Chapter II: A General Framework for Mathematical Fuzzy Logic

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1 Introduction

Mathematical Fuzzy Logic was born in the last decade of the XXth century as a systematical study of a particular kind of systems of non-classical many-valued logic with the works of Baaz, Cignoli, Esteva, Godo, Gottwald, Hájek, Montagna, Mundici, Novák, and others (see e.g. [2, 13, 49–52, 57, 58, 69, 72]). Because of their motivation in the theory of fuzzy sets, the first studied systems were those that admitted a semantics based on particular well-known continuous t-norms: Łukasiewicz, product, and minimum t-norm, which respectively corresponded to Łukasiewicz, Product, and Gödel–Dummett many-valued logics. The first comprehensive attempt at systematization of the studies on these logics was Hájek's celebrated monograph [53] published in 1998. This book studied both propositional and first-order formalisms for these logics and set the agenda for the area by considering all the usual issues in Mathematical Logic for these specific systems, including their algebraic semantics, proof theory, decidability and computational aspects, and applications. Moreover, in order to provide a common ground for the three aforementioned systems, the monograph presented Basic fuzzy Logic BL, which was conjectured (and later proved [14]) to be complete with respect to the semantics of all continuous t-norms, and was hence a common base logic that could be axiomatically extended to the three of them.

The main outcome of the monograph was that, by setting solid logical foundations, it gave rise to a flourishing new field of study, as witnessed by the prolific literature since 1998 (surveyed, though not exhaustively, by Chapter I of this handbook), in which an increasing number of researchers have contributed by proposing a growing collection of systems of fuzzy logics obtained by modifying the defining conditions of BL and its three main extensions. For instance, the divisibility condition of BL was removed in the logic MTL [32] which is complete with respect to the semantics of all left-continuous t-norms [64], many axiomatic extensions of MTL were studied (see e.g. [30, 71]), negation was removed when considering fuzzy logics based on hoops [33], commutativity of t-norms was disregarded in [54], and t-norms were replaced by *uninorms* in [68]. On the other hand, logics with a higher expressive power were introduced by considering expanded real-valued algebras (with projection $\triangle$, involution $\sim$, truth-constants, etc., see e.g. [16, 31, 34, 35, 37]). Coherently with their initial motivations, the proponents of all these systems have always borne in mind an intended (so called *standard*) semantics based on real-valued algebras, and tried to show soundness and completeness of the logics with respect to them. However, in recent works fuzzy logics have started emancipating from the real-valued algebras as the only intended semantics by considering systems complete with respect to rational, finite or hyperreal linearly ordered algebras [18, 30, 36, 38, 70].

When dealing with this huge variety of fuzzy logics, and in order to avoid a useless repetition of analogous results and proofs, one may want to have some tools to prove general results that apply not only to a particular logic, but to a whole class of logics. To some extent this has been achieved by means of the notions of core and $\triangle$-core fuzzy logics from [56] that have provided a useful framework for some papers such as [18, 70]. However, those classes contain only axiomatic expansions of MTL and $\mathrm{MTL}_\triangle$ logics, so they do not cover the aforementioned weaker systems. Therefore, we need to look for a more general framework able to cope with all known examples and with other new logics that may arise in the near future.

In doing so, one certainly needs some intuition about the class of objects one would like to mathematically determine, namely some intuition of what are the minimal properties that should be required for a logic to be fuzzy. The evolution outlined above shows that almost no property of these systems is essential as they have been step-by-step disregarded. Nevertheless, there is one that has remained untouched so far: *completeness with respect to a semantics based on linearly ordered algebras*. It actually corresponds to the main thesis of [5] that defends that *fuzzy logics are the logics of chains*. Such a claim must be read as a methodological statement, pointing at a roughly defined class of logics, rather than a precise mathematical description of what fuzzy logics are (or should be), for there could be many different ways in which a logic might enjoy a complete semantics based on chains.

On the other hand, Algebraic Logic is the branch of Mathematical Logic that studies logical systems by giving them a semantics based on some particular kind of algebraic structures. The development we have just outlined shows how Algebraic Logic has been fruitfully applied to fuzzy logics, and it has also been very useful in many other families of non-classical logics. Moreover, in the last decades, it has evolved to a more abstract discipline, Abstract Algebraic Logic, which aims at understanding the various ways in which a logical system can be endowed with an algebraic semantics and developing methods and results to deal with broad classes of those systems (see the survey [40] or the comprehensive monographs [7, 24, 39, 88]). Therefore, it is a reasonable candidate to provide the general framework we are looking for.

The aim of this chapter is to present a marriage of Mathematical Fuzzy Logic and (Abstract) Algebraic Logic. In other words, we want to use the notions and techniques from the latter to create a new framework where we can develop in a natural way a particular technical notion corresponding to the intuition of fuzzy logics as the logics of chains. Since the order relation in the algebraic counterparts of fuzzy logics is typically determined by an implication connective, we will present our framework in the context of *weakly implicative logics* (introduced in [17]) which generalize the well-known class of *implicative logics* studied by Rasiowa in [76]. These logics enjoy an implication connective $\to$ such that for any algebra $\boldsymbol{A}$ in the algebraic semantics one can define an order relation by setting for any pair of elements a, b in $\boldsymbol{A}$: $a \leq b$ iff $a \to^{\boldsymbol{A}} b \in F$, where F is the subset of designated elements of the algebra representing truth (in typical examples $F = \{\overline{1}^{\boldsymbol{A}}\}$ or $F = \{a \in A \mid a \wedge^{\boldsymbol{A}} \overline{1}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}\}$). This allows to characterize fuzzy logics, in this context, as those which are complete with respect to the class of algebras where implication defines a linear ordering or, equivalently, as those logics whose finitely subdirectly irreducible algebraic models are linearly ordered by the implication. We call them *weakly implicative semilinear logics* inspired by the tradition in Universal Algebra of calling a class of algebras 'semiX' whenever their subdirectly irreducible members are X. We choose the term 'semilinear' instead of 'fuzzy' in spite of the fact that a first step towards the general definition we are offering here had been done by the first author in [17], when he introduced the class under the name *weakly implicative fuzzy logics*, because the term 'fuzzy' is probably too heavily charged with many conflicting potential meanings. It needs to be stressed that by this mathematical definition we do not expect to capture the whole intuitive notion of arbitrary fuzzy logic. Even if we agree that the linear ordering in the semantics is crucial for a formal logic to be *fuzzy*, there might still be several other ways in which a logic might have a complete semantics somehow based on chains (see e.g. [8, 9]). But still, the notion of weakly implicative semilinear logic will be able to include (and provide a useful mathematical framework for) almost all the prominent examples of fuzzy logics known so far and exclude non-classical logics which are usually not recognized as fuzzy logics in the Logic community.

The chapter is structured as follows. In Section 2 we introduce the necessary notions from (Abstract) Algebraic Logic, the definition of weakly implicative logic and some refinements thereof and provide three increasingly stronger completeness theorems for them. Moreover, we present a very general notion of substructural logics as a particular family of weakly implicative logics, discuss their syntactical properties and deduction theorems, and we conclude with a rather general study of disjunction connectives. Section 3 presents and studies the main notion of this chapter: semilinearity. It characterizes semilinear logics in terms of properties of filters and properties of disjunctions, and gives methods to axiomatize semilinear logics. Section 4 studies first-order predicate systems built over weakly implicative semilinear logics. It gives axiomatizations, completeness theorems, and a general process of Skolemization. We conclude with Section 5 providing historical remarks to understand the genesis of the ideas and results presented in this chapter and many bibliographical references for further studies in related topics.

2 Weakly implicative logics

This section provides the general basis for the framework presented in this chapter. Subsection 2.1 gives the most elementary necessary syntactical and semantical notions and proves completeness of all logics with respect to the class of their models. Subsection 2.2 introduces the notion of weakly implicative logics and proves their completeness with respect to the class of their reduced models. Subsection 2.3 introduces other semantical notions, including relatively subdirectly irreducible models (RSI), and proves completeness of weakly implicative logics with respect to the class of their RSI reduced models. Subsection 2.4 studies the class of algebraically implicative logics, i.e. those weakly implicative logics enjoying a stronger link with their algebraic semantics. Subsection 2.5 studies particular kind of algebraically implicative logics: a wide class of substructural logics based on the non-associative version of full Lambek logic. Subsection 2.6 proves local forms of deduction theorems for associative substructural logics and shows their relation with proof by cases properties. Finally, Subsection 2.7 considers proof by cases properties in the framework of a general notion of disjunction and gives some characterizations that will be very useful in the rest of the chapter.

2.1 Basic notions and a first completeness theorem

In this preliminary subsection we give the most basic syntactic and semantic notions we need for a general framework to study propositional logics and we prove a first completeness theorem for them.

DEFINITION 2.1.1 (Language). *A* propositional language $\mathcal{L}$ *is a* countable *type, i.e. a function* $ar\colon C_{\mathcal{L}} \to \mathsf{N}$*, where* $C_{\mathcal{L}}$ *is a countable set of symbols called* connectives*, giving for each one its* arity*. Nullary connectives are also called* truth-constants*. We write* $\langle c, n\rangle \in \mathcal{L}$ *whenever* $c \in C_{\mathcal{L}}$ *and* $ar(c) = n$*.*

The restriction to countable languages is necessary for very few results and simplifies the formulation of many others. The same holds for the following restriction of the cardinality of the set of propositional variables. Note that, in particular, all the notions and results of this subsection do not rely on these restrictions.

DEFINITION 2.1.2 (Formula). *Let* Var *be a fixed infinite countable set of symbols called* (propositional) variables*. The set* $Fm_{\mathcal{L}}$ *of (propositional)* formulae *in a propositional language* $\mathcal{L}$ *is the least set containing* Var *and closed under connectives of* $\mathcal{L}$*, i.e. for each* $\langle c, n\rangle \in \mathcal{L}$ *and every* $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$*,* $c(\varphi_1, \dots, \varphi_n)$ *is a formula.*

In what follows, beginning with the next definition, it will be convenient to identify $Fm_{\mathcal{L}}$ with the domain of the absolutely free algebra $\boldsymbol{Fm}_{\mathcal{L}}$ of type $\mathcal{L}$ and generators Var.[1] The variables will be usually denoted by lower case Latin letters p, q, r, ... The formulae will be usually denoted by lower-case Greek letters φ, ψ, χ, ... and their sets by upper-case ones Γ, Δ, Σ, ... The set of all sequences (including infinite sequences) of formulae is denoted by $Fm_{\mathcal{L}}^{\leq\omega}$.

DEFINITION 2.1.3 (Substitution). *Let* $\mathcal{L}$ *be a propositional language. An* $\mathcal{L}$-substitution *is an endomorphism on the algebra* $\boldsymbol{Fm}_{\mathcal{L}}$*, i.e. a mapping* $\sigma\colon Fm_{\mathcal{L}} \to Fm_{\mathcal{L}}$*, such that* $\sigma(c(\varphi_1, \dots, \varphi_n)) = c(\sigma(\varphi_1), \dots, \sigma(\varphi_n))$ *holds for each* $\langle c, n\rangle \in \mathcal{L}$ *and every* $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$*.*

Since an $\mathcal{L}$-substitution is a mapping whose domain is a free $\mathcal{L}$-algebra, it is fully determined by its values on the generators (propositional variables).

DEFINITION 2.1.4 (Consecution). *A* consecution[2] *in a propositional language* $\mathcal{L}$ *is a pair* $\langle\Gamma, \varphi\rangle$*, where* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*.*

Instead of '$\langle\Gamma, \varphi\rangle$' we write '$\Gamma \rhd \varphi$'. To simplify matters we will identify a formula φ with the consecution of the form $\emptyset \rhd \varphi$. Clearly, each subset $\mathcal{X}$ of the set of all consecutions can be understood as a relation between sets of formulae and formulae. We will use an infix notation and write '$\Gamma \vdash_{\mathcal{X}} \varphi$' instead of '$\Gamma \rhd \varphi \in \mathcal{X}$'.

[1] Recall that $\boldsymbol{Fm}_{\mathcal{L}}$ has the domain $Fm_{\mathcal{L}}$ and operations: $c^{\boldsymbol{Fm}_{\mathcal{L}}}(\varphi_1, \dots, \varphi_n) = c(\varphi_1, \dots, \varphi_n)$.

[2] The term 'consecution' is taken from [1] (the term 'sequent' is sometimes used instead).

DEFINITION 2.1.5 (Logic). *Let $\mathcal{L}$ be a propositional language. A set* L *of consecutions in $\mathcal{L}$ is called a* logic *in the language $\mathcal{L}$ when it satisfies the following conditions for each $\Gamma \cup \Delta \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$:*

- *If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$.* *(Reflexivity)*
- *If $\Delta \vdash_{\mathrm{L}} \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$.* *(Cut)*
- *If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each $\mathcal{L}$-substitution σ.* *(Structurality)*

A logic L *is called* inconsistent *if* L *is the set of all consecutions.*

Observe that reflexivity implies that any logic is non-empty and together with cut it entails the following monotonicity condition:

- If $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Monotonicity)

Notice the difference between '$\Gamma \rhd \varphi$' (denoting an object) and '$\Gamma \vdash_{\mathrm{L}} \varphi$' (stating the fact $\Gamma \rhd \varphi \in \mathrm{L}$). When the language or logic are known from the context we omit the parameters L or $\mathcal{L}$; the same convention will be followed in any other case indexed by L or $\mathcal{L}$. Moreover, instead of '$\Gamma \cup \Delta \vdash \varphi$', '$\Gamma \cup \{\psi\} \vdash \varphi$', and '$\emptyset \vdash \varphi$' we respectively write just '$\Gamma, \Delta \vdash \varphi$', '$\Gamma, \psi \vdash \varphi$', and '$\vdash \varphi$'. Finally, we write '$\Gamma \vdash \Sigma$' instead of '$\Gamma \vdash \chi$ for each $\chi \in \Sigma$' and '$\Gamma \dashv\vdash \Sigma$' instead of '$\Gamma \vdash \Sigma$ and $\Sigma \vdash \Gamma$'. The formulae φ such that $\vdash \varphi$ are called *theorems* of the logic.

It is easy to observe that the intersection of an arbitrary class of logics in the same language is a logic as well. Let us introduce the notion of *theory*. The importance of this notion will become apparent later when we introduce Lindenbaum matrices.

DEFINITION 2.1.6 (Theory). *A* theory *of a logic* L *is a set of formulae T such that if $T \vdash_{\mathrm{L}} \varphi$ then $\varphi \in T$. By* $\mathrm{Th}(\mathrm{L})$ *we denote the set of all theories of* L.

Theories are sometimes also called *deductively closed sets of formulae* and will be usually denoted by upper case Latin letters T, S, R, ... Notice that for each set Γ of formulae, the set $\mathrm{Th}_{\mathrm{L}}(\Gamma) = \{\varphi \mid \Gamma \vdash_{\mathrm{L}} \varphi\}$ belongs to $\mathrm{Th}(\mathrm{L})$. Observe that $\mathrm{Th}_{\mathrm{L}}(\Gamma)$ is the least theory containing Γ; we call it the *theory generated by* Γ. Note that the set of theorems of L equals to $\mathrm{Th}_{\mathrm{L}}(\emptyset)$ and thus it is a subset of any theory T of the logic L.

Now we introduce the notion of axiomatic system as the same kind of objects as logics, i.e. sets of consecutions closed under substitutions; this will simplify the formulation of some upcoming results.

DEFINITION 2.1.7 (Axiomatic system). *Let $\mathcal{L}$ be a propositional language. An* axiomatic system $\mathcal{AS}$ *in the language $\mathcal{L}$ is a set $\mathcal{AS}$ of consecutions closed under arbitrary substitutions. The elements of $\mathcal{AS}$ of the form $\Gamma \rhd \varphi$ are called* axioms *if $\Gamma = \emptyset$,* finitary deduction rules *if Γ is a finite set, and* infinitary deduction rules *otherwise. An axiomatic system is said to be* finitary *if all its deduction rules are finitary.*

Notice that the convention we have made above identifying the consecution $\emptyset \rhd \varphi$ with the formula φ, allows to call φ an axiom of the axiomatic system. Of course, each axiomatic system can also be seen as a collection of schemata (by a schema we understand a consecution and all its substitution instances).

DEFINITION 2.1.8 (Proof). *Let $\mathcal{L}$ be a propositional language and $\mathcal{AS}$ an axiomatic system in $\mathcal{L}$. A* proof *of a formula φ from a set of formulae Γ in $\mathcal{AS}$ is a well-founded tree (with no infinitely-long branch) labeled by formulae such that*

- *its root is labeled by φ and leaves by axioms of $\mathcal{AS}$ or elements of Γ and*
- *if a node is labeled by ψ and $\Delta \neq \emptyset$ is the set of labels of its preceding nodes, then $\Delta \rhd \psi \in \mathcal{AS}$.*

We write $\Gamma \vdash_{\mathcal{AS}} \varphi$ if there is a proof of φ from Γ in $\mathcal{AS}$.

Observe that formal proofs can be seen as well-founded relations (with leaves as minimal elements and the root as a maximum), thus we can prove facts about formulae by induction over the complexity of their formal proofs. Notice that a deduction rule $\{\psi_1, \psi_2, \dots\} \rhd \varphi$ gives a way to construct a proof of φ from Γ if we know the proofs of ψ_1, ψ_2, ... from Γ: we just glue them together in a single tree using the rule $\{\psi_1, \psi_2, \dots\} \rhd \varphi$. In contrast, the meta-rule: from $\Gamma \vdash \psi_1$, $\Delta \vdash \psi_2, \dots$ obtain $\Sigma \vdash \varphi$ only tells us that if there are proofs of $\psi_1, \psi_2, \dots$ from $\Gamma, \Delta, \dots$, then there is a proof of φ from Σ as well, though it gives no hint to its construction. We could say that rules are inferences between *formulae*, whereas meta-rules are in fact inferences between *consecutions*. We will see prominent examples of meta-rules at the end of this section and in the next one.

LEMMA 2.1.9. *Let $\mathcal{L}$ be a propositional language and $\mathcal{AS}$ an axiomatic system in $\mathcal{L}$. Then $\vdash_{\mathcal{AS}}$ is the least logic containing $\mathcal{AS}$.*

Proof. Obviously $\vdash_{\mathcal{AS}}$ is a logic and $\mathcal{AS} \subseteq \vdash_{\mathcal{AS}}$. We prove that for each logic L, if $\mathcal{AS} \subseteq \mathrm{L}$, then $\vdash_{\mathcal{AS}} \subseteq \mathrm{L}$. Assume that $\Gamma \vdash_{\mathcal{AS}} \varphi$, i.e. there is a proof of φ from Γ. By induction over the complexity of the proof we can show that for each formula ψ which labels some node in the proof we have $\Gamma \vdash_{\mathrm{L}} \psi$, and hence in particular $\Gamma \vdash_{\mathrm{L}} \varphi$. □

DEFINITION 2.1.10 (Presentation, finitary logic). *Let $\mathcal{L}$ be a propositional language, $\mathcal{AS}$ an axiomatic system in $\mathcal{L}$, and* L *a logic in $\mathcal{L}$. We say that $\mathcal{AS}$ is an* axiomatic system *for (or a* presentation *of) the logic* L *if* $\mathrm{L} = \vdash_{\mathcal{AS}}$. *A logic is said to be* finitary *if it has some finitary presentation.*

Observe that each logic has a presentation, for L understood as an axiomatic system is a presentation of the logic L itself (due to Lemma 2.1.9). Next we show that our definition of finitary logics is equivalent to the usual one:

LEMMA 2.1.11. *Let* L *be a logic. Then* L *is finitary iff for each set of formulae $\Gamma \cup \{\varphi\}$ we have: if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}} \varphi$.*

Proof. Assume that L is finitary. Then, by definition, it has a finitary presentation $\mathcal{AS}$. Observe that proofs in a finitary axiomatic system are always finite (because by definition the tree has no infinite branches and, because of finitarity, each node has finitely many preceding nodes, thus by König's Lemma the tree is finite). This gives the implication from left to right. The reverse direction is straightforward. □

Observe that in the finitary case we can represent the tree as a linear sequence of formulae, obtaining thus the usual notion of finite proof.

DEFINITION 2.1.12 (Finitary companion). *The* finitary companion *of a logic* L *is the logic* $\mathcal{FC}(\mathrm{L})$ *defined as:* $\Gamma \vdash_{\mathcal{FC}(\mathrm{L})} \varphi$ *iff there is a finite subset* $\Gamma_0 \subseteq \Gamma$ *such that* $\Gamma_0 \vdash_{\mathrm{L}} \varphi$.

Note that the finitary companion of a logic L is the strongest finitary logic contained in L and it is naturally axiomatized by the set of all finitary consecutions provable in L.

DEFINITION 2.1.13 (Expansion). *Let* $\mathcal{L}_1 \subseteq \mathcal{L}_2$ *be propositional languages,* L_i *a logic in* $\mathcal{L}_i$*, and* $\mathcal{S}$ *a set of consecutions in* $\mathcal{L}_2$.

- L_2 *is the* expansion of L_1 by $\mathcal{S}$ *if it is the weakest logic in* $\mathcal{L}_2$ *containing* L_1 *and* $\mathcal{S}$*, i.e. the logic axiomatized by all* $\mathcal{L}_2$*-substitutional instances of consecutions from* $\mathcal{S} \cup \mathcal{AS}$*, for any presentation* $\mathcal{AS}$ *of* L_1.
- L_2 *is an* expansion of L_1 *if* $\mathrm{L}_1 \subseteq \mathrm{L}_2$*, i.e. it is the expansion of* L_1 *by* $\mathcal{S}$*, for some set of consecutions* $\mathcal{S}$.
- L_2 *is an* axiomatic expansion of L_1 *if it is an expansion obtained by adding a set of* formulae.
- L_2 *is a* conservative expansion of L_1 *if it is an expansion and for each consecution* $\Gamma \rhd \varphi$ *in* $\mathcal{L}_1$ *we have that* $\Gamma \vdash_{\mathrm{L}_2} \varphi$ *entails* $\Gamma \vdash_{\mathrm{L}_1} \varphi$.

If $\mathcal{L}_1 = \mathcal{L}_2$*, we use 'extension' instead 'expansion'.*[3]

Next we introduce the necessary basic semantical notions. Let us fix a propositional language $\mathcal{L}$. The logics in this language are given a semantical interpretation by means of the notion of logical matrix, which is a pair formed by an $\mathcal{L}$-algebra (which interprets the formulae capitalizing on the fact that $\mathcal{L}$ can also be seen as an algebraic language) and a filter (a subset of designated elements in the domain of the algebra which gives a notion of truth for the logic):

DEFINITION 2.1.14 (Logical matrix). *An* $\mathcal{L}$-matrix *is a pair* $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ *where* $\boldsymbol{A}$ *is an* $\mathcal{L}$*-algebra called the* algebraic reduct of $\mathbf{A}$*, and* F *is a subset of* A *called the* filter *of* $\mathbf{A}$. *The elements of* F *are called* designated elements *of* $\mathbf{A}$.

A matrix is trivial *if* $F = A$. *A matrix is* finite *if its underlying algebra has a finite domain. The matrices where* $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$ *are called* Lindenbaum *matrices.*

DEFINITION 2.1.15 (Evaluation). *Let* $\boldsymbol{A}$ *be an* $\mathcal{L}$*-algebra. An* $\boldsymbol{A}$-evaluation *is a homomorphism from* $\boldsymbol{Fm}_{\mathcal{L}}$ *to* $\boldsymbol{A}$*, i.e. a mapping* $e\colon Fm_{\mathcal{L}} \to A$*, such that for each* $\langle c, n \rangle \in \mathcal{L}$ *and each* n*-tuple of formulae* $\varphi_1, \ldots, \varphi_n$ *we have:* $e(c(\varphi_1, \ldots, \varphi_n)) = c^{\boldsymbol{A}}(e(\varphi_1), \ldots, e(\varphi_n))$.

As in the case of substitutions, since an $\boldsymbol{A}$-evaluation is a mapping whose domain is a free $\mathcal{L}$-algebra, it is fully determined by its values on the generators (propositional variables). By $e[p{\to}a]$ we denote the evaluation obtained from e by assigning the element $a \in A$ to the variable p and leaving the values of remaining variables unchanged. For a formula φ build from variables $p_1, \ldots, p_n$, an algebra $\boldsymbol{A}$, elements $a_1, \ldots, a_n \in A$ and an $\boldsymbol{A}$-evaluation e such that $e(p_i) = a_i$, we write $\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n)$ instead of $e(\varphi(p_1, \ldots, p_n))$. Given a matrix $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ and $\boldsymbol{A}$-evaluation e, we will also call e an $\mathbf{A}$*-evaluation*.

[3] Observe that any *conservative extension* of any logic is just the logic itself.

DEFINITION 2.1.16 (Semantical consequence). *A formula φ is a* semantical consequence *of a set Γ of formulae w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-matrices if for each $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e, we have $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$; we denote it by $\Gamma \models_{\mathbb{K}} \varphi$.*

We write $\models_{\mathbf{A}}$ instead of $\models_{\{\mathbf{A}\}}$. Obviously, $\models_{\mathbb{K}}$ is a set a of consecutions, but more can be proved (the second claim will be generalized in Proposition 2.3.13):

LEMMA 2.1.17. *Let $\mathbb{K}$ a class of $\mathcal{L}$-matrices. Then $\models_{\mathbb{K}}$ is a logic in $\mathcal{L}$. Furthermore if $\mathbb{K}$ is a finite class of finite matrices, then the logic $\models_{\mathbb{K}}$ is finitary.*

Proof. We need to check the three properties in the definition of logic. The first one is obvious. To show the second one fix $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and an $\boldsymbol{A}$-evaluation e such that $e[\Delta] \subseteq F$. Then clearly $e(\psi) \in F$ for each $\psi \in \Gamma$, i.e. $e[\Gamma] \subseteq F$, and so $e(\varphi) \in F$. The final condition: fix $\langle \boldsymbol{A}, F\rangle$ and e as before and assume that $e(\sigma[\Gamma]) \subseteq F$. Since $e' = e \circ \sigma$ is an $\boldsymbol{A}$-evaluation and $e'[\Gamma] \subseteq F$, we obtain $e(\sigma(\varphi)) = e'(\varphi) \in F$.

The second claim: if we prove it for $\mathbb{K} = \{\langle \boldsymbol{A}, F\rangle\}$ the proof is done by observing that: (1) $\models_{\mathbb{K}\cup\mathbb{L}} = \models_{\mathbb{K}} \cap \models_{\mathbb{L}}$, and (2) the intersection of two finitary logics is finitary. Assume that $\Gamma' \not\models_{\mathbb{K}} \varphi$ for each finite $\Gamma' \subseteq \Gamma$ and we want to show that $\Gamma \not\models_{\mathbb{K}} \varphi$.

Let us consider the finite set A endowed with discrete topology and its power A^{Var} with product (=weak) topology. Both spaces are compact (the first one trivially and the second one due to Tychonoff theorem). Clearly each evaluation e can be identified with an element of A^{Var} and vice versa. For each formula ψ we define a mapping $H_\psi \colon A^{Var} \to A$ as $H_\psi(e) = e(\psi)$. It can be easily shown that these mappings are continuous, thus $(H_\psi)^{-1}[F]$ is a closed set and so is the set $(H_\psi)^{-1}[F] \cap (H_\varphi)^{-1}[A \setminus F]$ (i.e. the set of evaluations which satisfy the formula ψ but not the formula φ). Let us now consider the system of closed sets $\{(H_\psi)^{-1} \cap (H_\varphi)^{-1}[A \setminus F]) \mid \psi \in \Gamma\}$. This is clearly a centered system (the intersection of any finite subsystem given by a set Γ' is non-empty, because it contains any evaluation which witnesses that $\Gamma' \not\models_{\mathbb{K}} \varphi$). Thus, due to the compactness of A^{Var}, the intersection of the whole system is non-empty and the proof is done (because any element of this intersection is an evaluation satisfying the set Γ but not the formula φ). □

DEFINITION 2.1.18 (L-matrix). *Let* L *be a logic in $\mathcal{L}$ and $\mathbf{A}$ an $\mathcal{L}$-matrix. We say that $\mathbf{A}$ is an* L-matrix *if* $\mathrm{L} \subseteq \models_{\mathbf{A}}$. *We denote the class of* L*-matrices by* $\mathbf{MOD}(\mathrm{L})$.

