C(\mathbf{A})$, namely their compatibility with taking subalgebras, and based on this, that on basic conditionals they agree with the initial conditional probability. Then we show that $\{0,1\}$-valued probabilities on $C(\mathbf{A})$ are in fact always canonical extensions, and as a consequence it follows that the set of canonical extensions on $C(\mathbf{A})$ is not a convex set of probabilities.

5.1. The canonical extension and subalgebras

Given the canonical extension μ_P to $C(\mathbf{A})$ of a conditional probability P on $C(\mathbf{A})$, and given a subalgebra $\mathbf{B}$ of $\mathbf{A}$, in this subsection we first examine the restriction of μ_P to the conditional subalgebra $C(\mathbf{B})$ of $C(\mathbf{A})$, and we show that such restriction coincides with the canonical extension of the restriction of P to $\mathbf{B}$. Then we use this result to show that μ_P is such that, for every basic conditional $(A|B) \in C(\mathbf{A})$, $\mu_P(A|B) = P(A|B)$. This in fact can be regarded as a slight generalisation of the Stalnaker thesis mentioned in Section 2 as in this case the antecedent need not have strictly positive probability.

To start with, let $\mathbf{A}$ be a finite Boolean algebra with $\mathrm{at}(\mathbf{A}) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$. If $\beta_1, \ldots, \beta_k$ are the atoms of $\mathbf{B}$, it means that the set of atoms of $\mathbf{A}$ can be partitioned in non-empty subsets $A_1, \ldots, A_k$ such that for all $j = 1, \ldots, k$, $\beta_j = \bigvee_{\alpha \in A_j} \alpha$. We will first consider the following particular case of a subalgebra $\mathbf{B}$ of $\mathbf{A}$: take an index $i < n$, and let $\mathbf{B}$ the subalgebra of $\mathbf{A}$ generated by $\alpha_1, \ldots, \alpha_{i-1}, \alpha_i \vee \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n$. In other words, for $j = 1, \ldots, n-1$, let

$$\beta_j = \begin{cases} \alpha_j, & \text{if } j < i \\ \alpha_i \vee \alpha_{i+1}, & \text{if } j = i \\ \alpha_{j+1}, & \text{if } j > i+1. \end{cases} \tag{15}$$

Then $\mathbf{B}$ is the subalgebra of $\mathbf{A}$ generated by $\beta_1, \ldots, \beta_{n-1}$ and $\mathrm{at}(\mathbf{B}) = \{\beta_1, \ldots, \beta_{n-1}\}$.

Now let us consider $P : \mathbb{A} \times \mathbb{A}' \to [0,1]$ a conditional probability and $\mu_P : C(\mathbf{A}) \to [0,1]$ its canonical extension to $C(\mathbf{A})$. Further, let $P' : \mathbb{B} \times \mathbb{B}' \to [0,1]$ be the restriction of P to $\mathbb{B} \times \mathbb{B}'$, and let $\mu_{P'} : C(\mathbf{B}) \to [0,1]$ be its canonical extension to $C(\mathbf{B})$. The question of interest is whether $\mu_{P'}$ is in fact the restriction of μ_P to $C(\mathbf{B})$. Next theorem shows this is actually the case. Indeed, given any permutation $(j_1, \ldots, j_{n-1})$ of $(1, \ldots, n-1)$, we let

$$\omega'_{j_1\cdots j_{n-2}} = (\beta_{j_1}|\top) \sqcap (\beta_{j_2}|\bar{\beta}_{j_1}) \sqcap \cdots \sqcap (\beta_{j_{n-2}}|\bar{\beta}_{j_1}\cdots\bar{\beta}_{j_{n-3}}) \in \mathrm{at}(C(\mathbf{B}))$$

and we recall that, by definition of the canonical extension of $\mu_{P'}$,

$$\mu_{P'}(\omega'_{j_1\cdots j_{n-2}}) = P(\beta_{j_1}|\top)P(\beta_{j_2}|\bar{\beta}_{j_1})\cdot\cdots\cdot P(\beta_{j_{n-2}}|\bar{\beta}_{j_1}\cdots\bar{\beta}_{j_{n-3}}).$$

In the next result we show that $\mu_P(\omega'_{j_1\cdots j_{n-2}}) = \mu_{P'}(\omega'_{j_1\cdots j_{n-2}})$. But first we state a preliminary remark that will be useful in the proof.

Remark 3. For any events A, B, and C in an algebra $\mathbf{A}$ with $A \leqslant B \leqslant C$ and $B \neq \bot$, and any conditional probability $P : \mathbb{A} \times \mathbb{A}' \to [0, 1]$, it holds that

$$P(A|B) = P(\bar{B}|C)P(A|B) + P(A|C). \tag{16}$$

Indeed, when $A \leqslant B \leqslant C$, from (CP3) one has $P(A|C) = P(A|B)P(B|C)$. Then one also has: $P(A|B) = P(A|B)P(\bar{B}|C) + P(A|B)P(B|C) = P(\bar{B}|C)P(A|B) + P(A|C)$.

Theorem 3. For each atom $\omega'_{j_1\cdots j_{n-2}} \in \mathrm{at}(C(\mathbf{B}))$, the following holds:

$$\begin{aligned} \mu_P(\omega'_{j_1\cdots j_{n-2}}) &= \mu_P((\beta_{j_1}|\top) \sqcap (\beta_{j_2}|\bar{\beta}_{j_1}) \sqcap \cdots \sqcap (\beta_{j_{n-2}}|\bar{\beta}_{j_1}\cdots\bar{\beta}_{j_{n-3}})) = \\ &= P(\beta_{j_1}|\top)P(\beta_{j_2}|\bar{\beta}_{j_1})\cdots P(\beta_{j_{n-2}}|\bar{\beta}_{j_1}\cdots\bar{\beta}_{j_{n-3}}) = \\ &= \mu_{P'}(\omega'_{j_1\cdots j_{n-2}}). \end{aligned} \tag{17}$$

Proof. Due to its length, and to easy the reading of the paper, the proof has been moved to Appendix A. □

As an illustration of Theorem 3, we examine the case of a simple example for $n = 4$.

Example 2. Let $\mathbf{A}$ be an algebra with four atoms $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Now let us consider the partition defined by the elements $\beta_1 = \alpha_1, \beta_2 = \alpha_2$ and $\beta_3 = \alpha_3 \vee \alpha_4$, and let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by these three elements so that $\{\beta_1, \beta_2, \beta_3\}$ become the atoms of $\mathbf{B}$. As above, let P be a conditional probability on $\mathbb{A} \times \mathbb{A}'$, and let P' be its restriction to $\mathbb{B} \times \mathbb{B}'$. According to Theorem 3, let us practically show that $\mu_{P'}$ is the restriction of μ_P on $C(\mathbf{B})$. We have to show that, for any pairwise different $i, j \in \{1, 2, 3\}$, the following condition holds:

$$\mu_P(\omega'_{ij}) = \mu_P((\beta_i|\top) \sqcap (\beta_j|\bar{\beta}_i)) = P(\beta_i) \cdot P(\beta_j|\bar{\beta}_i) = \mu_{P'}((\beta_i|\top) \sqcap (\beta_j|\bar{\beta}_i)) = \mu_{P'}(\omega'_{ij}).$$

The cases $(\beta_i|\top) \sqcap (\beta_3|\bar{\beta}_i)$ with $i \in \{1, 2\}$ can be easily verified by exploiting (13). Let us consider the case $(\beta_3|\top) \sqcap (\beta_1|\bar{\beta}_3)$, the other case $(\beta_3|\top) \sqcap (\beta_2|\bar{\beta}_3)$ is analogous. We have to compute

$$\mu_P((\beta_3|\top) \sqcap (\beta_1|\bar{\beta}_3)) = \mu_P((\alpha_3 \vee \alpha_4|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2)).$$

We observe that

$$(\alpha_3 \vee \alpha_4|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2) = (\alpha_3|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2) \sqcup (\alpha_4|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2).$$

In particular, concerning $(\alpha_3|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2)$, by applying the distributivity property and (C5), it holds that

$$\begin{aligned} &(\alpha_3|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2) = (\alpha_3|\top) \sqcap (\alpha_1 \vee \alpha_2 \vee \alpha_4|\alpha_1 \vee \alpha_2 \vee \alpha_4) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2) = \\ &= (\alpha_3|\top) \sqcap (\alpha_1 \vee \alpha_2|\alpha_1 \vee \alpha_2 \vee \alpha_4) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2) \sqcup (\alpha_3|\top) \sqcap (\alpha_4|\alpha_1 \vee \alpha_2 \vee \alpha_4) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2) = \\ &= (\alpha_3|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2 \vee \alpha_4) \sqcup (\alpha_3|\top) \sqcap (\alpha_4|\alpha_1 \vee \alpha_2 \vee \alpha_4) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2) = \omega_{31} \sqcup \omega_{341}. \end{aligned}$$

Then, by applying (16), with $A = \alpha_1$, $B = \alpha_1 \vee \alpha_2$, and $C = \alpha_1 \vee \alpha_2 \vee \alpha_4$, it holds that

$$P(\alpha_1|\alpha_1 \vee \alpha_2 \vee \alpha_4) + P(\alpha_4|\alpha_1 \vee \alpha_2 \vee \alpha_4)P(\alpha_1|\alpha_1 \vee \alpha_2) = P(\alpha_1|\alpha_1 \vee \alpha_2),$$

and hence

$$\mu_P[(\alpha_3|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2)] = \mu_P(\omega_{31} \sqcup \omega_{341}) = \mu_P(\omega_{31}) + \mu_P(\omega_{341}) = \\ = P(\alpha_3|\top)[P(\alpha_1|\alpha_1 \vee \alpha_2 \vee \alpha_4) + P(\alpha_4|\alpha_1 \vee \alpha_2 \vee \alpha_4)P(\alpha_1|\alpha_1 \vee \alpha_2)] = P(\alpha_3|\top)P(\alpha_1|\alpha_1 \vee \alpha_2).$$

Analogously, it holds that $(\alpha_4|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2) = \omega_{41} \sqcup \omega_{431}$ and hence

$$\mu_P[(\alpha_4|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2)] = P(\alpha_4|\top)P(\alpha_1|\alpha_1 \vee \alpha_2).$$

Thus $(\beta_3|\top) \sqcap (\beta_1|\bar{\beta}_3) = \omega_{31} \sqcup \omega_{341} \sqcup \omega_{41} \sqcup \omega_{431}$ and hence

$$\mu_P((\beta_3|\top) \sqcap (\beta_1|\bar{\beta}_3)) = [P(\alpha_3|\top) + P(\alpha_4|\top)]P(\alpha_1|\alpha_1 \vee \alpha_2) = P(\alpha_3 \vee \alpha_4|\top)P(\alpha_1|\alpha_1 \vee \alpha_2) = \\ = P(\beta_3|\top)P(\beta_1|\bar{\beta}_3) = \mu_{P'}((\beta_3|\top) \sqcap (\beta_1|\bar{\beta}_3)).$$

Notice that, by suitably reordering the subscripts, the result of Theorem 3, still holds for the case where $\beta_i = \alpha_i \vee \alpha_t$, with $t > i + 1$. More in general, for each conditional subalgebra $C(\mathbf{B})$ of $C(\mathbf{A})$, by a suitable iterated application of Theorem 3 it can be verified that (17) is satisfied. This yields the following result.

Theorem 4. Let **A** be a finite Boolean algebra. For any subalgebra **B** of **A**, and any conditional probability $P : \mathbb{A} \times \mathbb{A}' \to [0, 1]$, let P' be its restriction on $\mathbb{B} \times \mathbb{B}'$. Then,

(i) for every atom $\omega' \in \mathrm{at}(C(\mathbf{B}))$ it holds that $\mu_P(\omega') = \mu_{P'}(\omega')$, and hence,

(ii) for each $C \in C(\mathbf{B})$ it also holds that $\mu_P(C) = \mu_{P'}(C)$.

Proof. Indeed, as already observed above, the first item (i) can be proved by an iterated application of Theorem 3. Then, (ii) follows by observing that $C = \bigsqcup_{\omega' \leqslant C} \omega'$, and by (i) it follows that $\mu_P(C) = \sum_{\omega' \leqslant C} \mu_P(\omega') = \sum_{\omega' \leqslant C} \mu_{P'}(\omega') = \mu_{P'}(C)$. □

To exemplify the iterative procedure to prove (i) in the above theorem, let us consider the following simple example.

Example 3. Let us consider the partition π of an algebra $\mathbf{A}^\circ$ associated with a family of two conditional events $\{A|H, B|K\}$ from the conditional algebra $C(\mathbf{A}^\circ)$ and further consider the partition associated with the sub-family $\{B|K\}$, which we denote by π' defined as follows:

$$\pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9\}, \qquad \pi' = \{\beta_1, \beta_2, \beta_3\},$$

where

$$\alpha_1 = AHBK\,,\ \alpha_2 = AH\bar{B}K\,,\ \alpha_3 = AH\bar{K}\,,\ \alpha_4 = \bar{A}HBK\,,$$
$$\alpha_5 = \bar{A}H\bar{B}K\,,\ \alpha_6 = \bar{A}H\bar{K}\,,\ \alpha_7 = \bar{H}BK\,,\ \alpha_8 = \bar{H}\bar{B}K\,,\ \alpha_9 = \bar{H}\bar{K}\,,$$

and

$$\beta_1 = BK = \alpha_1 \vee \alpha_4 \vee \alpha_7\,,\ \beta_2 = \bar{B}K = \alpha_2 \vee \alpha_5 \vee \alpha_8\,,\ \beta_3 = \bar{K} = \alpha_3 \vee \alpha_6 \vee \alpha_9\,.$$

Let us denote by **A** the subalgebra of $\mathbf{A}^\circ$ generated by π, that is, where $\mathrm{at}(\mathbf{A}) = \pi$, and by **B** the subalgebra of **A** (and of $\mathbf{A}^\circ$) generated by π', that is, where $\mathrm{at}(\mathbf{B}) = \pi'$.

Moreover, let us introduce the following 'intermediate' partitions:

$$\pi^{(1)} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \vee \alpha_9, \alpha_7, \alpha_8\}\,,\ \ \pi^{(2)} = \{\alpha_1, \alpha_2, \alpha_3 \vee \alpha_6 \vee \alpha_9, \alpha_4, \alpha_5, \alpha_7, \alpha_8\}\,,$$

$$\pi^{(3)} = \{\alpha_1, \alpha_2, \alpha_3 \vee \alpha_6 \vee \alpha_9, \alpha_4, \alpha_5 \vee \alpha_8, \alpha_7\}\,, \quad \pi^{(4)} = \{\alpha_1, \alpha_2 \vee \alpha_5 \vee \alpha_8, \alpha_3 \vee \alpha_6 \vee \alpha_9, \alpha_4, \alpha_7\}\,,$$

$$\pi^{(5)} = \{\alpha_1, \alpha_2 \vee \alpha_5 \vee \alpha_8, \alpha_3 \vee \alpha_6 \vee \alpha_9, \alpha_4 \vee \alpha_7\}\,, \quad \pi^{(6)} = \{\alpha_1 \vee \alpha_4 \vee \alpha_7, \alpha_2 \vee \alpha_5 \vee \alpha_8, \alpha_3 \vee \alpha_6 \vee \alpha_9\}\,,$$

where $\pi^{(6)} = \pi' = \{\beta_1, \beta_2, \beta_3\}$. Note that each partition $\pi^{(k)}$ is the result of "fusing" two atoms in the previous partition $\pi^{(k-1)}$. Then, by iteratively applying the result of Theorem 3, it follows that the property (i) holds on $\pi^{(k)}$, $k = 1, 2, \ldots, 5$, and finally on $\pi^{(6)} = \pi' = \{\beta_1, \beta_2, \beta_3\}$. Therefore, we get that if P is a conditional probability on $\mathbb{A} \times \mathbb{A}'$ and P' its restriction on $\mathbb{B} \times \mathbb{B}'$, we have that

$$\mu_P(\omega') = \mu_{P'}(\omega')\,,$$

for every atom $\omega' \in \mathrm{at}(C(\mathbf{B}))$.

Theorem 4 shows that the restriction of the canonical extension μ_P on the conditional algebra $C(\mathbf{A})$ to the conditional subalgebra $C(\mathbf{B})$ coincides with the canonical extension $\mu_{P'}$ on the conditional subalgebra $C(\mathbf{B})$, see the commutative diagram in Figure 2.