Observe that for each presentation $\mathcal{AS}$ of a logic L we have: $\mathbf{A} \in \mathbf{MOD}(\mathrm{L})$ iff $\mathcal{AS} \subseteq \models_{\mathbf{A}}$ (one direction is obvious, the second one is Lemma 2.1.9).

LEMMA 2.1.19. *Let* L *be a logic in $\mathcal{L}$ and a mapping $g \colon \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism of $\mathcal{L}$-algebras $\boldsymbol{A}, \boldsymbol{B}$. Then:*

- $\langle \boldsymbol{A}, g^{-1}[G]\rangle \in \mathbf{MOD}(\mathrm{L})$*, whenever* $\langle \boldsymbol{B}, G\rangle \in \mathbf{MOD}(\mathrm{L})$.
- $\langle \boldsymbol{B}, g[F]\rangle \in \mathbf{MOD}(\mathrm{L})$*, whenever* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ *and g is surjective and $g(x) \in g[F]$ implies $x \in F$.*

Proof. The first claim is straightforward: assume that $\Gamma \vdash_{\mathrm{L}} \varphi$ and $e[\Gamma] \subseteq g^{-1}[G]$ for some $\boldsymbol{A}$-evaluation e. Thus, $g[e[\Gamma]] \subseteq G$ which, since $g \circ e$ is a $\boldsymbol{B}$-evaluation and $\langle \boldsymbol{B}, G\rangle \in \mathbf{MOD}(\mathrm{L})$, implies that $g(e(\varphi)) \in G$, i.e. $e(\varphi) \in g^{-1}[G]$.

The second claim: assume that $\Gamma \vdash_{\mathrm{L}} \psi$ and for a $\boldsymbol{B}$-evaluation f it is the case that $f[\Gamma] \subseteq g[F]$. Let us define an $\boldsymbol{A}$-evaluation e by setting $e(v) = a$ for some a such that $g(a) = f(v)$ (such a has to exist because g is surjective). Next we show by induction that $f(\varphi) = g(e(\varphi))$. The base is trivial. Let us assume that $\varphi = c(\varphi_1, \ldots, \varphi_n)$. Then:

$$\begin{aligned} f(c(\varphi_1, \ldots, \varphi_n)) &= c^{\boldsymbol{B}}(f(\varphi_1), \ldots, f(\varphi_n)) = \\ &= c^{\boldsymbol{B}}(g(e(\varphi_1)), \ldots, g(e(\varphi_n))) = \\ &= g(c^{\boldsymbol{A}}(e(\varphi_1), \ldots, e(\varphi_n))) = \\ &= g(e(c(\varphi_1, \ldots, \varphi_n))). \end{aligned}$$

From $g[e[\Gamma]] = f[\Gamma] \subseteq g[F]$ we obtain $e[\Gamma] \subseteq F$. Thus $e(\psi) \in F$ and so $f(\psi) = g(e(\psi)) \in g[F]$. □

DEFINITION 2.1.20 (Logical filter). *Given a logic* L *in $\mathcal{L}$ and an $\mathcal{L}$-algebra $\boldsymbol{A}$, a subset $F \subseteq A$ is an* L-filter *if $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all* L*-filters over $\boldsymbol{A}$.*

Observe that $A \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ and $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is closed under arbitrary intersections, i.e. $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system (we deal with closure systems in detail in Subsection 2.3) which allows us to endow it with a (complete) lattice structure:

DEFINITION 2.1.21 (Generated filters and lattice of logical filters). *Let* L *be a logic in $\mathcal{L}$ and $\boldsymbol{A}$ an $\mathcal{L}$-algebra. Given a set $X \subseteq A$, the* logical filter generated *by X is* $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = \bigcap\{F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) \mid X \subseteq F\}$. *$\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is given a lattice structure by defining for any $F, G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, $F \wedge G = F \cap G$ and $F \vee G = \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(F \cup G)$.*

Moreover, some results in Subsection 2.7 will need the following definition:

DEFINITION 2.1.22 (Filter-distributivity). *A logic* L *is* filter-distributive *if for each $\mathcal{L}$-algebra, the lattice $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is distributive.*

The elements of a filter generated by a set are characterized in the next proposition by means of the notion of *proof in algebra*. It consists in generalizing to any algebra the notion of proof introduced in Definition 2.1.8 for the algebra of formulae.

PROPOSITION 2.1.23 (Proof in algebra). *Let* L *be a logic, $\mathcal{AS}$ one of its presentations, $\boldsymbol{A}$ an $\mathcal{L}$-algebra, and $X \cup \{a\} \subseteq A$. Let us define a set $V_{\mathcal{AS}} \subseteq \mathcal{P}(A) \times A$ as $\{\langle e[\Gamma], e(\psi)\rangle \mid e$ is an $\boldsymbol{A}$-evaluation and $\Gamma \rhd \psi \in \mathcal{AS}\}$.[4] Then $a \in \mathrm{Fi}^{\boldsymbol{A}}(X)$ iff there is a well-founded tree (called* proof of a from X*) labeled by elements of A such that*

- *its root is labeled by a, and leaves are labeled by elements x such that $x \in X$ or $\langle \emptyset, x\rangle \in V_{\mathcal{AS}}$ and*

- *if a node is labeled by x and $Z \neq \emptyset$ is the set of labels of its preceding nodes, then $\langle Z, x\rangle \in V_{\mathcal{AS}}$.*

[4]Note that if $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$, then $V_{\mathcal{AS}} = \mathcal{AS}$.

Proof. Let $D(X)$ be the set of elements of A for which there exists a proof from X. We can easily show that $\mathcal{AS} \subseteq \models_{\langle \boldsymbol{A}, D(X)\rangle}$. Indeed, assume that $\Gamma \rhd \varphi \in \mathcal{AS}$ and $h[\Gamma] \subseteq D(X)$ for some evaluation h. Then for each $x \in h[\Gamma]$ there is a proof from X and, since $\langle h[\Gamma], h[\varphi]\rangle \in V_{\mathcal{AS}}$, we can connect these proofs so that they will form a proof of $h(\varphi)$. Thus $D(X) \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ and, since $X \subseteq D(X)$, we obtain $\mathrm{Fi}(X) \subseteq D(X)$. To prove the converse direction consider $x \in D(X)$ and notice that for each y appearing in its proof we can easily prove inductively that $y \in \mathrm{Fi}(X)$ (because $\mathrm{Fi}(X)$ is closed under all the rules of L, in particular those in $\mathcal{AS}$). □

Next we show that the filters of Lindenbaum matrices can be nicely characterized.

PROPOSITION 2.1.24. *For any logic* L *in a language* $\mathcal{L}$, $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm_{\mathcal{L}}}) = \mathrm{Th}(\mathrm{L})$.

Proof. Let $\Gamma \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm_{\mathcal{L}}})$, i.e. if $\Delta \vdash_{\mathrm{L}} \varphi$ then for each $\boldsymbol{Fm}_{\mathcal{L}}$-evaluation e we have $e(\varphi) \in \Gamma$ whenever $e[\Delta] \subseteq \Gamma$. Therefore, in the particular case where the evaluation e is the identity and $\Delta = \Gamma$, we obtain $\Gamma \in \mathrm{Th}(\mathrm{L})$.

Next assume that $T \in \mathrm{Th}(\mathrm{L})$, $\Delta \vdash_{\mathrm{L}} \varphi$, and e is an $\boldsymbol{Fm}_{\mathcal{L}}$-evaluation such that $e[\Delta] \subseteq T$, thus also $T \vdash_{\mathrm{L}} e[\Delta]$. By structurality, also $e[\Delta] \vdash_{\mathrm{L}} e(\varphi)$, and thus also $T \vdash_{\mathrm{L}} e(\varphi)$. Since T is a theory, we have $e(\varphi) \in T$. □

We close the subsection observing that the notions introduced so far are enough to obtain a first completeness theorem for any logic.

THEOREM 2.1.25 (Completeness w.r.t. all models). *Let* L *be a logic. Then for each set* Γ *of formulae and each formula* φ *the following holds:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\Gamma \models_{\mathbf{MOD}(\mathrm{L})} \varphi$.

Proof. Soundness is obvious. For the reverse direction assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$ and define $T = \mathrm{Th}_{\mathrm{L}}(\Gamma)$. We know that $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle \in \mathbf{MOD}(\mathrm{L})$ and then the identity mapping is the $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle$-evaluation we need to show that $\Gamma \not\models_{\mathbf{MOD}(\mathrm{L})} \varphi$. □

2.2 Weakly implicative logics and a second completeness theorem

In this subsection we first introduce the main defining notion for the framework of this chapter: the class of weakly implicative logics. Then we use the notions of Leibniz congruence and reduced model to prove a second completeness theorem. Although these notions can be introduced in general for any propositional logic and completeness with respect to its reduced models can be proved in general, we will restrict to weakly implicative logics for the sake of simplicity.

DEFINITION 2.2.1 (Weakly implicative logic). *Let* L *be a logic in a language* $\mathcal{L}$. *We say that* L *is a* weakly implicative logic *if there is a binary connective* $\to$ *(primitive or definable by a formula of two variables in language* $\mathcal{L}$*) such that:*

(R)	$\vdash_{\mathrm{L}} \varphi \to \varphi$
(MP)	$\varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi$
(T)	$\varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi$
(sCng)	$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \to c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n)$ *for each* $\langle c, n\rangle \in \mathcal{L}$ *and each* $0 \leq i < n$.

The acronyms respectively stand for 'reflexivity', '*modus ponens*', 'transitivity' and 'symmetrized congruence'. The connective $\to$ is called a *weak implication* of L. There could be, in principle, several different weak implications in a given logic. For the sake of simpler notation we will avoid indexing many of the upcoming notions with a weak implication by assuming from now on that each language comes with a fixed binary (primitive or derivable) connective $\to$, such that if a logic in this language is weakly implicative, then $\to$ is one of its weak implications and all notions are defined w.r.t. this particular implication. We will call $\to$ the *principal implication* of the logic. In some rare cases, when we may need to speak at once about notions corresponding to different weak implications, we will index these notions by their corresponding weak implication to avoid any confusion.

EXAMPLE 2.2.2. In classical logic the usual connectives of implication and equivalence, $\to$ and $\leftrightarrow$, are both actually weak implications in our sense, but observe they have a very different logical behavior (for instance, only the former satisfies $\varphi \vdash \psi \to \varphi$). More generally, all connectives $\to$ in the various logics mentioned in Chapter I are weak implications. Therefore, all these logics are examples of weakly implicative logics.

Now we consider the properties of the symmetrization of a weak implication $\to$ in a logic L. Given a pair of formulae φ, ψ, we use the expression '$\varphi \leftrightarrow \psi$' to denote the set of formulae $\{\varphi \to \psi, \psi \to \varphi\}$ (recall that, according to previous conventions, by $\Gamma \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$ we mean that $\Gamma \vdash_{\mathrm{L}} \varphi \to \psi$ and $\Gamma \vdash_{\mathrm{L}} \psi \to \varphi$). We can easily show that $\leftrightarrow$ behaves like a congruence.

THEOREM 2.2.3 (Congruence Property). *Let* L *be a weakly implicative logic and* φ, ψ, χ *formulae. Then:*

- $\vdash_{\mathrm{L}} \varphi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \psi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \hat{\chi}$,

where $\hat{\chi}$ *is obtained from* χ *by replacing some occurrences of* φ *in* χ *by* ψ.

By using the last part for $\chi = \varphi \to' \psi$ we obtain an important corollary:

COROLLARY 2.2.4. *Let* $\to$ *and* $\to'$ *be two weak implications in a logic* L. *Then:*

$$\varphi \leftrightarrow \psi \dashv\vdash_{\mathrm{L}} \varphi \leftrightarrow' \psi.$$

Therefore, if we had two different weak implications in a logic, their symmetrizations would behave exactly in the same way as far as provability is concerned. Now, aiming to obtain a finer complete semantics for weakly implicative logics, we introduce some further semantic notions.

DEFINITION 2.2.5 (Leibniz congruence). *Let* $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ *be an* L*-matrix for a weakly implicative logic* L. *The* matrix preorder $\leq_{\mathbf{A}}$ *of* $\mathbf{A}$ *is defined as* $a \leq_{\mathbf{A}} b$ *iff* $a \to^{\boldsymbol{A}} b \in F$. *Further we define the* Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$ *of* $\mathbf{A}$ *as* $\langle a, b \rangle \in \Omega_{\boldsymbol{A}}(F)$ *iff* $a \leq_{\mathbf{A}} b$ *and* $b \leq_{\mathbf{A}} a$.

DEFINITION 2.2.6 (Logical congruence). *A* logical congruence *in a matrix* $\langle \boldsymbol{A}, F\rangle$ *is a congruence* θ *of* $\boldsymbol{A}$ *compatible with* F*, i.e. such that for each* $a, b \in A$ *if* $a \in F$ *and* $\langle a, b\rangle \in \theta$*, then* $b \in F$*.*

THEOREM 2.2.7 (Characterization of Leibniz congruence). *Let* L *be a weakly implicative logic and* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$*. Then:*

- $\leq_{\mathbf{A}}$ *is a preorder.*
- $\Omega_{\boldsymbol{A}}(F)$ *is the largest logical congruence of* $\boldsymbol{A}$*.*
- $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ *if, and only if, for each formula* χ *and each* $\mathbf{A}$*-evaluation* e *it is the case that* $e[p{\to}a](\chi) \in F$ *iff* $e[p{\to}b](\chi) \in F$*.*

Proof. The fact that $\leq_{\mathbf{A}}$ is a preorder follows from (R) and (T). $\Omega_{\boldsymbol{A}}(F)$ is a congruence because of (sCng), and it is logical because of (MP). To see that it is the largest one, assume that θ is a logical congruence of $\boldsymbol{A}$ and $\langle a, b\rangle \in \theta$. Since $\langle a, a\rangle \in \theta$, we have $\langle a \to^{\boldsymbol{A}} a, a \to^{\boldsymbol{A}} b\rangle \in \theta$ and $a \to^{\boldsymbol{A}} a \in F$. Hence, by compatibility, $a \to^{\boldsymbol{A}} b \in F$. Analogously $b \to^{\boldsymbol{A}} a \in F$, and so $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$, i.e. $\theta \subseteq \Omega_{\boldsymbol{A}}(F)$.

Final claim: one direction is a straightforward corollary of (sCng). The converse direction: consider the formula $p \to q$ and the evaluation $e(q) = b$. Then we obtain that $a \to^{\boldsymbol{A}} b \in F$ iff $b \to^{\boldsymbol{A}} b \in F$. Thus $a \leq_{\mathbf{A}} b$ and, since using the evaluation $e(q) = a$ we can prove $b \leq_{\mathbf{A}} a$, the proof is done. □

DEFINITION 2.2.8 (Reduced matrix, $\mathbf{MOD}^*(\mathrm{L})$, and $\mathbf{ALG}^*(\mathrm{L})$). *Let* L *be a weakly implicative logic. An* L*-matrix* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ *is said to be* reduced *if* $\Omega_{\boldsymbol{A}}(F)$ *is the identity relation* $\mathrm{Id}_{\boldsymbol{A}}$*. The class of all reduced models of* L *is denoted by* $\mathbf{MOD}^*(\mathrm{L})$*, and the class of algebraic reducts of matrices of* $\mathbf{MOD}^*(\mathrm{L})$ *is denoted by* $\mathbf{ALG}^*(\mathrm{L})$*. The members of* $\mathbf{ALG}^*(\mathrm{L})$ *are called* L-algebras*.*

Observe that a reduced model of a logic is non-trivial if, and only if, its algebraic reduct has more than one element. We could alternatively define reduced matrices as those whose matrix preorder is an order. The next lemma shows us how to turn any model into a reduced one. Let $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ be a matrix, we use the following notation: $[a]_F = \{b \in A \mid \langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)\}$, $[F] = \{[a]_F \mid a \in F\}$, and $\mathbf{A}^* = \langle \boldsymbol{A}/\Omega_{\boldsymbol{A}}(F), [F]\rangle$.

LEMMA 2.2.9. *Let* L *be a weakly implicative logic and* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$*. Then:*

1. $\mathbf{A}^* \in \mathbf{MOD}(\mathrm{L})$*.*
2. $[a]_F \leq_{\mathbf{A}^*} [b]_F$ *iff* $a \to^{\boldsymbol{A}} b \in F$*, for every* $a, b \in A$*.*
3. $\mathbf{A}^* \in \mathbf{MOD}^*(\mathrm{L})$*.*

Proof. 1. Clearly $[\cdot]_F$ is a surjective homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)$. By Lemma 2.1.19, all we need to show is: $[a]_F \in [F]$ implies $a \in F$. The assumption gives us $[a]_F = [b]_F$ for some $b \in F$. Then $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ and, since $\Omega_{\boldsymbol{A}}(F)$ is a logical congruence, we obtain $a \in F$.

2. $[a]_F \leq_{\mathbf{A}^*} [b]_F$ iff $[a]_F \rightarrow^{\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)} [b]_F \in [F]$ iff $[a \rightarrow^{\boldsymbol{A}} b]_F \in [F]$ iff $a \rightarrow^{\boldsymbol{A}} b \in F$.

3. $[a]_F \leq_{\mathbf{A}^*} [b]_F$ and $[b]_F \leq_{\mathbf{A}^*} [a]_F$ entail $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ and so $[a]_F = [b]_F$. □

DEFINITION 2.2.10 (Leibniz operator). *Let* L *be a weakly implicative logic in a language $\mathcal{L}$, and $\boldsymbol{A}$ be an $\mathcal{L}$-algebra. The* Leibniz operator *associated to $\boldsymbol{A}$ is the function giving for each $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$.*

PROPOSITION 2.2.11. *Let* L *be a weakly implicative logic* L *and $\boldsymbol{A}$ an $\mathcal{L}$-algebra. Then*

1. *$\Omega_{\boldsymbol{A}}$ is monotone (i.e. if $F \subseteq G$ then $\Omega_{\boldsymbol{A}}(F) \subseteq \Omega_{\boldsymbol{A}}(G)$).*

2. *$\Omega_{\boldsymbol{A}}$ commutes with inverse images by homomorphisms, that is, for every $\mathcal{L}$-algebra $\boldsymbol{B}$, every homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{B})$, $\Omega_{\boldsymbol{A}}(h^{-1}[F]) = h^{-1}[\Omega_{\boldsymbol{B}}(F)] = \{\langle a, b\rangle \mid \langle h(a), h(b)\rangle \in \Omega_{\boldsymbol{B}}(F)\}$.*

3. *$\Omega_{\boldsymbol{A}}[\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})] = \boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$, where by $\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$ we denote the set ordered by inclusion[5] of congruences of $\boldsymbol{A}$ giving a quotient in $\mathbf{ALG}^*(\mathrm{L})$.*

Proof. The proofs of the first two claims are easy (by Lemma 2.1.19 $h^{-1}[F]$ is indeed a filter on $\boldsymbol{A}$). To prove the third first one observe that $\Omega_{\boldsymbol{A}}[\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})] \subseteq \boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$ (due to the Lemma 2.2.9). To show the second direction assume $\Theta \in \boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$. We know that $\boldsymbol{A}/\Theta \in \mathbf{ALG}^*(\mathrm{L})$, i.e. $\langle \boldsymbol{A}/\Theta, F_0\rangle \in \mathbf{MOD}^*(\mathrm{L})$ for some filter F_0. Let k be the canonical mapping from $\boldsymbol{A}$ onto $\boldsymbol{A}/\Theta$, we define $F = k^{-1}[F_0]$ and, again by Lemma 2.1.19, we know that $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. To complete the proof just observe that $\Omega_{\boldsymbol{A}}(F) = \Omega_{\boldsymbol{A}}(k^{-1}[F_0]) = k^{-1}[\Omega_{\boldsymbol{A}/\Theta}(F_0)] = k^{-1}[\mathrm{Id}_{\boldsymbol{A}/\Theta}] = \Theta$. □

Next we introduce the well-known notion of Lindenbaum–Tarski matrices, in the traditional way as it is usually done in the literature, and show how they are related to reduced matrices.

DEFINITION 2.2.12 (Lindenbaum–Tarski matrix). *Let* L *be a weakly implicative logic in $\mathcal{L}$ and $T \in Th(\mathrm{L})$. For every formula φ, we define the set*

$$[\varphi]_T = \{\psi \in Fm_{\mathcal{L}} \mid \varphi \leftrightarrow \psi \subseteq T\}.$$

The Lindenbaum–Tarski matrix with respect to L *and T, $\mathbf{LindT}_T$, is the $\mathcal{L}$-matrix whose designated set is $\{[\varphi]_T \mid \varphi \in T\}$, and whose $\mathcal{L}$-algebra has the domain $\{[\varphi]_T \mid \varphi \in Fm_{\mathcal{L}}\}$ and operations $c^{\mathbf{LindT}_T}([\varphi_1]_T, \ldots, [\varphi_n]_T) = [c(\varphi_1, \ldots, \varphi_n)]_T$.*

Clearly, for every $T \in \mathrm{Th}(\mathrm{L})$, the matrix $\mathbf{LindT}_T$ coincides with $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle^*$. Now we are ready to prove the main result of this subsection.

THEOREM 2.2.13 (Completeness w.r.t. reduced models). *Let* L *be a weakly implicative logic. Then for any set Γ of formulae and any formula φ the following holds: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi$.*

[5] Later, after Proposition 2.3.13, we show that $\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$ is actually a lattice.

Proof. Soundness is obvious. For the reverse direction, let T be the theory generated by Γ; clearly $\mathbf{LindT}_T \in \mathbf{MOD}^*(\mathrm{L})$ (Lemma 2.2.9). Consider a $\mathbf{LindT}_T$-evaluation e defined as $e(\psi) = [\psi]_T$ and observe that $e[\Gamma] \subseteq e[T] \subseteq [T]$. Thus from $\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi$ we obtain that $[\varphi]_T = e(\varphi) \in [T]$ and thus $T \vdash_{\mathrm{L}} \varphi$ and so finally $\Gamma \vdash_{\mathrm{L}} \varphi$. □

The proof shows how the theorem can be strengthened: every weakly implicative logic is complete w.r.t. the class of Lindenbaum–Tarski matrices.

2.3 Advanced semantics and a third completeness theorem

In this subsection, after recalling some further knowledge about closure systems, closure operators and logical matrices, we obtain a third completeness theorem for finitary weakly implicative logics.

DEFINITION 2.3.1 (Closure system). *A* closure system *over a set A is a collection of subsets $\mathcal{C} \subseteq \mathcal{P}(A)$ closed under arbitrary intersections and such that $A \in \mathcal{C}$. The elements of $\mathcal{C}$ are called* closed sets*.*

For example, we have seen that given a logic L and $\langle \mathbf{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$, $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system over A; in particular $Th(\mathrm{L})$ is a closure system over $Fm_{\mathcal{L}}$.

DEFINITION 2.3.2 (Closure operator). *Given a set A, a* closure operator *over A is a mapping $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that for every $X, Y \subseteq A$:*

1. $X \subseteq C(X)$,
2. $C(X) = C(C(X))$, *and*
3. *if $X \subseteq Y$, then $C(X) \subseteq C(Y)$.*

Every closure operator C defines a closure system: $\{X \subseteq A \mid C(X) = X\}$. Conversely, given a closure system $\mathcal{C}$ over A, one can define a closure operator as follows: $C(X) = \bigcap\{Y \in \mathcal{C} \mid X \subseteq Y\}$. This gives a one-to-one correspondence between closure operators and systems. The closure operator associated to the closure system $\mathrm{Th}(\mathrm{L})$ will be denoted as Th_{L}, analogously the one associated to $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ will be denoted as $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$; as usual, we omit the parameters when known from the context.

A closure operator C is *finitary* if for every $X \subseteq A$, $C(X) = \bigcup\{C(Y) \mid Y \subseteq X$ and Y is finite$\}$. A closure system $\mathcal{C}$ is called *inductive* if it is closed under unions of upwards directed families (i.e. families $\mathcal{D} \neq \emptyset$ such that for every $A, B \in \mathcal{D}$, there is $C \in \mathcal{D}$ such that $A \cup B \subseteq C$).

THEOREM 2.3.3 (Schmidt Theorem). *A closure operator C is finitary if, and only if, its associated closure system $\mathcal{C}$ is inductive.*

Proof. Assume that C is finitary and take an upwards directed family $\mathcal{D} \subseteq \mathcal{C}$. It suffices to show that $C(\bigcup \mathcal{D}) \subseteq \bigcup \mathcal{D}$. Take any $a \in C(\bigcup \mathcal{D})$. By finitarity, $a \in C(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in \bigcup \mathcal{D}$. Since $\mathcal{D}$ is upwards directed, there exists a $T_0 \in \mathcal{D}$ such that $a_1, \ldots, a_n \in T_0$ and, hence, $a \in C(T_0) = T_0 \in \mathcal{D}$, and so $a \in \bigcup \mathcal{D}$. Conversely, assume that $\mathcal{C}$ is inductive, take any $X \subseteq A$ and consider the family $\mathcal{D} = \{C(F) \mid F \subseteq X \text{ finite}\}$. Since $\mathcal{D}$ is clearly upwards directed we have $\bigcup \mathcal{D} \in \mathcal{C}$, and therefore $\bigcup \mathcal{D} = C(X)$. □

Note that the finitarity of a logic L is equivalent to the finitarity of the corresponding closure operator Th_{L}. The next corollary is the first example we meet in this chapter of the so-called *transfer theorems*: theorems which transfer a given property of a logic L (understood as the closure operator/system over the set of formulae) to the analogous property of closure operator/system of all L-filters over *any* algebra.

COROLLARY 2.3.4 (Transfer theorem for finitarity). *Given a logic* L *in a language* $\mathcal{L}$, *the following conditions are equivalent:*

1. L *is finitary.*
2. $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$ *is a finitary closure operator for any* $\mathcal{L}$*-algebra* $\boldsymbol{A}$.
3. $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ *is an inductive closure system for any* $\mathcal{L}$*-algebra* $\boldsymbol{A}$.

Proof. The equivalence of the last two claims is established by the previous theorem. It is clear that 2 implies 1 by taking $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$. Let us see that 1 implies 3. Take an upwards directed family $\mathcal{F} \subseteq \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ and define $F = \bigcup \mathcal{F}$. We need to show that $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. Assume that $\Gamma \vdash_{\mathrm{L}} \varphi$ and e is an $\boldsymbol{A}$-evaluation such that $e[\Gamma] \subseteq F$. Since L is finitary, there is a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{L}} \varphi$. Then, since the family is upwards directed, there has to be an $F_0 \in \mathcal{F}$ such that $e[\Gamma_0] \subseteq F_0$ and so $e(\varphi) \in F_0 \subseteq F$. □

A useful notion in the theory of closure systems is that of *base*, which is a distinguished family of closed sets allowing to describe all closed sets of the system.

DEFINITION 2.3.5 (Base). *A* base *of a closure system* $\mathcal{C}$ *over* A *is any* $\mathcal{B} \subseteq \mathcal{C}$ *satisfying one of the following equivalent conditions:*

1. $\mathcal{C}$ *is the finest closure system containing* $\mathcal{B}$.
2. *For every* $T \in \mathcal{C} \setminus \{A\}$, *there is a* $\mathcal{D} \subseteq \mathcal{B}$ *such that* $T = \bigcap \mathcal{D}$.
3. *For every* $T \in \mathcal{C} \setminus \{A\}$, $T = \bigcap \{B \in \mathcal{B} \mid T \subseteq B\}$.
4. *For every* $Y \in \mathcal{C}$ *and* $a \in A \setminus Y$ *there is* $Z \in \mathcal{B}$ *such that* $Y \subseteq Z$ *and* $a \notin Z$.

DEFINITION 2.3.6 (Maximal w.r.t. an element, saturated, and (finitely) $\cap$-irreducible closed sets). *An element* X *of a closure system* $\mathcal{C}$ *over* A *is called*

- maximal w.r.t. an element a *if it is a maximal element of the set* $\{Y \in \mathcal{C} \mid a \notin Y\}$ *w.r.t. the order given by inclusion,*
- saturated *if it is maximal w.r.t. some element* a,
- (finitely) $\cap$-irreducible *if for each (finite non-empty) set* $\mathcal{Y} \subseteq \mathcal{C}$ *such that* $X = \bigcap_{Y \in \mathcal{Y}} Y$, *there is* $Y \in \mathcal{Y}$ *such that* $X = Y$.