This result enables a local approach in order to study properties of basic and compound conditionals, as

$$\begin{tikzcd}
\mathbf{A}, P : \mathbb{A} \times \mathbb{A}' \to [0,1] \arrow[r, Rightarrow, "C(\cdot),\, \mu_{(\cdot)}", "\text{Canonical extension}"'] \arrow[d, dotted, "\mathbf{B} \subseteq \mathbf{A}"', "\text{Restriction}"] & C(\mathbf{A}), \mu_P : C(\mathbf{A}) \to [0,1] \arrow[d, dotted, "C(\mathbf{B}) \subseteq C(\mathbf{A})"', "\text{Restriction}"] \\
\mathbf{B}, P' : \mathbb{B} \times \mathbb{B}' \to [0,1] \arrow[r, Rightarrow, "C(\cdot),\, \mu_{(\cdot)}", "\text{Canonical extension}"'] & C(\mathbf{B}), \mu_{P'} : C(\mathbf{B}) \to [0,1]
\end{tikzcd}$$

Figure 2: Compatibility of the canonical extension construction with respect to taking subalgebras. Notice that in the above diagram, $\mu_{P'} = \mu_P \upharpoonright_{C(\mathbf{B})}$. That is to say, the canonical extension $\mu_{P'}$ of the conditional probability P' obtained by restricting P to $\mathbb{B} \times \mathbb{B}'$ is the restriction of the canonical extension μ_P to the subalgebra $C(\mathbf{B})$ of $C(\mathbf{A})$.

done in the next section.

Actually, the above Theorem 4 is very powerful, because when dealing with a set of conditional events $\mathcal{F} = \{(A_i|H_i)\}_{i\in I}$ from $C(\mathbf{A})$ and their probabilities, one needs not resort to probabilities defined on the whole algebra $C(\mathbf{A})$ (and thus specified on all the atoms of $C(\mathbf{A})$) but only on a relevant subalgebra of $C(\mathbf{A})$. Indeed, it is enough to consider the conditional subalgebra $C(\mathbf{B})$ where $\mathbf{B}$ is the subalgebra of $\mathbf{A}$ generated by a suitable partition determined by the family $\mathcal{F}$ of conditional events along the lines studied in [26, Sec. 2.1].

In the next result we extend a main result of [22] in the following sense. In [22, Theorem 6.13] the authors show that if P is a positive probability on $\mathbf{A}$, then its canonical extension on $C(\mathbf{A})$ is such that $\mu_P(A|H) = P(AH)/P(H)$. That is, for positive probabilities, conditional probability can be understood as a simple probability over conditionals, a result that it is related to the well-known *Stalnaker's thesis*. Thanks to Theorem 4, here we show that this still holds if we start with a conditional probability on $\mathbb{A} \times \mathbb{A}'$.

Theorem 5. Let P be a conditional probability on $\mathbb{A} \times \mathbb{A}'$ and μ_P its canonical extension to $C(\mathbf{A})$. Then, for every basic conditional $(A|H) \in C(\mathbf{A})$, it holds that $\mu_P(A|H) = P(A|H)$.

Proof. Given any $(A|H) \in \mathcal{C}(\mathbf{A})$, let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by the partition $\{\beta_1, \beta_2, \beta_3\} = \{AH, \bar{A}H, \bar{H}\}$. Let $P' : \mathbb{B} \times \mathbb{B}' \to [0,1]$ be the restriction of P to $\mathbb{B} \times \mathbb{B}'$, and let $\mu_{P'} : \mathcal{C}(\mathbf{B}) \to [0,1]$ be its canonical extension to $\mathcal{C}(\mathbf{B})$. Of course $P'(A|H) = P(A|H)$. From (C4) and (C5), it holds that $(A|H) \sqcap H = (AH|H) \sqcap H = AH$. Then,

$$(A|H) = ((A|H) \sqcap H) \sqcup ((A|H) \sqcap \bar{H}) = AH \sqcup ((A|H) \sqcap \bar{H}) = \omega'_1 \sqcup \omega'_{31}, \tag{18}$$

where $\omega'_1 = \beta_1 = AH$ and $\omega'_{31} = \beta_3 \sqcap (\beta_1|\bar{\beta}_3) = \bar{H} \sqcap (A|H)$. Therefore, from (14) and Theorem 4

$$\begin{aligned}\mu_P(A|H) &= \mu_{P'}(A|H) = \mu_{P'}(\omega'_1) + \mu_{P'}(\omega'_{31}) = P(\beta_1) + P(\beta_3)P(\beta_1|\bar{\beta}_3) = \\ &= P(AH) + P(\bar{H})P(A|H) = P(H)P(A|H) + P(\bar{H})P(A|H) = P(A|H).\end{aligned}$$

□

A direct consequence of the previous result is that, given two conditional probabilities P and P', with $P \neq P'$, it follows that $\mu_P \neq \mu_{P'}$. Indeed, from Theorem 5, if $\mu_P = \mu_{P'}$, then $P = P'$ because

$$P(A|H) = \mu_P(A|H) = \mu_{P'}(A|H) = P'(A|H)\,,\ \forall\, A|H \in \mathcal{C}(\mathbf{A})\,.$$

Remark 4. In light of Theorem 5, the result recalled in Remark 1 can be further extended so to involve also the canonical extensions of conditional probabilities. Indeed, we can now characterize the order $\leqslant$ between conditional events in the following way: for every conditional events $A|H$ and $B|K$, it holds that

$$A|H \leqslant B|K \Longleftrightarrow \mu_P(A|H) \leqslant \mu_P(B|K)\ \forall P.$$

Remark 5. Given three events A, B, C, with $A \leqslant B \leqslant C$, by (CP3) it holds that $P(A|C) = P(A|B)P(B|C)$. Moreover, by recalling $(C5)$, we observe that

$$(A|B) = [(A|B) \sqcap (B|C)] \sqcup [(A|B) \sqcap (\bar{B}|C)] = (A|C) \sqcup [(A|B) \sqcap (\bar{B}|C)],$$

and hence, from (16) and Theorem 5,

$$P(A|B) = P(A|C) + P(A|B)P(\bar{B}|C) = P(A|C) + \mu_P[(A|B) \sqcap (\bar{B}|C)]. \tag{19}$$

Thus,

$$\mu_P[(A|B) \sqcap (\bar{B}|C)] = P(A|B)P(\bar{B}|C). \tag{20}$$

As we can see, (20) shows that the "independence" between $A|B$ and $B|C$, when $A \leqslant B \leqslant C$, still holds between $A|B$ and $\bar{B}|C$. In particular, given any events E and H, by applying (20) with $A = EH, B = H$,and $C = \top$, as $\bar{H}|\top = \bar{H}$, we obtain

$$\mu_P((E|H) \sqcap \bar{H}) = P(E|H)P(\bar{H})\,. \tag{21}$$

Formula (21) will be generalized in Theorem 6, where $\bar{H}$ is replaced by any K such that $HK = \bot$.

5.2. On the canonical extensions of $\{0,1\}$*-valued conditional probabilities*

It is very well-known that homomorphisms from a Boolean algebra **A** into the two-element Boolean algebra $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$ are in fact the $\{0,1\}$-valued probabilities on the algebra **A**. We now show that the homomorphisms from $C(\mathbf{A})$ into to **2** are the canonical extensions of $\{0,1\}$-valued conditional probabilities on $\mathbb{A} \times \mathbb{A}'$. Specifically, we prove that for each $\omega \in \mathrm{at}(C(\mathbf{A}))$ the map $\mu : C(\mathbf{A}) \to \{0,1\}$, such that $\mu(s) = 1$, if $\omega \leqslant s$, and $\mu(s) = 0$, otherwise, is a canonical extension of a suitable conditional probability on $\mathbb{A} \times \mathbb{A}'$.

Lemma 1. For any atom $\omega \in \mathrm{at}(C(\mathbf{A}))$, there is a conditional probability $P_\omega : \mathbb{A} \times \mathbb{A}' \to [0,1]$ whose canonical extension μ_{P_ω} is such that $\mu_{P_\omega}(\omega) = 1$ and $\mu_{P_\omega}(\omega') = 0$ for any atom $\omega' \neq \omega$.

Proof. Assume $\omega = \langle \alpha_1, \ldots, \alpha_n \rangle$. For any event $B \in \mathbf{A}$, let $\mathrm{at}_\leqslant(B)$ be the set of atoms of **A** below B, and let $\min(B) = \min\{i \in \{1,...,n\} | \alpha_i \in \mathrm{at}_\leqslant(B)\}$. Then define $P_\omega : \mathbb{A} \times \mathbb{A}' \to [0,1]$ as follows:

$$P_\omega(A|B) = \begin{cases} 1, & \text{if } \alpha_j \leqslant A, \text{where } j = \min(B) \\ 0, & \text{otherwise} \end{cases}$$

Notice that, from the definition, it directly follows that $P_\omega(A|B) = 1$ if $\omega \leqslant (A|B)$ in $C(\mathbf{A})$, and $P_\omega(A|B) = 0$ otherwise. Moreover P_ω is indeed a $\{0,1\}$-valued mapping.

It is not difficult to check that, so defined, P_ω is a conditional probability:

(CP1) We have to show that for all $B \in \mathbb{A}'$, $P_\omega(\cdot|B) : \mathbb{A} \to [0,1]$ is a finitely additive probability on **A**. Indeed, it is clear from the definition that $P_\omega(\top|B) = 1$ and $P_\omega(\bot|B) = 0$. As for the additivity, assume $A \wedge C = \bot$. Then $P_\omega(A \vee C|B) = 1$ iff $\omega \leqslant (A \vee C|B) = (A|B) \sqcup (C|B)$ iff $\omega \leqslant (A|B)$ or $\omega \leqslant (C|B)$ but not both, that is, iff $P_\omega(A|B) = 1$ and $P_\omega(C|B) = 0$, or $P_\omega(C|B) = 1$ and $P_\omega(A|B) = 0$. Therefore, $P_\omega(A \vee C|B) = 1$ iff $P_\omega(A|B) + P_\omega(C|B) = 1$.

(CP2) For all $A \in \mathbb{A}$ and $B \in \mathbb{A}'$, we have to show that $P_\omega(A|B) = P_\omega(A \wedge B|B)$. As observed above, $P_\omega(A|B) = 1$ iff $\omega \leqslant (A|B)$ iff $\omega \leqslant (A \wedge B|B)$ iff $P_\omega(A \wedge B|B) = 1$.

(CP3) Finally, we show that for all $A \in \mathbb{A}$, $B, C \in \mathbb{A}'$, if $A \leqslant B \leqslant C$, then $P_\omega(A|C) = P_\omega(A|B) \cdot P_\omega(B|C)$. We have that $P_\omega(A|C) = 1$ iff $\omega \leqslant (A|C)$, and since $(A|C) = (A|B) \sqcap (B|C)$, $\omega \leqslant (A|C)$ iff $\omega \leqslant (A|B)$ and $\omega \leqslant (B|C)$, that is, iff $P_\omega(A|B) = P_\omega(B|C) = 1$.

Notice that, by direct application of the definition of $P_\omega(\cdot|\cdot)$, it is easy to check that, for every $i = 1, \ldots, n-1$, we have $P_\omega(\alpha_i|\alpha_i \vee \alpha_{i+1} ... \vee \alpha_n) = 1$ since $i = \min(\alpha_i \vee \alpha_{i+1} \vee \ldots \vee \alpha_n)$. Therefore,

$$\mu_{P_\omega}(\omega) = P_\omega(\alpha_1|\top) \cdot P_\omega(\alpha_2|\alpha_2 \vee ... \vee \alpha_n) \cdot \ldots \cdot P_\omega(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) = 1,$$

and hence $\mu_{P_\omega}(\omega') = 0$ for $\omega' \neq \omega$. □

In the last part of the proof of the above lemma we have proved that, for any atom $\omega = \langle \alpha_1, \ldots, \alpha_n \rangle \in \mathrm{at}(C(\mathbf{A}))$, μ_{P_ω} is such that

$$P_\omega(\alpha_1|\top) = P_\omega(\alpha_2|\alpha_2 \vee ... \vee \alpha_n) = \ldots = P_\omega(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) = 1,$$

and according to Remark 2, this means that the probability assessment $\mathcal{P} = (1, \ldots, 1)$ on the family

$$\mathcal{F} = \{(\alpha_1|\top), (\alpha_2|\bar{\alpha}_1), \ldots, (\alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2})\}$$

(as a restriction of P_ω) is coherent. In the rest of this subsection we show that this is in accordance with what can be obtained by applying the Algorithm 1 in [32, 25], in order to check coherence of $\mathcal{P}$, by considering a suitable sequence of linear systems and by verifying the solvability of each linear system.

Step 1 In the first step we consider the following system Σ_1 associated with the pair $(\mathcal{F}, \mathcal{P})$ in the unknowns $\lambda_1, \dots, \lambda_n$,

$$(\Sigma_1) \quad \begin{cases} \lambda_1 = P(\alpha_1|\top)(\lambda_1 + \cdots + \lambda_n), \\ \lambda_2 = P(\alpha_2|\bar{\alpha}_1)(\lambda_2 + \cdots + \lambda_n), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\ \lambda_{n-1} = P(\alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2})(\lambda_{n-1} + \lambda_n), \\ \lambda_1 + \cdots + \lambda_n = 1, \\ \lambda_h \geqslant 0, \forall h, \end{cases}$$

that is

$$(\Sigma_1) \quad \begin{cases} \lambda_1 = \lambda_1 + \cdots + \lambda_n, \\ \lambda_2 = \lambda_2 + \cdots + \lambda_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots , \\ \lambda_{n-1} = \lambda_{n-1} + \lambda_n, \\ \lambda_1 + \cdots + \lambda_n = 1, \\ \lambda_h \geqslant 0, \forall h. \end{cases}$$

The system Σ_1 has a unique solution given by the vector $(\lambda_1, \dots, \lambda_n) = (1, 0, \dots, 0)$, which can be interpreted as a (coherent) probability assessment π_1 on $\{\alpha_1, \dots, \alpha_n\}$, defined as

$$\pi_1 : \quad P(\alpha_1) = 1\,, \quad P(\alpha_2) = \cdots = P(\alpha_n) = 0\,.$$

By the algorithm, we have $I_0 = \{2, \dots, n-1\}$, $\mathcal{F}_0 = \{\alpha_2|\bar{\alpha}_1, \dots, \alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2}\}$ and $\mathcal{P}_0 = (1, \dots, 1)$. Then, the procedure continues with the second step.

Step 2 In the second step we consider the following system Σ_2, associated with the new pair $(\mathcal{F}, \mathcal{P}) = (\{\alpha_2|\bar{\alpha}_1, \dots, \alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2}\}, (1, \dots, 1))$, in the unknowns $\lambda_2, \dots, \lambda_n$,

$$(\Sigma_2) \quad \begin{cases} \lambda_2 = \lambda_2 + \cdots + \lambda_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots , \\ \lambda_{n-1} = \lambda_{n-1} + \lambda_n, \\ \lambda_2 + \cdots + \lambda_n = 1, \\ \lambda_h \geqslant 0, \forall h. \end{cases}$$

The system Σ_2 has a unique solution given by the vector $(\lambda_2, \dots, \lambda_n) = (1, 0, \dots, 0)$, which can be interpreted as a (coherent) probability assessment π_2 on $\{\alpha_2|\bar{\alpha}_1, \dots, \alpha_n|\bar{\alpha}_1\}$, defined as

$$\pi_2 : \quad P(\alpha_2|\bar{\alpha}_1) = 1,\ P(\alpha_3|\bar{\alpha}_1) = \cdots = P(\alpha_n|\bar{\alpha}_1) = 0\,.$$

By the algorithm, we have $I_0 = \{3, \dots, n-1\}$, $\mathcal{F}_0 = \{\alpha_3|\bar{\alpha}_1\bar{\alpha}_2, \dots, \alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2}\}$ and $\mathcal{P}_0 = (1, \dots, 1)$. Then, the procedure continues with the third step.

. . . .

Step $n-1$ In the last step we consider the following system Σ_{n-1} associated with the pair $(\mathcal{F}, \mathcal{P}) = (\{\alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2}\}, (1))$, in the unknowns λ_{n-1}, λ_n

$$(\Sigma_{n-1}) \quad \begin{cases} \lambda_{n-1} = \lambda_{n-1} + \lambda_n, \\ \lambda_{n-1} + \lambda_n = 1, \\ \lambda_{n-1} \geqslant 0, \lambda_n \geqslant 0. \end{cases}$$

The system Σ_{n-1} has a unique solution $(\lambda_{n-1}, \lambda_n) = (1, 0)$, which can be interpreted as a (coherent) probability assessment π_{n-1} on $\{\alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2}, \alpha_n|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2}\}$, defined as

$$\pi_{n-1}: \quad P(\alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2}) = 1,\ P(\alpha_n|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2}) = 0\,.$$

By the algorithm, we have $I_0 = \emptyset$; then, the procedure ends by declaring $\mathcal{P}$ coherent.

We observe that the conditional probability P on $\mathbb{A} \times \mathbb{A}'$, extension of the assessment $\mathcal{P}$ on $\mathcal{F}$, is such that $\mu_P(\omega) = 1$ and $\mu_P(\omega') = 0$ for every atom $\omega' \neq \omega$. Then, since $\mu_P = \mu_{P_\omega}$ and, as we have observed above, the canonical extensions are unique, it holds that $P = P_\omega$, that is P_ω is the unique extension of the assessment $\mathcal{P}$ on $\mathcal{F}$ as a (full) conditional probability to $\mathbb{A} \times \mathbb{A}'$. Notice that, for each $r = 1, \ldots, n-1$, the probability assessment π_r is a restriction of P_ω on the family $\{\alpha_r|\bar{\alpha}_1 \cdots \bar{\alpha}_{r-1}, \ldots, \alpha_n|\bar{\alpha}_1 \cdots \bar{\alpha}_{r-1}\}$, that is on the family $\{\alpha_r|\alpha_r \vee \cdots \vee \alpha_n, \ldots, \alpha_n|\alpha_r \vee \cdots \vee \alpha_n\}$.