Note that the set A is finitely $\cap$-irreducible but is not $\cap$-irreducible, because it is the intersection of the empty set. Also observe that finite-$\cap$-irreducibility of X can be equivalently defined by the following condition: for each $Y_1, Y_2 \in \mathcal{C}$ such that $X = Y_1 \cap Y_2$ we have $X = Y_1$ or $X = Y_2$.

PROPOSITION 2.3.7. *Let $\mathcal{C}$ be a closure system over a set A and $T \in \mathcal{C}$. Then, T is saturated if, and only if, T is $\cap$-irreducible.*

Proof. Assume that T is not $\cap$-irreducible, i.e. there is a family $\{T_i \mid i \in I\} \subseteq \mathcal{C}$ such that $T = \bigcap_{i \in I} T_i$ and $T \subsetneq T_i$ for every $i \in I$. Therefore, for every $i \in I$ we can choose $b_i \in T_i \setminus T$, and thus $T \subsetneq C(T, b_i) \subseteq T_i$; this gives: $T = \bigcap\{C(T, b_i) \mid i \in I\}$ and hence $T = \bigcap\{C(T, b) \mid b \notin T\}$. Assume, in search of a contradiction, that T is maximal w.r.t. an element $a \in A$. Then for every $b \notin T$, we have $T \subsetneq C(T, b)$, and the maximality implies $a \in C(T, b)$. Thus $a \in \bigcap\{C(T, b) \mid b \notin T\} = T$; a contradiction.

Conversely, assume that T is $\cap$-irreducible. Clearly, $T \subsetneq \bigcap\{C(T, b) \mid b \notin T\}$ and thus there is $a \in \bigcap\{C(T, b) \mid b \notin T\} \setminus T$, which means that T is maximal w.r.t. a. Indeed: if $T' \in \mathcal{C}$ and $T \subsetneq T'$, then there is $b \in T' \setminus T$, and thus $a \in C(T, b) \subseteq T'$. □

The next lemma allows to prove that in finitary closure systems, the $\cap$-irreducible sets always form a base. This will be later used, as a particular consequence, to obtain a refined completeness theorem for weakly implicative logics.

LEMMA 2.3.8 (Abstract Lindenbaum Lemma). *Let C be a finitary closure operator and $\mathcal{C}$ its corresponding closure system. If $T \in \mathcal{C}$ and $a \notin T$, then there is $T' \in \mathcal{C}$ such that $T \subseteq T'$ and T' is maximal with respect to a.*

Proof. The proof is an easy application of Zorn's Lemma. Observe that the set $\mathcal{A} = \{S \in \mathcal{C} \mid T \subseteq S, a \notin S\}$ is clearly non-empty because $T \in \mathcal{A}$. Take any chain $\{S_i \mid i \in I\} \subseteq \mathcal{A}$. By Schmidt Theorem $\bigcup_{i \in I} S_i \in \mathcal{C}$ and it is obvious that it contains T and it does not have a as an element, hence $\bigcup_{i \in I} S_i \in \mathcal{A}$ and it is an upper bound of the chain. By Zorn's Lemma $\mathcal{A}$ has some maximal element T' which satisfies the desired property. □

COROLLARY 2.3.9. *Let C be a finitary closure operator and $\mathcal{C}$ its associated closure system. Then the class of $\cap$-irreducible (i.e. saturated) sets of $\mathcal{C}$ forms a base of $\mathcal{C}$.*

On the other hand, it is also interesting to consider the maximal closed sets in a closure system:

DEFINITION 2.3.10 (Maximal closed set). *Let $\mathcal{C}$ be a closure system over A. A closed set $T \in \mathcal{C} \setminus \{A\}$ is called* maximal *or* maximally consistent *if it is a maximal element in $\mathcal{C} \setminus \{A\}$ with respect to the order given by inclusion.*

The following characterization is straightforward:

PROPOSITION 2.3.11. *Let $\mathcal{C}$ be a closure system over A and $T \in \mathcal{C} \setminus \{A\}$. The following are equivalent:*

1. *T is maximally consistent.*
2. *T is maximal w.r.t. every $a \in A \setminus T$.*

As another consequence of the abstract Lindenbaum Lemma one can show that every closed set can be extended to a maximally consistent one:

PROPOSITION 2.3.12. *Let $\mathcal{C}$ be a finitary closure system over A with an inconsistent element (i.e. an element $a \in A$ such that $C(a) = A$). Then every $T \in \mathcal{C} \setminus \{A\}$ can be extended to a maximally consistent $T' \in \mathcal{C}$.*

Proof. Since a does not belong to T (otherwise we would have $T = A$), by the Lindenbaum Lemma, there is $T' \supseteq T$ maximal w.r.t. a. Then T' is actually maximally consistent. Indeed, if $T' \subsetneq T''$, then $a \in T''$ and thus $T'' = A$. □

However, this last result does not entail that maximally consistent sets form a base. Although it is well-known that this is the case in classical logic, it is not generally true. For instance, the basic fuzzy logic BL introduced in the previous chapter provides a counterexample.

Now we introduce some further necessary notions on matrix theory. Observe that an $\mathcal{L}$-matrix $\langle \boldsymbol{A}, F\rangle$ can be regarded as a first-order structure in the equality-free predicate language with function symbols from $\mathcal{L}$ and a unique unary predicate symbol, with domain A, function symbols interpreted as the operations of $\boldsymbol{A}$, and the predicate interpreted by F. From this perspective, one can define the usual notions of substructure (now called *submatrix*), *homomorphism* (if $a \in F_1$, then $h(a) \in F_2$), *strict homomorphism* ($a \in F_1$ iff $h(a) \in F_2$), *isomorphism*, *direct product*, *reduced product* and *ultraproduct* for matrices. Given a class of matrices $\mathbb{K}$, we will denote by $\mathbf{S}(\mathbb{K})$, $\mathbf{H}(\mathbb{K})$, $\mathbf{H}_{\mathrm{S}}(\mathbb{K})$, $\mathbf{I}(\mathbb{K})$, $\mathbf{P}(\mathbb{K})$, $\mathbf{P}_{\mathrm{R}}(\mathbb{K})$ and $\mathbf{P}_{\mathrm{U}}(\mathbb{K})$ the closure of $\mathbb{K}$ under the mentioned operations. Another special operation on the classes of matrices we will need later is the operator of reduced products over countably complete filters (i.e. filters closed under countable intersections) which we will denote as $\mathbf{P}_{\sigma\text{-}f}$. Note that obviously $\mathbf{P}(\mathbb{K}) \subseteq \mathbf{P}_{\sigma\text{-}f}(\mathbb{K})$. It should also be noted that a bijective matrix homomorphism is not necessarily an isomorphism (because its inverse need not be a matrix homomorphism). An *embedding* of matrices is an injective strict homomorphism.

From the results in [24] one can obtain the following properties about the behavior of these operators on models and reduced models (observe that the third claim generalizes Lemma 2.1.17):

PROPOSITION 2.3.13. *Let* L *be a weakly implicative logic. Then:*

1. $\mathbf{SP}(\mathbf{MOD}(\mathrm{L})) \subseteq \mathbf{MOD}(\mathrm{L})$.
2. $\mathbf{SP}_{\sigma\text{-}f}(\mathbf{MOD}^*(\mathrm{L})) \subseteq \mathbf{MOD}^*(\mathrm{L})$.
3. *If* $\mathbb{K} \subseteq \mathbf{MOD}(\mathrm{L})$*,* $\mathbf{P}_{\mathrm{U}}\mathbf{I}(\mathbb{K}) \subseteq \mathbf{I}(\mathbb{K})$*, and* $\mathrm{L} = \models_{\mathbb{K}}$*, then* L *is finitary.*
4. $\mathbf{P}_{\mathrm{U}}(\mathbf{MOD}^*(\mathrm{L})) \subseteq \mathbf{MOD}^*(\mathrm{L})$ *iff* L *is finitary.*

As a consequence we obtain that, for every weakly implicative logic L, $\mathbf{ALG}^*(\mathrm{L})$ is closed under subalgebras and direct products; moreover for every $\mathcal{L}$-algebra $\boldsymbol{A}$ the set of relative congruences, $\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$ is a complete lattice w.r.t. the inclusion order (indeed, given a family $\mathcal{X} \subseteq \boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$ the quotient of $\boldsymbol{A}$ by $\bigcap \mathcal{X}$ embeds into the direct product of the quotients of $\boldsymbol{A}$ by the elements of $\mathcal{X}$ and hence, since $\mathbf{ALG}^*(\mathrm{A})$ is closed under $\mathbf{S}$ and $\mathbf{P}$, we are done).

The notion of subdirect product from Universal Algebra is also generalized to matrices. A matrix $\mathbf{A}$ is said to be *representable as a subdirect product* of the family of matrices $\{\mathbf{A}_i \mid i \in I\}$ if there is an embedding homomorphism α from $\mathbf{A}$ into the direct product $\prod_{i\in I} \mathbf{A}_i$ such that for every $i \in I$, the composition of α with the i-th projection, $\pi_i \circ \alpha$, is a surjective homomorphism. In this case, α is called a *subdirect representation*, and it is called *finite* if I is finite.

Let L be a logic and $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. By $\mathbf{P}_{\mathrm{SD}}(\mathbb{K})$ we denote the closure of $\mathbb{K}$ under subdirect products. A non-trivial matrix $\mathbf{A} \in \mathbb{K}$ is *(finitely) subdirectly irreducible relative to* $\mathbb{K}$ if for every (finite non-empty) subdirect representation α of $\mathbf{A}$ with a family $\{\mathbf{A}_i \mid i \in I\} \subseteq \mathbb{K}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. The class of all (finitely) subdirectly irreducible matrices relative to $\mathbb{K}$ is denoted as $\mathbb{K}_{\mathrm{R(F)SI}}$. Of course $\mathbb{K}_{\mathrm{RSI}} \subseteq \mathbb{K}_{\mathrm{RFSI}}$. When $\mathbb{K} = \mathbf{MOD}^*(\mathrm{L})$ these classes are characterized in the following way:

THEOREM 2.3.14 (Characterization of RSI and RFSI reduced models). *Given a weakly implicative logic* L *and* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$*, we have:*

1. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ *if, and only if,* F *is* $\cap$*-irreducible in* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.
2. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ *if, and only if,* F *is finitely* $\cap$*-irreducible in* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.

Proof. Let us first solve the case when $\mathbf{A}$ is a trivial reduced matrix, i.e. $F = A = \{a\}$. Recall that in this case, F is finitely $\cap$-irreducible but not $\cap$-irreducible in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. Obviously, $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ and $\mathbf{A} \notin \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$, because the product of the empty system of matrices is a trivial matrix.

We write only the proof for the first claim (the second one is completely analogous). Suppose that $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ and, in search of a contradiction, that F is not $\cap$-irreducible in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, i.e. $F = \bigcap_{i\in I} F_i$ where $F \subsetneq F_i \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ for every $i \in I$. We use these filters to define reduced matrices $\mathbf{A}_i = \langle \boldsymbol{A}, F_i\rangle^* \in \mathbf{MOD}^*(\mathrm{L})$ and show that $\alpha\colon \mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}_i$, defined as $\alpha(a) = \langle [a]_{F_i} \mid i \in I\rangle$, is a subdirect representation of $\mathbf{A}$. The homomorphisms $\pi_i \circ \alpha$ are indeed surjective, α is clearly is a strict homomorphism, so it remains to show the injectivity of α. Assume that $a \neq b$. Then, since $\mathbf{A}$ is reduced, we can (without loss of generality) assume that $a \to^{\boldsymbol{A}} b \notin F$ and so $a \to^{\boldsymbol{A}} b \notin F_i$ for some $i \in I$. Thus $[a]_{F_i} \neq [b]_{F_i}$ and so $\alpha(a) \neq \alpha(b)$. Since $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$, there must be $j \in I$ such that $\pi_j \circ \alpha$ is an isomorphism. Assume now that $a \in F_j$, this implies $\pi_j(\alpha(a)) = [a]_{F_j} \in [F_j]$ and, since $\pi_j \circ \alpha$ is isomorphism it is a also a strict homomorphism of $\mathbf{A}$ and $\mathbf{A}_j$ and so $a \in F$, and hence $F_j = F$—a contradiction.

We prove the converse direction contrapositively: assume that $\mathbf{A} \notin \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$, i.e. there is a family of reduced models of the logic $\{\mathbf{A}_i = \langle \boldsymbol{A}_i, F_i\rangle \mid i \in I\}$ and a subdirect representation $\alpha\colon \mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}_i$ where no projection gives an isomorphism. This will allow us to define a collection of filters giving a decomposition of F. Indeed, take $\bar{F}_i = (\pi_i \circ \alpha)^{-1}[F_i]$ and so by Lemma 2.1.19 $\bar{F}_i \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. Due to the strictness of α we have $F = \bigcap_{i\in I} \bar{F}_i$. If $F = \bar{F}_j$ for some $j \in I$, then $\pi_j \circ \alpha$ would be an isomorphism, contradicting the hypothesis. Therefore F is not $\cap$-irreducible in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. □

THEOREM 2.3.15 (Subdirect representation). *If* L *is a finitary weakly implicative logic, then* $\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P}_{\mathrm{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}})$*, thus in particular every matrix in* $\mathbf{MOD}^*(\mathrm{L})$ *is representable as a subdirect product of matrices in* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$.

Proof. One inclusion is relatively easy: $\mathbf{P}_{\mathrm{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}) \subseteq \mathbf{SP}(\mathbf{MOD}^*(\mathrm{L})) \subseteq \mathbf{SP}_{\sigma\text{-}f}(\mathbf{MOD}^*(\mathrm{L})) \subseteq \mathbf{MOD}^*(\mathrm{L})$, (the last inclusion is due to claim 2 in Proposition 2.3.13, the others are trivial). To prove the converse inclusion consider $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$. By Corollary 2.3.4, $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$ is finitary and, by Corollary 2.3.9, there exists a family $\{F_i \mid i \in I\}$ of $\cap$-irreducible filters such that $F = \bigcap_{i\in I} F_i$. Take $\mathbf{A}_i = \langle \boldsymbol{A}, F_i\rangle^*$. So we have a subdirect representation $\alpha\colon \mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}_i$ taking $\alpha(a) = \langle [a]_i \mid i \in I\rangle$ for every $a \in A$. Because F_i is $\cap$-irreducible, Theorem 2.3.14 tells us that $\langle \boldsymbol{A}, F_i\rangle^* \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ for every $i \in I$. □

As a consequence of this theorem and Theorem 2.2.13 we obtain a third completeness theorem (this time restricted to finitary logics):

THEOREM 2.3.16 (Completeness w.r.t. RSI reduced models). *Let* L *be a finitary weakly implicative logic. Then* $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}$.

2.4 Algebraically implicative logics

In this subsection we consider the relation between weakly implicative logics and the equational consequence on their corresponding classes of algebras. Let us fix a propositional language $\mathcal{L}$.

DEFINITION 2.4.1 (Equation). *An equation in the language* $\mathcal{L}$ *is a formal expression of the form* $\varphi \approx \psi$*, where* $\varphi, \psi \in Fm_{\mathcal{L}}$.

DEFINITION 2.4.2 (Equational consequence). *We say that an equation* $\varphi \approx \psi$ *is a consequence of a set of equations* Π *w.r.t. a class* $\mathbb{K}$ *of* $\mathcal{L}$*-algebras if for each* $\boldsymbol{A} \in \mathbb{K}$ *and each* $\boldsymbol{A}$*-evaluation* e *we have* $e(\varphi) = e(\psi)$ *whenever* $e(\alpha) = e(\beta)$ *for each* $\alpha \approx \beta \in \Pi$*; when this is the case, we denote it by* $\Pi \models_{\mathbb{K}} \varphi \approx \psi$.

Given any weakly implicative logic L, the equational consequence given by the class of L-algebras can be translated into the logic in the following way:

PROPOSITION 2.4.3. *Let* L *be a weakly implicative logic and* $\Pi \cup \{\varphi \approx \psi\}$ *a set of equations. Then* $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ *iff* $\{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$.

Proof. We show the proof of one implication, the proof of the converse one is similar. Assume $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$. To check that $\{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$ it is enough (due to the completeness theorem 2.2.13) to check the equivalent semantical statement $\{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi \leftrightarrow \psi$. Take any $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ and an $\boldsymbol{A}$-evaluation v satisfying the premises, i.e. for every $\alpha \approx \beta \in \Pi$ we have $v(\alpha) \to^{\boldsymbol{A}} v(\beta), v(\beta) \to^{\boldsymbol{A}} v(\alpha) \in F$, and hence (since the matrix is reduced) $v(\alpha) = v(\beta)$. By the assumption (using that $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$) we know that $v(\varphi) = v(\psi)$ and thus $v(\varphi) \to^{\boldsymbol{A}} v(\psi), v(\psi) \to^{\boldsymbol{A}} v(\varphi) \in F$. □

However, if we want to obtain a better connection between the logic and the equational consequence enjoying also a translation from the former to the latter, we need to restrict to a special subclass of weakly implicative logics.

DEFINITION 2.4.4 (Algebraically implicative logic). *We say that a logic* L *is* algebraically implicative *if it is weakly implicative and there is a set of equations $\mathcal{E}$ in one variable such that for each $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ and each $a \in A$ holds: $a \in F$ if, and only if, $\mu^{\boldsymbol{A}}(a) = \nu^{\boldsymbol{A}}(a)$ for every $\mu \approx \nu \in \mathcal{E}$. In this case, $\mathbf{ALG}^*(\mathrm{L})$ is called the* equivalent algebraic semantics *of* L.

For a set Γ of formulae, by $\mathcal{E}[\Gamma]$ we denote the set $\bigcup\{\mathcal{E}(\gamma) \mid \gamma \in \Gamma\}$ of equations.

THEOREM 2.4.5 (Characterizations of algebraically implicative logics). *Given any weakly implicative logic* L*, the following are equivalent:*

1. L *is algebraically implicative.*
2. *There is a set of equations $\mathcal{E}$ in one variable such that*

   (Alg) $\quad p \dashv\vdash_{\mathrm{L}} \{\mu(p) \leftrightarrow \nu(p) \mid \mu \approx \nu \in \mathcal{E}\}$.

3. *There is a set of equations $\mathcal{E}$ in one variable such that:*
   - *for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\mathcal{E}[\Gamma] \models_{\mathbf{ALG}^*(\mathrm{L})} \mathcal{E}(\varphi)$ and*
   - *$p \approx q \models_{\mathbf{ALG}^*(\mathrm{L})} \mathcal{E}[p \leftrightarrow q]$ and $\mathcal{E}[p \leftrightarrow q] \models_{\mathbf{ALG}^*(\mathrm{L})} p \approx q$.*
4. *For every $\mathcal{L}$-algebra $\boldsymbol{A}$, the Leibniz operator $\Omega_{\boldsymbol{A}}$ is a lattice isomorphism from $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ to $\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$.*
5. *For every $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$, F is the least* L*-filter on $\boldsymbol{A}$.*

In the first three items the sets $\mathcal{E}$ can be taken the same.

Proof. First, we prove the equivalence of the first three claims, then the equivalence of the last two claims, and finally we prove the implications 1→4 and 4→2.
2→1: It follows immediately from the completeness theorem and the definition of algebraically implicative logic.
1→3: The first condition again is easily checked by using the completeness theorem and the definition of algebraically implicative logic. To prove $p \approx q \models_{\mathbf{ALG}^*(\mathrm{L})} \mathcal{E}[p \leftrightarrow q]$ take $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ and an evaluation e on $\boldsymbol{A}$ such that $e(p) = e(q)$. Then $e(p) \to^{\boldsymbol{A}} e(q) \in F$ (by (R)) and so $\mu^{\boldsymbol{A}}(e(p) \to^{\boldsymbol{A}} e(q)) = \nu^{\boldsymbol{A}}(e(p) \to^{\boldsymbol{A}} e(q))$ for every $\mu \approx \nu \in \mathcal{E}$, and the same for the reverse implication. To prove $\mathcal{E}[p \leftrightarrow q] \models_{\mathbf{ALG}^*(\mathrm{L})} p \approx q$ take $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ and an evaluation e on $\boldsymbol{A}$ satisfying the equations in the premises. Then $e(p) \to^{\boldsymbol{A}} e(q), e(q) \to^{\boldsymbol{A}} e(p) \in F$, i.e. $\langle e(p), e(q)\rangle \in \Omega_{\boldsymbol{A}}(F)$ and since the matrix is reduced, $e(p) = e(q)$.
3→2: We want to check $p \dashv\vdash_{\mathrm{L}} \{\mu(p) \leftrightarrow \nu(p) \mid \mu \approx \nu \in \mathcal{E}\}$. By the first condition in 3, this is equivalent to a (double) equational consequence w.r.t. $\mathbf{ALG}^*(\mathrm{L})$ which, using the second condition in 3 becomes a trivial statement.
4→5: Just observe that $\Omega_{\boldsymbol{A}}(F) = \mathrm{Id}_{\boldsymbol{A}}$ and use the isomorphism of $\Omega_{\boldsymbol{A}}$.
5→4: Recall that for every $\mathcal{L}$-algebra $\boldsymbol{A}$, $\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$ is a complete lattice (see the comments after Proposition 2.3.13). From Proposition 2.2.11 we know that $\Omega_{\boldsymbol{A}}$ is surjective and it preserves meets. We show that it is one-to-one. Suppose $\Omega_{\boldsymbol{A}}(F) =$

$\Omega_{\boldsymbol{A}}(G)$ for some $F, G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. Then, by the assumption 5, $F/\Omega_{\boldsymbol{A}}(F)$ is the least L-filter on $\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)$, and $G/\Omega_{\boldsymbol{A}}(G)$ is the least L-filter on $\boldsymbol{A}/\Omega_{\boldsymbol{A}}(G) = \boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)$, thus: $F/\Omega_{\boldsymbol{A}}(F) = G/\Omega_{\boldsymbol{A}}(G)$. Now take any $a \in F$. $[a]_F \in F/\Omega_{\boldsymbol{A}}(F) = G/\Omega_{\boldsymbol{A}}(G)$, and, since $\Omega_{\boldsymbol{A}}(F) = \Omega_{\boldsymbol{A}}(G)$, $[a]_F = [a]_G$, and thus $[a]_G \in G/\Omega_{\boldsymbol{A}}(G)$, which gives $a \in G$. By symmetry, we have $F = G$. We show now that $\Omega_{\boldsymbol{A}}$ is order-reflecting: if $\Omega_{\boldsymbol{A}}(F) \subseteq \Omega_{\boldsymbol{A}}(G)$ then $\Omega_{\boldsymbol{A}}(F \cap G) = \Omega_{\boldsymbol{A}}(F) \cap \Omega_{\boldsymbol{A}}(G) = \Omega_{\boldsymbol{A}}(F)$, so $F \cap G = F$, by injectivity, and thus $F \subseteq G$. Therefore, $\Omega_{\boldsymbol{A}}$ is an order-preserving and order-reflecting bijection, and hence it is a lattice isomorphism.
1→4: We first show that $\Omega_{\boldsymbol{A}}$ is one-one. Suppose $\Omega_{\boldsymbol{A}}(F) = \Omega_{\boldsymbol{A}}(G)$ for some $F, G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. Given any $a \in A$, we have the following chain of equivalencies: $a \in F$ iff $[a]_F \in F/\Omega_{\boldsymbol{A}}(F)$ iff $\mu([a]_F) = \nu([a]_F)$ for every $\mu \approx \nu \in \mathcal{E}$ iff $\mu([a]_G) = \nu([a]_G)$ for every $\mu \approx \nu \in \mathcal{E}$ iff $[a]_G \in G/\Omega_{\boldsymbol{A}}(G)$ iff $a \in G$. In a very similar way we can check that it is order-reflecting. From Proposition 2.2.11 we know that $\Omega_{\boldsymbol{A}}(F)$ is onto and order-preserving, and thus it is a lattice isomorphism.
4→2: First we prove $T = \mathrm{Th}_{\mathrm{L}}(\{\alpha \leftrightarrow \beta \mid \langle \alpha, \beta\rangle \in \Omega_{\boldsymbol{Fm}}(T)\})$ for every $T \in \mathrm{Th}(\mathrm{L})$. Define $T' = \mathrm{Th}_{\mathrm{L}}(\{\alpha \leftrightarrow \beta \mid \langle \alpha, \beta\rangle \in \Omega_{\boldsymbol{Fm}}(T)\})$. On the one hand, $T' \subseteq T$, so by monotonicity $\Omega_{\boldsymbol{Fm}}(T') \subseteq \Omega_{\boldsymbol{Fm}}(T)$. On the other hand, if $\langle \alpha, \beta\rangle \in \Omega_{\boldsymbol{Fm}}(T)$, then $\alpha \leftrightarrow \beta \in T'$, so $\langle \alpha, \beta\rangle \in \Omega_{\boldsymbol{Fm}}(T')$. Therefore, we have $\Omega_{\boldsymbol{Fm}}(T') = \Omega_{\boldsymbol{Fm}}(T)$ and, by injectivity, $T = T'$. Thus, in particular we have shown that

$$p \dashv\vdash \{\alpha \leftrightarrow \beta \mid \langle \alpha, \beta\rangle \in \Omega_{\boldsymbol{Fm}}(\mathrm{Th}_{\mathrm{L}}(p))\}.$$

Let σ be the substitution mapping all variables to p. Then

$$p \dashv\vdash \{\sigma(\alpha) \leftrightarrow \sigma(\beta) \mid \langle \alpha, \beta\rangle \in \Omega_{\boldsymbol{Fm}}(\mathrm{Th}_{\mathrm{L}}(p))\}.$$

Therefore the set $\mathcal{E}(p) = \{\sigma(\alpha) \approx \sigma(\beta) \mid \langle \alpha, \beta\rangle \in \Omega_{\boldsymbol{Fm}}(\mathrm{Th}_{\mathrm{L}}(p))\}$ clearly satisfies the condition (Alg). □

Observe that, due to Corollary 2.2.4, the definition of algebraically implicative logics is intrinsic because it does not depend on the chosen implication.

EXAMPLE 2.4.6. In many cases of interest, one equation is enough to satisfy condition (Alg). For instance, classical logic and, in general, all the expansions of MTL mentioned in the previous chapter are algebraically implicative by using the set $\{p \approx \overline{1}\}$, and UL is algebraically implicative by using $\{p \wedge \overline{1} \approx \overline{1}\}$.

PROPOSITION 2.4.7. *If* L *is a finitary algebraically implicative logic, then* $\mathbf{ALG}^*(\mathrm{L})$ *is a quasivariety and the set* $\mathcal{E}$ *can be taken finite.*

Proof. To prove that $\mathbf{ALG}^*(\mathrm{L})$ is a quasivariety (i.e. a quasiequational class of algebras) it is enough to take an arbitrary $\mathcal{L}$-algebra $\boldsymbol{A}$ such that all the quasiequations valid in $\mathbf{ALG}^*(\mathrm{L})$ hold in $\boldsymbol{A}$, and prove that then $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$. Define the filter of $\boldsymbol{A}$ as the set $F = \{a \in A \mid \mu^{\boldsymbol{A}}(a) = \nu^{\boldsymbol{A}}(a)$ for every $\mu \approx \nu \in \mathcal{E}\}$. Let us see that $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$. Suppose that $\Gamma \vdash_{\mathrm{L}} \varphi$. By finitarity, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{L}} \varphi$. Assume that, for some evaluation e on $\boldsymbol{A}$, $e[\Gamma] \subseteq F$. By the first condition in part 3 of the previous theorem we have $\mathcal{E}[\Gamma_0] \models_{\mathbf{ALG}^*(\mathrm{L})} \mathcal{E}(\varphi)$, which can be seen as a quasiequation and hence also valid in $\boldsymbol{A}$. On the other hand we have $e[\Gamma_0] \subseteq F$, therefore $e(\varphi) \in F$. Finally, the second condition in part 3 of the previous theorem implies that the matrix is reduced.

The fact that the set $\mathcal{E}$ can be taken finite is a straightforward corollary of claim 2 of Theorem 2.4.5. □

Analogously to the convention for weakly implicative logics, which always come with a fixed principal implication, for each algebraically implicative logic we fix the default set of equations providing algebraicity and denote it as $\mathcal{E}$ (in fact, this set is unique up to interderivability in $\mathbf{ALG}^*(\mathrm{L})$). Observe that these equations can be identified with their corresponding pairs of formulae; then we call them *algebraizing pairs*. The fact that filters in reduced matrices can be defined by equations has several interesting straightforward consequences.