We remark that, in order to verify the coherence of the assessment $\mathcal{P}$ on $\mathcal{F}$, an equivalent procedure is given in [31] which exploits a suitable *class of (unconditional) probability assessements $\{P_r\}$ agreeing with $\mathcal{P}$*. For each conditional event $(\alpha_h|\alpha_h \vee \cdots \vee \alpha_n)$ in $\mathcal{F}$, its probability is represented as a ratio by using an element of the class. In our case the (unique) agreeing class is $\{P_0, P_1, \ldots, P_{n-1}\}$, where, for each h, the assessment P_{h-1} is defined as

$$P_{h-1}(\alpha_h) = 1\,,\ P_{h-1}(\alpha_j) = 0\,,\ j > h.$$

Indeed, we have

$$P(\alpha_h|\alpha_h \vee \cdots \vee \alpha_n) = \frac{P_{h-1}(\alpha_h)}{P_{h-1}(\alpha_h \vee \cdots \vee \alpha_n)}\,,\ h = 1, \ldots, n\,.$$

We recall that in [31], given any event $E \in \mathbf{A}$, with $E \neq \bot$, the *zero-layer* of E with respect to an agreeing class $\{P_r\}$ is the first index k such that $P_k(E) > 0$, denoted $o(E) = k$. In our case, for each atom α_h the first index k such that $P_k(\alpha_h) > 0$ is $k = h-1$ and hence $o(\alpha_h) = h-1$. Thus, the events $\alpha_1, \alpha_2, \ldots, \alpha_n$ belong to different zero-layers, because

$$o(\alpha_1) = 0\,,\ o(\alpha_2) = 1\,,\ \ldots, o(\alpha_{n-1}) = n-2\,,\ o(\alpha_n) = n-1\,.$$

Finally, in [31] the zero-layer of a conditional event $E|H$, with respect to an agreeing class $\{P_r\}$, is defined as $o(E|H) = o(EH) - o(H)$. Then, by observing that for every h it holds that $o(\alpha_h) = o(\alpha_h \vee \cdots \vee \alpha_n)$, with respect to $\{P_0, P_1, \ldots, P_{n-1}\}$ we obtain

$$o(\alpha_h|\alpha_h \vee \cdots \vee \alpha_n) = 0,\ \forall\, (\alpha_h|\alpha_h \vee \cdots \vee \alpha_n) \in \mathcal{F}\,.$$

5.3. On the non convexity of the set of canonical extensions

Based on Lemma 1 we can verify that the set of canonical extensions μ_P on $\mathcal{C}(\mathbf{A})$ is not convex. Indeed, as we have seen, for every $\omega \in \mathrm{at}(\mathcal{C}(\mathbf{A}))$, the probability μ_{P_ω} as in Lemma 1 is a homomorphism of $\mathcal{C}(\mathbf{A})$ to the two element Boolean algebra $\{0, 1\}$. In other words, the set of all probability measures μ_P on $\mathcal{C}(\mathbf{A})$ that are canonical extensions of some conditional probability P contains all the homomorphisms of $\mathcal{C}(\mathbf{A})$ to $\{0, 1\}$. Thus, if the set of canonical extensions were convex, this set would coincide with the set of all probability measures of $\mathcal{C}(\mathbf{A})$ and this is known not to be the case, since there are probabilities on algebras $\mathcal{C}(\mathbf{A})$ that are not conditional probabilities and hence are not canonical extensions, see for instance [22, Example 6.3]. Next we provide another example.

Example 4. Let us consider an algebra $\mathbf{A}$ with atoms $\{\alpha_1, \alpha_2, \dots, \alpha_5\}$, and let $A = \alpha_1 \vee \alpha_2$, $H = \alpha_1 \vee \cdots \vee \alpha_4$ and $B = \alpha_1 \vee \alpha_3 \vee \alpha_5$. Notice that $ABH = \alpha_1$, $A\bar{B}H = \alpha_2$, $\bar{A}BH = \alpha_3$, $\bar{A}\,\bar{B}H = \alpha_4$, $\bar{H} = \alpha_5$. Then, consider in $C(\mathbf{A})$ the two atoms $\omega' = \omega_{1234}$ and $\omega'' = \omega_{2314}$. We also consider the conditional events:

$$A|H = (\alpha_1 \vee \alpha_2|\alpha_1 \vee \cdots \vee \alpha_4),\ \bar{B}|H = (\alpha_2 \vee \alpha_4|\alpha_1 \vee \cdots \vee \alpha_4),\ A|BH = (\alpha_1|\alpha_1 \vee \alpha_3).$$

As it can be verified, it holds that: $\omega' \leqslant (A|H)$, $\omega' \nleqslant (\bar{B}|H)$, $\omega' \leqslant (A|BH)$, $\omega'' \leqslant (A|H)$, $\omega'' \leqslant (\bar{B}|H)$ and $\omega'' \nleqslant (A|BH)$. Therefore: $\omega' \sqcup \omega'' \leqslant (A|H)$, $\omega' \sqcup \omega'' \nleqslant (\bar{B}|H)$ and $\omega' \sqcup \omega'' \nleqslant (A|BH)$. We can show that the set of canonical extensions μ_P on $C(\mathbf{A})$ is not convex. For instance, by considering $\mu_{P_{\omega'}}$ and $\mu_{P_{\omega''}}$ defined as in Lemma 1, and by setting $\mu_a = a\mu_{P_{\omega'}} + (1-a)\mu_{P_{\omega''}}$, we can show that μ_a is a canonical extension of a conditional probability if and only if $a = 1$ or $a = 0$, with trivially $\mu_1 = \mu_{P_{\omega'}}$ and $\mu_0 = \mu_{P_{\omega''}}$. Indeed, we observe that

$$\mu_a(\omega' \sqcup \omega'') = a \cdot \mu_{P_{\omega'}}(\omega' \sqcup \omega'') + (1-a) \cdot \mu_{P_{\omega''}}(\omega' \sqcup \omega'') = a + 1 - a = 1.$$

and moreover that $\mu_a(A|H) = \mu_a(\omega' \sqcup \omega'') = 1$, $\mu_a(\bar{B}|H) = \mu_a(\omega'') = 1 - a$, and $\mu_a(A|BH) = \mu_a(\omega') = a$. If μ_a coincides with the canonical extension μ_P of some conditional probability P, then by Theorem 5 we have that $P(A|H) = 1$, $P(\bar{B}|H) = 1 - a$, and $P(A|BH) = a$. We can verify that the assessment $\mathcal{P} = (1, 1-a, a)$ on the family $\{A|H, \bar{B}|H, A|BH\}$, where

$$P(A|H) = 1\,,\quad P(\bar{B}|H) = 1 - a\,,\quad P(A|BH) = a\,,$$

is coherent if and only if $a = 0$ or $a = 1$. Indeed, the constituents generated by $\{A|H, \bar{B}|H, A|BH\}$ are

$$C_0 = \bar{H}\,,\ C_1 = ABH\,,\ C_2 = A\bar{B}H\,,\ C_3 = \bar{A}BH\,,\ C_4 = \bar{A}\bar{B}H\,.$$

We set $P(C_h|H) = \lambda_h$, $h = 1, 2, 3, 4$. Then we observe that

$$\begin{array}{l} 1 = P(A|H) = P(C_1|H) + P(C_2|H) = \lambda_1 + \lambda_2, \\ 1 - a = P(\bar{B}|H) = P(C_2|H) + P(C_4|H) = \lambda_2 + \lambda_4, \\ a^2 = P(A|BH)P(B|H) = P(AB|H) = \lambda_1, \\ 1 = P(H|H) = P(C_1|H) + \cdots + P(C_4|H) = \lambda_1 + \cdots + \lambda_4, \end{array} \tag{22}$$

or equivalently

$$\begin{cases} \lambda_1 = a^2, \\ \lambda_2 = 1 - a = 1 - a^2, \\ \lambda_3 = \lambda_4 = 0, \\ \lambda_h \geqslant 0\,,\ \forall\, h\,. \end{cases}$$

One can check that this linear system of equations on the unknowns λ_i's is solvable if and only if $a = 1$ or $a = 0$. Therefore, for every $0 < a < 1$, there does not exist any conditional probability P which is the restriction of μ_a to the basic conditional events, that is, such that $\forall (A|H) \in C(\mathbf{A})$, $\mu_a(A|H) = P(A|H)$. Thus, by Theorem 5, for every $0 < a < 1$ the map μ_a cannot be the canonical extension of any conditional probability P and hence the set of canonical extensions μ_P on $C(\mathbf{A})$ is not convex.

6. Probability of the conjunction and the disjunction of conditionals

In this section we start by showing that the probability of the conjunction $K \sqcap (A|H)$, when $HK = \bot$, is the product of the probabilities of the conjuncts. Then, we represent the conjunction of two conditional events as a suitable disjunction. Finally, based on the canonical extension, we obtain the probability for the conjunction and the disjunction of two conditional events $A|H$ and $B|K$, which are related with analogous results given in the setting of coherence in [24, 25, 26, 29, 33]. In the next result we generalize formula (21).

Theorem 6. Given an algebra $\mathbf{A}$ and any events $A, H, K \in \mathbb{A}$, with $H \neq \bot$ and $HK = \bot$, given a conditional probability P on $\mathbb{A} \times \mathbb{A}'$ and its canonical extension μ_P to $\mathcal{C}(\mathbf{A})$, it holds

$$\mu_P[K \sqcap (A|H)] = P(K)P(A|H)\,. \tag{23}$$

Proof. As $HK = \bot$, it holds that $H\bar{K} = H$, $H \vee \bar{K} = \bar{K}$, and $\bar{H}K = K$; then

$$\top = (AH \vee \bar{A}H \vee \bar{H}) \wedge (K \vee \bar{K}) = AH\bar{K} \vee \bar{H}K \vee \bar{A}H\bar{K} \vee \bar{H}\bar{K}.$$

We consider the partition $\{\beta_1, \dots, \beta_4\}$, where

$$\beta_1 = AH\bar{K} = AH\,,\ \beta_2 = \bar{H}K = K\,,\ \beta_3 = \bar{A}H\bar{K} = \bar{A}H\,,\ \beta_4 = \bar{H}\bar{K},$$

and the associated subalgebra $\mathbf{B}$; moreover we consider the atoms $\omega'_{i_1 i_2 i_3}$'s of $\mathcal{C}(\mathbf{B})$. As $\omega'_{213} \sqcup \omega'_{214} = \omega'_{21}$, it holds that

$$K \sqcap (A|H) = \omega'_{213} \sqcup \omega'_{214} \sqcup \omega'_{241} = \omega'_{21} \sqcup \omega'_{241}\,.$$

Let P' be the restriction of P to $\mathbb{B} \times \mathbb{B}'$ and $\mu_{P'}$ its canonical extension to $\mathcal{C}(\mathbf{B})$. As $H\bar{K} = H$ it holds that $P(AH|\bar{K}) = P(A|H\bar{K})P(H|\bar{K}) = P(A|H)P(H|\bar{K})$. Then, from Theorem 5 and from (13) we obtain

$$\begin{aligned}
&\mu_P[K \sqcap (A|H)] = \mu_{P'}[K \sqcap (A|H)] = \mu_{P'}(\omega'_{21}) + \mu_{P'}(\omega'_{241}) = \\
&= P(\beta_2)P(\beta_1|\bar{\beta}_2) + P(\beta_2)P(\beta_4|\bar{\beta}_2)P(\beta_1|\bar{\beta}_2\bar{\beta}_4) = P(K)P(AH|\bar{K}) + P(K)P(\bar{H}|\bar{K})P(A|H) = \\
&= P(K)[P(AH|\bar{K}) + P(\bar{H}|\bar{K})P(A|H)] = P(K)[P(A|H)P(H|\bar{K}) + P(A|H)P(\bar{H}|\bar{K})] = P(K)P(A|H)\,.
\end{aligned}$$

□

Using the above result, we can now provide a suitable representation of the conjunction of two conditionals based on which we will show in Theorem 8 how to compute the probability of such a compound conditional object.

Theorem 7. For any conditional events $A|H, B|K \in \mathcal{C}(\mathbf{A})$ it holds that

$$(A|H) \sqcap (B|K) = [(AHBK|H \vee K)] \sqcup [(A|H) \sqcap (\bar{H}BK|H \vee K)] \sqcup [(B|K) \sqcap (AH\bar{K}|H \vee K)]. \tag{24}$$

Proof. From $(C5)$ it follows that $(AH|H \vee K) = (A|H) \sqcap (H|H \vee K)$ and $(BK|H \vee K) = (B|K) \sqcap (K|H \vee K)$. Then

$$\begin{aligned}
&(A|H) \sqcap (B|K) \sqcap (HK|H \vee K) = (A|H) \sqcap (H|H \vee K) \sqcap (B|K) \sqcap (K|H \vee K) = \\
&= (AH|H \vee K) \sqcap (BK|H \vee K) = AHBK|H \vee K,
\end{aligned}$$

$$\begin{aligned}
&(A|H) \sqcap (B|K) \sqcap (H\bar{K}|H \vee K) = (A|H) \sqcap (H|H \vee K) \sqcap (B|K) \sqcap (\bar{K}|H \vee K) = \\
&= (AH|H \vee K) \sqcap (B|K) \sqcap (\bar{K}|H \vee K) = (B|K) \sqcap (AH\bar{K}|H \vee K),
\end{aligned}$$

and

$$\begin{aligned}
&(A|H) \sqcap (B|K) \sqcap (\bar{H}K|H \vee K) = (A|H) \sqcap (\bar{H}|H \vee K) \sqcap (B|K) \sqcap (K|H \vee K) = \\
&= (A|H) \sqcap (\bar{H}|H \vee K) \sqcap (BK|H \vee K) = (A|H) \sqcap (\bar{H}BK|H \vee K).
\end{aligned}$$

Then,

$$\begin{aligned}&(A|H) \sqcap (B|K) = (A|H) \sqcap (B|K) \sqcap (H \vee K|H \vee K) = \\ &= [(A|H) \sqcap (B|K) \sqcap (HK|H \vee K)] \sqcup [(A|H) \sqcap (B|K) \sqcap (H\bar{K}|H \vee K)] \sqcup [(A|H) \sqcap (B|K) \sqcap (\bar{H}K|H \vee K)] = \\ &= [(AHBK|H \vee K)] \sqcup [(A|H) \sqcap (\bar{H}BK|H \vee K)] \sqcup [(B|K) \sqcap (AH\bar{K}|H \vee K)].\end{aligned}$$

□

In the next result we finally obtain the probability for the conjunction of two conditionals $A|H \sqcap B|K$ in terms of conditional probabilities of events related to the partition obtained from the family $\{A|H, B|K\}$.

Theorem 8. Given an algebra $\mathbf{A}$ and a conditional probability P on $\mathbb{A} \times \mathbb{A}'$, let μ_P be the canonical extension to $C(\mathbf{A})$. For any conditional events $A|H, B|K \in C(\mathbf{A})$ it holds that

$$\mu_P[(A|H) \sqcap (B|K)] = P(AHBK|H \vee K) + P(A|H)P(\bar{H}BK|H \vee K) + P(B|K)P(\bar{K}AH|H \vee K). \quad (25)$$

Proof. From (24) it follows that

$$\mu_P[(A|H) \sqcap (B|K)] = P(AHBK|H \vee K) + \mu_P[(A|H) \sqcap (\bar{H}BK|H \vee K)] + \mu_P[(B|K) \sqcap (\bar{K}AH|H \vee K)]. \quad (26)$$

In order to obtain (25) we need to show that

$$\mu_P[(A|H) \sqcap (\bar{H}BK|H \vee K)] = P(A|H)P(\bar{H}BK|H \vee K) \quad (27)$$

and

$$\mu_P[(B|K) \sqcap (AH\bar{K}|H \vee K)] = P(B|K)P(AH\bar{K}|H \vee K). \quad (28)$$

We first assume that the uncertain events A, H, B, K logically independent and we consider the subalgebra $\mathbf{B}$ generated by the partition $\{\beta_1, \ldots, \beta_9\}$, where

$$\begin{aligned}&\beta_1 = AHBK,\ \beta_2 = AH\bar{B}K,\ \beta_3 = AH\bar{K},\ \beta_4 = \bar{A}HBK, \\ &\beta_5 = \bar{A}H\bar{B}K,\ \beta_6 = \bar{A}H\bar{K},\ \beta_7 = \bar{H}BK,\ \beta_8 = \bar{H}\bar{B}K,\ \beta_9 = \bar{H}\bar{K}.\end{aligned} \quad (29)$$

Notice that, by logical independence, $\beta_j \neq \bot$, for each $j = 1, \ldots, 9$. Moreover, we consider the compound conditionals $\omega'_{i_1 \cdots i_k}$'s of $C(\mathbf{B})$, $1 \leqslant k \leqslant 8$. Let P' be the restriction of P to $\mathbb{B} \times \mathbb{B}'$ and $\mu_{P'}$ its canonical extension to $C(\mathbf{B})$. We recall that from Theorem 4, $\mu_P(C) = \mu_{P'}(C)$, for every $C \in C(\mathbf{B})$. By exploiting the distributivity property, we decompose the conjunction $(A|H) \sqcap (\bar{H}BK|H \vee K)$ as

$$(A|H) \sqcap (\bar{H}BK|H \vee K) = [(A|H) \sqcap (\bar{H}BK|H \vee K) \sqcap (H \vee K)] \sqcup [(A|H) \sqcap (\bar{H}BK|H \vee K) \sqcap (\bar{H}\,\bar{K})]. \quad (30)$$

For the compound $(A|H) \sqcap (\bar{H}BK|H \vee K) \sqcap (H \vee K)$ it holds that

$$(A|H) \sqcap (\bar{H}BK|H \vee K) \sqcap (H \vee K) = (A|H) \sqcap \bar{H}BK,$$

and, by Theorem 6, $\mu_P[(A|H) \sqcap (\bar{H}BK|H \vee K) \sqcap (H \vee K)] = \mu_P[(A|H) \sqcap \bar{H}BK] = P(A|H)P(\bar{H}BK)$. Moreover, as $P(\bar{H}BK) = P(\bar{H}BK \wedge (H \vee K)) = P(\bar{H}BK|H \vee K)P(H \vee K)$, it follows that

$$\mu_P[(A|H) \sqcap (\bar{H}BK|H \vee K) \sqcap (H \vee K)] = P(A|H)P(\bar{H}BK|H \vee K)P(H \vee K). \quad (31)$$

For the compound $(A|H) \sqcap (\bar{H}BK|H \vee K) \sqcap (\bar{H}\,\bar{K})$ it holds that

$$(A|H) \sqcap (\bar{H}BK|H \vee K) \sqcap (\bar{H}\,\bar{K}) = \omega'_{971} \sqcup \omega'_{972} \sqcup \omega'_{973} \sqcup \omega'_{9781} \sqcup \omega'_{9782} \sqcup \omega'_{9783}, \quad (32)$$

$$\begin{array}{lcl}
\omega'_{971} & = & \bar{H}\bar{K} \sqcap (\bar{H}BK|H\vee K) \sqcap (AHBK|H\vee \bar{B}K), \\
\omega'_{972} & = & \bar{H}\bar{K} \sqcap (\bar{H}BK|H\vee K) \sqcap (AH\bar{B}K|H\vee \bar{B}K) \\
\omega'_{973} & = & \bar{H}\bar{K} \sqcap (\bar{H}BK|H\vee K) \sqcap (AH\bar{K}|H\vee \bar{B}K), \\
\omega'_{9781} & = & \bar{H}\bar{K} \sqcap (\bar{H}BK|H\vee K) \sqcap (\bar{H}\bar{B}K|H\vee \bar{B}K) \sqcap (AHBK|H), \\
\omega'_{9782} & = & \bar{H}\bar{K} \sqcap (\bar{H}BK|H\vee K) \sqcap (\bar{H}\bar{B}K|H\vee \bar{B}K) \sqcap (AH\bar{B}K|H), \\
\omega'_{9783} & = & \bar{H}\bar{K} \sqcap (\bar{H}BK|H\vee K) \sqcap (\bar{H}\bar{B}K|H\vee \bar{B}K) \sqcap (AH\bar{K}|H).
\end{array}$$

Table 1: List of ω' such that $\omega' \leqslant (A|H) \sqcap (\bar{H}BK|H\vee K) \sqcap (\bar{H}\,\bar{K})$.

where the list of ω''s which appear in the previous formula are given in Table 1.

As

$$\begin{array}{l}
P(AHBK|H\vee \bar{B}K) + P(AH\bar{B}K|H\vee \bar{B}K) + P(AH\bar{K}|H\vee \bar{B}K) = \\
= P(AH|H\vee \bar{H}\bar{B}K) = P(A|H)P(H|H\vee \bar{H}\bar{B}K),
\end{array}$$

from (13) it holds that

$$\begin{array}{l}
\mu_P(\omega'_{971} \sqcup \omega'_{972} \sqcup \omega'_{973}) = \mu_P(\omega'_{971}) + \mu_P(\omega'_{972}) + \mu_P(\omega'_{973}) = \cdots = \\
= P(\bar{H}\bar{K})P(\bar{H}BK|H\vee K)P(AH|H\vee \bar{H}\bar{B}K) = \\
= P(\bar{H}\bar{K})P(\bar{H}BK|H\vee K)P(A|H)P(H|H\vee \bar{H}\bar{B}K).
\end{array}$$

Likewise,

$$\begin{array}{l}
\mu_P(\omega'_{9781} \sqcup \omega'_{9782} \sqcup \omega'_{9783}) = \mu_P(\omega'_{9781}) + \mu_P(\omega'_{9782}) + \mu_P(\omega'_{9783}) = \cdots = \\
= P(\bar{H}\bar{K})P(\bar{H}BK|H\vee K)P(\bar{H}\bar{B}K|H\vee \bar{H}\bar{B}K)P(A|H).
\end{array}$$

Then, by recalling (32),

$$\begin{array}{l}
\mu_P[(A|H) \sqcap (\bar{H}BK|H\vee K) \sqcap (\bar{H}\,\bar{K})] = \mu_P(\omega'_{971} \sqcup \omega'_{972} \sqcup \omega'_{973}) + \mu_P(\omega'_{9781} \sqcup \omega'_{9782} \sqcup \omega'_{9783}) = \\
= P(\bar{H}\bar{K})P(\bar{H}BK|H\vee K)P(A|H)P(H|H\vee \bar{H}\bar{B}K) + P(\bar{H}\bar{K})P(\bar{H}BK|H\vee K)P(\bar{H}\bar{B}K|H\vee \bar{H}\bar{B}K)P(A|H) = \\
= P(A|H)P(\bar{H}BK|H\vee K)P(\bar{H}\bar{K}).
\end{array} \tag{33}$$

Thus, from (30), (31), and (33) we obtain

$$\begin{array}{l}
\mu_P[(A|H) \sqcap (\bar{H}BK|H\vee K)] = \\
\quad = \mu_P[(A|H) \sqcap (\bar{H}BK|H\vee K) \sqcap (H\vee K)] + \mu_P[(A|H) \sqcap (\bar{H}BK|H\vee K) \sqcap (\bar{H}\,\bar{K})] = \\
\quad = P(A|H)P(\bar{H}BK|H\vee K)P(H\vee K) + P(A|H)P(\bar{H}BK|H\vee K)P(\bar{H}\bar{K}) = \\
\quad = P(A|H)P(\bar{H}BK|H\vee K),
\end{array}$$

that is formula (27). By a similar reasoning we can prove formula (28) and hence formula (25) holds in the case where A, B, H, K are logically independent.

Concerning the case where there are some logical dependencies among the events A, B, H, K, it may happen in (29) that $\beta_i = \bot$, for some i. In this case the proof can be adapted by considering the partition given by the β_i's different from $\bot$. $\square$

Remark 6. Notice that formula (25) coincides with the prevision of the conjunction $C = (A|H) \wedge (B|K)$, introduced in the setting of coherence as the following conditional random quantity (see, e.g.,[24, 26])

$$(A|H) \wedge (B|K) = (AHBK|H\vee K) + P(A|H)(\bar{H}BK|H\vee K) + P(B|K)(AH\bar{K}|H\vee K), \tag{34}$$

that is

$$\mu_P[(A|H) \sqcap (B|K)] = \mathbb{P}(C) = P(AHBK|H \vee K) + P(A|H)P(\bar{H}BK|H \vee K) + P(B|K)P(\bar{K}AH|H \vee K), \quad (35)$$

where $\mathbb{P}$ denotes the prevision and we use the same symbol for (conditional) events and their indicators. Moreover, when $P(H \vee K) > 0$, formula (25) becomes

$$\mu_P[(A|H) \sqcap (B|K)] = \frac{P(AHBK) + P(A|H)P(\bar{H}BK) + P(B|K)P(\bar{K}AH)}{P(H \vee K)},$$

that is the formula obtained by McGee ([34]) and Kaufmann ([8]).

In the next result we first examine the conjunction between $(A|H) \sqcap (B|K)$ and $H \vee K$, by showing the factorization of its probability. Then, we verify that the same property holds for the probability of the conjunction between $(A|H) \sqcap (B|K)$ and $\bar{H}\,\bar{K}$.

Theorem 9. Given two conditional events $A|H$ and $B|K$ it holds that

$$\begin{aligned} \mu_P[(A|H) \sqcap (B|K) \sqcap (H \vee K)] &= P(AHBK) + P(A|H)P(\bar{H}BK) + P(B|K)P(\bar{K}AH) \\ &= \mu_P[(A|H) \sqcap (B|K)]P(H \vee K), \end{aligned} \quad (36)$$

and

$$\mu_P[(A|H) \sqcap (B|K) \sqcap (\bar{H}\,\bar{K})] = \mu_P[(A|H) \sqcap (B|K)]P(\bar{H}\,\bar{K}). \quad (37)$$

Proof. We observe that from (24) it directly follows that

$$(A|H) \sqcap (B|K) \sqcap (H \vee K) = AHBK \sqcup [(A|H) \sqcap (\bar{H}BK)] \sqcup [(B|K) \sqcap (AH\bar{K})], \quad (38)$$

since $(AHBK|H \vee K) \sqcap (H \vee K) = AHBK$, $(\bar{H}BK|H \vee K) \sqcap (H \vee K) = \bar{H}BK$ and $(AH\bar{K}|H \vee K) \sqcap (H \vee K) = AH\bar{K}$. Moreover, as by Theorem 6

$$\mu_P[(A|H) \sqcap (B|K) \sqcap \bar{H}K)] = \mu_P((A|H) \sqcap \bar{H}BK) = P(A|H)P(\bar{H}BK),$$

and

$$\mu_P((A|H) \sqcap (B|K) \sqcap H\bar{K}) = P(B|K)P(AH\bar{K}),$$

it follows that

$$\begin{aligned} &\mu_P[(A|H) \sqcap (B|K) \sqcap (H \sqcup K)] = \mu_P[AHBK \sqcup (A|H) \sqcap (\bar{H}BK) \sqcup (B|K) \sqcap (AH\bar{K})] = \\ &= P(ABHK) + P(\bar{H}BK)P(A|H) + P(AH\bar{K})P(B|K) = \\ &= [P(ABHK|H \vee K) + P(\bar{H}BK|H \vee K)P(A|H) + P(AH\bar{K}|H \vee K)P(B|K)]P(H \vee K) = \\ &= \mu_P[(A|H) \sqcap (B|K)]P(H \vee K), \end{aligned}$$

that is, (36) is satisfied.
For the compound $(A|H) \sqcap (B|K) \sqcap \bar{H}\,\bar{K}$, as

$$(A|H) \sqcap (B|K) = [(A|H) \sqcap (B|K) \sqcap (H \vee K)] \sqcup [(A|H) \sqcap (B|K) \sqcap (\bar{H}\,\bar{K})],$$

it follows that

$$\begin{aligned} &\mu_P[(A|H) \sqcap (B|K) \sqcap \bar{H}\,\bar{K}] = \mu_P[(A|H) \sqcap (B|K)] - \mu_P[(A|H) \sqcap (B|K) \sqcap (H \sqcup K)] = \\ &= \mu_P[(A|H) \sqcap (B|K)] - \mu_P[(A|H) \sqcap (B|K)]P(H \vee K) = \mu_P[(A|H) \sqcap (B|K)]P(\bar{H}\,\bar{K}), \end{aligned}$$

that is, (37) is satisfied. □

The next result collects some particular expressions for the probability of the conjunction of two conditionals as direct consequences of Theorem 8.

Corollary 2. Given an algebra $\mathbf{A}$ and a conditional probability P on $\mathbb{A}\times\mathbb{A}'$, let μ_P be the canonical extension to $\mathcal{C}(\mathbf{A})$. It holds that:

1. if $HK = \bot$, then $\mu_P[(A|H) \sqcap (B|K)] = P(A|H)P(B|K)$;
2. if $H\vee K = \top$, then $\mu_P[(A|H) \sqcap (B|K)] = P(AHBK) + P(A|H)P(\bar{H}BK) + P(B|K)P(\bar{K}AH)$;
3. if $H = K$, then $\mu_P[(A|H) \sqcap (B|K)] = P(AB|H)$;
4. if $\bar{H}BK = \bar{K}AH = \bot$, then $\mu_P[(A|H) \sqcap (B|K)] = P(AHBK|H\vee K)$;
5. if $A|H \sqsubseteq B|K$, then $\mu_P[(A|H) \sqcap (B|K)] = P(A|H)$;
6. if $HB = \bot$ and $H \leqslant K$, then $\mu_P[(A|H) \sqcap (B|K)] = P(A|H)P(B|K)$.
7. it holds that $(\bar{A}|H) \sqcap (A|H \vee K) = (\bar{A}|H) \sqcap (A\bar{H}K|H \vee K)$ and

$$\mu_P[(\bar{A}|H) \sqcap (A|H \vee K)] = P(\bar{A}|H)P(A\bar{H}K|H \vee K)\,.$$

Proof. Assertion 1. As $HK = \bot$, and hence $\bar{H}BK = BK$, $AH\bar{K} = AH$, it holds that

$$P(AHBK|H\vee K) = 0\,,\quad P(\bar{H}BK|H\vee K) = P(BK|H\vee K)\,,\quad P(\bar{K}AH|H\vee K) = P(AH|H\vee K)\,.$$

Then, by observing that

$$P(H|H\vee K) + P(K|H\vee K) = P(H\vee K|H\vee K) = 1\,,$$

formula (25) becomes

$$\begin{aligned}&\mu_P[(A|H) \sqcap (B|K)] = P(A|H)P(BK|(H\vee K) + P(B|K)P(AH|(H\vee K) =\\&= P(A|H)P(B|K)P(K|(H\vee K) + P(A|H)P(B|K)P(H|(H\vee K) = P(A|H)P(B|K)\,.\end{aligned}$$

Assertion 2. As $H\vee K = \top$, formula (25) directly becomes

$$\mu_P[(A|H) \sqcap (B|K)] = P(AHBK) + P(A|H)P(\bar{H}BK) + P(B|K)P(\bar{K}AH)\,.$$

This formula, by observing that $H \sqcup K = HK \sqcup \bar{H}K \sqcup H\bar{K}$ and that

$$\begin{aligned}&(A|H) \sqcap (B|K) = (A|H) \sqcap (B|K) \sqcap (H \sqcup K) =\\&= [(A|H) \sqcap (B|K) \sqcap HK] \sqcup [(A|H) \sqcap (B|K) \sqcap \bar{H}K] \sqcup [(A|H) \sqcap (B|K) \sqcap H\bar{K}] =\\&= AHBK \sqcup [(A|H) \sqcap \bar{H}BK] \sqcup [(B|K) \sqcap AH\bar{K}]\,,\end{aligned}$$

also follows by exploiting Theorem 6.

Assertion 3. As $H = K$, it holds that $(A|H) \sqcap (B|K) = (A|H) \sqcap (B|H) = AB|H$ (i.e., the conjunction is a conditional event), with

$$P(\bar{H}BK|H\vee K) = P(\bar{H}B|H) = 0\,,\quad P(\bar{K}AH|H\vee K) = P(\bar{H}A|H) = 0\,.$$

Then, formula (25) becomes

$$\mu_P[(A|H) \sqcap (B|K)] = P[(A|H) \wedge (B|H)] = P(AB|H)\,.$$

Assertion 4. As $\bar{H}BK = \bar{K}AH = \bot$, it holds that

$$(A|H) \sqcap (B|K) \sqcap \bar{H}K = (A|H) \sqcap \bar{H}BK = \bot\,,\quad (A|H) \sqcap (B|K) \sqcap H\bar{K} = (B|K) \sqcap \bar{K}AH = \bot\,.$$

Then, by recalling (24), we obtain $(A|H) \sqcap (B|K) = (AHBK|H \vee K)$, i.e., the conjunction is a conditional event, which in this case also coincides with the quasi conjunction of $A|H$ and $B|K$, and with the conjunction $(A|H) \wedge_{df} (B|K)$ in the trivalent logic of de Finetti. Indeed the quasi conjunction of $A|H$ and $B|K$ is the conditional event

$$((\bar{H} \vee AH) \wedge (\bar{K} \vee BK)|H \vee K) = (AHBK \vee \bar{H}BK \vee \bar{K}AH|H \vee K) = (AHBK|H \vee K)$$

and

$$(A|H) \wedge_{df} (B|K) = AHBK|(AHBK \vee \bar{A}H \vee \bar{B}K) = (ABHK|H \vee K),$$

because

$$H \vee K = \ldots = AHBK \vee \bar{A}H \vee \bar{B}K.$$

Then, formula (25) becomes $\mu_P[(A|H) \sqcap (B|K)] = P(AHBK|H \vee K)$.