PROPOSITION 2.4.8. *Let* L *be an algebraically implicative logic,* $\boldsymbol{A}, \boldsymbol{B} \in \mathbf{ALG}^*(\mathrm{L})$*, and* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$*. Then:*

1. $F \subseteq G$ *for any* $G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$*.*
2. *If* $\langle \boldsymbol{A}, G\rangle \in \mathbf{MOD}^*(\mathrm{L})$ *then* $F = G$*, i.e.* $\boldsymbol{A}$ *is the algebraic reduct of a unique reduced matrix.*
3. *A mapping* $h\colon A \to B$ *is a homomorphism of algebras from* $\boldsymbol{A}$ *to* $\boldsymbol{B}$ *iff it is a homomorphism between the corresponding matrices.*
4. *A mapping* $h\colon A \to B$ *is an embedding of algebras from* $\boldsymbol{A}$ *to* $\boldsymbol{B}$ *iff it is a one-to-one strict homomorphism between the corresponding matrices.*
5. $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})_{\mathrm{R(F)SI}}$ *iff* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{R(F)SI}}$*.*

Finally, we close our excursion to Abstract Algebraic Logic by introducing two special subclasses of algebraically implicative logics.

DEFINITION 2.4.9 (Rasiowa-implicative and regularly implicative logics). *We say that a logic* L *is* Rasiowa-implicative *if it is weakly implicative and*

(W) $\quad \varphi \vdash_{\mathrm{L}} \psi \to \varphi.$

We use the term regularly implicative *if* L *satisfies only this weaker condition:*

(Reg) $\quad \varphi, \psi \vdash_{\mathrm{L}} \psi \to \varphi.$

PROPOSITION 2.4.10. *A weakly implicative logic* L *is regularly implicative iff all the filters of the matrices in* $\mathbf{MOD}^*(\mathrm{L})$ *are singletons.*

Proof. Elementary check. □

PROPOSITION 2.4.11. *Each Rasiowa-implicative logic is regularly implicative and each regularly implicative logic is algebraically implicative.*

Proof. The first claim is obvious. For the second one it suffices to check that any regularly implicative logic satisfies the condition (Alg) (see Theorem 2.4.5) for the set of equations $\mathcal{E}(p) = \{p \approx p \to p\}$. □

Symbol	Arity	Name	Alternative names
$\to$	2	principal implication	right residuum
$\&$	2	residuated conjunction	fusion, multiplicative/strong conj.
$\rightsquigarrow$	2	dual implication	left residuum
$\wedge$	2	lattice protoconjunction	additive/weak/lattice conjunction
$\vee$	2	lattice protodisjunction	additive/weak/lattice disjunction
$\overline{1}$	0	*verum*	multiplicative truth, unit
$\overline{0}$	0	*falsum*	multiplicative falsum
$\top$	0	top	additive/lattice truth
$\bot$	0	bottom	additive/lattice falsum

Table 1. The language of SL

The four classes of implicative logics that we have defined (weakly, algebraically, regularly, Rasiowa-implicative) are mutually distinct. Indeed:

- The equivalence fragment of classical logic is a regularly implicative but not Rasiowa-implicative logic (to be more precise it is easy to see that the equivalence connective $\leftrightarrow$ does not satisfy the condition (W); moreover, in [21] it is proved that no weak implication satisfying this condition is definable in this logic).
- The uninorm logic UL is algebraically, but not regularly, implicative (because of Proposition 2.4.10).
- The logic BCI is weakly, but not algebraically, implicative (see [7]).

2.5 Substructural logics

In this section we introduce an important broad family of weakly implicative logics, the *substructural logics*, starting from a very weak one which we call SL. We present this basic logic in an implicit way as the least logic with a certain desired behavior of connectives. Then we define substructural logics as expansions of the corresponding fragment of SL. We will study some syntactical and semantical properties, and algebraization of these logics. Then we will be able to identify them among the substructural logics studied under this label in the literature; indeed we will show that SL actually coincides with the bounded non-associative full Lambek logic.

The language $\mathcal{L}_{\mathrm{SL}}$ consists of the connectives listed in Table 1, i.e. most of the usual connectives in substructural logics (we will comment on the names and rôle played by these connectives after the next definition). When writing formulae in this language (or its fragments) we will assume that the increasing binding order is: first $\&$, then $\{\wedge, \vee\}$, and finally $\{\to, \rightsquigarrow\}$. For the sake of consistency with the general convention in this chapter that every logic comes with a fixed principal implication $\to$, we keep on using this notation along with $\rightsquigarrow$ as the dual implication (soon we will prove a duality theorem that shows that the choice between the principal and the dual implication is in a way arbitrary). When identifying SL with the bounded non-associative full Lambek

Consecution	Symbol	Name
$\varphi \to (\psi \to \chi) \dashv\vdash \psi \,\&\, \varphi \to \chi$	(Res)	residuation
$\varphi \to (\psi \to \chi) \dashv\vdash \psi \to (\varphi \rightsquigarrow \chi)$	($\mathrm{E}_{\rightsquigarrow}$)	$\rightsquigarrow$-exchange
$\varphi \to \psi \dashv\vdash \varphi \rightsquigarrow \psi$	(symm)	symmetry
$\vdash \varphi \wedge \psi \to \varphi$	($\wedge 1$)	lower bound
$\vdash \varphi \wedge \psi \to \psi$	($\wedge 2$)	lower bound
$\chi \to \varphi, \chi \to \psi \vdash \chi \to \varphi \wedge \psi$	($\wedge 3$)	infimality
$\vdash \varphi \to \varphi \vee \psi$	($\vee 1$)	upper bound
$\vdash \psi \to \varphi \vee \psi$	($\vee 2$)	upper bound
$\varphi \to \chi, \psi \to \chi \vdash \varphi \vee \psi \to \chi$	($\vee 3$)	supremality
$\varphi \vdash \overline{1} \to \varphi$	(Push)	push
$\overline{1} \to \varphi \vdash \varphi$	(Pop)	pop
$\vdash \varphi \to \top$	(Veq)	*verum ex quolibet*
$\vdash \bot \to \varphi$	(Efq)	*ex falso quodlibet*

Table 2. Consecutions for SL

logic we will also show how our notation relates to the standard one for substructural logics in the literature which uses $\backslash$ and $/$ instead of $\to$ and $\rightsquigarrow$, graphically denoting that these implications are respectively the right and left residua of the conjunction $\&$.

DEFINITION 2.5.1 (The logic SL). SL *is the weakest weakly implicative logic in the language* $\mathcal{L}_{\mathrm{SL}}$ *satisfying the consecutions from Table 2.*

Observe that SL is a weakly implicative logic and $\to$ is its principal implication. Even though we do not explicitly postulate any additional properties of $\to$, we will see in Proposition 2.5.5 that its interplay with other connectives entails some rather strong properties usually possessed by implications in known (non-)classical logics. The connective $\&$ is a *residuated conjunction* whose rôle could be described as 'aggregation of premises in a chain of implications' as shown by residuation rules (Res). In fact, it must be noted that the order of arguments in the formulation of (Res) is arbitrary (for any connective $\&$ we could always define its transposition $\&^t$ as $\varphi \,\&^t\, \psi = \psi \,\&\, \varphi$); we have decided to formulate it in this way to have a more straightforward connection with a stronger axiomatic formulation of (Res) which is equivalent to associativity (see Theorem 2.5.7). While (Res) allows us to aggregate premises, ($\mathrm{E}_{\rightsquigarrow}$) allows us to swap them but at the price of replacing the inner occurrence of the principal implication by its dual version $\rightsquigarrow$ (the rule (symm) ensures that $\rightsquigarrow$ can be seen as another principal implication in SL interderivable with $\to$). However, we cannot replace ($\mathrm{E}_{\rightsquigarrow}$) by a simpler form involving only one implication, because it would entail commutativity of $\&$ (which can be refuted by a simple semantic counterexample). On the other hand, the semantics of these connectives is quite simple. Indeed, if we fix $\&^{\boldsymbol{A}}$ in any reduced SL-matrix $\mathbf{A}$ then $\to^{\boldsymbol{A}}$ has to be its *right residuum* and $\rightsquigarrow^{\boldsymbol{A}}$ the *left residuum* (see part 8 of Proposition 2.5.10) and both $\to^{\boldsymbol{A}}$ and $\rightsquigarrow^{\boldsymbol{A}}$ define the same matrix order $\leq^{\mathbf{A}}$. For more details on residuated structures and their logics see [43].

The remaining binary connectives are easily understood: the rules for $\wedge$ and $\vee$ ensure that these connectives correspond to the operations of infimum and supremum in the lattice order given by the principal implication. Note however that we do not call them 'conjunction' and 'disjunction' in Table 1 but add the prefix 'proto'. The reason is that these rules are not enough to enforce by themselves a *proper behavior* of these connectives:

1. in the case of $\wedge$, the adjunction rule $\varphi, \psi \vdash_{\mathrm{SL}} \varphi \wedge \psi$, essential in the intended behavior of conjunctions, holds due to the presence of the truth constant $\overline{1}$ and fails in the least weakly implicative logic satisfying all consecutions from Table 2 but (Push) and (Pop),
2. in the case of $\vee$, this protodisjunction does not enjoy the Proof by Cases Property in SL: $\Gamma, \varphi \vdash \chi$ and $\Gamma, \psi \vdash \chi$ entail $\Gamma, \varphi \vee \psi \vdash \chi$ (in Section 2.6 we will see how to recover this property in some extensions of SL and Section 2.7 studies its characterizations and consequences).

The meaning of $\top$ and $\bot$ and their defining rules is self-explanatory as maximum and minimum elements of the order induced by the principal implication. The rôle of $\overline{1}$ is to be the 'least designated truth value'. Finally, the rôle of $\overline{0}$, although its value is left unspecified (note that there is no consecution involving $\overline{0}$ among those in Table 2), is to define negations by $\varphi \to \overline{0}$ and $\varphi \rightsquigarrow \overline{0}$.

Of course we could immediately design one specific axiomatic system for SL (consisting of reflexivity, transitivity, *modus ponens*, the congruence rules for all connectives and consecutions from Table 2). Later (Theorem 2.5.13) we will present a more natural axiomatic system for SL. The idea behind our definition of SL, and behind the convention for substructural logics that we will introduce soon, is to pick a short list of rules that connectives must satisfy to have the minimal usual behavior in substructural logics. Moreover, as we will soon see (Proposition 2.5.4), the connectives are uniquely determined by these rules. The axiomatic system mentioned above allows us to prove quite easily the following duality theorem:

DEFINITION 2.5.2 (Mirror image). *Given a formula χ of $\mathcal{L}_{\mathrm{SL}}$ its* mirror image *χ' is obtained by replacing in χ all occurrences of $\to$ by $\rightsquigarrow$, and vice versa, and by replacing all subformulae of the form $\alpha \mathbin{\&} \beta$ by $\beta \mathbin{\&} \alpha$. The mirror image of a set T of formulae of $\mathcal{L}_{\mathrm{SL}}$ is $T' = \{\psi' \mid \psi \in T\}$.*

THEOREM 2.5.3 (Duality Theorem). *For each set of formulae $T \cup \{\varphi\}$ of $\mathcal{L}_{\mathrm{SL}}$ we have:*

$$T \vdash_{\mathrm{SL}} \varphi \qquad \textit{iff} \qquad T' \vdash_{\mathrm{SL}} \varphi'.$$

Proof. We show only one direction (the second one immediately follows from the fact that $(\varphi')' = \varphi$). We prove the claim for axioms and rules from the axiomatic system described in the last paragraph above with formulae replaced by variables and then, by structurality and the notion of proof, we are done.

The case of (symm) is trivial. From $p \leftrightarrow q \vdash_{\mathrm{SL}} p \mathbin{\&} r \to q \mathbin{\&} r$ and $p \leftrightarrow q \vdash_{\mathrm{SL}} r \mathbin{\&} p \to r \mathbin{\&} q$ we obtain $p \rightsquigarrow q, q \rightsquigarrow p \vdash_{\mathrm{SL}} p \mathbin{\&} r \rightsquigarrow q \mathbin{\&} r$ and $p \rightsquigarrow q, q \rightsquigarrow p \vdash_{\mathrm{SL}} r \mathbin{\&} p \rightsquigarrow r \mathbin{\&} q$, and so we have the mirror version of congruence for $\&$. The mirror versions of congruence rules for both implications are proved analogously.

Next observe: $\varphi \rightsquigarrow (\psi \rightsquigarrow \chi) \dashv\vdash_{\mathrm{SL}} \varphi \to (\psi \rightsquigarrow \chi) \dashv\vdash_{\mathrm{SL}} \psi \to (\varphi \to \chi) \dashv\vdash_{\mathrm{SL}} \psi \rightsquigarrow (\varphi \to \chi)$ and $\varphi \rightsquigarrow (\psi \rightsquigarrow \chi) \dashv\vdash_{\mathrm{SL}} \varphi \to (\psi \rightsquigarrow \chi) \dashv\vdash_{\mathrm{SL}} \psi \to (\varphi \to \chi) \dashv\vdash_{\mathrm{SL}} \varphi \,\&\, \psi \to \chi \dashv\vdash_{\mathrm{SL}} \varphi \,\&\, \psi \rightsquigarrow \chi$, thus also mirror versions of ($\mathrm{E}_{\rightsquigarrow}$) and (Res) are proved.

Let $T \rhd \varphi$ be any of the remaining rules. We know that neither $\&$ nor $\rightsquigarrow$ appears in any formula from $T \cup \{\varphi\}$ and all of these formulae are either variables or contain $\to$ exactly once and as principal connective. Thus the rule (symm) gives us straightforwardly the mirror versions of these rules. □

The following proposition shows that connectives are uniquely determined by the rules we have introduced.

PROPOSITION 2.5.4. *Let* L *be a weakly implicative expansion of* SL *with the same principal implication* $\to$ *and* c *a connective of* $\mathcal{L}_{\mathrm{SL}} \setminus \{\overline{0}\}$*. Suppose that* $\hat{c}$ *is a connective (primitive or definable) of* L *of the same arity as* c *obeying the rules for* c *in Table 2. Then the two connectives are equivalent in* L*, i.e.* $\vdash_{\mathrm{L}} \varphi \; c \; \psi \leftrightarrow \varphi \; \hat{c} \; \psi$*, or* $\vdash_{\mathrm{L}} c \leftrightarrow \hat{c}$*, according to the arity of* c*.*

Proof. The only non-trivial case is for $\rightsquigarrow$. From $\vdash_{\mathrm{L}} (\varphi \rightsquigarrow \psi) \to (\varphi \rightsquigarrow \psi)$ we use ($\mathrm{E}_{\rightsquigarrow}$) to obtain $\vdash_{\mathrm{L}} \varphi \to ((\varphi \rightsquigarrow \psi) \to \psi)$. Using ($\mathrm{E}_{\rightsquigarrow}$) this time for $\hat{\rightsquigarrow}$ gives us $\vdash_{\mathrm{L}} (\varphi \rightsquigarrow \psi) \to (\varphi \mathbin{\hat{\rightsquigarrow}} \psi)$. The second implication is completely analogous. □

PROPOSITION 2.5.5. *The following are derivable in* SL*:*

$(\mathrm{P_{SL}}1)$	$\vdash \varphi \to ((\varphi \to \psi) \rightsquigarrow \psi)$	
$(\mathrm{P_{SL}}2)$	$\vdash \varphi \,\&\, (\varphi \to \psi) \to \psi$	
$(\mathrm{P_{SL}}3)$	$\vdash \varphi \to ((\varphi \rightsquigarrow \psi) \to \psi)$	
$(\mathrm{P_{SL}}4)$	$\varphi \vdash (\varphi \to \psi) \to \psi$	
$(\mathrm{P_{SL}}5)$	$\varphi \to \psi \vdash (\psi \to \chi) \to (\varphi \to \chi)$	*(Suffixing)*
$(\mathrm{P_{SL}}6)$	$\psi \to \chi \vdash (\varphi \to \psi) \to (\varphi \to \chi)$	*(Prefixing)*
$(\mathrm{P_{SL}}7)$	$\vdash \varphi \to (\psi \to \psi \,\&\, \varphi)$	
$(\mathrm{P_{SL}}8)$	$\varphi \to \psi \vdash \chi \,\&\, \varphi \to \chi \,\&\, \psi$	
$(\mathrm{P_{SL}}9)$	$\varphi \to \psi \vdash \varphi \,\&\, \chi \to \psi \,\&\, \chi$	
$(\mathrm{P_{SL}}10)$	$\varphi_1 \to \psi_1, \varphi_2 \to \psi_2 \vdash \varphi_1 \,\&\, \varphi_2 \to \psi_1 \,\&\, \psi_2$	
$(\mathrm{P_{SL}}11)$	$\varphi, \psi \vdash \varphi \wedge \psi$	
$(\mathrm{P_{SL}}12)$	$\vdash (\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$	
$(\mathrm{P_{SL}}13)$	$\vdash (\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$	
$(\mathrm{P_{SL}}14)$	$\vdash \overline{1}$	
$(\mathrm{P_{SL}}15)$	$\vdash \overline{1} \to (\varphi \to \varphi)$	
$(\mathrm{P_{SL}}16)$	$\vdash \varphi \leftrightarrow (\overline{1} \to \varphi)$	
$(\mathrm{P_{SL}}17)$	$\vdash \varphi \,\&\, \overline{1} \leftrightarrow \varphi$	
$(\mathrm{P_{SL}}18)$	$\vdash \overline{1} \,\&\, \varphi \leftrightarrow \varphi$	

(P_{SL}19) $\vdash \top \leftrightarrow (\bot \to \bot)$

(P_{SL}20) $\vdash \chi \mathbin{\&} (\varphi \vee \psi) \leftrightarrow \chi \mathbin{\&} \varphi \vee \chi \mathbin{\&} \psi$

(P_{SL}21) $\vdash (\varphi \vee \psi) \mathbin{\&} \chi \leftrightarrow \varphi \mathbin{\&} \chi \vee \psi \mathbin{\&} \chi$

(P_{SL}22) $\vdash (\varphi \wedge \overline{1}) \mathbin{\&} (\psi \wedge \overline{1}) \to \varphi \wedge \overline{1}$

(P_{SL}23) $\vdash (\varphi \wedge \overline{1}) \mathbin{\&} (\psi \wedge \overline{1}) \to \psi \wedge \overline{1}$

(P_{SL}24) $\vdash (\varphi \to \psi) \wedge \overline{1} \to (\varphi \wedge \overline{1} \to \psi \wedge \overline{1})$.

For $* \in \{\wedge, \vee\}$*,* SL *also proves:*

(C_*) $\vdash \varphi * \psi \to \psi * \varphi$

(I_*) $\vdash \varphi * \varphi \leftrightarrow \varphi$

(A_*) $\vdash (\varphi * \psi) * \chi \leftrightarrow \varphi * (\psi * \chi)$.

Proof. The proof of the second part is straightforward. We give hints of the proofs of the more complicated statements in the first part:

(P_{SL}1) From $(\varphi \to \psi) \to (\varphi \to \psi)$ using ($E_\rightsquigarrow$).

(P_{SL}4) From (P_{SL}1) using (MP) and (symm).

(P_{SL}5) From $\varphi \to \psi, \psi \to ((\psi \to \chi) \rightsquigarrow \chi) \vdash \varphi \to ((\psi \to \chi) \rightsquigarrow \chi)$ using ($E_\rightsquigarrow$).

(P_{SL}6) From (P_{SL}2) we obtain $\psi \to \chi \vdash \varphi \mathbin{\&} (\varphi \to \psi) \to \chi$ and (Res) completes the proof.

(P_{SL}8) From $\psi \to (\chi \to \chi \mathbin{\&} \psi)$ we obtain $\varphi \to \psi \vdash \varphi \to (\chi \to \chi \mathbin{\&} \psi)$ and (Res) completes the proof.

(P_{SL}9) Use the previous claim together with the duality theorem and (symm) twice.

(P_{SL}11) $\varphi \vdash \overline{1} \to \varphi$ and $\psi \vdash \overline{1} \to \psi$ and thus $\varphi, \psi \vdash \overline{1} \to \varphi \wedge \psi$. The rest is trivial.

(P_{SL}12) Using ($\wedge$1), (P_{SL}2), and (P_{SL}8) we prove $\chi \mathbin{\&} ((\chi \to \varphi) \wedge (\chi \to \psi) \to \varphi)$ and analogously also $\chi \mathbin{\&} ((\chi \to \varphi) \wedge (\chi \to \psi) \to \psi)$, ($\wedge$3) and (Res) complete the proof.

(P_{SL}13) First, we obtain $((\varphi \to \chi) \rightsquigarrow \chi) \to ((\varphi \to \chi) \wedge (\psi \to \chi) \rightsquigarrow \chi)$ from ($\wedge$1) and the dual of (P_{SL}5). Then from $\varphi \to ((\varphi \to \chi) \rightsquigarrow \chi)$ (P_{SL}1) we obtain $\varphi \to ((\varphi \to \chi) \wedge (\psi \to \chi) \rightsquigarrow \chi)$. Analogously we can prove that $\psi \to ((\varphi \to \chi) \wedge (\psi \to \chi) \rightsquigarrow \chi)$. Finally, ($\vee$3) and ($E_\rightsquigarrow$) complete the proof.

(P_{SL}16) From $\overline{1} \to (\varphi \rightsquigarrow \varphi)$ using ($E_\rightsquigarrow$) we obtain $\varphi \to (\overline{1} \to \varphi)$. The converse implication follows from (P_{SL}4) and (P_{SL}14).

(P_{SL}20) From ($\vee$1) (or ($\vee$2) respectively) and (P_{SL}8) we obtain: $\chi \mathbin{\&} \varphi \to \chi \mathbin{\&} (\varphi \vee \psi)$ and $\chi \mathbin{\&} \psi \to \chi \mathbin{\&} (\varphi \vee \psi)$, and so ($\vee$3) completes the proof of one implication. The converse one: from (Res) and ($\vee$1) (or ($\vee$2) respectively) we obtain $\varphi \to (\chi \to \chi \mathbin{\&} \varphi \vee \chi \mathbin{\&} \psi)$ and $\psi \to (\chi \to \chi \mathbin{\&} \varphi \vee \chi \mathbin{\&} \psi)$. ($\vee$3) and (Res) complete the proof.

(P_{SL}21) Analogous (or using duality theorem and (symm)).

(P_{SL}22) We apply (P_{SL}8) to $\psi \wedge \overline{1} \to \overline{1}$ and obtain $(\varphi \wedge \overline{1}) \mathbin{\&} (\psi \wedge \overline{1}) \to (\varphi \wedge \overline{1}) \mathbin{\&} \overline{1}$ and (P_{SL}17) completes the proof. □

Next we study some notable extensions of SL.

DEFINITION 2.5.6. *Let us consider the following consecutions:*

a_1	$\varphi \mathbin{\&} (\psi \mathbin{\&} \chi) \to (\varphi \mathbin{\&} \psi) \mathbin{\&} \chi$	*re-associate to the left*
a_2	$(\varphi \mathbin{\&} \psi) \mathbin{\&} \chi \to \varphi \mathbin{\&} (\psi \mathbin{\&} \chi)$	*re-associate to the right*
e	$\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi)$	*exchange*
c	$\varphi \to (\varphi \to \psi) \vdash \varphi \to \psi$	*contraction*
i	$\vdash \psi \to (\varphi \to \psi)$	*left weakening*
o	$\overline{0} \to \varphi.$	*right weakening*

Given any $X \subseteq \{\mathrm{a}_1, \mathrm{a}_2, \mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$ *and any weakly implicative logic* L *in a sufficiently expressive language, by* L_X *we denote the expansion of* L *by* X*. If both* a_1 *and* a_2 *are in* X *we replace them by the symbol* a*. Analogously if both* i *and* o *are in* X *we replace them by the symbol* w.

The next theorem shows how the axioms of exchange, contraction and weakening can be described as properties of $\&$. Moreover, it also shows under which conditions the conjunction $\&$ is associative. It turns out that both halves of associativity are equivalent to other interesting logical laws, usually resulting from strengthening rules of SL into an axiomatic form.

THEOREM 2.5.7. *Each one of the following axiomatic extensions of* SL *is axiomatized by any one of the corresponding indicated rules:*

$\mathrm{SL}_{\mathrm{a}_1}$
1. $\vdash (\varphi \mathbin{\&} \psi \to \chi) \to (\psi \to (\varphi \to \chi))$
2. $\vdash (\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$
3. $\vdash (\varphi \to (\psi \rightsquigarrow \chi)) \to (\psi \rightsquigarrow (\varphi \to \chi))$.

$\mathrm{SL}_{\mathrm{a}_2}$
1. $\vdash (\psi \to (\varphi \to \chi)) \to (\varphi \mathbin{\&} \psi \to \chi)$
2. $\vdash (\psi \rightsquigarrow (\varphi \to \chi)) \to (\varphi \to (\psi \rightsquigarrow \chi))$.

SL_{e}
1. $\vdash \varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$
2. $\vdash (\varphi \rightsquigarrow \psi) \to (\varphi \to \chi)$
3. $\vdash (\varphi \to \psi) \to (\varphi \rightsquigarrow \chi)$.

SL_{c}
1. $\vdash \varphi \to \varphi \mathbin{\&} \varphi$
2. $\vdash \varphi \wedge \psi \to \varphi \mathbin{\&} \psi$.

SL_{i}
1. $\vdash \varphi \mathbin{\&} \psi \to \psi$
2. $\psi \vdash \varphi \to \psi$
3. $\vdash \varphi \to \overline{1}$
4. $\vdash \varphi \mathbin{\&} \psi \to \varphi$
5. $\vdash \varphi \mathbin{\&} \psi \to \varphi \wedge \psi$.

Proof. Recall that axioms are regarded as rules with empty set of hypotheses. The claims of this theorem will be proved by showing (a chain of) implications of the form 'the extension of SL by the rule x derives the rule y' ($[x \vdash y]$ in symbols), where x and y are either the names of the rules or numbers denoting the formulae in question.

$\mathrm{SL}_{\mathrm{a}_1}$ [1⊢2] From $\chi \,\&\, (\chi \to \varphi) \to \varphi$ we obtain $(\varphi \to \psi) \to (\chi \,\&\, (\chi \to \varphi) \to \psi)$ (by Suffixing). Thus 1 (in the form $(\chi \,\&\, (\chi \to \varphi) \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi)))$ and transitivity complete the proof.

[2⊢3] From $\psi \to ((\psi \rightsquigarrow \chi) \to \chi)$ (an instance of $(\mathrm{P_{SL}}3)$) and 2 we obtain $\psi \to ((\varphi \to (\psi \rightsquigarrow \chi)) \to (\varphi \to \chi))$. $(\mathrm{E}_{\rightsquigarrow})$ finishes the proof.

[3⊢1] $(\varphi \,\&\, \psi \to \chi) \to (\varphi \,\&\, \psi \to \chi)$ and so $\varphi \,\&\, \psi \to ((\varphi \,\&\, \psi \to \chi) \rightsquigarrow \chi)$ by $(\mathrm{E}_{\rightsquigarrow})$. Thus $\psi \to (\varphi \to ((\varphi \,\&\, \psi \to \chi) \rightsquigarrow \chi))$, thus by 3 and transitivity $\psi \to ((\varphi \,\&\, \psi \to \chi) \rightsquigarrow (\varphi \to \chi))$. $(\mathrm{E}_{\rightsquigarrow})$ finishes the proof.

[1⊢a_1] From $(\varphi \,\&\, \psi) \,\&\, \chi \to (\varphi \,\&\, \psi) \,\&\, \chi$ we get $\chi \to (\varphi \,\&\, \psi \to (\varphi \,\&\, \psi) \,\&\, \chi)$ by (Res); by 1 and transitivity we obtain $\chi \to (\psi \to (\varphi \to (\varphi \,\&\, \psi) \,\&\, \chi))$; (Res) used twice completes the proof.

[a_1⊢2] $\chi \,\&\, (\chi \to \varphi) \to \varphi$ thus by Suffixing $(\varphi \to \psi) \to (\chi \,\&\, (\chi \to \varphi) \to \psi)$; (Res) gives us $(\chi \,\&\, (\chi \to \varphi)) \,\&\, (\varphi \to \psi) \to \psi$ and so using a_1 we obtain $\chi \,\&\, ((\chi \to \varphi) \,\&\, (\varphi \to \psi)) \to \psi$; (Res) completes the proof.

$\mathrm{SL}_{\mathrm{a}_2}$ [1⊢a_2] From $\varphi \,\&\, (\psi \,\&\, \chi) \to \varphi \,\&\, (\psi \,\&\, \chi)$ by (Res) used twice we obtain $\chi \to (\psi \to (\varphi \to \varphi \,\&\, (\psi \,\&\, \chi)))$; by 1 and transitivity we obtain $\chi \to (\varphi \,\&\, \psi \to \varphi \,\&\, (\psi \,\&\, \chi))$; (Res) completes the proof.

[2⊢1] From $(\psi \to (\varphi \to \chi)) \to (\psi \to (\varphi \to \chi))$ and $(\mathrm{E}_{\rightsquigarrow})$ we obtain $\psi \to ((\psi \to (\varphi \to \chi)) \rightsquigarrow (\varphi \to \chi))$. Thus also $\psi \to (\varphi \to ((\psi \to (\varphi \to \chi)) \rightsquigarrow \chi))$ (by 2 and transitivity). Using (Res) and $(\mathrm{E}_{\rightsquigarrow})$ complete the proof.

[a_2⊢2] From $(\psi \rightsquigarrow (\varphi \to \chi)) \to (\psi \rightsquigarrow (\varphi \to \chi))$ and $(\mathrm{E}_{\rightsquigarrow})$ we obtain $\psi \to ((\psi \rightsquigarrow (\varphi \to \chi)) \to (\varphi \to \chi))$ and so by the (Res) used twice we have $\varphi \,\&\, ((\psi \rightsquigarrow (\varphi \to \chi)) \,\&\, \psi) \to \chi$. Thus also $(\varphi \,\&\, (\psi \rightsquigarrow (\varphi \to \chi))) \,\&\, \psi \to \chi$ by a_2. Using (Res) and $(\mathrm{E}_{\rightsquigarrow})$ we obtain $\varphi \,\&\, (\psi \rightsquigarrow (\varphi \to \chi)) \to (\psi \rightsquigarrow \chi)$; (Res) completes the proof.

SL_{e} [1⊢e] is obvious using (Res); [e⊢2] follows from Proposition 2.5.4. To prove [2⊢3] we start with $(\mathrm{P_{SL}}1)$: $\varphi \to ((\varphi \to \psi) \rightsquigarrow \psi)$, then 2 and $(\mathrm{E}_{\rightsquigarrow})$ complete the proof. To prove [3⊢1] we start with $(\mathrm{P_{SL}}7)$ and by 3 we obtain $\varphi \to (\psi \rightsquigarrow \psi \& \varphi)$, then $(\mathrm{E}_{\rightsquigarrow})$ and (Res) complete the proof.

SL_{c} [1⊢c] is obvious using (Res); for [c⊢2] observe that from $(\wedge 1)$ and $(\wedge 2)$ be obtain $(\varphi \wedge \psi) \,\&\, (\varphi \wedge \psi) \to \varphi \,\&\, \psi$ by $(\mathrm{P_{SL}}10)$ and so (Res) and c complete the proof. The final claim [2⊢1] follows easily using $(\mathrm{I}_{\wedge})$.

SL_{i} The proofs of [1⊢i], [i⊢2], [2⊢3], [3⊢1], [3⊢4], and [4⊢3] are almost straightforward. To conclude the proof observe that from the fact that [i⊢4] we obtain [i⊢5] using $(\wedge 3)$; the final implication [5⊢i] is trivial. □

Now we can easily obtain the duality theorem (cf. Theorem 2.5.3) for notable extensions of SL. Recall that by χ' we denote the mirror image of χ.

THEOREM 2.5.8 (Duality theorem for SL_X). *Let $X \subseteq \{a, e, c, i, o\}$. Then for each set of formulae $T \cup \{\varphi\}$ we have:*

$$T \vdash_{\mathrm{SL}_X} \varphi \qquad \textit{iff} \qquad T' \vdash_{\mathrm{SL}_X} \varphi'.$$

Based on the logic SL we introduce now a general notion of substructural logic. By doing so we do not expect to encompass all logics that may have been labeled in this manner in the literature, but we only intend to introduce a broad class of substructural logics in the framework of weakly implicative logics to which our methods will usefully apply. We could achieve a greater level of generality by means of a more complex, and probably less natural, definition, however we think the following convention is broad enough for the purposes of the present text.

CONVENTION 2.5.9 ((Associative) substructural logic). *A weakly implicative logic in a language $\mathcal{L}$ is* substructural *if it is an expansion of the $\mathcal{L} \cap \mathcal{L}_{\mathrm{SL}}$-fragment of* SL. *A substructural logic is* associative *if it expands the $\mathcal{L} \cap \mathcal{L}_{\mathrm{SL}}$-fragment of* SL_{a}.

Note that this convention clearly covers many well-known systems such as BCK and BCI, all fuzzy logics introduced in Chapter I, intuitionistic and classical logic. Later, in Theorem 2.5.13, we identify SL with a well studied logical system: a bounded non-associative variant of *full Lambek logic* FL [43]. Then it will become apparent that our convention also covers most logics referred to as substructural logics in the literature. In particular it covers all *substructural logics over* FL as deeply studied in [43] (axiomatic extensions of FL) or *substructural logics as logics of residuated structures* as proposed in the final remarks (Section 6) of [74] (fragments of axiomatic extensions of FL).

Our design choices make the definition of substructural logic a normative one in the sense that, when using a traditional symbol for a connective of a given logic, we are postulating that it must at least satisfy the logical rules derivable in SL. This may have some unexpected consequences. For instance, the logic $\mathrm{BCK}_\wedge$ of BCK-semilattices [62] in the language $\{\to, \wedge\}$ is not a substructural logic in the sense of our convention (it does not satisfy $(\mathrm{P_{SL}}12)$ from Proposition 2.5.5); however, if we formulated it in the language $\{\to, \overline{\wedge}\}$, then it would indeed satisfy the convention (because then the only $\mathcal{L}_{\mathrm{SL}}$ connective present in its language, namely $\to$, behaves as it should). Other examples which illustrate this situation are Avron's logics RMI_{min} and RMI, excluded from our notion because they do not satisfy $(\mathrm{P_{SL}}11)$.

The previously proved syntactical properties of SL and its prominent extensions (Proposition 2.5.5 and Theorem 2.5.7) clearly hold for all substructural logics in a sufficiently expressive language. Let us list several further observations on substructural logics (L stands for an arbitrary weakly implicative substructural logic in a sufficiently expressive language):

- In $\mathrm{L_o}$ the truth constants $\overline{0}$ and $\bot$ coincide ($\vdash_{\mathrm{SL_o}} \bot \leftrightarrow \overline{0}$) using Proposition 2.5.4.
- In $\mathrm{L_i}$ the truth constants $\overline{1}$ and $\top$ coincide ($\vdash_{\mathrm{SL_i}} \top \leftrightarrow \overline{1}$) using Proposition 2.5.4, and furthermore $\vdash_{\mathrm{SL_i}} \overline{1} \leftrightarrow (\varphi \to \varphi)$.

- L_{ae} is axiomatized (relative to L) by $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$.
- L_{ac} proves $(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$ (but, in general, this axiom is not sufficient to axiomatize L_{ac} relative to L).
- L is Rasiowa-implicative iff it proves i (and thus all these logics are algebraically implicative). Furthermore, in Rasiowa-implicative substructural logics we prove $\overline{1} \leftrightarrow (\varphi \to \varphi)$ and so $\overline{1}$ can be viewed as a defined connective.

The following are some basic semantic properties of the connectives in substructural logics, which can be easily checked.

PROPOSITION 2.5.10. *Let* L *be a substructural logic in a sufficiently expressive language and* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$. *Then:*

1. $\overline{1}^{\boldsymbol{A}} = \min_{\leq_{\mathbf{A}}} F$.
2. $\top^{\boldsymbol{A}} = \max_{\leq_{\mathbf{A}}} A$ *and* $\top^{\boldsymbol{A}} \in F$.
3. $\bot^{\boldsymbol{A}} = \min_{\leq_{\mathbf{A}}} A$ *and* $\bot^{\boldsymbol{A}} \notin F$ *if* $\mathbf{A}$ *is not trivial.*
4. $\leq_{\mathbf{A}}$ *is a* $\vee$*-semilattice order.*
5. $\leq_{\mathbf{A}}$ *is a* $\wedge$*-semilattice order.*
6. $\to^{\boldsymbol{A}}$ *is antitone in the first argument and monotone in the second one w.r.t.* $\leq_{\mathbf{A}}$.
7. $\&^{\boldsymbol{A}}$ *is monotone in both arguments w.r.t.* $\leq_{\mathbf{A}}$ *and* $\overline{1}^{\boldsymbol{A}}$ *is its unit.*
8. *For every* $x, y, z \in A$, $x \,\&^{\boldsymbol{A}}\, y \leq_{\mathbf{A}} z$ *iff* $y \leq_{\mathbf{A}} x \to^{\boldsymbol{A}} z$ *iff* $x \leq_{\mathbf{A}} y \rightsquigarrow^{\boldsymbol{A}} z$.
9. *For every* $x, y, z \in A$, $x \to^{\boldsymbol{A}} y = \max\{z \mid x \,\&^{\boldsymbol{A}}\, z \leq_{\mathbf{A}} y\}$.
10. *For every* $x, y, z \in A$, $x \rightsquigarrow^{\boldsymbol{A}} y = \max\{z \mid z \,\&^{\boldsymbol{A}}\, x \leq_{\mathbf{A}} y\}$.
11. *For every* $x, y, z \in A$, $x \,\&^{\boldsymbol{A}}\, y = \min\{z \mid y \leq_{\mathbf{A}} x \to^{\boldsymbol{A}} z\} =$
$= \min\{z \mid x \leq_{\mathbf{A}} y \rightsquigarrow^{\boldsymbol{A}} z\}$.

Proof. All claims are easily checked. The eighth item follows from (Res) and the duality theorem, and the last three items follow from this one. □

THEOREM 2.5.11. *Any substructural logic with* $\vee$ *or* $\wedge$ *in its language is algebraically implicative.*

Proof. The proof would be simpler if we further assumed that $\overline{1}$ is in the language of our logic, because then the algebraizing pair would be simply $\langle \chi \wedge \overline{1}, \overline{1}\rangle$ (or $\langle \chi \vee \overline{1}, \chi\rangle$), which can be proved in a straightforward way or using the previous proposition. Let us give a more general proof not assuming the presence of $\overline{1}$.

Assume that our logic has $\wedge$ in its language; we show that $\langle (\chi \to \chi) \wedge \chi, \chi \to \chi\rangle$ is an algebraizing pair (in the second case we would analogously prove that claim for $\langle (\chi \to \chi) \vee \chi, \chi\rangle$). One direction is easy: trivially $\chi \vdash (\chi \to \chi) \wedge \chi \to (\chi \to \chi)$ and $\chi \vdash (\chi \to \chi) \to (\chi \to \chi) \wedge \chi$ (because $\chi \vdash (\chi \to \chi) \to \chi$). The second direction: clearly $(\chi \to \chi) \leftrightarrow (\chi \to \chi) \wedge \chi \vdash (\chi \to \chi) \wedge \chi$ and so $(\chi \to \chi) \leftrightarrow (\chi \to \chi) \wedge \chi \vdash \chi$ (as obviously $(\chi \to \chi) \wedge \chi \vdash \chi$). □

$$\varphi \setminus \varphi \qquad \varphi, \varphi \setminus \psi \vdash \psi \qquad \varphi \vdash (\varphi \setminus \psi) \setminus \psi$$
$$\varphi \setminus \psi \vdash (\psi \setminus \chi) \setminus (\varphi \setminus \chi) \qquad \psi \setminus \chi \vdash (\varphi \setminus \psi) \setminus (\varphi \setminus \chi)$$
$$\varphi \setminus ((\psi / \varphi) \setminus \psi) \qquad \varphi \setminus (\psi \setminus \chi) \vdash \psi \setminus (\chi / \varphi) \qquad \psi / \varphi \vdash \varphi \setminus \psi$$
$$\varphi \wedge \psi \setminus \varphi \qquad \varphi \wedge \psi \setminus \psi \qquad (\chi \setminus \varphi) \wedge (\chi \setminus \psi) \setminus (\chi \setminus \varphi \wedge \psi) \qquad \varphi, \psi \vdash \varphi \wedge \psi$$
$$\varphi \setminus \varphi \vee \psi \qquad \psi \setminus \varphi \vee \psi \qquad (\varphi \setminus \chi) \wedge (\psi \setminus \chi) \setminus (\varphi \vee \psi \setminus \chi) \qquad (\chi / \varphi) \wedge (\chi / \psi) \setminus (\chi / \varphi \vee \psi)$$
$$\psi \setminus (\varphi \setminus \varphi\psi) \qquad \psi \setminus (\varphi \setminus \chi) \vdash \varphi\psi \setminus \chi$$
$$\overline{1} \qquad \overline{1} \setminus (\varphi \setminus \varphi) \qquad \varphi \setminus (\overline{1} \setminus \varphi)$$

Table 3. The axiomatic system for the 'non-associative' full Lambek logic

Our next aim is to identify our basic logic SL and its extensions SL_X among well-known substructural logics. Since many substructural logics are introduced in the literature in their unbounded form, we need the following convention:

CONVENTION 2.5.12. *Let* L *be a weakly implicative logic. The logic called* bounded L*, denoted as* $\mathrm{L}_\perp$*, is the expansion of* L *in the language with the additional truth constant* $\perp$ *satisfying the axiom* $\perp \to \varphi$*.*

Reputably, the most prominent substructural logic is the full Lambek calculus [43], denoted as FL. FL is formulated in a variant of our language $\mathcal{L}_{\mathrm{SL}}$, which we will call here $\mathcal{L}_{\mathrm{FL}}$, with truth constants $\overline{0}$ and $\overline{1}$, two implication connectives $\setminus$ and $/$ (in the commutative extension of FL, denoted as $\mathrm{FL_e}$, these two connectives coincide and then they are denoted by the symbol $\to$), residuated conjunction $\&$,[6] and lattice connectives $\wedge, \vee$. An axiomatic system for FL (taken from [43, Figure 2.10]) is presented in Table 4.

Another important substructural logic is a non-associative variant of full Lambek calculus, also in the language $\mathcal{L}_{\mathrm{FL}}$, given by the axiomatic system in Table 3 (presented in [45, Figure 5]). The name and the symbol for this logic are not yet settled and thus we avoid any explicit reference.

Before we show how these two logics are related with SL we provide a translation between their languages. We actually present two possible translations (the duality theorem ensures the validity of the following theorem no matter which one we use).[7]

$\mathcal{L}_{\mathrm{FL}}$ notation	direct translation	indirect translation
$\varphi\psi$	$\varphi \,\&\, \psi$	$\psi \,\&\, \varphi$
$\varphi \setminus \psi$	$\varphi \to \psi$	$\varphi \rightsquigarrow \psi$
ψ / φ	$\varphi \rightsquigarrow \psi$	$\varphi \to \varphi$

[6]There is a usual convention in the papers on this logic and its variants to omit the symbol $\&$ and write just $\varphi\psi$ instead of $\varphi \,\&\, \psi$.

[7]This independence from the choice of the translation also ensures that any fragment of (non-associative) full Lambek logic containing at least $/$ or $\setminus$ is substructural in the sense of Convention 2.5.9.

(id_l)	$\varphi\backslash\varphi$	(identity)
(pf_l)	$(\varphi\backslash\psi)\backslash[(\chi\backslash\varphi)\backslash(\chi\backslash\psi)]$	(prefixing)
(as_{ll})	$\varphi\backslash[(\psi/\varphi)\backslash\psi]$	(assertion)
(a)	$[(\psi\backslash\chi)/\varphi]\backslash[\psi\backslash(\chi/\varphi)]$	(associativity)
($\&\backslash/$)	$[\psi(\psi\backslash\varphi)/\psi]\backslash(\varphi/\psi)$	(fusion divisions)
($\&\wedge$)	$[(\varphi\wedge\overline{1})(\psi\wedge\overline{1})]\backslash(\varphi\wedge\psi)$	(fusion conjunction)
($\wedge\backslash$)	$(\varphi\wedge\psi)\backslash\varphi$	(conjunction division)
($\wedge\backslash$)	$(\varphi\wedge\psi)\backslash\psi$	(conjunction division)
($\backslash\wedge$)	$[(\varphi\backslash\psi)\wedge(\varphi\backslash\chi)]\backslash[\varphi\backslash(\psi\wedge\chi)]$	(division conjunction)
($\backslash\vee$)	$\varphi\backslash(\varphi\vee\psi)$	(division disjunction)
($\backslash\vee$)	$\psi\backslash(\varphi\vee\psi)$	(division disjunction)
($\vee\backslash$)	$[(\varphi\backslash\chi)\wedge(\psi\backslash\chi)]\backslash[(\varphi\vee\psi)\backslash\chi]$	(disjunction division)
($\backslash\&$)	$\psi\backslash(\varphi\backslash\varphi\psi)$	(division fusion)
($\&\backslash$)	$[\psi\backslash(\varphi\backslash\chi)]\backslash(\varphi\psi\backslash\chi)$	(fusion division)
($\overline{1}$)	$\overline{1}$	(unit)
($\overline{1}\backslash$)	$\overline{1}\backslash(\varphi\backslash\varphi)$	(unit division)
($\backslash\overline{1}$)	$\varphi\backslash(\overline{1}\backslash\varphi)$	(division unit)
(mp_l)	$\varphi, \varphi\backslash\psi \vdash \psi$	(*modus ponens*)
(adj_u)	$\varphi \vdash \varphi\wedge\overline{1}$	(adjunction unit)
(pn_l)	$\varphi \vdash \psi\backslash\varphi\psi$	(product normality)
(pn_r)	$\varphi \vdash \psi\varphi/\psi$	(product normality)

Table 4. An axiomatic system for FL

THEOREM 2.5.13. *The logic* $\mathrm{SL_a}$ *is termwise equivalent to bounded full Lambek logic using any of the translations given above. Analogously, the logic* SL *is termwise equivalent to bounded non-associative full Lambek logic.*

Proof. All the axioms and rules of (non-associative) full Lambek logic are among the consecutions in Table 2, those proved in Proposition 2.5.5 or are equivalent to associativity as shown in Theorem 2.5.7. We leave the converse direction as an exercise. □

Therefore, all the prominent extensions of $\mathrm{SL_a}$ that we have considered here are the bounded versions of the well-known extensions of FL studied in the mainstream literature on substructural logics, namely for every $X \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i},\mathrm{o}\}$ $\mathrm{SL}_{\mathrm{a},X}$ coincides with the bounded version of FL_X (modulo language translation). The logics FL_X are called *basic substructural logics* in [43].

Since in any substructural logic extending $\mathrm{SL_e}$ only one implication is needed, we obtain a simplified axiomatic system for $\mathrm{FL_e}$ in Table 5 (taken from [43, Figure 2.9]). One can easily observe that the axiomatic system for $\mathrm{FL_{ew}}$ can be simplified by taking the one for $\mathrm{FL_e}$ and replacing (&∧) with $\varphi \,\&\, \psi \to \varphi\wedge\psi$ and removing the axioms ($\overline{1}$), ($\overline{1}\to$), and the rule (adj_u).

(id)	$\varphi \to \varphi$	(identity)
(pf)	$(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$	(prefixing)
(per)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	(permutation)
(&∧)	$[(\varphi \wedge \overline{1})(\psi \wedge \overline{1})] \to (\varphi \wedge \psi)$	(fusion conjunction)
(∧→)	$(\varphi \wedge \psi) \to \varphi$	(conjunction implication)
(∧→)	$(\varphi \wedge \psi) \to \psi$	(conjunction implication)
(→∧)	$[(\varphi \to \psi) \wedge (\varphi \to \chi)] \to [\varphi \to (\psi \wedge \chi)]$	(implication conjunction)
(→∨)	$\varphi \to (\varphi \vee \psi)$	(implication disjunction)
(→∨)	$\psi \to (\varphi \vee \psi)$	(implication disjunction)
(∨→)	$[(\varphi \to \chi) \wedge (\psi \to \chi)] \to [(\varphi \vee \psi) \to \chi]$	(disjunction implication)
(→&)	$\psi \to (\varphi \to \varphi\psi)$	(division fusion)
(&→)	$[\psi \to (\varphi \to \chi)] \to (\varphi\psi \to \chi)$	(fusion implication)
($\overline{1}$)	$\overline{1}$	(unit)
($\overline{1}$→)	$\overline{1} \to (\varphi \to \varphi)$	(unit implication)
(mp)	$\varphi, \varphi \to \psi \vdash \psi$	(*modus ponens*)
(adj$_u$)	$\varphi \vdash \varphi \wedge \overline{1}$	(adjunction unit)

Table 5. An axiomatic system for $\mathrm{FL_e}$

2.6 Deduction theorems and proof by cases in substructural logics

In this subsection we deal with various forms of deduction theorems and use them to obtain proof by cases properties for prominent substructural logics. Recall that we work with a fixed set of propositional variables Var. A $\star$-formula is built using Var and a fixed propositional variable $\star$ not occurring in Var, a $\star$-substitution is defined for the extended language as expected. These notions will play a technical rôle in the upcoming definitions. Let φ be a formula, δ be a $\star$-formula, and σ a $\star$-substitution defined as $\sigma(\star) = \varphi$ and $\sigma p = p$ for $p \in Var$. By $\delta(\varphi)$ we denote the formula (in the original set of variables) $\sigma\delta$.

DEFINITION 2.6.1. *Given a set of $\star$-formulae Γ, we define the set Γ^* of $\star$-formulae as the smallest set such that*

- $\star \in \Gamma^*$ *and*
- $\delta(\chi) \in \Gamma^*$ *for each* $\delta \in \Gamma$ *and each* $\chi \in \Gamma^*$.

If L *has* & *and* $\overline{1}$ *in its language we define the set* $\Pi(\Gamma)$ *of $\star$-formulae as the smallest set of formulae containing* $\Gamma \cup \{\overline{1}\}$ *and closed under* &.

Note that the elements of $\Pi(\Gamma)$ can be uniquely described by finite trees labeled by elements of $\Gamma \cup \{\overline{1}\}$.[8]

[8] A problem could arise here if Γ contains a conjunction of some of its elements; then there are (at least) two trees 'representing' this formula as conjunction of elements of Γ. Clearly there has to be a tree containing all the possible tree-representation as subtrees; we take this maximal one as the unique representation.

DEFINITION 2.6.2 ((Almost) (MP)-based logic, basic deduction terms). *Let* bDT *be a set of* $\star$*-formulae. A substructural logic* L *is* almost (MP)-based w.r.t. the set of basic deduction terms bDT *if:*

- *the set* bDT *is closed under all* $\star$*-substitutions* σ *such that* $\sigma(\star) = \star$*,*
- L *has a presentation where the only deduction rules are* modus ponens *and those from* $\{\varphi \rhd \chi(\varphi) \mid \varphi \in Fm_{\mathcal{L}}, \chi \in \mathrm{bDT}\}$*, and*
- *for each* $\beta \in \mathrm{bDT}$ *and each formulae* φ, ψ*, there exist* $\beta_1, \beta_2 \in \mathrm{bDT}$ *such that:*

$$\vdash_{\mathrm{L}} \beta_1(\varphi \to \psi) \to (\beta_2(\varphi) \to \beta(\psi)).$$

L *is called* (MP)-based *if it admits the empty set as a set of basic deduction terms.*

Note that the described axiomatic system is indeed closed under all substitutions. $\mathrm{FL_e}$ is almost (MP)-based with $\mathrm{bDT} = \{\star \wedge \overline{1}\}$ (recall the axiomatic system in Table 5 and $(\mathrm{P_{SL}}24)$), while $\mathrm{FL_{ew}}$ is (MP)-based. Also notice that any axiomatic extension of an almost (MP)-based logic is almost (MP)-based too. Finally observe that $\varphi \vdash_{\mathrm{L}} \chi(\varphi)$ for each $\chi \in \Pi(\mathrm{bDT}^*)$, and, if L is (MP)-based (i.e. has $\mathrm{bDT} = \emptyset$), then $\mathrm{bDT}^* = \{\star\}$.

THEOREM 2.6.3 (Almost-Implicational Deduction Theorem for almost (MP)-based logics). *Let* L *be a substructural logic with* $\&$ *and* $\overline{1}$ *in the language, and assume that it is almost* (MP)*-based with a set of basic deduction terms* bDT*. Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae the following holds:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \delta(\varphi) \to \psi \textit{ for some } \delta \in \Pi(\mathrm{bDT}^*).$$

Proof. To prove the right-to-left direction just recall $\varphi \vdash_{\mathrm{L}} \delta(\varphi)$ for any $\delta \in \Pi(\mathrm{bDT}^*)$ and use *modus ponens*. To prove the left-to-right direction we show that for each χ in the proof of ψ from the assumptions $\Gamma \cup \{\varphi\}$ there is $\delta_\chi \in \Pi(\mathrm{bDT}^*)$ such that $\Gamma \vdash_{\mathrm{L}} \delta_\chi(\varphi) \to \chi$. If $\chi = \varphi$ we set $\delta_\chi = \star$; if $\chi \in \Gamma$ or it is an axiom we set $\delta_\chi = \overline{1}$.

Assume that χ is obtained by *modus ponens* from η and $\eta \to \chi$. By induction hypothesis, we have $\Gamma \vdash_{\mathrm{L}} \delta_\eta(\varphi) \to \eta$ and $\Gamma \vdash_{\mathrm{L}} \delta_{\eta\to\chi}(\varphi) \to (\eta \to \chi)$. From the former we derive $\Gamma \vdash_{\mathrm{L}} (\eta \to \chi) \to (\delta_\eta(\varphi) \to \chi)$, and so, by using the latter, $\Gamma \vdash_{\mathrm{L}} \delta_{\eta\to\chi}(\varphi) \to (\delta_\eta(\varphi) \to \chi)$, and thus $\Gamma \vdash_{\mathrm{L}} \delta_\eta(\varphi) \,\&\, \delta_{\eta\to\chi}(\varphi) \to \chi$, and it suffices to set $\delta_\chi = \delta_\eta \,\&\, \delta_{\eta\to\chi}$ to complete the proof.

Assume that χ is obtained by a rule of the form $\eta \rhd \chi$, where $\chi = \beta(\eta)$ for some $\beta \in \mathrm{bDT}$. By the induction hypothesis, we know that $\Gamma \vdash_{\mathrm{L}} \delta_\eta(\varphi) \to \eta$. We first observe one simple claim and prove another two:

Claim 1: For each $\beta \in \mathrm{bDT}$ and formulae φ, ψ, there exists $\hat{\beta} \in \mathrm{bDT}$ such that

$$\varphi \to \psi \vdash_{\mathrm{L}} \hat{\beta}(\varphi) \to \beta(\psi).$$

Claim 2: For each $\beta \in \mathrm{bDT}$ and formulae φ, ψ, there exist $\beta_1, \beta_2 \in \mathrm{bDT}$ such that:

$$\vdash_{\mathrm{L}} \beta_1(\varphi) \,\&\, \beta_2(\psi) \to \beta(\varphi \,\&\, \psi).$$

Proof of Claim 2: From the assumption of the theorem we obtain:

$$\vdash_{\mathrm{L}} \bar{\beta}_2(\varphi \to \varphi \mathbin{\&} \psi) \to (\beta_1(\varphi) \to \beta(\varphi \mathbin{\&} \psi)) \text{ for some } \beta_1, \bar{\beta}_2 \in \mathrm{bDT}.$$

Using Claim 1 for $\beta = \bar{\beta}_2$ and the fact that $\vdash_{\mathrm{L}} \psi \to (\varphi \to \varphi \mathbin{\&} \psi)$ we obtain:

$$\vdash_{\mathrm{L}} \beta_2(\psi) \to \bar{\beta}_2(\varphi \to \varphi \mathbin{\&} \psi) \text{ for some } \beta_2 \in \mathrm{bDT}.$$

The rest of the proof is simple.