Assertion 5. If $(A|H) \subseteq (B|K)$, then from Theorem 1, $(A|H) \sqcap (B|K) = (A|H)$; therefore $\mu_P[(A|H) \sqcap (B|K)] = P(A|H)$.

Assertion 6. If $HB = \bot$ and $H \leqslant K$, then $AHBK = AH\bar{K} = \bot$, $(\bar{H}BK|H \vee K) = (BK|K) = (B|K)$. Therefore formula (25) becomes

$$\mu_P[(A|H) \sqcap (B|K)] = P(A|H)P(B|K).$$

Assertion 7. We observe that

$$\begin{aligned} (A|H \vee K) &= (AH \vee A\bar{H}K|H \vee K) = (AH|H \vee K) \sqcup (A\bar{H}K)|H \vee K) = \\ &= ((A|H) \sqcap (H|H \vee K)) \sqcup (A\bar{H}K)|H \vee K), \end{aligned}$$

and hence $(\bar{A}|H) \sqcap (A|H \vee K) = (\bar{A}|H) \sqcap (A\bar{H}K|H \vee K)$. Then, from Assertion 6 it follows that

$$\mu_P[(\bar{A}|H) \sqcap (A|H \vee K)] = \mu_P[(\bar{A}|H) \sqcap (A\bar{H}K|H \vee K)] = P(\bar{A}|H)P(A\bar{H}K|H \vee K).$$

□

We now move to the disjunction operation of conditionals. In the next result we give a suitable representation of the disjunction of two conditionals $(A|H) \sqcup (B|K)$, similar to the one for the conjunction provided in Theorem 7, and that will allow us later in Theorem 11 to give an operational formulation for the probability of the disjunction of two conditionals.

Theorem 10. For any conditional events $A|H, B|K \in C(\mathbf{A})$ it holds that

$$(A|H) \sqcup (B|K) = [(AH \vee BK|H \vee K)] \sqcup [(A|H) \sqcap (\bar{H}\bar{B}K|H \vee K)] \sqcup [(B|K) \sqcap (\bar{A}H\bar{K}|H \vee K)]. \quad (39)$$

Proof. We observe that

$$(A|H) \sqcup (B|K) = [(A|H) \sqcap (B|K)] \sqcup [(\bar{A}|H) \sqcap (B|K)] \sqcup [(A|H) \sqcap (\bar{B}|K)]$$

where, by recalling Theorem 7,

$$(A|H) \sqcap (B|K) = [(AHBK|H \vee K)] \sqcup [(A|H) \sqcap (\bar{H}BK|H \vee K)] \sqcup [(B|K) \sqcap (AH\bar{K}|H \vee K)],$$

$$(\bar{A}|H) \sqcap (B|K) = [(\bar{A}HBK|H \vee K)] \sqcup [(\bar{A}|H) \sqcap (\bar{H}BK|H \vee K)] \sqcup [(B|K) \sqcap (\bar{A}H\bar{K}|H \vee K)],$$

and

$$(A|H) \sqcap (\bar{B}|K) = [(AH\bar{B}K|H \vee K)] \sqcup [(A|H) \sqcap (\bar{H}\bar{B}K|H \vee K)] \sqcup [(\bar{B}|K) \sqcap (AH\bar{K}|H \vee K)].$$

By observing that

$$(A|H) \sqcap (\bar{H}BK|H \vee K)] \sqcup (\bar{A}|H) \sqcap (\bar{H}BK|H \vee K)] = \bar{H}BK|H \vee K,$$

$$(B|K) \sqcap (AH\bar{K}|H \vee K)] \sqcup (\bar{B}|K) \sqcap (AH\bar{K}|H \vee K)] = AH\bar{K}|H \vee K,$$

and

$$(AHBK|H \vee K)] \sqcup (\bar{A}HBK|H \vee K) \sqcup (AH\bar{B}K|H \vee K) \sqcup (\bar{H}BK|H \vee K) \sqcup (AH\bar{K}|H \vee K) = (AH \vee BK)|H \vee K),$$

it follows that (39) is satisfied. □

In the next result we obtain the probability of the disjunction $(A|H) \sqcup (B|K)$.

Theorem 11. Given an algebra $\mathbf{A}$ and a conditional probability P on $\mathbb{A} \times \mathbb{A}'$, let μ_P be the canonical extension to $C(\mathbf{A})$. For any conditional events $A|H, B|K \in C(\mathbf{A})$ it holds that

$$\mu_P[(A|H) \sqcup (B|K)] = P(AH \vee BK|(H \vee K)) + P(A|H)P(\bar{H}\,\bar{B}K|H \vee K) + P(B|K)P(\bar{A}H\bar{K}|H \vee K). \quad (40)$$

Proof. By suitably applying Corollary 2, Assertion 6, it holds that

$$\mu_P[(A|H) \sqcap (\bar{H}\bar{B}K|H \vee K)] = P(A|H)P(\bar{H}\bar{B}K|H \vee K)$$

and

$$\mu_P[(B|K) \sqcap (\bar{A}H\bar{K}|H \vee K)] = P(B|K)P(\bar{A}H\bar{K}|H \vee K).$$

Then, by recalling (39) it holds that

$$\begin{aligned}\mu_P[(A|H) \sqcup (B|K) &= \mu_P(AH \vee BK|H \vee K) + \mu_P[(A|H) \sqcap (\bar{H}\bar{B}K|H \vee K)] + \mu_P[(B|K) \sqcap (\bar{A}H\bar{K}|H \vee K)] = \\ &= P(AH \vee BK|(H \vee K)) + P(A|H)P(\bar{H}\,\bar{B}K|H \vee K) + P(B|K)P(\bar{A}H\bar{K}|H \vee K),\end{aligned}$$

that is, (40) is satisfied. □

We observe that (40) coincides with the prevision of the disjunction of two conditional events $\mathcal{D} = (A|H) \vee (B|K)$, obtained in the framework of conditional random quantities in [24], where

$$\mathcal{D} = (AH \vee BK|H \vee K) + P(A|H)\,(\bar{H}\,\bar{B}K|H \vee K) + P(B|K)\,(\bar{A}H\bar{K}|H \vee K). \quad (41)$$

We also observe that De Morgan Laws are satisfied in $C(\mathbf{A})$, therefore, $\mu_P[\overline{(A|H) \sqcup (B|K)}] = \mu_P[(\bar{A}|H) \sqcap (\bar{B}|K)]$ and $\mu_P[\overline{(A|H) \sqcap (B|K)}] = \mu_P[(\bar{A}|H) \sqcup (\bar{B}|K)]$ in agreement with formulas (25) and (40). Based on (34) and (41), as $\mu_P[(A|H) \sqcap (B|K)] = \mathbb{P}(C)$ and $\mu_P[(A|H) \sqcup (B|K)] = \mathbb{P}(\mathcal{D})$, by recalling the prevision sum rule $\mathbb{P}(\mathcal{D}) = P(A|H) + P(B|K) - \mathbb{P}(C)$ obtained in [35, 24], it holds that

$$\mu_P[(A|H) \sqcup (B|K)] = P(A|H) + P(B|K) - \mu_P[(A|H) \sqcap (B|K)]. \quad (42)$$

Indeed,

$$\mu_P[(A|H) \sqcup (B|K)] = P(A|H) + \mu_P[(\bar{A}|H) \sqcap (B|K)] = P(A|H) + \mu_P(B|K) - \mu_P[(A|H) \sqcap (B|K)].$$

7. Algebraic and probabilistic entailment with conditionals and nonmonotonic reasoning

In this section we first consider an entailment relation $\models$ among conditionals of a conditional algebra $C(\mathbf{A})$ defined in terms of the lattice order in $C(\mathbf{A})$. We show that this algebraic definition can be probabilistically characterised by means of canonical extensions as a generalisation of Adams' p-entailment. Then we show that these two equivalent entailments induce a nonmonotonic consequence relation on the algebra of plain events $\mathbf{A}$ satisfying the well-known rules of the system P and we discuss the Rational Monotony rule. The underlying algebraic nature of $\models$ allows us to provide simple algebraic proofs based on results given in the paper, like the decomposition property of the conjunction of Theorem 7.

Let us say that an atom $\omega \in C(\mathbf{A})$ *satisfies* a compound t when $\omega \leqslant t$. We also say that t is *satisfiable* if $t \neq \perp$, i.e. if there exists $\omega \in C(\mathbf{A})$ such that $\omega \leqslant t$. Then, it is clear that for every atom ω and every compound t, either ω satisfies t or ω satisfies $\bar{t}$. In particular, if $t = (A|H)$, then either ω satisfies $(A|H)$, or falsifies it (i.e. it satisfies $(\bar{A}|H)$). This reflects the fact that, by construction, conditionals are Boolean objects in conditional algebras $C(\mathbf{A})$. But this is compatible with the 3-valued nature of basic conditionals when viewed from the original algebra of (plain) events $\mathbf{A}$. Indeed, for every atom α of $\mathbf{A}$, exactly one of the following three conditions holds: either $\alpha \leqslant AH$, or $\alpha \leqslant \bar{A}H$, or $\alpha \leqslant \bar{H}$, that respectively correspond to the cases in which either α satisfies $(A|H)$, or α falsifies $(A|H)$, or α makes $(A|H)$ undefined or void.

Definition 4. Given any set of (compound) conditionals $\mathcal{F} \subseteq C(\mathbf{A})$, we say that $\mathcal{F}$ is *consistent* if $\sqcap\{s|s \in \mathcal{F}\}$ is satisfiable, that is, if $\sqcap\{s|s \in \mathcal{F}\} \neq \perp$.

We define a consequence relation between sets of conditionals and conditionals from a conditional algebra $C(\mathbf{A})$ by means of the order relation $\leqslant$.

Definition 5. Let $\mathcal{F} \cup \{t\} \subseteq C(\mathbf{A})$ be a set of (compound) conditionals, with $\mathcal{F}$ consistent. Then we say that t is a consequence of $\mathcal{F}$, written $\mathcal{F} \models t$, whenever $\sqcap\{s|s \in \mathcal{F}\} \leqslant t$.

Note that, by definition, $\mathcal{F} \models t$ holds iff every atom of $C(\mathbf{A})$ which is below every compound conditional of $\mathcal{F}$ is also below t. Note that if $\mathcal{F}$ is not consistent, then $\mathcal{F} \models t$ trivially holds.

The consequence relation $\models$ among (compound) conditionals can also be characterized in probabilistic terms.

Proposition 4. For any set of (compound) conditionals $\mathcal{F} \cup \{t\} \subseteq C(\mathbf{A})$, with $\mathcal{F}$ consistent, it holds that $\mathcal{F} \models t$ iff, for all conditional probability P, $\mu_P(\sqcap\{r|r \in \mathcal{F}\}) \leqslant \mu_P(t)$.

Proof. By letting $s = \sqcap\{r|r \in \mathcal{F}\}$, it amounts to prove that $s \leqslant t$ iff $\mu_P(s) \leqslant \mu_P(t)$ for any conditional probability P. From consistency of $\mathcal{F}$ it holds that $s > \perp$. The left-to-right direction is direct. For the converse direction, suppose $\perp < s \nleqslant t$. Then there is an atom $\omega \in C(\mathbf{A})$ such that $\omega \leqslant s$ but $\omega \nleqslant t$. By Lemma 1, μ_{P_ω} is such that $\mu_{P_\omega}(s) = 1$ while $\mu_{P_\omega}(t) = 0$. □

As an immediate consequence of this proposition and its proof, in the following corollary we have a stronger version of the previous result, that in fact shows that $\models$ coincides with a generalised form of Adams' p-entailment for (basic) conditionals [2]. In the following we will say that a conditional probability P is a *p-model* of a (compound) conditional s when $\mu_P(s) = 1$, and we will say that a set $\mathcal{F}$ of (compound) conditionals *p-entails* another (compound) conditional t, written $\mathcal{F} \vdash_p t$, when every conditional probability P that is a p-model of all conditionals $s \in \mathcal{F}$ is a p-model of t as well.

Definition 6. For any set of (compound) conditionals $\mathcal{F} \cup \{t\} \subseteq C(\mathbf{A})$, with $\mathcal{F}$ consistent, we say that $\mathcal{F}$ *p-entails* t, written $\mathcal{F} \vdash_p t$, when every p-model of $\mathcal{F}$ is a p-model of t as well.

Lemma 2. For any set of consistent (compound) conditionals $\mathcal{F} \subseteq \mathcal{C}(\mathbf{A})$, it holds that

$$\mathcal{F} \vdash_p \sqcap\{s|s \in \mathcal{F}\}. \tag{43}$$

Proof. Let P be a p-model of $\mathcal{F}$, that is $\mu_P(s) = 1$ for all $s \in \mathcal{F}$. If P were not a p-model of $\sqcap\{s|s \in \mathcal{F}$, that is $\mu_P(\sqcap\{s|s \in \mathcal{F}) < 1$, then there would exist an atom ω of $\mathcal{C}(\mathbf{A})$ such that $\omega \not\leqslant \sqcap\{s|s \in \mathcal{F}\}$ and $\mu_P(\omega) > 0$. Moreover, there would exist $s \in \mathcal{F}$ such that $\omega \not\leqslant s$ and hence $\mu_P(s) < 1$ contradicting the assumption. □

Proposition 5. For any set of (compound) conditionals $\mathcal{F} \cup \{t\} \subseteq \mathcal{C}(\mathbf{A})$, it holds that $\mathcal{F} \models t$ iff $\mathcal{F} \vdash_p t$.

Proof. ($\Longrightarrow$). If $\mathcal{F} \models t$, then by Proposition 4, $\mu_P(\sqcap\{s|s \in \mathcal{F}) \leqslant \mu_P(t)$ for every P. Then, from (43) for every p-model P of $\mathcal{F}$ it holds that $\mu_P(\sqcap\{s|s \in \mathcal{F}) = 1$ and hence $\mu_P(t) = 1$. Thus $\mathcal{F} \vdash_p t$.
($\Longleftarrow$). Assume that $\mathcal{F} \vdash_p t$. If t were not a consequence of $\mathcal{F}$, then there would exist an atom ω of $\mathcal{C}(\mathbf{A})$ such that $\omega \leqslant \sqcap\{s|s \in \mathcal{F}\}$ and $\omega \not\leqslant t$. Then, from Lemma 1, it would be $\mu_{P_\omega}(\sqcap\{s|s \in \mathcal{F}) = 1$ and $\mu_{P_\omega}(t) = 0$. Moreover, as $\mu_{P_\omega}(\sqcap\{s|s \in \mathcal{F}) \leqslant \mu_{P_\omega}(s)$ for all $s \in \mathcal{F}$, it would follow that $\mu_{P_\omega}(s) = 1$ for all $s \in \mathcal{F}$, with $\mu_{P_\omega}(t) = 0$. Thus, P_ω would be a p-model of $\mathcal{F}$, but not a p-model of t, which contradicts the assumption. □

Now we turn to some properties of the entailment $\models$ related to core properties of the well-known System P for nonmonotonic inference relations.

Proposition 6. The following inferences for $\models$ hold:

(i) $\{(A|H), (B|H)\} \models (AB|H)$ (related to And)

(ii) $\{(A|H), (B|AH)\} \models (B|H)$ (related to Cut)

(iii) $\{(A|H), (A|K)\} \models (A|H \vee K)$ (related to Or)

(iv) $\{(A|H), (B|H)\} \models (B|AH)$ (related to CM)

(v) $\{(A|H \vee K), (\bar{A}|H)\} \models (A|K)$ (related to the Or rule)

Proof. (i) It directly follows from the definition of $\mathcal{C}(\mathbf{A})$.

(ii) We have $(A|H) \sqcap (B|AH) = (AH|H) \sqcap (ABH|AH) = (ABH|H) \leqslant (B|H)$, where in the second equality we have applied condition (C5), and thus (ii) is satisfied.

(iii) By (24) we get: $(A|H) \sqcap (A|K) = (AHK|H \vee K) \sqcup ((A|H) \sqcap (A\bar{H}K|H \vee K)) \sqcup ((A|K) \sqcap (AH\bar{K}|H \vee K)) \leqslant (AHK|H \vee K) \sqcup (A\bar{H}K|H \vee K) \sqcup (AH\bar{K}|H \vee K)$, but each of the three disjuntcs is less or equal than $(A|H \vee K)$, hence $(A|H) \sqcap (A|K) \leqslant (A|H \vee K)$.

(iv) In this case, we have $(A|H) \sqcap (B|H) = (AB|H) = (ABH|H)$, but by the property (C5), $(ABH|H) = (ABH|AH) \sqcap (AH|H) \leqslant (ABH|AH) = (B|AH)$, and hence $(A|H) \sqcap (B|H) \leqslant (B|AH)$.