Claim 3: For each $\beta \in \mathrm{bDT}$, $\delta \in \Pi(\mathrm{bDT}^*)$, and formula φ there exists $\hat{\delta} \in \Pi(\mathrm{bDT}^*)$ such that:

$$\vdash_{\mathrm{L}} \hat{\delta}(\varphi) \to \beta(\delta(\varphi)).$$

Proof of Claim 3. We proceed by induction via the depth of the tree representing δ. If $\delta \in \mathrm{bDT}^*$ or $\delta = \bar{1}$ the proof is done by setting $\hat{\delta} = \beta(\delta)$ or $\hat{\delta} = \bar{1}$ respectively. Next assume that $\delta = \eta_1 \mathbin{\&} \eta_2$ for some $\eta_1, \eta_2 \in \Pi(\mathrm{bDT}^*)$. By Claim 2 we obtain $\beta_1, \beta_2 \in \mathrm{bDT}$ such that $\vdash_{\mathrm{L}} \beta_1(\eta_1(\varphi)) \mathbin{\&} \beta_2(\eta_2(\varphi)) \to \beta(\eta_1(\varphi) \mathbin{\&} \eta_2(\varphi))$. Then by the induction assumption we obtain $\hat{\delta}_1, \hat{\delta}_2 \in \Pi(\mathrm{bDT}^*)$ such that $\vdash_{\mathrm{L}} \hat{\delta}_1(\varphi) \to \beta_1(\eta_1(\varphi))$ and $\vdash_{\mathrm{L}} \hat{\delta}_2(\varphi) \to \beta_2(\eta_2(\varphi))$. Setting $\hat{\delta} = \hat{\delta}_1 \mathbin{\&} \hat{\delta}_2$ completes the proof using $(\mathrm{P_{SL}}10)$.

The rest of the proof of the theorem. Recall that we have $\Gamma \vdash_{\mathrm{L}} \delta_\eta(\varphi) \to \eta$. Therefore via Claim 1 we obtain $\hat{\beta} \in \mathrm{bDT}$ such that $\Gamma \vdash_{\mathrm{L}} \hat{\beta}(\delta_\eta(\varphi)) \to \beta(\eta)$. Claim 3 gives us δ_χ such that $\Gamma \vdash_{\mathrm{L}} \delta_\chi(\varphi) \to \chi$. □

DEFINITION 2.6.4 (Almost-Implicational Deduction Theorem, deduction terms). *Let* DT *be a set of ⋆-formulae. A logic* L *has the* Almost-Implicational Deduction Theorem *w.r.t. the set of* deduction terms DT*, if for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \delta(\varphi) \to \psi \textit{ for some } \delta \in \mathrm{DT}.$$

The previous theorem says that all almost (MP)-based logics enjoy the Almost-Implicational Deduction Theorem with $\mathrm{DT} = \Pi(\mathrm{bDT}^*)$. We improve this result in two ways: first we notice under which assumption we can simplify the set of deduction terms, and second we show that the condition that L is almost (MP)-based in Theorem 2.6.3 is in fact necessary.

THEOREM 2.6.5. *Let* L *be a substructural with* & *and* $\bar{1}$ *in the language satisfying the Almost-Implicational Deduction Theorem w.r.t. a set* DT.

- L *has the Almost-Implicational Deduction Theorem w.r.t. a set* $\mathrm{DT}' \subseteq \mathrm{DT}$ *if, and only if, for every* $\chi \in \mathrm{DT}$ *and every formula* φ *there is* $\delta \in \mathrm{DT}'$ *such that* $\vdash_{\mathrm{L}} \delta(\varphi) \to \chi(\varphi)$.
- *If* L *is finitary, then* L *is almost* (MP)*-based with the set*

$$\mathrm{bDT} = \{\sigma\delta \mid \delta \in \mathrm{DT}, \sigma \textit{ a } \star\textit{-substitution such that } \sigma(\star) = \star\}.$$

Proof. The proof of the right-to-left direction of the first claim is straightforward. The converse one is also easy: from $\vdash_{\rm L} \chi(\varphi) \to \chi(\varphi)$ we obtain (using Almost-Implicational Deduction Theorem w.r.t. DT) $\varphi \vdash_{\rm L} \chi(\varphi)$ and so (using Almost-Implicational Deduction Theorem w.r.t. DT$'$) we obtain $\vdash_{\rm L} \delta(\varphi) \to \chi(\varphi)$ for some $\delta \in {\rm DT}'$.

For the proof of the second claim, let us define the logic $\rm L'$ axiomatized by all the theorems of L, *modus ponens* and the rules $\{\varphi \rhd \delta(\varphi) \mid \varphi \in Fm_{\mathcal{L}}, \delta \in {\rm bDT}\}$ (note that this set is closed under substitutions). From $\vdash_{\rm L} \delta(\varphi) \to \delta(\varphi)$ and the right-to-left direction of the Deduction Theorem we obtain that $\varphi \vdash_{\rm L} \delta(\varphi)$, as L is substructural, it has *modus ponens* and so $\rm L' \subseteq L$. Assume that $\Gamma \vdash_{\rm L} \psi$. Due to the finitarity we have $\varphi_1, \dots, \varphi_n \vdash_{\rm L} \psi$ for $\varphi_i \in \Gamma$. By repeatedly using the left-to-right direction of the Deduction Theorem we obtain $\vdash_{\rm L} \delta_1(\varphi_1) \to (\delta_2(\varphi_2) \to (\cdots \to (\delta_n(\varphi_n) \to \psi) \dots)$ for $\delta_i \in {\rm DT}$. Thus we obviously have $\varphi_1, \dots, \varphi_n \vdash_{\rm L'} \psi$. The last defining condition of a set of basic deduction terms is easily obtained by a double application of the Deduction Theorem to $\varphi \to \psi, \varphi \vdash_{\rm L} \sigma\delta(\psi)$. □

Let us recall the standard notation $\varphi^n = \varphi^{n-1} \,\&\, \varphi$ (where $\varphi^0 = \overline{1}$). Note that in associative substructural logics the bracketing in φ^n is irrelevant.

COROLLARY 2.6.6 (Implicational Deduction Theorem for associative (MP)-based logics). *Let* L *be an associative substructural logic with* $\&$ *and* $\overline{1}$ *in the language. Then* L *is* (MP)*-based iff* L *is finitary and for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae the following holds:*

$$\Gamma, \varphi \vdash_{\rm L} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\rm L} \varphi^n \to \psi \textit{ for some } n \geq 0.$$

We have seen that $\rm FL_{ew}$ is an example of (MP)-based logic, thus we have just shown that it enjoys this form of deduction theorem (and obviously the same holds for its axiomatic extensions). In contrast, we can use the previous theorem to show that $\rm FL_e$ is not (MP)-based: indeed, $\varphi \vdash_{\rm FL_e} \varphi \wedge \overline{1}$ would entail provability of the formula $\varphi^n \to \varphi \wedge \overline{1}$ for some n which can be refuted by a simple semantical counterexample. On the other hand, since $\rm FL_e$ is associative and $\vdash_{\rm FL_e} (p \wedge \overline{1}) \leftrightarrow (p \wedge \overline{1}) \wedge \overline{1}$, we know that for each $\chi \in \Pi({\rm bDT}^*_{\rm FL_e})$ there is some n such that $\vdash_{\rm FL_e} (p \wedge \overline{1})^n \to \chi$, therefore for each set $\Gamma \cup \{\varphi, \psi\}$ of formulae the following holds:

$$\Gamma, \varphi \vdash_{\rm FL_e} \psi \qquad \text{iff} \qquad \Gamma \vdash_{\rm FL_e} (\varphi \wedge \overline{1})^n \to \psi \text{ for some } n \geq 0.$$

The situation in FL is more complicated. First we introduce the notion of conjugate.

DEFINITION 2.6.7 (Left, right and iterated conjugates). *Given formula* α*, we define* left and right conjugates *w.r.t.* α *as* $\lambda_\alpha(\star) = (\alpha \to \star \,\&\, \alpha) \wedge \overline{1}$ *and* $\rho_\alpha(\star) = (\alpha \rightsquigarrow \alpha \,\&\, \star) \wedge \overline{1}$. *An* iterated conjugate *is a formula of the form* $\gamma(\star) = \gamma_{\alpha_1}(\gamma_{\alpha_2}(\dots(\gamma_{\alpha_n}(\star))\dots)$*, where each* γ_{α_i} *is either* λ_{α_i} *or* ρ_{α_i}.

A formula of the form $\gamma(\varphi)$ *where* γ *is a left, right, or iterated conjugate is called* left*,* right*, or* iterated *(resp.)* conjugate *of* φ.

THEOREM 2.6.8 (Almost-Implicational Deduction Theorem for FL). *The logic* FL *is almost* (MP)*-based with the set of basic deduction terms* $\{\lambda_\alpha(\star), \rho_\alpha(\star) \mid \alpha \in Fm_{\mathcal{L}}\}$. *Therefore, for every set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae the following holds:*

$$\Gamma, \varphi \vdash_{\rm FL} \psi \quad \textit{iff} \quad \Gamma \vdash_{\rm FL} \chi(\varphi) \to \psi \textit{ for some conjunction } \chi \textit{ of iterated conjugates.}$$

Proof. Let $\mathrm{bDT_{FL}}$ be the set of all left and right conjugates. Note that this set is closed under every substitution for which $\sigma(\star) = \star$ and that the following derivations are valid in FL: $\lambda_{\overline{1}}(\varphi) \dashv\vdash \rho_{\overline{1}}(\varphi) \dashv\vdash \varphi \wedge \overline{1}$, $\varphi \vdash \rho_\alpha(\varphi)$, $\varphi \vdash \lambda_\alpha(\varphi)$. Therefore the product normality and adjunction unit rules could be equivalently replaced by the rules $\varphi \vdash \chi(\varphi)$ for $\chi \in \mathrm{bDT_{FL}}$. To complete the proof that $\mathrm{bDT_{FL}}$ is a set of basic deductive terms of FL, it is enough to prove the following:

$$\vdash \rho_\alpha(\varphi \to \psi) \to (\rho_\alpha(\varphi) \to \rho_\alpha(\psi))$$

$$\vdash \lambda_\alpha(\varphi \to \psi) \to (\lambda_\alpha(\varphi) \to \lambda_\alpha(\psi)).$$

The proof is heavily based on associativity; we use its variant forms introduced in Theorem 2.5.7. First we prove $(\varphi \to \psi) \to (\alpha \,\&\, \varphi \to \alpha \,\&\, \psi)$: from $(\mathrm{P_{SL}}7)$ in the form $(\psi \to (\alpha \to \alpha \,\&\, \psi))$ and prefixing get $(\varphi \to \psi) \to (\varphi \to (\alpha \to \alpha \,\&\, \psi))$ and so associativity finishes the proof. Now we prove the first claim:

a	$(\varphi \rightsquigarrow \psi) \to ((\alpha \rightsquigarrow \varphi) \rightsquigarrow (\alpha \rightsquigarrow \psi))$	mirror of associativity
b	$(\psi \rightsquigarrow (\varphi \rightsquigarrow \alpha)) \to (\psi \,\&\, \varphi \rightsquigarrow \alpha)$	mirror of associativity
c	$(\varphi \to \psi) \to (\alpha \,\&\, \varphi \to \alpha \,\&\, \psi)$	proved above
d	$\alpha \,\&\, \varphi \to ((\varphi \to \psi) \rightsquigarrow \alpha \,\&\, \psi)$	c and $(\mathrm{E}_\rightsquigarrow)$
e	$(\alpha \rightsquigarrow \alpha \,\&\, \varphi) \to [\alpha \rightsquigarrow ((\varphi \to \psi) \rightsquigarrow \alpha \,\&\, \psi)]$	d and mirror of $(\mathrm{P_{SL}}6)$
f	$(\alpha \rightsquigarrow \alpha \,\&\, \varphi) \to [\alpha \,\&\, (\varphi \to \psi) \rightsquigarrow \alpha \,\&\, \psi]$	e and an instance of b
g	$(\alpha \rightsquigarrow \alpha \,\&\, \varphi) \to [(\alpha \rightsquigarrow \alpha \,\&\, (\varphi \to \psi)) \rightsquigarrow (\alpha \rightsquigarrow \alpha \,\&\, \psi)]$	f and an instance of a
h	$(\alpha \rightsquigarrow \alpha \,\&\, (\varphi \to \psi)) \to [(\alpha \rightsquigarrow \alpha \,\&\, \varphi) \to (\alpha \rightsquigarrow \alpha \,\&\, \psi)]$	g and $(\mathrm{E}_\rightsquigarrow)$
i	$(\alpha \rightsquigarrow \alpha \,\&\, (\varphi \to \psi)) \wedge \overline{1} \to [(\alpha \rightsquigarrow \alpha \,\&\, \varphi) \wedge \overline{1} \to (\alpha \rightsquigarrow \alpha \,\&\, \psi) \wedge \overline{1}]$	h and $(\mathrm{P_{SL}}24)$ twice.

Proof of the second claim:

a′	$(\varphi \rightsquigarrow \psi) \to (\varphi \,\&\, \alpha \rightsquigarrow \psi \,\&\, \alpha)$	mirror of c
b′	$\varphi \to ((\varphi \to \psi) \rightsquigarrow \psi))$	$(\mathrm{P_{SL}}1)$
c′	$\varphi \to ((\varphi \to \psi) \,\&\, \alpha \rightsquigarrow \psi \,\&\, \alpha)$	a′ and an instance of b′
d′	$(\varphi \to \psi) \,\&\, \alpha \to (\varphi \to \psi \,\&\, \alpha)$	c′ and $(\mathrm{E}_\rightsquigarrow)$
e′	$(\alpha \to (\varphi \to \psi) \,\&\, \alpha) \to [\alpha \to (\varphi \to \psi \,\&\, \alpha)]$	d′ and $(\mathrm{P_{SL}}6)$
f′	$(\alpha \to (\varphi \to \psi) \,\&\, \alpha) \to (\varphi \,\&\, \alpha \to \psi \,\&\, \alpha)$	e′ and associativity
g′	$(\alpha \to (\varphi \to \psi) \,\&\, \alpha) \to [(\alpha \to \varphi \,\&\, \alpha) \to (\alpha \to \psi \,\&\, \alpha)]$	f′ and associativity
h′	$(\alpha \to (\varphi \to \psi) \,\&\, \alpha) \wedge \overline{1} \to [(\alpha \to \varphi \,\&\, \alpha) \wedge \overline{1} \to (\alpha \to \psi \,\&\, \alpha) \wedge \overline{1}]$	g′ and $(\mathrm{P_{SL}}24)$ twice.

□

Interestingly enough, the deductions theorems studied in this section yield a connection with a variant of the classical proof by cases property. Recall that classical logic enjoys this meta-rule:

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \vee \psi \vdash \chi}\,.$$

We will see now how a similar property can be obtained for almost (MP)-based substructural logics with a more complex form of disjunction built from their sets of basic deduction terms.

THEOREM 2.6.9 (Proof by Cases Property). *Let* L *be a substructural logic with* $\&$ *and* $\overline{1}$ *in its language and assume that it is almost* (MP)*-based with a set of basic deduction terms* bDT *such that*

- *for each* $\beta \in \mathrm{bDT}^* \setminus \{\star\}$ *we have* $\vdash_{\mathrm{L}} \beta(\varphi) \to \overline{1}$ *for any formula* φ *and*
- *there is* $\beta_0 \in \mathrm{bDT}$ *such that* $\vdash_{\mathrm{L}} \beta_0(\varphi) \to \varphi$ *for any formula* φ*.*

Then the following meta-rule is valid in L*:*

$$\frac{\Gamma, \varphi \vdash \chi \qquad\qquad \Gamma, \psi \vdash \chi}{\Gamma \cup \{\alpha(\varphi) \vee \beta(\psi) \mid \alpha, \beta \in \mathrm{bDT}^*\} \vdash \chi}\,.$$

Proof. We start by showing that L enjoys the Almost-Implicational Deduction Theorem w.r.t. the set $\mathrm{DT} = \{\delta \in \Pi(\mathrm{bDT}^*) \mid \vdash_{\mathrm{L}} \delta(\varphi) \to \overline{1}\}$. We use Theorem 2.6.5: for each $\delta \in \Pi(\mathrm{bDT}^*)$ we consider $\hat{\delta} \in \Pi(\mathrm{bDT}^*)$ resulting from δ understood as a tree (see footnote 8) by replacing the node-labels $\star$ by $\beta_0(\star)$; then using $(\mathrm{P_{SL}}8)$ and $(\mathrm{P_{SL}}9)$ and the assumptions of this theorem we can easily show that for each formula φ we have $\vdash_{\mathrm{L}} \hat{\delta}(\varphi) \to \delta(\varphi)$ and $\vdash_{\mathrm{L}} \hat{\delta}(\varphi) \to \overline{1}$. Note that for all $\delta \in \mathrm{DT}$ we can easily prove $\vdash_{\mathrm{L}} \delta(\varphi) \,\&\, \psi \to \psi$ and $\vdash_{\mathrm{L}} \psi \,\&\, \delta(\varphi) \to \psi$.

Next, assume that $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$. From the Almost-Implicational Deduction Theorem we obtain $\delta_\varphi, \delta_\psi \in \mathrm{DT}$ such that $\Gamma \vdash_{\mathrm{L}} \delta_\varphi(\varphi) \to \chi$ and $\Gamma \vdash_{\mathrm{L}} \delta_\psi(\psi) \to \chi$ and so $\Gamma \vdash_{\mathrm{L}} \delta_\varphi(\varphi) \vee \delta_\psi(\psi) \to \chi$. The proof is done by showing by induction over the sum of the depths of the trees representing $\delta_\varphi, \delta_\psi$ that:

$$\{\alpha(\varphi) \vee \beta(\psi) \mid \alpha, \beta \in \mathrm{bDT}^*\} \vdash_{\mathrm{L}} \delta_\varphi(\varphi) \vee \delta_\psi(\psi).$$

The base of induction (when $\delta_\varphi, \delta_\psi \in \mathrm{bDT}^*$) is trivial. For the induction step assume that $\delta_\psi = \delta_1 \,\&\, \delta_2$. Using $(\mathrm{P_{SL}}20)$, $(\mathrm{P_{SL}}21)$, $(\vee 1)$, $(\vee 2)$, and $(\vee 3)$ we obtain the following chain of implications:

$$(\delta_\varphi(\varphi) \vee \delta_1(\psi)) \,\&\, (\delta_\varphi(\varphi) \vee \delta_2(\psi)) \to$$

$$\to [\delta_\varphi(\varphi) \,\&\, \delta_\varphi(\varphi)] \vee [\delta_\varphi(\varphi) \,\&\, \delta_2(\psi)] \vee [\delta_1(\psi) \,\&\, \delta_\varphi(\varphi)] \vee [\delta_1(\psi) \,\&\, \delta_2(\psi)] \to$$

$$\to \delta_\varphi(\varphi) \vee \delta_\varphi(\varphi) \vee \delta_\varphi(\varphi) \vee [\delta_1(\psi) \,\&\, \delta_2(\psi)] \to \delta_\varphi(\varphi) \vee \delta_\psi(\psi).$$

The induction assumption used for $\delta_\varphi(\varphi) \vee \delta_1(\psi)$ and $\delta_\varphi(\varphi) \vee \delta_2(\psi)$ together with $(\mathrm{P_{SL}}7)$ completes the proof. $\square$

COROLLARY 2.6.10 (Proof by Cases Property for logics with weakening). *Let* L *be a substructural Rasiowa-implicative logic with* $\&$ *and* $\overline{1}$ *in its language. Assume that it is almost* (MP)*-based with a set of basic deduction terms* bDT*. Then the following meta-rule is valid in* L:

$$\frac{\Gamma, \varphi \vdash \chi \qquad\qquad \Gamma, \psi \vdash \chi}{\Gamma \cup \{\alpha(\varphi) \vee \beta(\psi) \mid \alpha, \beta \in \mathrm{bDT}^*\} \vdash \chi}\,.$$

Proof. Let us define $\mathrm{bDT}_\star = \mathrm{bDT} \cup \{\star\}$ and note that $\mathrm{bDT}_\star$ satisfies the conditions of the previous theorem. As clearly, $\mathrm{bDT}_\star^* = \mathrm{bDT}^*$ the proof is done. □

As a corollary we obtain the proof by cases property for three prominent substructural logics (note that in the case of $\mathrm{FL_e}$ we can simplify the form given by the theorem above using the provability of $\vdash_{\mathrm{FL_e}} (p \wedge \overline{1}) \leftrightarrow (p \wedge \overline{1}) \wedge \overline{1}$).

COROLLARY 2.6.11 (Proof by Cases Property for extensions of FL). *The following meta-rule is valid in* FL:

$$\frac{\Gamma, \varphi \vdash \chi \qquad\qquad \Gamma, \psi \vdash \chi}{\Gamma \cup \{\gamma_1(\varphi) \vee \gamma_1(\psi) \mid \gamma_1, \gamma_2 \text{ iterated conjugates}\} \vdash \chi}\,.$$

The following meta-rule is valid in $\mathrm{FL_e}$:

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1}) \vdash \chi}\,.$$

The following meta-rule is valid in $\mathrm{FL_{ew}}$:

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \vee \psi \vdash \chi}\,.$$

2.7 Generalized disjunctions

As we have seen in the previous subsection, substructural logics may retain a form of the classical proof by cases property at the price of using a more complex disjunction. For instance, in the case of FL we have used a rather complicated set $\{\gamma_1(\varphi) \vee \gamma_2(\psi) \mid \gamma_1, \gamma_2 \text{ iterated conjugates}\}$ of infinitely many formulae involving two variables and parameters. The proof by cases property will play an important rôle in the following sections where we study the interplay of disjunctions and implications (in particular, disjunctions will be used to provide a powerful characterization of semilinear implications and, moreover, as we will see in Section 4, they are crucial for first-order logics). In order to prepare the ground for that, in this subsection we provide an abstract analysis of disjunction connectives general enough to cover their possible complicated forms, as the one we have seen in FL. Although a number of results we prove in this section hold in general, for the sake of simplicity here we will mostly restrict ourselves to the case of *finitary* logics which will allow us to provide easier proofs. We will indicate in which results the finitarity assumption is not actually used.

DEFINITION 2.7.1 (Notation for generalized disjunctions). *Let $\nabla(p, q, \vec{r})$ be a set of formulae in two variables p, q and a sequence (possibly empty, finite or infinite) of further variables $\vec{r}$ called* parameters. *We define:*

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \vec{\alpha}) \mid \vec{\alpha} \in Fm_{\mathcal{L}}^{\leq\omega}\}.$$

Given sets $\Phi, \Psi \subseteq Fm_{\mathcal{L}}$, $\Phi \nabla \Psi$ denotes the set $\bigcup\{\varphi \nabla \psi \mid \varphi \in \Phi, \psi \in \Psi\}$. When there are no parameters in the set $\nabla(p, q)$ and it is a singleton, we write $\varphi \vee \psi$ instead of $\varphi \nabla \psi$.

CONVENTION 2.7.2 (Protodisjunction and p-protodisjunction). *A parameterized set of formulae $\nabla(p, q, \vec{r})$ will be called a* p-protodisjunction *in* L *whenever it satisfies:*

$$(\mathrm{PD}) \quad \varphi \vdash_{\mathrm{L}} \varphi \nabla \psi \quad \textit{and} \quad \psi \vdash_{\mathrm{L}} \varphi \nabla \psi.$$

If ∇ has no parameters we drop the prefix 'p-'.

This convention does not define an interesting notion on its own because, actually, any theorem (or set of theorems) in two variables of a given logic would be a protodisjunction in this logic; we only introduce it as a useful means to shorten the formulation of many upcoming definitions and results. In contrast, requiring the property of proof by cases results in more interesting notions of disjunction.

DEFINITION 2.7.3 (p-disjunction, disjunction). *Given a logic* L*, a (p-)protodisjunction ∇ is called a* (p-)disjunction *(in* L*) whenever it satisfies the* Proof by Cases Property, PCP *for short:*

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi}\,.$$

Observe that if ∇ is a disjunction in a logic it remains a disjunction in all its axiomatic extensions and in its fragments containing the connectives used in ∇. Later we will give sufficient and necessary conditions for the preservation of the Proof by Cases Property in expansions. Interestingly enough, all p-disjunctions in a given logic are interderivable as we can easily prove:

LEMMA 2.7.4. *Let* L *be a logic and ∇, ∇' parameterized sets of formulae. Assume that ∇ is a p-disjunction in* L*. Then ∇' is a p-disjunction in* L *iff $\varphi \nabla \psi \dashv\vdash_{\mathrm{L}} \varphi \nabla' \psi$.*

Thus the notion of p-disjunction is intrinsic for a given logic and in the upcoming definition it does not matter which p-disjunction we choose since all of them are interderivable.

DEFINITION 2.7.5 (p-disjunctional logic, disjunctional logic, disjunctive logic). *Let* L *be a logic. We say that* L *is a* (p-)disjunctional *logic if it has a (p-)disjunction. We use the term* disjunctive *instead if the disjunction is just a single parameter-free formula.*

As we will see in Example 2.7.9, in substructural logics the connective $\vee$ need not be a disjunction in the technical sense just defined. Therefore, we introduce the following terminology:

DEFINITION 2.7.6 (Lattice-disjunctive logic). *Let* L *be weakly implicative disjunctive logic* L *with principal implication* $\to$ *and disjunction* $\vee$*. Then* L *is* lattice disjunctive *if:*

$(\vee 1)$ $\vdash_{\mathrm{L}} \varphi \to \varphi \vee \psi$
$(\vee 2)$ $\vdash_{\mathrm{L}} \psi \to \varphi \vee \psi$
$(\vee 3)$ $\varphi \to \chi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \vee \psi \to \chi$.

Thus a substructural logic with $\vee$ in its language is lattice-disjunctive iff $\vee$ satisfies PCP. Note that if a logic L satisfies the conditions $(\vee 1)$–$(\vee 3)$ for two different (primitive or derivable) connectives $\vee$ and $\vee'$, then we can easily prove a stronger version of Lemma 2.7.4: $\vdash_{\mathrm{L}} \varphi \vee \psi \leftrightarrow \varphi \vee' \psi$. Also note that for any lattice-disjunctive logic L and $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$, the algebra $\langle A, \vee^{\boldsymbol{A}} \rangle$ is a join-semilattice with semi-lattice order $\leq_{\mathbf{A}}$.

Clearly the classes of lattice-disjunctive, disjunctive, disjunctional and p-disjunctional logics form an increasing chain under inclusion. Next, we present several examples of lattice-disjunctive logics to show that all these inclusions are proper, thus demonstrating the non-triviality of our hierarchy.

LEMMA 2.7.7. *Any* (MP)*-based substructural Rasiowa-implicative logic with* $\&$ *and* $\overline{1}$ *in its language (e.g. any axiomatic extension of* $\mathrm{FL_{ew}}$*) is a lattice-disjunctive logic.*

Proof. The PCP for these logics is shown in Corollary 2.6.10 by using the connective $\vee$ and hence they are lattice-disjunctive. □

THEOREM 2.7.8. *There is a p-disjunctional logic which is not disjunctional, a disjunctional logic which is not disjunctive, and a disjunctive substructural logic which is not lattice-disjunctive.*

The proof of this theorem is established by the following three examples.

EXAMPLE 2.7.9. *The logic* $\mathrm{FL_e}$ *is disjunctive but not lattice-disjunctive.* In Corollary 2.6.11 we have seen that $\nabla(p, q) = (p \wedge \overline{1}) \vee (q \wedge \overline{1})$ satisfies the PCP. Since it clearly satisfies (PD) too, it is a p-disjunction.

Assume now that the lattice connective $\vee$ satisfies the PCP. From $\varphi \vdash \varphi \wedge \overline{1}$ we easily get $\varphi \vdash (\varphi \wedge \overline{1}) \vee \psi$. As also $\psi \vdash (\varphi \wedge \overline{1}) \vee \psi$, we could use the PCP of $\vee$ to obtain $\varphi \vee \psi \vdash (\varphi \wedge \overline{1}) \vee \psi$. Consider the $\mathrm{FL_e}$-matrix with the domain $\{\bot, a, b, \overline{1}, \top\}$; designated set $\{\overline{1}, \top\}$; the lattice connectives defined in the way that the elements form the non-distributive lattice *diamond* where $\bot$ is the minimum element and $\top$ is the maximum; residual conjunction: $x \mathbin{\&} \overline{1} = \overline{1} \mathbin{\&} x = x$ and $x \mathbin{\&} y = x \wedge y$ for $x, y \neq \overline{1}$; and implication: $x \to y = \max\{z \mid z \mathbin{\&} x \leq y\}$. Then, we reach a contradiction from these simple observations: $a \vee b = \top$ but $(a \wedge \overline{1}) \vee b = \bot \vee b = b$.