(v) Applying (24), we get: $(A|H \vee K) \sqcap (\bar{A}|H) =$
$= (A(H \vee K)\bar{A}H|H \vee K) \sqcup [(A|H \vee K) \sqcap (\bar{H}\,\bar{K}\,\bar{A}H|H \vee K)] \sqcup [(\bar{A}|H) \sqcap (A(H \vee K)\bar{H}|H \vee K)] =$
$= \bot \sqcup \bot \sqcup [(\bar{A}|H) \sqcap (AK\bar{H}|H \vee K)] \leqslant (AK\bar{H}|H \vee K) = (AK\bar{H}|K) \sqcap (K|H \vee K) \leqslant (AK\bar{H}|K) \leqslant (A|K)$.
□

Note that the property (iv) above can be equivalently expressed under the form

(vi) $(A|H) \models (\bar{B}|H) \sqcup (A|BH)$,

since, obviously $(A|H) \leqslant (\bar{B}|H) \sqcup (A|BH)$ iff $(A|H) \sqcap (B|H) \leqslant (A|BH)$.

Now, each set of (compound) conditionals defines a nonmonotonic consequence relation on the algebra of events $\mathbf{A}$.

Definition 7. Let $\mathcal{F}$ be a consistent set of (compound) conditionals. Then we define the consequence relation $\mid\!\sim_{\mathcal{F}} \subseteq 2^{\mathbf{A}} \times \mathbf{A}$ on events from $\mathbf{A}$ as follows:

$$\{B_1, \ldots, B_n\} \mid\!\sim_{\mathcal{F}} A \text{ if } \mathcal{F} \models (A|B_1...B_n).$$

Equivalently, by Corollary 5, $\{B_1, \ldots, B_n\} \mid\!\sim_{\mathcal{F}} A$ if $\mathcal{F} \vdash_p (A|B_1...B_n)$, that is, if $P(A|B_1,..,B_n) = 1$ for every conditional probability P model of $\mathcal{F}$. Hence $\mid\!\sim_{\mathcal{F}}$ is in fact the p-entailment *relative* to $\mathcal{F}$.

Theorem 12. The consequence relation $\mid\!\sim_{\mathcal{F}}$ satisfies the core properties of System P:

- Reflexivity: $A \mid\!\sim_{\mathcal{F}} A$
- And: if $H \mid\!\sim_{\mathcal{F}} A$ and $H \mid\!\sim_{\mathcal{F}} B$ then $H \mid\!\sim_{\mathcal{F}} AB$
- Cut: if $H \mid\!\sim_{\mathcal{F}} A$ and $AH \mid\!\sim_{\mathcal{F}} B$ then $H \mid\!\sim_{\mathcal{F}} B$
- Or: if $H \mid\!\sim_{\mathcal{F}} A$ and $K \mid\!\sim_{\mathcal{F}} A$ then $H \vee K \mid\!\sim_{\mathcal{F}} A$
- CM: if $H \mid\!\sim_{\mathcal{F}} A$ and $H \mid\!\sim_{\mathcal{F}} B$ then $BH \mid\!\sim_{\mathcal{F}} A$

Moreover it satisfies the following additional property related to disjunction:

- Orm: if $H \vee K \mid\!\sim_{\mathcal{F}} A$ and $H \mid\!\sim_{\mathcal{F}} \bar{A}$ then $K \mid\!\sim_{\mathcal{F}} A$

Proof. Reflexivity trivially holds, and the rest of properties directly follow from properties (i)-(vi) in Prop. 6. □

As a consequence, $\mid\!\sim_{\mathcal{F}}$ is a *preferential consequence relation* in the sense of [15]. We recall that a probabilistic analysis of System P inference rules has been given, in the setting of coherence, in [6]; moreover, the p-validity of such rules has been verified in [25].

It has been shown elsewhere, see e.g. [22], that the Rational Monotony rule

- RM: $H \mid\!\approx A$ and $H \not\mid\!\approx \bar{B}$ then $BH \mid\!\approx A$,

or its equivalent disjunctive variant

- dRM: $H \mid\!\approx A$ then $H \mid\!\approx \bar{B}$ or $BH \mid\!\approx A$,

for entailments $\mid\!\approx$ similar to $\mid\!\sim_{\mathcal{F}}$ is not valid. In fact RM or dRM is not valid in general for $\mid\!\sim_{\mathcal{F}}$, unless the set $\mathcal{F}$ consists of only one atom of $\mathcal{C}(\mathbf{A})$, since in such a case $H \not\mid\!\sim_{\mathcal{F}} \bar{B}$ is equivalent to $H \mid\!\sim_{\mathcal{F}} B$ and then RM becomes just CM. The following is a counter-example of the validity of the rule already in the case that $\mathcal{F}$ is a disjunction of two atoms of $\mathcal{C}(\mathbf{A})$.

Example 5. Consider an algebra $\mathbf{A}$ of events, three (uncertain) logically independent events A, B and H, and the subalgebra $\mathbf{B}$ generated by the partition $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ of $\mathbf{A}$, where

$$\alpha_1 = ABH, \alpha_2 = A\bar{B}H, \alpha_3 = \bar{A}BH, \alpha_4 = \bar{A}\bar{B}H, \text{ and } \alpha_5 = \bar{H}$$

Further, consider the following two conditionals of $C(\mathbf{B})$:

$$\omega_1 = (\alpha_1|\top) \text{ and } \omega_{23} = (\alpha_2|\top) \sqcap (\alpha_3|\bar{\alpha}_2),$$

and let $\mathcal{F} = \{\omega_1 \sqcup \omega_{23}\}$. Then we can check that:

1) $\omega_1 = (ABH|\top) \leqslant (ABH|H) \leqslant (A|H)$, $\omega_1 \leqslant (B|H)$ and hence $\omega_1 \nleqslant (\bar{B}|H)$; similarly $\omega_1 \leqslant A|BH$;

2) $\omega_{23} \leqslant (\alpha_2|\top) = (A\bar{B}H|\top) \leqslant (A\bar{B}H|H) \leqslant (A|H)$; similarly $\omega_{23} \leqslant \bar{B}|H$;

3) $\omega_{23} \leqslant (\alpha_3|\bar{\alpha}_2) = (\bar{A}BH|\bar{A} \vee B \vee \bar{H}) \leqslant (\bar{A}BH|BH) = (\bar{A}|BH)$, hence $\omega_{23} \nleqslant (A|BH)$.

Therefore, we have $\omega_1 \sqcup \omega_{23} \leqslant (A|H)$, $\omega_1 \sqcup \omega_{23} \nleqslant (\bar{B}|H)$ and $\omega_1 \sqcup \omega_{23} \nleqslant (A|BH)$. In other words, $\mathcal{F} \models (A|H)$, $\mathcal{F} \not\models (\bar{B}|H)$ and $\mathcal{F} \not\models (A|BH)$, or equivalently

$$H \mid\!\sim_{\mathcal{F}} A,\ H \mid\!\not\sim_{\mathcal{F}} \bar{B}, \text{ and } BH \mid\!\not\sim_{\mathcal{F}} A.$$

Thus, we have shown a counter-example of the validity of the RM rule for the consequence relation $\mid\!\sim_{\mathcal{F}}$.

However, a weaker probabilistic formulation of the dRM rule, called dWRM (for disjunctive Weak Rational Monotony) [36, 37]) holds, in accordance with [6, 38].

Proposition 7. For any events $A, B, H \in \mathbf{A}$ with $BH \neq \bot$ and for any conditional probability P on $\mathbb{A} \times \mathbb{A}'$, the following rule

- dWRM: if $P(A|H) = 1$, then either $P(\bar{B}|H) = 1$ or $P(A|BH) = 1$

holds.

Proof. By (vi) of Proposition 6, it holds that $(A|H) \models (\bar{B}|H) \sqcup (A|BH)$, and therefore, for any P, $P(A|H) = \mu_P(A|H) \leqslant \mu_P((\bar{B}|H) \sqcup (A|BH))$. Now, by Assertion 6 of Corollary 2, we have that $\mu_P((\bar{B}|H) \sqcap (A|BH)) = P(\bar{B}|H)P(A|BH)$. Therefore, since for any $s, t \in C(\mathbf{A})$ it holds that $\mu_P(s \sqcup t) = \mu_P(s) + \mu_P(t) - \mu_P(s \sqcap t)$, we have the following decomposition expression:

$$\mu_P((\bar{B}|H) \sqcup (A|BH)) = \mu_P(\bar{B}|H) + \mu_P(A|BH) - \mu_P(\bar{B}|H)\mu_P(A|BH) = P(\bar{B}|H) + P(A|BH) - P(\bar{B}|H)P(A|BH),$$

and thus, $\mu_P((\bar{B}|H) \sqcup (A|BH)) = 1$ iff $P(\bar{B}|H) = 1$ or $P(A|BH) = 1$. Indeed, for every x, y, it holds that

$$x + y - xy = 1 \iff (1 - x)(1 - y) = 0 \iff x = 1, \text{ or } y = 1. \tag{44}$$

$\square$

In the following remark, we deepen some probabilistic aspects of the RM and dWRM rules

Remark 7. We first examine the RM and dWRM rules in the light of Example 5; let $\mathcal{F} = \{\omega_1 \sqcup \omega_{23}\}$, and let ω' and ω'' be two atoms such that $\omega' \leqslant \omega_1$ and $\omega'' \leqslant \omega_{23}$, and consider the associated conditional probabilities $P_{\omega'}$ and $P_{\omega''}$. Then, clearly, $\mu_{P_{\omega'}}(\omega_1) = 1$ and $\mu_{P_{\omega''}}(\omega_{23}) = 1$. Now let $P = P_{\omega'}$ or $P = P_{\omega''}$. It holds that

$$P(A|H) = \mu_P(\omega_1 \sqcup \omega_{23}) = 1\,, \quad P(\bar{B}|H) = \mu_P(\omega_{23}), \quad P(A|BH) = \mu_P(\omega_1).$$

Of course $\mathcal{F} \vdash_p A|H$. Moreover, as $P_{\omega'}$ is a p-model of $\mathcal{F}$ but not of $\bar{B}|H$, then $\mathcal{F} \nvdash_p \bar{B}|H$. Similarly, as $P_{\omega''}$ is a p-model of $\mathcal{F}$ but not of $A|BH$, then $\mathcal{F} \nvdash_p A|BH$. Thus, $\mathcal{F} \vdash_p A|H$, $\mathcal{F} \nvdash_p \bar{B}|H$, and $\mathcal{F} \nvdash_p A|BH$, that is RM rule is not valid.

Concerning dWRM rule we observe that: (i) if $\mu_P(\omega_1) = 1$ then $P(A|H) = 1$, $P(\bar{B}|H) = 0$, and $P(A|BH) = 1$; (ii) if $\mu_P(\omega_{23}) = 1$ then $P(A|H) = 1$, $P(\bar{B}|H) = 1$, and $P(A|BH) = 0$. Both cases (i) and (ii) are in agreement with dWRM rule. Notice that there are no conditional probabilities P such that $\mu_P(\omega_1 \sqcup \omega_{23}) = 1$, with $0 < \mu_P(\omega_1) = a < 1$. Indeed, in this case it would be $P(A|H) = 1$, $P(\bar{B}|H) = 1-a$, $P(A|BH) = a$, with $0 < a < 1$. But, as shown in Example 4, the assessment $P(A|H) = 1$, $P(\bar{B}|H) = 1-a$, $P(A|BH) = a$ is coherent if and only if $a = 0$ or $a = 1$.

We conclude by providing some further probabilistic details about the validity of the dWRM rule and the non-validity of the RM rule. First of all, observe that the dWRM rule can be equivalently written as ([38, Equation (3)]):

- if $P(A|H) = 1$ and $P(\bar{B}|H) < 1$, then $P(A|BH) = 1$.

Let be given any set of (compound) conditionals $\mathcal{F}$ and the conditional events $A|H, \bar{B}|H, A|BH$. We denote by $(\mathcal{P}, x, y, z)$ a coherent probability assessment on $\mathcal{F} \cup \{A|H, \bar{B}|H, A|BH\}$, where $\mathcal{P}$ is a probability assessment on $\mathcal{F}$ (possibly $\mathcal{P}$ is the restriction on $\mathcal{F}$ of some μ_P), with $x = P(A|H), y = P(\bar{B}|H)$, and $z = P(A|BH)$. We recall that ([6]) the probability assessment (x, y, z) on $\{A|H, \bar{B}|H, A|BH\}$ is coherent for every $(x, y) \in [0, 1]^2$ and $z' \leqslant z \leqslant z''$, where

$$z' = \begin{cases} \frac{x-y}{1-y}, & \text{if } x > y, \\ 0, & \text{if } x \leqslant y, \end{cases} \qquad z'' = \begin{cases} \frac{x}{1-y}, & \text{if } x + y < 1, \\ 1, & \text{if } x + y \geqslant 1. \end{cases} \tag{45}$$

We observe that when $x = 1$ and $y < 0$ it holds that $z = 1$; when $x = 1$ and $y = 1$ it holds that $z \in [0, 1]$.

Now, let us further assume that $\mathcal{F}$ is consistent and such that

$$(i)\; H \mid\!\sim_{\mathcal{F}} A, \quad (ii)\; H \mid\!\not\sim_{\mathcal{F}} \bar{B}, \quad (iii)\; BH \mid\!\not\sim_{\mathcal{F}} A,$$

whose existence is ensured by Example 5. In what follows we denote by $\mathcal{P}_1$ the assessment on $\mathcal{F}$ such that $\mathcal{P}_1(t) = 1$ for all $t \in \mathcal{F}$. Then, the following assertions are valid:

(a) from the condition (i), the assessment $(\mathcal{P}_1, x)$ on $\mathcal{F} \cup \{A|H\}$ is coherent only if $x = 1$;

(b) the imprecise assessment $(\mathcal{P}_1, 1, 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 1)$ is g-coherent, that is there exist (at least) two values y, z such that the assessment $(\mathcal{P}_1, 1, y, z)$ on $\mathcal{F} \cup \{A|H, \bar{B}|H, A|BH\}$ is coherent;

(c) from the conditions (i) and (ii) there exists a value $y^* < 1$ such that the assessment $(\mathcal{P}_1, 1, y^*)$ on $\mathcal{F} \cup \{A|H, \bar{B}|H\}$ is coherent; then, by dWRM rule, the assessment $(\mathcal{P}_1, 1, y^*, z)$, is coherent only if $z = 1$;

(d) if the assessment $(\mathcal{P}_1, 1, 1)$ on $\mathcal{F} \cup \{A|H, \bar{B}|H\}$ is coherent, then the assessment $(\mathcal{P}_1, 1, 1, 0 \leqslant z \leqslant 1)$ is g-coherent.

Therefore, as shown by the previous assertions, if we assess $\mathcal{P}_1$ on $\mathcal{F}$, we can only derive that $x = 1$ while we can infer on the value of z only once the value of y is also specified.

We end this section with an example related to the failure of the transitive property for the consequence relation $\mid\!\sim_{\mathcal{F}}$.

Example 6. Consider three (uncertain) logically independent events A, B and C in an algebra $\mathbf{A}$ of events. Let $\mathbf{B}$ be the subalgebra generated by the partition $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ of $\mathbf{A}$, where

$$\alpha_1 = ABC,\, \alpha_2 = AB\bar{C},\, \alpha_3 = A\bar{B}C,\, \alpha_4 = A\bar{B}\,\bar{C},\, \alpha_5 = \bar{A}BC,\, \alpha_6 = \bar{A}B\bar{C},\, \alpha_7 = \bar{A}\,\bar{B}.$$

Further, consider the following conjoined conditional of $C(\mathbf{B})$:

$$\omega_{52} = \alpha_5 \sqcap (\alpha_2|\bar{\alpha}_5) = \bar{A}BC \sqcap (AB\bar{C}|(\bar{A}\vee\bar{B}\vee C)),$$

and let $\mathcal{F} = \{\omega_{52}\}$. Then we can check that:

1) $\omega_{52} \leqslant (C|B)$, $\omega_{52} \leqslant (B|A)$ and hence $\omega_{52} \leqslant (C|B) \sqcap (B|A)$;

2) $\omega_{52} \leqslant (\bar{C}|A)$ and hence $\omega_{52} \nleqslant (C|A)$.

Therefore, $\{C|B, B|A\}$ is consistent and we have $\omega_{52} \leqslant (C|B)$, $\omega_{52} \leqslant (B|A)$, but $\omega_{52} \nleqslant (C|A)$. In other words, $\mathcal{F} \models (C|B)$, $\mathcal{F} \models (B|A)$ and $\mathcal{F} \not\models (C|A)$, or equivalently

$$B \mathrel{|\!\sim}_{\mathcal{F}} C,\; A \mathrel{|\!\sim}_{\mathcal{F}} B, \text{ and } A \mathrel{|\!\not\sim}_{\mathcal{F}} C.$$

Thus, we have shown a counter-example of the validity of the *transitivity* rule for the consequence relation $\mathrel{|\!\sim}_{\mathcal{F}}$. Moreover, we observe that

$$\{C|B, B|A, (A|A\vee B)\} \models C|A$$

or equivalently that

$$\{C|B, B|A\} \models (C|A) \sqcup (\bar{A}|A\vee B). \tag{46}$$

Indeed, by applying (24),

$$\begin{aligned}
&(C|B) \sqcap (B|A) \sqcap (A|A\vee B) = \\
&= [(ABC|A \vee B) \sqcup (C|B) \sqcap (\bar{B}AB|A\vee B) \sqcup (B|A) \sqcap (\bar{A}BC|A\vee B)] \sqcap (A|A \vee B) = \\
&= (ABC|A\vee B) \sqcup \bot \sqcup \bot = (ABC|A\vee B) = (BC|A) \sqcap (A|A\vee B) \leqslant (BC|A) \leqslant (C|A).
\end{aligned}$$

Then, we obtain a weaker version of transitivity (see [25]):

$$\text{if } B \mathrel{|\!\sim}_{\mathcal{F}} C,\, A \mathrel{|\!\sim}_{\mathcal{F}} B \text{ and } A\vee B \mathrel{|\!\sim}_{\mathcal{F}} B, \text{ then } A \mathrel{|\!\sim}_{\mathcal{F}} C.$$

Finally, we can verify the following probabilistic version of weak transitivity

"if $P(C|B) = 1$ and $P(B|A) = 1$, then either $P(C|A) = 1$, or $P(\bar{A}|A \vee B) = 1$",

or equivalently (see [38])

"if $P(C|B) = 1$, $P(B|A) = 1$, and $P(A|A \vee B) > 0$, then $P(C|A) = 1$".

Indeed, as $A\bar{A} = \bot$ and $A \leqslant A \vee B$, from Assertion 6 of Corollary 2 it holds that

$$\mu_P((C|A) \sqcap (\bar{A}|A\vee B)) = P(C|A)P(\bar{A}|A\vee B).$$

Then, based on (42)

$$\mu_P((C|A) \sqcup (\bar{A}|A\vee B)) = P(C|A) + P(\bar{A}|A\vee B) - P(C|A)P(\bar{A}|A\vee B).$$

Moreover, by recalling (44), $\mu_P((C|A) \sqcup (\bar{A}|A\vee B)) = 1$ iff $P(C|A) = 1$ or $P(\bar{A}|A\vee B) = 1$. Thus, if $P(C|B) = P(B|A) = 1$, then by Lemma 2 $\mu_P((C|B) \sqcap (B|A)) = 1$. Finally, from (46), $\mu_P[(C|A) \sqcup (\bar{A}|A \vee$

$B)] = 1$ and hence $P(C|A) = 1$, or $P(\bar{A}|A \vee B) = 1$.
Notice that the implication

$$P(C|B) = P(B|A) = 1 \implies \mu_P[(C|B) \sqcap (B|A)] = 1$$

also follows by observing that $\mu_P[(C|B) \sqcap (B|A)] \geqslant \max\{P(C|B) + P(B|A) - 1, 0\}$. More in general, given two conditional events $A|H$ and $B|K$, the Fréchet-Hoeffding bounds for μ_P are satisfied, that is

$$\max\{\mu_P(A|H) + \mu_P(B|K) - 1, 0\} \leqslant \mu_P[(A|H) \sqcap (B|K)] \leqslant \min\{\mu_P(A|H), \mu_P(B|K)\}, \tag{47}$$

or equivalently

$$\max\{P(A|H) + P(B|K) - 1, 0\} \leqslant \mu_P[(A|H) \sqcap (B|K)] \leqslant \min\{P(A|H), P(B|K)\}. \tag{48}$$

Indeed, as $(A|H) \sqcap (B|K) \leqslant (A|H)$ and $(A|H) \sqcap (B|K) \leqslant (B|K)$, the rightmost inequality in (47) is satisfied. Moreover, $\mu_P([(A|H) \sqcap (B|K)] \geqslant 0$ and, from (42),

$$\mu_P([(A|H) \sqcap (B|K)]) = P(A|H) + P(B|K) - \mu_P[(A|H) \sqcup (B|K)] \geqslant P(A|H) + P(B|K) - 1,$$

because $\mu_P[(A|H) \sqcup (B|K)] \leqslant 1$. Thus, the leftmost inequality in (47) is satisfied. The inequalities in (48) are in agreement with the approach given in [24] where the conjunction is defined as the conditional random quantity $(A|H) \wedge (B|K)$ recalled in (34).

8. Conclusions and future work

In this paper we have advanced the study of conditionals in the setting of the Boolean algebras of conditionals as proposed in [22]. More precisely, after a first analysis on the lattice order of our algebras and the known Goodman and Nguyen order relation, we have considered the canonical extension μ_P of a conditional probability P on $\mathbb{A} \times \mathbb{A}'$ to the Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$. Our first main result establishes that for every basic conditional $(A|H)$, $P(A|H) = \mu_P(A|H)$ and hence the conditional probability P coincides with the restriction of μ_P to basic conditionals.

In turn we get an operational computation of the probability of a conjunction and a disjunction of conditionals, in agreement with previous approaches in the literature, in particular with the one developed by Gilio and Sanfilippo by formalising conditionals as random quantities [24].

Finally, we have discussed relations of our approach with nonmonotonic reasoning. First we have introduced a (monotonic) entailment relation among conditionals defined in terms of the lattice order of $\mathcal{C}(\mathbf{A})$ and then we have examined a nonmonotonic consequence relation on the algebra $\mathbf{A}$, which satisfies the well-known rules of the system P. Moreover, we have discussed the Rational Monotony and the disjunctive Weak Rational Monotony rules.

As for future work, an aspect to be deepened concerns the notion of iterated conditional, say $(B|K)|(A|H)$, and its probability in the realm of Boolean algebras of conditionals. Indeed, if we define $\mu_P((B|K)|(A|H)) =_{def} \frac{\mu_P(A|H) \sqcap (B|K))}{\mu_P(A|H)}$, then, under the hypothesis $P(A|H) > 0$, it holds that

$$\mu_P((B|K)|(A|H)) = \frac{P(AHBK|(H \vee K)) + P(A|H)P(\bar{H}BK|(H \vee K)) + P(B|K)P(\bar{K}AH|(H \vee K))}{P(A|H)}, \tag{49}$$

which is the prevision of the iterated conditional $(B|K)|(A|H)$ obtained in the setting of coherence in [35, Section 6] (see also [39]). Under the further assumption $P(H \vee K) > 0$, formula (49) coincides with the result given in [8, Thm. 3]. For some applications of iterated conditionals see e.g. [40, 41].

Encouraged by the above obtained results, we also plan to deepen into the relationship between the approach based on Boolean algebras of conditionals, together with canonical extensions of conditional probabilities on events, and the approach based on interpreting compound and iterated conditionals as random quantities, see [23] for promising first results.

Appendix A. Proof of Theorem 3

Theorem 3. For each atom $\omega'_{j_1\cdots j_{n-2}} \in \mathrm{at}(C(\mathbf{B}))$, the following holds:

$$\begin{aligned}\mu_P(\omega'_{j_1\cdots j_{n-2}}) &= \mu_P((\beta_{j_1}|\top) \sqcap (\beta_{j_2}|\bar\beta_{j_1}) \sqcap \cdots \sqcap (\beta_{j_{n-2}}|\bar\beta_{j_1}\cdots\bar\beta_{j_{n-3}})) = \\ &= P(\beta_{j_1}|\top)P(\beta_{j_2}|\bar\beta_{j_1})\cdots P(\beta_{j_{n-2}}|\bar\beta_{j_1}\cdots\bar\beta_{j_{n-3}}) = \\ &= \mu_{P'}(\omega'_{j_1\cdots j_{n-2}}).\end{aligned} \tag{17}$$

Proof. We first examine the simple case where $\beta_{j_{n-1}} = \beta_i = \alpha_i \vee \alpha_{i+1}$, and $\beta_{j_h} = \alpha_{t_h}$, $h = 1, \ldots n-2$, where $(t_1, \ldots, t_{n-2})$ is a permutation of $(1, \ldots, i-1, i+2, \ldots, n\}$. In this case, we have:

$$\begin{aligned}&\omega'_{j_1\cdots j_{n-2}} = (\beta_{j_1}|\top) \sqcap (\beta_{j_2}|\bar\beta_{j_1}) \sqcap \cdots \sqcap (\beta_{j_{n-2}}|\bar\beta_{j_1}\cdots\bar\beta_{j_{n-3}}) = \\ &= (\alpha_{t_1}|\top) \sqcap (\alpha_{t_2}|\bar\alpha_{t_1}) \sqcap \cdots \sqcap (\alpha_{t_{n-2}}|\bar\alpha_{t_1}\cdots\bar\alpha_{t_{n-3}}) = \omega_{t_1\cdots t_{n-2}}.\end{aligned}$$

Then, by recalling (13),

$$\begin{aligned}&\mu_P(\omega'_{j_1\cdots j_{n-2}}) = \mu_P(\omega_{t_1\cdots t_{n-2}}) = P(\alpha_{t_1}|\top)P(\alpha_{t_2}|\bar\alpha_{t_1})\cdots P(\alpha_{t_{n-2}}|\bar\alpha_{t_1}\cdots\bar\alpha_{t_{n-3}}) = \\ &= P(\beta_{j_1}|\top)P(\beta_{j_2}|\bar\beta_{j_1})\cdots P(\beta_{j_{n-2}}|\bar\beta_{j_1}\cdots\bar\beta_{j_{n-3}}) = \mu_{P'}(\omega'_{j_1\cdots j_{n-2}}).\end{aligned}$$

We now consider the case where $\beta_{j_{n-1}} \neq \beta_i$. Without loss of generality, we prove (17) when $(j_1, \ldots, j_{n-1}) = (1, \ldots, n-1)$ and hence $\beta_{n-1} \neq \beta_i$, so that $i \in \{1, \ldots, n-2\}$. By recalling (15) it holds that

$$(\beta_i|\beta_i \vee \cdots \vee \beta_{n-1}) = (\alpha_i \vee \alpha_{i+1}|\alpha_i \vee \cdots \vee \alpha_n) = (\alpha_i|\alpha_i \vee \cdots \vee \alpha_n) \sqcup (\alpha_{i+1}|\alpha_i \vee \cdots \vee \alpha_n).$$

By defining

$$r_{k\cdots n-1} = (\alpha_k|\alpha_k \vee \cdots \vee \alpha_n) \sqcap \cdots \sqcap (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n), \quad \text{for } k = 1, \ldots, n-1,$$

it holds that $r_{1\cdots n-1} = \omega_{1\cdots n-1} = \omega_{1\cdots k-1} \sqcap r_{k\cdots n-1}$, $k = 2, \ldots, n-1$. Moreover

$$\omega_{1\cdots n-1} = \omega_{1\cdots k-1} \sqcap (\alpha_k|\alpha_k \vee \cdots \vee \alpha_n) \sqcap r_{k+1\cdots n-1}, \quad k = 2 \ldots n-2. \tag{A.1}$$

Then,

$$\begin{aligned}\omega'_{12\cdots n-2} &= (\beta_1|\top) \sqcap \cdots \sqcap (\beta_{n-2}|\bar\beta_1 \wedge \cdots \wedge \bar\beta_{n-3}) = \\ &= (\alpha_1|\top) \sqcap \cdots \sqcap (\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_i \vee \alpha_{i+1}|\alpha_i \vee \cdots \vee \alpha_n) \sqcap \\ &\quad \sqcap (\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n) \sqcap \cdots \sqcap (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) = \\ &= \omega_{1\cdots i-1} \sqcap (\alpha_i \vee \alpha_{i+1}|\alpha_i \vee \cdots \vee \alpha_n) \sqcap r_{i+2\cdots n-1} = \\ &= [\omega_{1\cdots i-1} \sqcap (\alpha_i|\alpha_i \vee \cdots \vee \alpha_n) \sqcup \omega_{1\cdots i-1} \sqcap (\alpha_{i+1}|\alpha_i \vee \cdots \vee \alpha_n)] \sqcap r_{i+2\cdots n-1} = \\ &= \omega_{1\cdots i} \sqcap r_{i+2\cdots n-1} \sqcup \omega_{1\cdots i-1 i+1} \sqcap r_{i+2\cdots n-1} = \\ &= W_{1\cdots i, i+2\cdots n-1} \sqcup W_{1\cdots i-1\, i+1, i+2\cdots n-1},\end{aligned} \tag{A.2}$$

where

$$W_{1\cdots i, i+2\cdots n-1} = \omega_{1\cdots i} \sqcap r_{i+2\cdots n-1}, \quad W_{1\cdots i-1\, i+1, i+2\cdots n-1} = \omega_{1\cdots i-1 i+1} \sqcap r_{i+2\cdots n-1}.$$

Notice that, when $i = 1$, the symbol $\omega_{1\cdots i-1}$ disappears; moreover the symbols $\omega_{1\cdots i}$ and $\omega_{1\cdots i-1 i+1}$ coincide with $\omega_1 = \alpha_1|\top$ and $\omega_2 = \alpha_2|\top$, respectively. From (A.1), it holds that

$$W_{1\cdots i\,,\,i+2\cdots n-1} \sqcap (\alpha_{i+1}|\alpha_{i+1}\vee\cdots\vee\alpha_n) = \omega_{1\cdots i} \sqcap (\alpha_{i+1}|\alpha_{i+1}\vee\cdots\vee\alpha_n) \sqcap r_{i+2\cdots n-1} = \omega_{1\cdots n-1},$$

and

$$W_{1\cdots i-1\,i+1\,,\,i+2\cdots n-1} \sqcap (\alpha_i|\alpha_i\vee\alpha_{i+2}\vee\cdots\vee\alpha_n) = \omega_{1\cdots i-1 i+1} \sqcap (\alpha_i|\alpha_i\vee\alpha_{i+2}\vee\cdots\vee\alpha_n) \sqcap r_{i+2\cdots n-1} = \omega_{1\cdots i-1 i+1 i i+2\cdots n}.$$

We now first examine the term $W_{1\cdots i\,,\,i+2\cdots n-1} = \omega_{1\cdots i} \sqcap r_{i+2\cdots n-1}$. By applying (C5) with $A = \alpha_{i+2}$, $B = \alpha_{i+2}\vee\cdots\vee\alpha_n$, and $C = \alpha_{i+1}\vee\cdots\vee\alpha_n$, it holds that $(B|C) \sqcap (A|B) = (A|B) \sqcap (B|C) = A|C$, that is

$$(\alpha_{i+2}\vee\cdots\vee\alpha_n|\alpha_{i+1}\vee\cdots\vee\alpha_n) \sqcap (\alpha_{i+2}|\alpha_{i+2}\vee\cdots\vee\alpha_n) = (\alpha_{i+2}|\alpha_{i+1}\vee\cdots\vee\alpha_n). \tag{A.3}$$

Then, by using (A.1) and (A.3), it follows that

$$\begin{aligned}
&W_{1\cdots i\,,\,i+2\cdots n-1} = \omega_{1\cdots i} \sqcap r_{i+2\cdots n-1} = \omega_{1\cdots i} \sqcap r_{i+2\cdots n-1} \sqcap (\alpha_{i+1}\vee\cdots\vee\alpha_n|\alpha_{i+1}\vee\cdots\vee\alpha_n) = \\
&= [\omega_{1\cdots i} \sqcap (\alpha_{i+1}|\alpha_{i+1}\vee\cdots\vee\alpha_n) \sqcap r_{i+2\cdots n-1}] \sqcup [\omega_{1\cdots i} \sqcap (\alpha_{i+2}\vee\cdots\vee\alpha_n|\alpha_{i+1}\vee\cdots\vee\alpha_n) \sqcap r_{i+2\cdots n-1}] = \\
&= \omega_{1\cdots n-1} \sqcup \omega_{1\cdots i} \sqcap (\alpha_{i+2}\vee\cdots\vee\alpha_n|\alpha_{i+1}\vee\cdots\vee\alpha_n) \sqcap (\alpha_{i+2}|\alpha_{i+2}\vee\cdots\vee\alpha_n) \sqcap r_{i+3\cdots n-1} = \\
&= \omega_{1\cdots n-1} \sqcup \omega_{1\cdots i} \sqcap (\alpha_{i+2}|\alpha_{i+1}\vee\cdots\vee\alpha_n) \sqcap r_{i+3\cdots n-1} = \omega_{1\cdots n-1} \sqcup \omega_{1\cdots i i+2} \sqcap r_{i+3\cdots n-1}.
\end{aligned}$$