EXAMPLE 2.7.10. *The implicational fragment of Gödel–Dummett logic* $\mathrm{G}_{\to}$ *(introduced in Chapter I) is disjunctional but not disjunctive.* First we show that the set $\varphi \nabla \psi = \{(\varphi \to \psi) \to \psi, (\psi \to \varphi) \to \varphi\}$ is a disjunction. Since G is an axiomatic extension of $\mathrm{FL_{ew}}$ it satisfies: $\varphi \vdash (\varphi \to \psi) \to \psi$ and $\psi \vdash (\varphi \to \psi) \to \psi$ and so ∇ satisfies (PD). Now observe that $\Gamma, \varphi \to \psi, (\varphi \to \psi) \to \psi, (\psi \to \varphi) \to \varphi \vdash \psi$ and as we assume that $\Gamma, \psi \vdash \chi$ thus $\Gamma, \varphi \to \psi, \varphi \nabla \psi \vdash \chi$ and so by the Deduction Theorem $\Gamma, \varphi \nabla \psi \vdash (\varphi \to \psi) \to \chi$. Analogously we prove $\Gamma, \varphi \nabla \psi \vdash (\psi \to \varphi) \to \chi$

and as $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$ is a theorem of Gödel–Dummett logic we obtain $\Gamma, \varphi \nabla \psi \vdash \chi$ as needed.

Assume that $\varphi(p,q)$ is a disjunction. As a consequence of the standard completeness theorem for G, we know that $G_\to$ is complete with respect to the matrix $\mathbf{A}$ whose universe is the real unit interval $[0,1]$, the filter is $\{1\}$ and the only operation is:

$$a \to^{\mathbf{A}} b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

By Lemma 2.7.4, the formula $\varphi(p,q)$ and the set $\varphi \nabla \psi$ are mutually derivable in $G_\to$. As in Gödel logic we have: $\varphi \vee \psi \leftrightarrow ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$, we can use the Deduction Theorem (Theorem 2.6.3) we obtain that $\varphi(p,q)$ is interpreted in $\mathbf{A}$ as the function maximum. So, in particular, for every $a,b \in [0,1)$ we have $\varphi^{\mathbf{A}}(a,b) = \max\{a,b\}$. We show by an infinite descent argument that this is impossible. Since $\to$ is the only connective in the language, the formula must be $\varphi(p,q) = \alpha(p,q) \to \beta(p,q)$. Take any $a,b \in [0,1)$. If $a \leq b$, $\varphi^{\mathbf{A}}(a,b) = \alpha^{\mathbf{A}}(a,b) \to^{\mathbf{A}} \beta^{\mathbf{A}}(a,b) = b$, which implies $\beta^{\mathbf{A}}(a,b) = b$. Analogously, if $a > b$, we have $\beta^{\mathbf{A}}(a,b) = a$. Thus, $\beta(p,q)$ would be a strictly shorter formula with the same property. Following this line of reasoning we would derive that for each $a,b \in [0,1)$ we have either $a \to^{\mathbf{A}} b = \max\{a,b\}$ or $b \to^{\mathbf{A}} a = \max\{a,b\}$—a contradiction.

EXAMPLE 2.7.11. *The implicational fragment of intuitionistic logic* $\mathrm{IPC}_\to$ *is p-disjunctional but not disjunctional.* The fact that $\mathrm{IPC}_\to$ is p-disjunctional follows from Theorem 2.7.20 (the filter-distributivity of this logic was proved in [27]). Assume, for contradiction, that a set ∇ is a disjunction in $\mathrm{IPC}_\to$. Thus by Corollary 2.7.17 it also a disjunction in the full intuitionistic logic IPC. From Lemma 2.7.7 we know that IPC is lattice-disjunctive and so, by Lemma 2.7.4, we have $p \nabla q \dashv\vdash_{\mathrm{L}} p \vee q$. Using finitarity, the presence of the lattice conjunction $\wedge$ in the language of IPC and the Deduction Theorem we obtain a formula $\vee'$ of two variables p, q built using only implication and lattice conjunction such that $\vdash_{\mathrm{IPC}} p \vee' q \leftrightarrow p \vee q$—which is known to be impossible (see e.g. [67]).

On the other hand, we have seen in Corollary 2.6.11 that in FL the set $\nabla(p,q,\vec{r}) = \{\gamma_1(p) \vee \gamma_2(q) \mid \gamma_1, \gamma_2 \text{ iterated conjugates}\}$ satisfies the PCP. As it clearly satisfies also (PD) it is a p-disjunction. Sato in [79, Proposition 6.9] showed that there is no protodisjunction in FL which would satisfy the PCP, i.e. that FL is not disjunctive. Thanks to the presence of the lattice conjunction in FL we could derive that not even a finite set of formulae would suffice to define a disjunction in FL, but, unfortunately, we are not able to extend this result to the non-existence of disjunctions defined by infinite sets. However we conjecture:

CONJECTURE 2.7.12. *The logic* FL *is another example of a p-disjunctional logic which is not disjunctional.*

The Proof by Cases Property implies other interesting syntactical properties that a disjunction is expected to satisfy: commutativity, idempotency, and associativity.[9] The proof of the next lemma is straightforward.

[9]Observe that commutativity, idempotency, and associativity are typically also satisfied by conjunction, whereas (PD) and the PCP are typically satisfied only by disjunction connectives.

LEMMA 2.7.13. *If* L *is a logic and* ∇ *is a p-protodisjunction satisfying the* PCP*, then it also satisfies the following conditions:*

(C_∇) $\varphi \nabla \psi \vdash_{\mathrm{L}} \psi \nabla \varphi$
(I_∇) $\varphi \nabla \varphi \vdash_{\mathrm{L}} \varphi$
(A_∇) $\varphi \nabla (\psi \nabla \chi) \dashv\vdash_{\mathrm{L}} (\varphi \nabla \psi) \nabla \chi$.

Next we provide a characterization of (p-)disjunctions, essentially by showing what needs to be added to the properties of the previous lemma to obtain a (p-)disjunction.

DEFINITION 2.7.14 (∇-form). *Let* ∇ *be a p-protodisjunction and* $R = \Gamma \rhd \varphi$ *be a consecution. We define the* ∇*-form of* R*, denoted by* R^∇*, as the set of consecutions* $\{\Gamma \nabla \chi \rhd \delta \mid \chi \in \mathit{Fm}_{\mathcal{L}}$ *and* $\delta \in \varphi \nabla \chi\}$.

THEOREM 2.7.15 (Syntactic characterization of p-disjunctions). *Let* L *be a finitary logic with a presentation* $\mathcal{AS}$ *and* ∇ *a (p-)protodisjunction. Then the following are equivalent:*

1. ∇ *is a (p-)disjunction.*
2. ∇ *enjoys the* strong Proof by Cases Property*,* sPCP *for short,*

$$\frac{\Gamma, \Phi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \Psi \vdash_{\mathrm{L}} \chi}{\Gamma, \Phi \nabla \Psi \vdash_{\mathrm{L}} \chi}.$$

3. ∇ *satisfies* (C_∇)*,* (I_∇)*, and* $R^\nabla \subseteq \mathrm{L}$ *for each* $R \in \mathrm{L}$.
4. ∇ *satisfies* (C_∇)*,* (I_∇)*, and* $R^\nabla \subseteq \mathrm{L}$ *for each* $R \in \mathcal{AS}$.

Proof. We start by showing the equivalence of the first three properties, and then we complete the proof by showing the implication 4→3 (the implication (3→4 is trivial).

1→2: we proceed by using induction. First observe that due the finitary of L we can assume that both Φ, Ψ are finite. Call a pair $\Gamma, \Phi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \Psi \vdash_{\mathrm{L}} \chi$ a *situation*; define the *complexity* of a situation as a pair $\langle n, m\rangle$ where n and m are respectively the cardinalities of $\Phi \setminus \Psi$ and $\Psi \setminus \Phi$. We show by induction on $k = n + m$ that in each situation we obtain $\Gamma, \Phi \nabla \Psi \vdash_{\mathrm{L}} \chi$.

First assume $k \leq 2$. If $n = 0$, i.e. $\Phi \subseteq \Psi$, we obtain $\Phi \nabla \Phi \subseteq \Phi \nabla \Psi$ and since $\Gamma, \Phi \nabla \Phi \vdash_{\mathrm{L}} \Gamma \cup \Phi$ the proof is done. The proof for $m = 0$ is analogous. If $n = m = 1$ we use the PCP. The induction step: consider a situation with complexity $\langle n, m\rangle$, where $n + m > 2$. We can assume without loss of generality that $n \geq 2$, take a formula $\varphi \in \Phi \setminus \Psi$ and define $\Phi' = \Phi \setminus \{\varphi\}$. We know that $\Gamma, \Phi', \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \Psi \vdash_{\mathrm{L}} \chi$. Thus we also know that $\Gamma, \Phi', \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \Phi', \Psi \vdash_{\mathrm{L}} \chi$; notice that the complexity of this situation is $\langle 1, m\rangle$ and so we can use the induction assumption to obtain $\Gamma, \Phi', \varphi \nabla \Psi \vdash_{\mathrm{L}} \chi$.

Thus we have the situation $\Gamma, \varphi \nabla \Psi, \Phi' \vdash_{\mathrm{L}} \chi$ and $\Gamma, \varphi \nabla \Psi, \Psi \vdash_{\mathrm{L}} \chi$ (the second claim is trivial); the complexity of this situation is $\langle n', m'\rangle$, where $n' \leq n - 1$ and $m' \leq m$, and so by the induction assumption we obtain $\Gamma, \varphi \nabla \Psi, \Phi' \nabla \Psi \vdash_{\mathrm{L}} \chi$ (which is exactly what we wanted).

2→3: from $\Gamma \vdash_{\mathrm{L}} \varphi$ we obtain and $\Gamma \vdash_{\mathrm{L}} \varphi \nabla \chi$ using (PD). By (PD) we also obtain $\chi \vdash_{\mathrm{L}} \varphi \nabla \chi$ and the sPCP completes the proof.

3→1: assume than $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$. Using the assumption we obtain $\Gamma \nabla \psi, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi \nabla \psi$ and $\Gamma \nabla \chi, \psi \nabla \chi \vdash_{\mathrm{L}} \chi \nabla \chi$. Using ($\mathrm{C}_\nabla$) and ($\mathrm{I}_\nabla$) we obtain $\Gamma \nabla \psi, \Gamma \nabla \chi, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$. By (PD) we known that $\Gamma \vdash_{\mathrm{L}} \Gamma \nabla \psi$ and $\Gamma \vdash_{\mathrm{L}} \Gamma \nabla \chi$ and so the proof is done.

4→3: assume that $\Gamma \vdash_{\mathrm{L}} \varphi$ and we show $\Gamma \nabla \chi \vdash_{\mathrm{L}} \delta \nabla \chi$ for each formula χ and each δ appearing in the proof of φ from Γ. If $\delta \in \Gamma$ or δ is an axiom, the proof is trivial. Now assume that $R = \Sigma \rhd \delta$ is the deduction rule we use to get δ. From the induction assumption we have $\Gamma \nabla \chi \vdash_{\mathrm{L}} \Sigma \nabla \chi$. As we know that $R^\nabla \in \mathrm{L}$ the proof is done. □

REMARK 2.7.16. *Notice that the equivalence of conditions* $2, 3,$ *and* 4 *could be established even without the assumption of finitarity. Moreover, any of these conditions implies in general condition* 1.

In Example 2.7.9 we have seen a connective $\vee$ which clearly satisfies (PD), (C_∇), (I_∇), and (A_∇) but it is not a disjunction. This demonstrates that the condition $R^\nabla \subseteq \mathrm{L}$ for each $R \in \mathrm{L}$ (resp. $R \in \mathcal{AS}$) is necessary in parts 3 and 4 of the previous theorem (in fact to prove this we have shown that the $\vee$-form of the rule (adj_u) is not valid in $\mathrm{FL_e}$).

As a corollary of the previous theorem we can answer the question when ∇ remains a p-disjunction in an expansion of a given logic.

COROLLARY 2.7.17. *Let* ∇ *be a p-disjunction in a finitary logic* L_1 *and let* L_2 *be an expansion of* L_1 *by a set* $\mathcal{C}$ *of finitary consecutions. Then* ∇ *is a p-disjunction in* L_2 *iff* $R^\nabla \subseteq \mathrm{L}_2$ *for each* $R \in \mathcal{C}$. *In particular,* ∇ *is a p-disjunction in any* axiomatic *expansion of* L_1.

Proof. The left-to-right direction is a straightforward application of the previous theorem. For the reverse direction take a presentation $\mathcal{AS}$ of L_1. We know that L_2 has a presentation $\mathcal{AS}' = \{\sigma[\Gamma] \rhd \sigma\varphi \mid \sigma \text{ is an } \mathcal{L}_2\text{-substitution}, \Gamma \rhd \varphi \in \mathcal{AS} \cup \mathcal{C}\}$. Thus we need to prove that for each $\Gamma \rhd \varphi \in \mathcal{AS} \cup \mathcal{C}$ and for each $\mathcal{L}_2$-substitution σ we have $(\sigma[\Gamma] \rhd \sigma\varphi)^\nabla \subseteq \mathrm{L}_2$, i.e. for each $\mathcal{L}_2$-formula χ, each $\delta(p, q, r_1, \dots, r_n) \in \nabla$ and each sequence $\alpha_1, \dots, \alpha_n$ of $\mathcal{L}_2$-formulae we have $\sigma[\Gamma] \nabla \chi \vdash_{\mathrm{L}_2} \delta(\sigma\varphi, \chi, \alpha_1, \dots, \alpha_n)$. If $\Gamma \rhd \varphi \in \mathcal{C}$, this is the assumption; we solve the remaining case.

Consider any enumeration of the propositional variables such that $p_0 = q$, $p_i = r_i$, and $\mathcal{L}_1$-substitutions ρ, ρ^{-1} and $\mathcal{L}_2$-substitution $\bar{\sigma}$ defined as:

- $\rho p_i = p_{i+n+1}$,
- $\rho^{-1} p_i = p_{i-n-1}$ for $i > n$ and p_i otherwise,
- $\bar{\sigma} p_i = \sigma(p_{i-n-1})$ for $i > n$, $\bar{\sigma} p_i = \alpha_i$ for $1 \leq i \leq n$ and $\bar{\sigma} p_0 = \chi$.

Observe that $\rho^{-1}\rho\psi = \psi$ and $\bar{\sigma}\rho\psi = \sigma\psi$. From $\Gamma \rhd \varphi \in \mathcal{AS}$ we get $\rho[\Gamma] \rhd \rho\varphi \in \mathcal{AS}$ and because $(\rho[\Gamma] \rhd \rho\varphi)^\nabla \subseteq \mathrm{L}_1 \subseteq \mathrm{L}_2$ we obtain: $\rho[\Gamma] \nabla q \vdash_{\mathrm{L}_2} \delta(\rho\varphi, q, r_1, \dots, r_n)$ and so $\bar{\sigma}[\rho[\Gamma] \nabla q] \vdash_{\mathrm{L}_2} \bar{\sigma}\delta(\rho\varphi, q, r_1, \dots, r_n)$. Because obviously, $\bar{\sigma}\delta(\rho\varphi, q, r_1, \dots, r_n) = \delta(\sigma\varphi, \chi, \alpha_1, \dots, \alpha_n)$, if we prove $\bar{\sigma}[\rho[\Gamma] \nabla q] \subseteq \sigma[\Gamma] \nabla \chi$ the proof is done. It is enough to observe that the formulae in $\bar{\sigma}[\rho[\Gamma] \nabla q]$ are of the form $\hat{\delta}(\sigma\psi, \chi, \bar{\sigma}\alpha_1, \dots, \bar{\sigma}\alpha_k) \in \nabla$ for some $\psi \in \Gamma$, $\hat{\delta}(p, q, r_1, \dots, r_k) \in \nabla$ and a sequence of $\mathcal{L}_2$-formulae $\alpha_1, \dots, \alpha_k$. □

Next, we prove that PCP also enjoys a transfer theorem (recall the commentary before Corollary 2.3.4).

THEOREM 2.7.18 (Transfer theorem for PCP). *Let* L *be a finitary logic with a presentation* $\mathcal{AS}$ *and* ∇ *a (p-)protodisjunction. Then the following are equivalent:*

1. ∇ *is a (p-)disjunction.*

2. $\mathrm{Fi}(Y) \cap \mathrm{Fi}(Z) = \mathrm{Fi}(Y \nabla^{\boldsymbol{A}} Z)$ *for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and each* $Y, Z \subseteq A$.

3. $\mathrm{Fi}(X, x) \cap \mathrm{Fi}(X, y) = \mathrm{Fi}(X, x \nabla^{\boldsymbol{A}} y)$ *for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and each* $X \cup \{x, y\} \subseteq A$.

Proof. 1→2: The inclusion $\mathrm{Fi}(X \nabla^{\boldsymbol{A}} Y) \subseteq \mathrm{Fi}(X) \cap \mathrm{Fi}(Y)$ follows easily from (PD). To prove the converse one, we start by showing that for each $x \in \mathrm{Fi}(X)$ we have $x \nabla^{\boldsymbol{A}} y \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$ for each y. If $x \in \mathrm{Fi}(X)$, then (due to Proposition 2.1.23) there is a proof of x from X in some presentation $\mathcal{AS}$ of L. We show that $z \nabla^{\boldsymbol{A}} y \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$ for each z labeling any node of that proof, i.e. for each $\chi(p, q, r_1, \ldots, r_n) \in \nabla$ and each sequence $u_1, \ldots, u_n$ of elements of A we have $\chi^{\boldsymbol{A}}(z, y, u_1, \ldots, u_n) \in \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$.

It is trivial if $z \in X$. Otherwise there is a set Z of labels of the preceding nodes (possible empty), a consecution $\Gamma \rhd \varphi \in \mathcal{AS}$, and an evaluation h, such that $h[\Gamma] = Z$ and $h(\varphi) = z$. Without loss of generality we could assume that variables $q, r_1, \ldots, r_n$ do not occur[10] in $\Gamma \cup \{\varphi\}$ and so we can set $h(q) = y$ and $h(r_i) = u_i$ for every $i \in \{1, \ldots, n\}$. Thus $h[\Gamma \nabla q] \subseteq Z \nabla^{\boldsymbol{A}} y \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$ (the last inclusion follows from the induction assumption). From Theorem 2.7.15 we know that $\Gamma \nabla q \vdash_{\mathrm{L}} \chi(\varphi, q, r_1, \ldots, r_n)$ and so $\chi^{\boldsymbol{A}}(z, y, u_1, \ldots, u_n) = h(\chi(\varphi, q, r_1, \ldots, r_n)) \in \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$.

Now we can finally prove that $\mathrm{Fi}(X) \cap \mathrm{Fi}(Y) \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} Y)$. If $z \in \mathrm{Fi}(X)$ then by the just proved claim for each $y \in Y$ holds: $z \nabla^{\boldsymbol{A}} y \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$ and so, by (C_∇), $y \nabla^{\boldsymbol{A}} z \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} y)$. This can be more compactly written as: $Y \nabla^{\boldsymbol{A}} z \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} Y)$. Analogously we obtain $z \nabla^{\boldsymbol{A}} z \subseteq \mathrm{Fi}(Y \nabla^{\boldsymbol{A}} z)$ from $z \in \mathrm{Fi}(Y)$. Thus $z \in \mathrm{Fi}(Y \nabla^{\boldsymbol{A}} z)$ (by (I_∇)) and so $z \in \mathrm{Fi}(X \nabla^{\boldsymbol{A}} Y)$.

2→3: One inclusion again easily follows from (PD). To prove the other one we use 2 to obtain $\mathrm{Fi}(X, x) \cap \mathrm{Fi}(X, y) \subseteq \mathrm{Fi}(X \nabla^{\boldsymbol{A}} X, X \nabla^{\boldsymbol{A}} y, x \nabla^{\boldsymbol{A}} X, x \nabla^{\boldsymbol{A}} y)$ and (PD) completes the proof.

The final implication is trivial. □

REMARK 2.7.19. *We have actually proved transfer of* sPCP *in all (not necessarily finitary) logics. Furthermore the equivalence of conditions 1 and 3 (but not 2) holds in all (not necessarily finitary) weakly implicative logics [20]. Later (in Theorem 3.2.13) we will prove equivalence of all these three conditions (and those from Theorem 2.7.15) for a special subclass of (not necessarily finitary) weakly implicative logics.*

[10] We could define a new suitable $\Gamma \rhd \varphi$ with the same properties using a Hilbert-hotel style argument: consider any enumeration of the propositional variables such that $p_0 = q$, $p_i = r_i$, a substitution $\sigma(p_i) = p_{i+n+1}$, and any evaluation h' such that $h'(\sigma p) = h(p)$. Then $\sigma[\Gamma] \rhd \sigma\varphi$ is the consecution we need: indeed $\sigma[\Gamma] \rhd \sigma\varphi \in \mathcal{AS}$, $h'[\sigma[\Gamma]] = Z$, and $h'(\sigma\varphi) = z$. Note that here we have used our assumption that the axiomatic systems are closed under substitutions.

The following theorem gives an intrinsic characterization of p-disjunctional logics in terms of (filter-)distributivity (recall Definition 2.1.22), as opposed to Theorem 2.7.15 characterizing particular p-disjunctions.

THEOREM 2.7.20 (Characterization of p-disjunctional logics). *For any finitary weakly implicative logic* L *the following are equivalent:*

1. L *is p-disjunctional.*
2. L *is filter-distributive.*
3. *The lattice* $\mathrm{Th}(\mathrm{L})$ *is distributive.*

Proof. 1→2: Given a p-disjunction ∇ and $F, G, H \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, we can write the following chain of equations (the non-trivial ones are due to part 2 of Theorem 2.7.18):

$$\begin{aligned} F \cap (G \vee H) &= \mathrm{Fi}(F) \cap \mathrm{Fi}(G \cup H) = \\ &= \mathrm{Fi}(F \nabla (G \cup H)) = \\ &= \mathrm{Fi}((F \nabla G) \cup (F \nabla H)) = \\ &= \mathrm{Fi}(F \nabla G) \vee \mathrm{Fi}(F \nabla H) = \\ &= (\mathrm{Fi}(F) \cap \mathrm{Fi}(G)) \vee (\mathrm{Fi}(F) \cap \mathrm{Fi}(H)) = \\ &= (F \cap G) \vee (F \cap H). \end{aligned}$$

2→3: Trivial.

3→1: Define $\nabla = \mathrm{Th}_{\mathrm{L}}(p) \cap \mathrm{Th}_{\mathrm{L}}(q)$. Given a pair of formulae φ, ψ, take a surjective substitution σ such that $\sigma p = \varphi$ and $\sigma q = \psi$. If we knew that $\mathrm{Th}_{\mathrm{L}}(\sigma p) \cap \mathrm{Th}_{\mathrm{L}}(\sigma q) = \mathrm{Th}_{\mathrm{L}}(\sigma[p \nabla q])$, then we could write the following chain of equations demonstrating that ∇ is a p-disjunction (we use the definition of join in $\mathrm{Th}(\mathrm{L})$, the distributivity of $\mathrm{Th}(\mathrm{L})$, the definition of σ, the fact mentioned above, and we observe that, due to the surjectivity of σ, we have $\sigma[p \nabla q] = \varphi \nabla \psi$):

$$\begin{aligned} \mathrm{Th}_{\mathrm{L}}(\Gamma, \varphi) \cap \mathrm{Th}_{\mathrm{L}}(\Gamma, \psi) &= (\mathrm{Th}_{\mathrm{L}}(\Gamma) \vee \mathrm{Th}_{\mathrm{L}}(\varphi)) \cap (\mathrm{Th}_{\mathrm{L}}(\Gamma) \vee \mathrm{Th}_{\mathrm{L}}(\psi)) = \\ &= \mathrm{Th}_{\mathrm{L}}(\Gamma) \vee (\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi)) = \\ &= \mathrm{Th}_{\mathrm{L}}(\Gamma) \vee \mathrm{Th}_{\mathrm{L}}(\sigma[p \nabla q]) = \\ &= \mathrm{Th}_{\mathrm{L}}(\Gamma) \vee \mathrm{Th}_{\mathrm{L}}(\varphi \nabla \psi) = \\ &= \mathrm{Th}_{\mathrm{L}}(\Gamma, \varphi \nabla \psi). \end{aligned}$$

The rest of the proof is dedicated to proving $\mathrm{Th}_{\mathrm{L}}(\sigma p) \cap \mathrm{Th}_{\mathrm{L}}(\sigma q) = \mathrm{Th}_{\mathrm{L}}(\sigma[p \nabla q])$. We define a theory $Y = \sigma^{-1}[\mathrm{Th}_{\mathrm{L}}(\emptyset)]$ (clearly $Y \in \mathrm{Th}(\mathrm{L})$ by Lemma 2.1.19), an interval in the lattice of theories $[Y, Fm_{\mathcal{L}}] = \{T \in \mathrm{Th}(\mathrm{L}) \mid Y \subseteq T\}$, and mappings $\boldsymbol{\sigma}\colon [Y, Fm_{\mathcal{L}}] \to \mathrm{Th}(\mathrm{L})$ and $\boldsymbol{\sigma}^{-1}\colon \mathrm{Th}(\mathrm{L}) \to [Y, Fm_{\mathcal{L}}]$ defined as $\boldsymbol{\sigma}(T) = \sigma[T]$ and $\boldsymbol{\sigma}^{-1}(T) = \sigma^{-1}[T]$. Clearly $\sigma^{-1}[T] \in [Y, Fm_{\mathcal{L}}]$ for each $T \in \mathrm{Th}(\mathrm{L})$ (by Lemma 2.1.19 and the fact that $\mathrm{Th}_{\mathrm{L}}(\emptyset) \subseteq T$). To prove that $\boldsymbol{\sigma}(T) \in \mathrm{Th}(\mathrm{L})$ for each $T \in [Y, Fm_{\mathcal{L}}]$, taking into account Lemma 2.1.19, all we need to show is $\sigma(\alpha) \in \sigma[T]$ implies $\alpha \in T$: the assumption gives us $\sigma\alpha = \sigma\beta$ for some $\beta \in T$, therefore $\vdash_{\mathrm{L}} \sigma\beta \to \sigma\alpha$, hence $\beta \to \alpha \in T$ (because $\beta \to \alpha \in Y$ and $Y \subseteq T$), and thus by (MP) we obtain $\alpha \in T$.

Claim 1: $\boldsymbol{\sigma}$ is an isomorphism. Clearly both $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{-1}$ are order-preserving, thus all we need to show is $\boldsymbol{\sigma}^{-1}(\boldsymbol{\sigma}(T)) = T$ and $\boldsymbol{\sigma}(\boldsymbol{\sigma}^{-1}(S)) = S$ for each $T \in [Y, Fm_{\mathcal{L}}]$ and $S \in \mathrm{Th}(\mathrm{L})$. The two non-trivial inclusions are $\boldsymbol{\sigma}^{-1}(\boldsymbol{\sigma}(T)) \subseteq T$ (which follows from the already proved fact: $\sigma(\alpha) \in \sigma[T]$ implies $\alpha \in T$) and $\boldsymbol{\sigma}(\boldsymbol{\sigma}^{-1}(S)) \supseteq S$ (which follows from the surjectivity of σ).

Claim 2: $\mathrm{Th}_{\mathrm{L}}(\sigma[\Sigma]) = \boldsymbol{\sigma}(Y \vee \mathrm{Th}_{\mathrm{L}}(\Sigma))$ *for each set of formulae* Σ. The first inclusion follows from: $\sigma[\Sigma] \subseteq \boldsymbol{\sigma}(Y \vee \mathrm{Th}_{\mathrm{L}}(\Sigma))$ and $\boldsymbol{\sigma}(Y \vee \mathrm{Th}_{\mathrm{L}}(\Sigma)) \in \mathrm{Th}(\mathrm{L})$. The second inclusion: if $\chi \in \boldsymbol{\sigma}(Y \vee \mathrm{Th}_{\mathrm{L}}(\Sigma))$, then $\chi = \sigma\delta$ and $Y, \Sigma \vdash_{\mathrm{L}} \delta$. Thus $\sigma[Y], \sigma[\Sigma] \vdash_{\mathrm{L}} \sigma(\delta)$, i.e. $\sigma[\Sigma] \vdash_{\mathrm{L}} \chi$.