Then, we obtain

$$W_{1\cdots i\,,\,i+2\cdots n-1} = \omega_{1\cdots i} \sqcap r_{i+2\cdots n-1} = \omega_{1\cdots n-1} \sqcup W_{1\cdots i\,i+2\,,\,i+3\cdots n-1}, \tag{A.4}$$

where $W_{1\cdots i\,i+2\,,\,i+3\cdots n-1} = \omega_{1\cdots i i+2} \sqcap r_{i+3\cdots n-1}$. Concerning $W_{1\cdots i\,i+2\,,\,i+3\cdots n-1}$, we observe that

$$\begin{aligned}
&W_{1\cdots i\,i+2\,,\,i+3\cdots n-1} = \omega_{1\cdots i i+2} \sqcap r_{i+3\cdots n-1} = \\
&= \omega_{1\cdots i i+2} \sqcap (\alpha_{i+1}\vee\alpha_{i+3}\vee\cdots\vee\alpha_n|\alpha_{i+1}\vee\alpha_{i+3}\vee\cdots\vee\alpha_n) \sqcap r_{i+3\cdots n-1} = \omega_{1\cdots i\,i+2\,i+1\,i+3\cdots n-1} \sqcup \\
&\sqcup \omega_{1\cdots i i+2} \sqcap (\alpha_{i+3}\vee\cdots\vee\alpha_n|\alpha_{i+1}\vee\alpha_{i+3}\vee\cdots\vee\alpha_n) \sqcap (\alpha_{i+3}|\alpha_{i+3}\vee\cdots\vee\alpha_n) \sqcap r_{i+4\cdots n-1} = \\
&= \omega_{1\cdots i\,i+2\,i+1\,i+3\cdots n-1} \sqcup \omega_{1\cdots i i+2} \sqcap (\alpha_{i+3}|\alpha_{i+1}\vee\alpha_{i+3}\vee\cdots\vee\alpha_n) \sqcap r_{i+4\cdots n-1} = \\
&= \omega_{1\cdots i\,i+2\,i+1\,i+3\cdots n-1} \sqcup \omega_{1\cdots i i+2 i+3} \sqcap r_{i+4\cdots n-1} = \omega_{1\cdots i\,i+2\,i+1\,i+3\cdots n-1} \sqcup W_{1\cdots i i+2 i+3\,,\,i+4\cdots n-1},
\end{aligned}$$

where $W_{1\cdots i i+2 i+3\,,\,i+4\cdots n-1} = \omega_{1\cdots i i+2 i+3} \sqcap r_{i+4\cdots n-1}$. Then, by iteration, we obtain

$$\begin{aligned}
&W_{1\cdots i\,,\,i+2\cdots n-1} = \omega_{1\cdots n-1} \sqcup W_{1\cdots i\,i+2\,,\,i+3\cdots n-1} = \omega_{1\cdots n-1} \sqcup \omega_{1\cdots i\,i+2\,i+1\,i+3\cdots n-1} \sqcup W_{1\cdots i i+2 i+3\,,\,i+4\cdots n-1} = \\
&= \cdots = \omega_{1\cdots n-1} \sqcup \omega_{1\cdots i\,i+2\,i+1\,i+3\cdots n-1} \sqcup \cdots \sqcup \omega_{1\cdots i\,i+2\,\cdots\,n-2 i+1 n-1} \sqcup W_{1\cdots i\,i+2\cdots n-1},
\end{aligned} \tag{A.5}$$

where $W_{1\cdots i\,i+2\cdots n-1} = \omega_{1\cdots i\,i+2\,\cdots n-1} = \omega_{1\cdots i\,i+2\,\cdots n-1 i+1} \sqcup \omega_{1\cdots i\,i+2\,\cdots n-1 n}$.
Notice that (A.5) has been obtained by exploiting (A.4) and the relation

$$W_{1\cdots i\,i+2\cdots i+k\,,\,i+k+1\cdots n-1} = \omega_{1\cdots i\,i+2\cdots i+k\,i+1\,i+k+1\cdots n-1} \sqcup W_{1\cdots i\,i+2\cdots i+k+1\,,\,i+k+2\cdots n-1}, \quad k = 2,\ldots,n-i-2, \tag{A.6}$$

which, for $k = n-i-2$, becomes $W_{1\cdots i\,i+2\cdots n-2\,,\,n-1} = \omega_{1\cdots i\,i+2\,\cdots\,n-2 i+1 n-1} \sqcup W_{1\cdots i\,i+2\cdots n-1}$. Thus,

$$W_{1\cdots i\,,\,i+2\cdots n-1} = \omega_{1\cdots n-1} \sqcup \omega_{1\cdots i\,i+2\,i+1\,i+3\cdots n-1} \sqcup \cdots \sqcup \omega_{1\cdots i\,i+2\,\cdots\,n-2 i+1 n-1} \sqcup \omega_{1\cdots i\,i+2\,\cdots n-1}. \tag{A.7}$$

We recall, from (13), that $\mu_P(\omega_{i_1\ldots i_k}) = P(\alpha_{i_1})P(\alpha_{i_2}|\alpha_{i_2}\vee\cdots\vee\alpha_{i_n})\cdots P(\alpha_{i_k}|\alpha_{i_k}\vee\cdots\vee\alpha_{i_n})$; then

$$\begin{aligned}
&\mu_P(W_{1\cdots i\,i+2\cdots n-1}) = \mu_P(\omega_{1\cdots i\,i+2\,\cdots n-1}) = \\
&= P(\alpha_1)\cdots P(\alpha_i|\alpha_i\vee\cdots\vee\alpha_n)P(\alpha_{i+2}|\alpha_{i+1}\vee\alpha_{i+2}\vee\cdots\vee\alpha_n)\cdots P(\alpha_{n-1}|\alpha_{i+1}\vee\alpha_{n-1}\vee\alpha_n) = \\
&= P(\omega_{1\cdots i\,i+2\,\cdots n-2})P(\alpha_{n-1}|\alpha_{i+1}\vee\alpha_{n-1}\vee\alpha_n).
\end{aligned}$$

Moreover, as

$$W_{1\cdots i\, i+2\cdots n-2,n-1} = \omega_{1\cdots i\, i+2\cdots n-2\, i+1\, n-1} \sqcup W_{1\cdots i\, i+2\cdots n-1},$$

it holds that

$$\begin{aligned}&\mu_P(W_{1\cdots i\, i+2\cdots n-2,n-1}) = \mu_P(\omega_{1\cdots i\, i+2\cdots n-2\, i+1\, n-1}) + \mu_P(W_{1\cdots i\, i+2\cdots n-1}) = \\ &= \mu_P(\omega_{1\cdots i\, i+2\, \cdots n-2})[P(\alpha_{i+1}|\alpha_{i+1}\vee\alpha_{n-1}\vee\alpha_n)P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n) + P(\alpha_{n-1}|\alpha_{i+1}\vee\alpha_{n-1}\vee\alpha_n)].\end{aligned} \tag{A.8}$$

By applying (16), with $A = \alpha_{n-1}$, $B = \alpha_{n-1}\vee\alpha_n$, and $C = \alpha_{i+1}\vee\alpha_{n-1}\vee\alpha_n$, as $\bar{B}C|C = \bar{B}|C$, it holds that

$$\begin{aligned}&P(\alpha_{i+1}|\alpha_{i+1}\vee\alpha_{n-1}\vee\alpha_n)P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n) + P(\alpha_{n-1}|\alpha_{i+1}\vee\alpha_{n-1}\vee\alpha_n) = P(\bar{B}C|C)P(A|B) + P(A|C) = \\ &= P(\bar{B}|C)P(A|B) + P(A|C) = P(A|B) = P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n).\end{aligned}$$

Therefore

$$\mu_P(W_{1\cdots i\, i+2\cdots n-2,n-1}) = \mu_P(\omega_{1\cdots i\, i+2\, \cdots n-2})P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n). \tag{A.9}$$

Now, as $W_{1\cdots i\, i+2\cdots n-3,n-2n-1} = \omega_{1\cdots i\, i+2\cdots n-3\, i+1\, n-2n-1} \sqcup W_{1\cdots i\, i+2\cdots n-2,n-1}$, by taking into account (A.9) it holds that

$$\begin{aligned}&\mu_P(W_{1\cdots i\, i+2\cdots n-3,n-2n-1}) = \mu_P(\omega_{1\cdots i\, i+2\cdots n-3\, i+1\, n-2n-1}) + \mu_P(W_{1\cdots i\, i+2\cdots n-2,n-1}) = \\ &\mu_P(\omega_{1\cdots i\, i+2\cdots n-3\, i+1\, n-2n-1}) + \mu_P(\omega_{1\cdots i\, i+2\, \cdots n-2})P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n) = \\ &= \mu_P(\omega_{1\cdots i\, i+2\cdots n-3\, i+1\, n-2n-1}) + \mu_P(\omega_{1\cdots i\, i+2\, \cdots n-3})P(\alpha_{n-2}|\alpha_{i+1}\vee\alpha_{n-2}\vee\alpha_{n-1}\vee\alpha_n)P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n) = \\ &= \mu_P(\omega_{1\cdots i\, i+2\, \cdots n-3})[P(\alpha_{i+1}|\alpha_{i+1}\vee\alpha_{n-2}\vee\alpha_{n-1}\vee\alpha_n)P(\alpha_{n-2}|\alpha_{n-2}\vee\alpha_{n-1}\vee\alpha_n) + \\ &+P(\alpha_{n-2}|\alpha_{i+1}\vee\alpha_{n-2}\vee\alpha_{n-1}\vee\alpha_n)]P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n).\end{aligned}$$

By applying (16), with $A = \alpha_{n-2}$, $B = \alpha_{n-2}\vee\alpha_{n-1}\vee\alpha_n$, and $C = \alpha_{i+1}\vee\alpha_{n-2}\vee\alpha_{n-1}\vee\alpha_n$, it holds that

$$\begin{aligned}&P(\alpha_{i+1}|\alpha_{i+1}\vee\alpha_{n-2}\vee\alpha_{n-1}\vee\alpha_n)P(\alpha_{n-2}|\alpha_{n-2}\vee\alpha_{n-1}\vee\alpha_n) + P(\alpha_{n-2}|\alpha_{i+1}\vee\alpha_{n-2}\vee\alpha_{n-1}\vee\alpha_n) = \\ &= P(\alpha_{n-2}|\alpha_{n-2}\vee\alpha_{n-1}\vee\alpha_n).\end{aligned}$$

Then,

$$\mu_P(W_{1\cdots i\, i+2\cdots n-3,n-2n-1}) = \mu_P(\omega_{1\cdots i\, i+2\, \cdots n-3})P(\alpha_{n-2}|\alpha_{n-2}\vee\alpha_{n-1}\vee\alpha_n)P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n).$$

By iterating the previous reasoning, for every $k = 2, \dots, n-i-2$ it holds that

$$\mu_P(W_{1\cdots i\, i+2\cdots i+k\,,\, i+k+1\cdots n-1}) = \mu_P(\omega_{1\cdots i\, i+2\cdots i+k})P(\alpha_{i+k+1}|\alpha_{i+k+1}\vee\cdots\vee\alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n).$$

In particular, for $k = 2$, one has

$$\begin{aligned}&\mu_P(W_{1\cdots i\, i+2\,,\, i+3\cdots n-1}) = \mu_P(\omega_{1\cdots i\, i+2})P(\alpha_{i+3}|\alpha_{i+3}\vee\cdots\vee\alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n) = \\ &= \mu_P(\omega_{1\cdots i})P(\alpha_{i+2}|\alpha_{i+1}\vee\alpha_{i+2}\vee\cdots\vee\alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n).\end{aligned}$$

Then, by recalling (A.4), we obtain

$$\begin{aligned}&\mu_P(W_{1\cdots i\,,\, i+2\cdots n-1}) = \mu_P(\omega_{1\cdots n-1}) + \mu_P(W_{1\cdots i\, i+2\,,\, i+3\cdots n-1}) = \\ &= \mu_P(\omega_{1\cdots i})[P(\alpha_{i+1}|\alpha_{i+1}\vee\alpha_{i+2}\vee\cdots\vee\alpha_n)P(\alpha_{i+2}|\alpha_{i+2}\vee\cdots\vee\alpha_n) + P(\alpha_{i+2}|\alpha_{i+1}\vee\alpha_{i+2}\vee\cdots\vee\alpha_n)]\cdot \\ &\cdot P(\alpha_{i+3}|\alpha_{i+3}\vee\cdots\vee\alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n)\end{aligned} \tag{A.10}$$

By applying (16), with $A = \alpha_{i+2}$, $B = \alpha_{i+2}\vee\cdots\vee\alpha_n$, and $C = \alpha_{i+1}\vee\alpha_{i+2}\vee\cdots\vee\alpha_n$, it holds that

$$\begin{aligned}&P(\alpha_{i+1}|\alpha_{i+1}\vee\alpha_{i+2}\vee\cdots\vee\alpha_n)P(\alpha_{i+2}|\alpha_{i+2}\vee\cdots\vee\alpha_n) + P(\alpha_{i+2}|\alpha_{i+1}\vee\alpha_{i+2}\vee\cdots\vee\alpha_n) = \\ &= P(\alpha_{i+2}|\alpha_{i+2}\vee\cdots\vee\alpha_n).\end{aligned}$$

Then, (A.10) becomes

$$\begin{aligned}&\mu_P(W_{1\cdots i\,,\,i+2\cdots n-1}) = \mu_P(\omega_{1\cdots i})P(\alpha_{i+2}|\alpha_{i+2}\vee\cdots\vee\alpha_n)P(\alpha_{i+3}|\alpha_{i+3}\vee\cdots\vee\alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n) = \\ &= P(\alpha_1|\top)\cdots P(\alpha_i|\alpha_i\vee\alpha_n)P(\alpha_{i+2}|\alpha_{i+2}\vee\cdots\vee\alpha_n)P(\alpha_{i+3}|\alpha_{i+3}\vee\cdots\vee\alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n)\,,\end{aligned} \tag{A.11}$$

which shows that the factorization property of μ_P holds for $W_{1\cdots i\,,\,i+2\cdots n-1}$.
We now examine the term $W_{1\cdots i-1\,i+1\,,\,i+2\cdots n-1}$ introduced in (A.2). By applying a similar reasoning as from (A.3) to (A.11), the term $W_{1\cdots i-1\,i+1\,,\,i+2\cdots n-1}$ can be represented as

$$\begin{aligned}&W_{1\cdots i-1\,i+1\,,\,i+2\cdots n-1} = \omega_{1\cdots i-1\,i+1\;i\,i+2\cdots n-1} \sqcup \omega_{1\cdots i-1\,i+1 i+2\,i\,i+3\cdots n-1} \sqcup \cdots \\ &\cdots \sqcup \omega_{1\cdots i-1\,i+1\cdots n-2\,i\,n-1} \sqcup \omega_{1\cdots i-1\,i+1\;\cdots n-1},\end{aligned} \tag{A.12}$$

and the factorization property is satisfied, that is

$$\begin{aligned}&\mu_P(W_{1\cdots i-1\,i+1\,,\,i+2\cdots n-1}) = \\ &\mu_P[(\alpha_1|\top) \sqcap \ldots \sqcap (\alpha_{i-1}|\alpha_{i-1}\vee\ldots\vee\alpha_n) \sqcap (\alpha_{i+1}|\alpha_{i+1}\vee\ldots\vee\alpha_n) \sqcap \ldots \sqcap (\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n)] = \\ &= P(\alpha_1|\top)\cdots P(\alpha_{i-1}|\alpha_{i-1}\vee\ldots\vee\alpha_n)P(\alpha_{i+1}|\alpha_i\vee\ldots\vee\alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n).\end{aligned} \tag{A.13}$$

Finally, concerning (A.2), from (A.11) and (A.13) we obtain

$$\begin{aligned}&\mu_P(\omega'_{12\cdots n-2}) = \mu_P(W_{1\cdots i\,,\,i+2\cdots n-1}) + \mu_P(W_{1\cdots i-1\,i+1\,,\,i+2\cdots n-1}) = \\ &= P(\alpha_1|\top)\cdots P(\alpha_{i-1}|\alpha_{i-1}\vee\cdots\vee\alpha_n)\cdot[P(\alpha_i|\alpha_i\vee\cdots\vee\alpha_n) + P(\alpha_{i+1}|\alpha_i\vee\cdots\vee\alpha_n)]\cdot \\ &\cdot P(\alpha_{i+2}|\alpha_{i+2}\vee\cdots\vee\alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n) = \\ &= P(\alpha_1|\top)\cdots P(\alpha_{i-1}|\alpha_{i-1}\vee\cdots\vee\alpha_n)\cdot P(\alpha_i\vee\alpha_{i+1}|\alpha_i\vee\cdots\vee\alpha_n)\cdot \\ &\cdot P(\alpha_{i+2}|\alpha_{i+2}\vee\cdots\vee\alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1}\vee\alpha_n) = \\ &= P(\beta_1|\top)\cdots P(\beta_i|\bar{\beta}_1\wedge\cdots\wedge\bar{\beta}_{i-1})\cdots P(\beta_{n-2}|\bar{\beta}_1\wedge\cdots\wedge\bar{\beta}_{n-3}) = \mu_{P'}(\omega'_{12\cdots n-2}),\end{aligned} \tag{A.14}$$

which shows that (17) holds for the sequence $(j_1,\ldots,j_{n-2}) = (1,\ldots,n-2)$, with $i \in \{1,\ldots,n-2\}$.
□

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