Now we can finish the proof of the theorem by a series of equations (we use Claim 2, Claim 1, distributivity of $\mathrm{Th}(\mathrm{L})$, and Claim 2 again):

$$\begin{aligned} \mathrm{Th}_{\mathrm{L}}(\sigma p) \cap \mathrm{Th}_{\mathrm{L}}(\sigma q) &= \boldsymbol{\sigma}(Y \vee \mathrm{Th}_{\mathrm{L}}(p)) \cap \boldsymbol{\sigma}(Y \vee \mathrm{Th}_{\mathrm{L}}(q)) = \\ &= \boldsymbol{\sigma}((Y \vee \mathrm{Th}_{\mathrm{L}}(p)) \cap (Y \vee \mathrm{Th}_{\mathrm{L}}(q))) = \\ &= \boldsymbol{\sigma}(Y \vee (\mathrm{Th}_{\mathrm{L}}(p) \cap \mathrm{Th}_{\mathrm{L}}(q))) = \\ &= \mathrm{Th}_{\mathrm{L}}(\sigma[\mathrm{Th}_{\mathrm{L}}(p) \cap \mathrm{Th}_{\mathrm{L}}(q)]) = \\ &= \mathrm{Th}_{\mathrm{L}}(\sigma[p \nabla q]). \end{aligned} \qquad \square$$

Next, we introduce the notion of ∇-prime filter by generalizing the classical notion of prime filter in Boolean algebras.

DEFINITION 2.7.21 (Prime filter). *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ *be a logic,* ∇ *a (possibly parameterized) set of formulae in two variables,* $\boldsymbol{A}$ *an* $\mathcal{L}$*-algebra, and* $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$*. Then,* F *is called* ∇-prime *if for every* $a, b \in A$*,* $a \nabla^{\boldsymbol{A}} b \subseteq F$ *iff* $a \in F$ *or* $b \in F$*.*

Notice that when ∇ defines a disjunction connective $\vee$, the previous definition gives just the usual notion of prime filter.

DEFINITION 2.7.22 (Prime Extension Property). *A logic* L *has the* prime extension property*,* PEP *for short, with respect to a set* ∇ *if* ∇*-prime theories form a base of the closure system* $\mathrm{Th}(\mathrm{L})$.

In the parameter-free case, we have the following characterizations of disjunctions in terms of prime filters and their properties:

THEOREM 2.7.23 (Characterizations of disjunctions). *Let* L *be a finitary logic and* ∇ *a protodisjunction. Then the following are equivalent:*

1. ∇ *is a disjunction.*
2. *For every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and every* $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$*,* F *is finitely* $\cap$*-irreducible iff it is* ∇*-prime.*
3. *For every* $\langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$*,* $\langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ *iff* F *is* ∇*-prime.*
4. *For every* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*,* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\Gamma \models_{\{\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L}) \mid F \textit{ is } \nabla\textit{-prime}\}} \varphi$.
5. *For every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$*,* ∇*-prime filters form a base of* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$*.*
6. L *has the* PEP *w.r.t.* ∇.

Proof. 1→2: Consider any $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, assume first that F is not ∇-prime, i.e. there are $x \notin F$ and $y \notin F$ such that $x \nabla^{\boldsymbol{A}} y \subseteq F$ (the second implication in the definition of prime filter holds always due to (PD)). Thus from Theorem 2.7.15 we know that $F = \mathrm{Fi}(F, x \nabla^{\boldsymbol{A}} y) = \mathrm{Fi}(F, x) \cap \mathrm{Fi}(F, y)$, i.e. F is the intersection of two strictly bigger filters. Next assume that F is finitely reducible, i.e. $F = F_1 \cap F_2$ and $F \subsetneq F_i$. Let us consider $a_i \in F_i \setminus F$. Thus, by (PD), we know that $a_1 \nabla^{\boldsymbol{A}} a_2 \subseteq F_i$ and so $a_1 \nabla^{\boldsymbol{A}} a_2 \subseteq F$, i.e. F is not ∇-prime.

The implication 2→3 is a direct consequence of Theorem 2.3.14 and 3→4 follows directly from Theorem 2.3.16.

4→1: We will show that ∇ has the PCP. Assume that $\Gamma, \varphi \nabla \psi \nvdash_{\mathrm{L}} \chi$. Thus there is an $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ where F is ∇-prime and an $\boldsymbol{A}$-evaluation e such that $e[\Gamma, \varphi \nabla \psi] \subseteq F$ and $e(\chi) \notin F$. Since ∇ is parameter-free, we obtain $e(\varphi) \nabla e(\psi) = e[\varphi \nabla \psi] \subseteq F$, and since F is ∇-prime, we have $e(\varphi) \in F$ or $e(\psi) \in F$ and so $\Gamma, \varphi \nvdash_{\mathrm{L}} \chi$ or $\Gamma, \psi \nvdash_{\mathrm{L}} \chi$.

We have established the equivalence of the first four claims. To complete the proof we first observe that 2→5 follows immediately from the fact that (finitely) $\cap$-irreducible filters form a base of $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ (Corollary 2.3.9). The implication 5→6 is trivial. We complete the proof by showing 6→1: assume that $\Gamma, \varphi \nabla \psi \nvdash_{\mathrm{L}} \chi$, then using the PEP there is a ∇-prime theory $\Gamma' \supseteq \Gamma \cup \varphi \nabla \psi$ such that $\Gamma' \nvdash_{\mathrm{L}} \chi$. Thus $\Gamma' \vdash_{\mathrm{L}} \varphi$ or $\Gamma' \vdash_{\mathrm{L}} \psi$. And so clearly $\Gamma, \varphi \nvdash_{\mathrm{L}} \chi$ or $\Gamma, \psi \nvdash_{\mathrm{L}} \chi$. □

REMARK 2.7.24. *Notice that the hypothesis in the previous theorem about the absence of parameters in ∇ has only been used to show 4→1. In fact, in the general case, condition 1 (∇ is a p-disjunction) still implies all the rest, and conditions 1, 5, 6 are still equivalent.*

We consider now the problem of finding the minimal p-disjunctional extension of a given logic. Observe, on the one hand, that if ∇ is a p-disjunction in a family of logics, it remains a p-disjunction in their intersection. On the other hand, any ∇ is a p-disjunction in the inconsistent logic. Thus, the following definition is sound:

DEFINITION 2.7.25 (Logic L^∇). *Let* L *be a finitary logic and ∇ a p-protodisjunction. We denote by L^∇ the least logic extending* L *where ∇ is a p-disjunction.*

PROPOSITION 2.7.26. *Let* L *be a finitary logic and ∇ a p-protodisjunction. Then L^∇ is finitary and it is the intersection of all finitary extensions of* L *where ∇ has the* PCP.

Proof. Recall the notion of finitary companion of a logic S, denoted as $\mathcal{FC}(\mathrm{S})$, which is the largest finitary logic contained in S. Thus, since L is finitary, we know that $\mathrm{L} \subseteq \mathcal{FC}(\mathrm{L}^\nabla) \subseteq \mathrm{L}^\nabla$. If we show that ∇ has the PCP in $\mathcal{FC}(\mathrm{L}^\nabla)$, we obtain $\mathcal{FC}(\mathrm{L}^\nabla) = \mathrm{L}^\nabla$ and hence L^∇ is finitary. Actually, one can easily show in general that if ∇ has the PCP in S, then it has the PCP in $\mathcal{FC}(\mathrm{S})$ as well. □

Moreover, in the parameter-free case, we can easily present a simple axiomatization and a complete semantics for L^∇.

THEOREM 2.7.27 (Axiomatization of L^∇). *Let* L *be a logic with a finitary presentation $\mathcal{AS}$ and let ∇ be a p-protodisjunction satisfying* (C_∇), (I_∇), *and* (A_∇). *Then L^∇ is axiomatized by $\mathcal{AS} \cup \bigcup\{R^\nabla \mid R \in \mathcal{AS}\}$.*

Proof. Let $\hat{\mathrm{L}}$ denote the logic axiomatized by $\mathcal{AS}' = \mathcal{AS} \cup \bigcup\{R^{\nabla} \mid R \in \mathcal{AS}\}$ (note that this set is closed under all substitutions because we assume ∇ to be parameter-free). First observe that for each $R \in \mathcal{AS}'$ we have $R^{\nabla} \subseteq \hat{\mathrm{L}}$ (obviously if $R \in \mathcal{AS}'$, otherwise we use (A_{∇})) hence we can use Theorem 2.7.15 to obtain that ∇ has the PCP in $\hat{\mathrm{L}}$ and thus $\mathrm{L}^{\nabla} \subseteq \hat{\mathrm{L}}$. On the other hand clearly $\mathcal{AS} \subseteq \mathrm{L}^{\nabla}$ and thus also $R^{\nabla} \subseteq \mathrm{L}^{\nabla}$ for each $R \in \mathcal{AS}$ (by Theorem 2.7.15 which we can use because Proposition 2.7.26 tells us that L^{∇} is finitary) and so $\hat{\mathrm{L}} \subseteq \mathrm{L}^{\nabla}$. □

THEOREM 2.7.28 (Semantics of L^{∇}). *Let* L *be a finitary logic and* ∇ *a protodisjunction. Define* $\mathbf{MOD}^*_p(\mathrm{L}) = \{\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L}) \mid F \textit{ is } \nabla\textit{-prime}\}$. *Then:*

$$\vdash_{\mathrm{L}^{\nabla}} = \models_{\mathbf{MOD}^*_p(\mathrm{L})}.$$

Proof. We prove that $\models_{\mathbf{MOD}^*_p(\mathrm{L})}$ is the least extension of L where ∇ is a disjunction. It is clear that it is an extension of L and that ∇ is a disjunction there (because of claim 4 in Theorem 2.7.23 and the fact that $\mathbf{MOD}^*(\mathrm{L}^{\nabla}) \subseteq \mathbf{MOD}^*(\mathrm{L})$). Finally, assume that L' is another extension of L where ∇ is a disjunction and $\Gamma \models_{\mathbf{MOD}^*_p(\mathrm{L})} \varphi$, then $\Gamma \models_{\mathbf{MOD}^*_p(\mathrm{L}')} \varphi$, and so $\Gamma \vdash_{\mathrm{L}'} \varphi$. □

Let us recall that matrices can be regarded as first-order structures where the filter corresponds to a unary predicate F, i.e. all atomic formulae in the corresponding classical first-order language are of the form $F(\varphi)$ where φ is a formula. A set of positive clauses $\mathcal{C} = \{\bigvee_{\varphi \in \Sigma_C} F(\varphi) \mid C \in \mathcal{C}\}$ is said to be valid in a matrix $\mathbf{M} = \langle \boldsymbol{A}, F\rangle$, written as $\mathbf{M} \models \mathcal{C}$, if for each $C \in \mathcal{C}$ and each $\mathbf{M}$-evaluation e there is a $\varphi \in \Sigma_C$ such that $e(\varphi) \in F$. A positive universal class of matrices is the collection of all models of a set of universal closures of positive clauses.[11] The next theorem presents an axiomatization, by means of a p-disjunction, of any logic given by a positive universal class of matrices.

THEOREM 2.7.29 (Axiomatization of the logic of a positive universal class of matrices). *Let* L *be a finitary logic with a p-disjunction* ∇ *and* $\mathcal{C}$ *a set of positive clauses. Then:*

$$\models_{\{\mathbf{B} \in \mathbf{MOD}^*(\mathrm{L}) \mid \mathbf{B} \models \mathcal{C}\}} = \mathrm{L} + \bigcup\{\nabla_{\psi \in \mathcal{I}} \psi \mid \bigvee_{\psi \in \mathcal{I}} F(\psi) \in \mathcal{C}\}.$$

Proof. Let us first denote the formula $\nabla_{\psi \in \mathcal{I}} \psi$ as C_{∇} for a clause $C = \bigvee_{\psi \in \mathcal{I}} F(\psi) \in \mathcal{C}$ and observe that for each matrix $\mathbf{M} = \langle \boldsymbol{A}, F\rangle$ we have: if $\mathbf{M}$ as a first-order structure satisfies C, then $\mathbf{M}$ as a matrix satisfies the propositional formula C_{∇}. Moreover, if F is ∇-prime, the reverse implication holds as well.

We denote the left-hand side logic as L^h and the right-hand side one as L^a. Clearly $\mathrm{L} \subseteq \mathrm{L}^h$ and due to the observation above also $\vdash_{\mathrm{L}^h} C_{\nabla}$ for each $C \in \mathcal{C}$. Thus $\mathrm{L}^a \subseteq \mathrm{L}^h$.

The converse direction $\mathrm{L}^h \subseteq \mathrm{L}^a$ will be proven counterpositively. Assume that there is a set $\Gamma \cup \{\varphi\}$ of formulae such that: $\Gamma \nvdash_{\mathrm{L}^a} \varphi$. Since ∇ is a p-disjunction

[11] Positive universal classes are usually defined as the collection of all models of a set of *positive universal formulae*, i.e. the universal closures of formulae built from atoms using conjunction and disjunction. Clearly each formula of this kind can be written as the universal closure of a conjunction of positive clauses and so its generated positive universal class is just the positive universal class generated by the collection of these positive clauses.

in L^a (by Corollary 2.7.17), we know that there is $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L}^a)$ where F is ∇-prime (by Theorem 2.7.23) such that $\Gamma \not\models_{\mathbf{A}} \varphi$. If we show that $\mathbf{A} \models \mathcal{C}$, the proof is done. Assume that $\mathbf{A} \not\models C$ for some $C \in \mathcal{C}$. Then, by the observation at the beginning of the proof, $\mathbf{A}$ as matrix would not satisfy the propositional formula C_∇—a contradiction. □

As a consequence we obtain a general method to axiomatize the intersection of axiomatic extensions of a common base logic by means of a generalized disjunction.

THEOREM 2.7.30 (Axiomatization of intersections of axiomatic extensions). *Let* L *be a finitary logic with a p-disjunction* ∇*, and let* L_1*,* L_2 *be axiomatic extensions of* L *respectively given by the sets of axioms* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *(without loss of generality we can assume that* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *share no propositional variables). Then:*

$$\mathrm{L}_1 \cap \mathrm{L}_2 = \mathrm{L} + \bigcup\{\varphi \nabla \psi \mid \varphi \in \mathcal{A}_1, \psi \in \mathcal{A}_2\}.$$

Proof. It is easy to see that $\mathrm{L}_1 \cap \mathrm{L}_2 = \models_{\mathbf{MOD}^*(\mathrm{L}_1)\cup\mathbf{MOD}^*(\mathrm{L}_2)}$. Consider the set of positive clauses $\mathcal{C} = \{F(\varphi) \vee F(\psi) \mid \varphi \in \mathcal{A}_1, \psi \in \mathcal{A}_2\}$.

If we show that $\mathbf{MOD}^*(\mathrm{L}_1) \cup \mathbf{MOD}^*(\mathrm{L}_2) = \{\mathbf{B} \in \mathbf{MOD}^*(\mathrm{L}) \mid \mathbf{B} \models \mathcal{C}\}$, the proof is done by Theorem 2.7.29. One inclusion is trivial. We prove the converse one counterpositively: consider $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ such that $\mathbf{A} \notin \mathbf{MOD}^*(\mathrm{L}_1) \cup \mathbf{MOD}^*(\mathrm{L}_2)$, i.e. there is $\varphi_i \in \mathcal{A}_i$ such that $\not\models_{\boldsymbol{A}} \varphi_i$; consider evaluations e_i witnessing those facts. As φ_1 and φ_2 do not share any propositional variable there is an evaluation e witnessing both facts (e is defined as e_i in the variables occurring in φ_i and arbitrarily elsewhere). This evaluation also shows that $\mathbf{A} \not\models F(\varphi_1) \vee F(\varphi_2)$. □

As another consequence of Theorem 2.7.29 the following example shows that in some cases one can easily axiomatize the logic defined by linearly ordered models of a given logic in terms of disjunction. Logics complete with respect to linearly ordered matrices are the central topic of the next section.

EXAMPLE 2.7.31. Since $\vee$ is a disjunction in $\mathrm{FL}_{\mathrm{ew}}$ and any matrix $\mathbf{M} \in \mathbf{MOD}(\mathrm{FL}_{\mathrm{ew}})$ is linearly ordered iff $\mathbf{M} \models F(\varphi \to \psi) \vee F(\psi \to \varphi)$, we can apply Theorem 2.7.29 and obtain:

$$\models_{\{\mathbf{B}\in\mathbf{MOD}^*(\mathrm{FL}_{\mathrm{ew}}) \mid \mathbf{B} \text{ is linearly ordered}\}} = \mathrm{FL}_{\mathrm{ew}} + (\varphi \to \psi) \vee (\psi \to \varphi).$$

3 Semilinear logics

This section is devoted to the central topic of the chapter: logics complete with respect to linearly ordered matrices, which we call *semilinear*. We aim to encompass by this notion the vast majority of fuzzy logics in the literature. Always in the context of weakly implicative logics, in the first subsection we provide the technical definition of semilinear logic and some auxiliary notions which allow for an approach to a large extent analogous to that we have followed for disjunctions. In the second subsection, we study the strong interplay between semilinear implications and disjunctions and obtain several interesting consequences. In the last subsection we focus on completeness properties with respect to finer semantics, i.e. distinguished subclasses of linearly ordered matrices.

3.1 Basic definitions, properties, and examples

We want to study a general notion of logic complete with respect to a semantics of linearly ordered matrices. A natural design choice is to restrict to the context of weakly implicative logics presented in the previous section because in these logics the implication connective induces a preorder in the matrices, which is actually an order relation in reduced models.

DEFINITION 3.1.1 (Linear filter and linear model). *Let* L *be a weakly implicative logic. Take any non-trivial* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$. *The filter* F *is called* linear *if* $\leq_{\mathbf{A}}$ *is a total preorder, i.e. for every* $a, b \in A$, $a \to^{\boldsymbol{A}} b \in F$ *or* $b \to^{\boldsymbol{A}} a \in F$. *Furthermore, we say that* $\mathbf{A}$ *is a* linearly ordered model *(or just a* linear model*) if* $\leq_{\mathbf{A}}$ *is a linear order (equivalently:* F *is linear and* $\mathbf{A}$ *is reduced). We denote the class of all linear models as* $\mathbf{MOD}^{\ell}(\mathrm{L})$.

Now, based on linear models, we can introduce the central concept of this section:

DEFINITION 3.1.2 (Semilinear implication, semilinear logic). *Let* L *be a weakly implicative logic. We say that* $\to$ *is a* semilinear implication *if the linear models it defines are a complete semantics for* L, *i.e.* $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\ell}(\mathrm{L})}$. *A weakly implicative logic is* semilinear *if it has a semilinear implication.*

Observe that the class of linear models and the notion of semilinearity thereof are not intrinsically defined for a given logic: they depend on which possible implication has been chosen as principal. For instance, in classical logic both $\to$ and $\equiv$ are weak implications, but only $\to$ makes the logic semilinear (linear models w.r.t. to $\to$ are the trivial model and that based on the two-element Boolean algebra, while the only linear model w.r.t. $\equiv$ is the trivial one).

The following simple lemma has an important corollary and will be used later to provide some useful counterexamples.

LEMMA 3.1.3. *Let* L *be a weakly implicative logic,* $\boldsymbol{A}$ *an* $\mathcal{L}$*-algebra, and* F *a linear filter. Then the set* $[F, A] = \{G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) \mid F \subseteq G\}$ *is linearly ordered by inclusion.*[12]

Proof. Assume that there are two incomparable filters $G_1, G_2 \in [F, A]$ and take elements $a_1 \in G_1 \setminus G_2$ and $a_2 \in G_2 \setminus G_1$. Assume (without a loss of generality) that $a_1 \leq_{\langle \boldsymbol{A}, F\rangle} a_2$. Thus also $a_1 \to^{\boldsymbol{A}} a_2 \in F \subseteq G_1$ and so by (MP) also $a_2 \in G_1$—a contradiction. □

PROPOSITION 3.1.4. *Let* L *be a weakly implicative logic. Then, all linear filters are finitely* $\cap$*-irreducible, and thus* $\mathbf{MOD}^{\ell}(\mathrm{L}) \subseteq \mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}}$.

Proof. If $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a linear filter, by the previous lemma we know that $[F, A]$ is linearly ordered by inclusion. Assume that $F = G_1 \cap G_2$, for some $G_1, G_2 \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. Then we must have $G_1 \subseteq G_2$, and hence $F = G_1$, or $G_2 \subseteq G_1$, and so $F = G_2$; therefore F is finitely $\cap$-irreducible. The second claim follows immediately from Theorem 2.3.14. □

[12]Observe that if L is algebraically implicative and $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^{\ell}(\mathrm{L})$, then $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) = [F, A]$.

In particular, linear theories are linear filters over the algebra $\boldsymbol{Fm}_{\mathcal{L}}$, and they are finitely $\cap$-irreducible. Recall (from Corollary 2.3.9) that in finitary logics the *finitely $\cap$-irreducible* theories form a base of the closure system $\mathrm{Th}(\mathrm{L})$. Thus, an interesting question is to determine under which conditions *linear* theories form a base of $\mathrm{Th}(\mathrm{L})$. We can characterize (see [17]) it by means of a generalization of the so-called *prelinearity property* which we rename to *semilinearity property*. This change of terminology follows the tradition from Universal Algebra of calling a class of algebras 'semi$X$' whenever its subdirectly irreducible members have the property X because, indeed, as will see in Theorem 3.1.8, finitary semilinear logics are characterized as those where all subdirectly irreducible models are linearly ordered.

DEFINITION 3.1.5 (Linear Extension Property, Semilinearity Property). *We say that a weakly implicative logic* L *has the*

- Linear Extension Property LEP *if linear theories form a base of* $\mathrm{Th}(\mathrm{L})$*, i.e. for every theory* $T \in Th(\mathrm{L})$ *and every formula* $\varphi \in Fm_{\mathcal{L}} \setminus T$*, there is a linear theory* $T' \supseteq T$ *such that* $\varphi \notin T'$.
- Semilinearity Property SLP *if the following meta-rule is valid:*

$$\frac{\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}.$$

Next we prove a transfer theorem for the SLP. Recall our standing assumption that the set *Var* of propositional variables is denumerable.

THEOREM 3.1.6 (Transfer of SLP). *Assume that a weakly implicative logic* L *satisfies the* SLP. *Then for each $\mathcal{L}$-algebra $\boldsymbol{A}$ and each set $X \cup \{a, b\} \subseteq A$ the following holds:*

$$\mathrm{Fi}(X, a \to b) \cap \mathrm{Fi}(X, b \to a) = \mathrm{Fi}(X).$$

Proof. To prove the non-trivial direction we show that for each $t \notin \mathrm{Fi}(X)$ we have $t \notin \mathrm{Fi}(X, a \to b)$ or $t \notin \mathrm{Fi}(X, b \to a)$. We distinguish two cases:

1) *Firstly assume that A is countable.* We can assume that the set *Var* of propositional variables contains (or is equal to) the set $\{v_z \mid z \in A\}$ (where $v_z \neq v_w$ whenever $z \neq w$). Consider the following set of formulae:

$$\Gamma = \{v_z \mid z \in \mathrm{Fi}(X)\} \cup \bigcup_{\langle c,n\rangle \in \mathcal{L}} \{c(v_{z_1}, \dots, v_{z_n}) \leftrightarrow v_{c^{\boldsymbol{A}}(z_1,\dots,z_n)} \mid z_i \in A\}.$$

Clearly, $\Gamma \nvdash_{\mathrm{L}} v_t$ (because $\langle \boldsymbol{A}, \mathrm{Fi}(X)\rangle \in \mathbf{MOD}(\mathrm{L})$ and for the $\boldsymbol{A}$-evaluation $e(v_z) = z$: $e[\Gamma] \subseteq \mathrm{Fi}(X)$ and $e(v_t) \notin \mathrm{Fi}(X)$). Thus by the SLP we have $\Gamma, v_a \to v_b \nvdash_{\mathrm{L}} v_t$ or $\Gamma, v_b \to v_a \nvdash_{\mathrm{L}} v_t$. Assume (without loss of generality) the former case and denote $T' = \mathrm{Th}_{\mathrm{L}}(\Gamma, v_a \to v_b)$. We show that the mapping $h\colon A \to Fm_{\mathcal{L}}/\Omega T'$ defined as $h(z) = [v_z]_{T'}$ is a homomorphism by a simple chain of equalities:

$$\begin{aligned} h(c^{\boldsymbol{A}}(z_1,\dots,z_n)) &= [v_{c^{\boldsymbol{A}}(z_1,\dots,z_n)}]_{T'} = [c(v_{z_1},\dots,v_{z_n})]_{T'} \\ &= c^{\boldsymbol{Fm}_{\mathcal{L}}/\Omega T'}([v_{z_1}]_{T'},\dots,[v_{z_n}]_{T'}) = c^{\boldsymbol{Fm}_{\mathcal{L}}/\Omega T'}(h(z_1),\dots,h(z_n)). \end{aligned}$$

Thus $F = h^{-1}([T']) \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ and, since clearly $X \cup \{a \to b\} \subseteq F$ and $t \notin F$, we have established that $t \notin \mathrm{Fi}(X, a \to b)$.

2) *Secondly assume that A is uncountable.* We introduce a new set of propositional variables[13] $Var' = \{v_z \mid z \in A\}$; we can safely assume that it contains the original set Var. We define a new logic L′ in the language $\mathcal{L}'$ which has the same connectives as $\mathcal{L}$ and variables from Var'. If we show that this logic has the SLP we can repeat the constructions from the first part of this proof. From our assumption we know that there is a presentation $\mathcal{AS}$ of L such that each of its rules has countably many premises.

Let us define $\mathcal{AS}' = \{\sigma[X] \rhd \sigma(\varphi) \mid X \rhd \varphi \in \mathcal{AS}$ and σ is an $\mathcal{L}'$-substitution$\}$ and $\mathrm{L}' = \vdash_{\mathcal{AS}'}$. Observe that $\Gamma \vdash_{\mathrm{L}'} \varphi$ iff there is a countable set $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}'} \varphi$ (clearly any proof in $\mathcal{AS}'$ has countably many leaves, because all of its rules have countably many premises). Next observe that L′ is a conservative expansion of L (consider the substitution σ sending all variables from Var to themselves and the rest to a fixed $p \in Var$, take any proof of φ from Γ in $\mathcal{AS}'$ and observe that the same tree with labels ψ replaced by $\sigma\psi$ is a proof of φ from Γ in L).

We show that L′ has the SLP: assume that $\Gamma, \varphi \to \psi \vdash_{\mathrm{L}'} \chi$ and $\Gamma, \psi \to \varphi \vdash_{\mathrm{L}'} \chi$. There is a countable subset $\Gamma' \subseteq \Gamma$ such that $\Gamma', \varphi \to \psi \vdash_{\mathrm{L}'} \chi$ and $\Gamma', \psi \to \varphi \vdash_{\mathrm{L}'} \chi$. Consider the set Var_0 of variables occurring in $\Gamma' \cup \{\varphi, \psi, \chi\}$ and a bijection g on the set Var' such that the image Var_0 is a subset of Var (such bijection clearly exists). Thus for the $\mathcal{L}'$-substitution σ induced by g exists an inverse substitution σ^{-1} and $\sigma[\Gamma'] \cup \{\sigma\varphi, \sigma\psi, \sigma\chi\} \subseteq Fm_{\mathcal{L}}$. Clearly also $\sigma[\Gamma'], \sigma\varphi \to \sigma\psi \vdash_{\mathrm{L}'} \sigma\chi$ and $\sigma[\Gamma'], \sigma\psi \to \sigma\varphi \vdash_{\mathrm{L}'} \sigma\chi$. Using the fact that L′ expands L conservatively, we obtain $\sigma[\Gamma'], \sigma\varphi \to \sigma\psi \vdash_{\mathrm{L}} \sigma\chi$ and $\sigma[\Gamma'], \sigma\psi \to \sigma\varphi \vdash_{\mathrm{L}} \sigma\chi$. From the SLP of L we know that $\sigma[\Gamma'] \vdash_{\mathrm{L}} \sigma\chi$ and $\sigma[\Gamma'] \vdash_{\mathrm{L}'} \sigma\chi$ so by structurality for the inverse substitution σ^{-1} also $\Gamma' \vdash_{\mathrm{L}'} \chi$. □

THEOREM 3.1.7 (Properties of semilinear logics). *Let* L *be a semilinear logic in language $\mathcal{L}$. Then:*

1. L *has the* LEP.
2. L *has the* SLP.
3. L *has the transferred* SLP, *i.e.* $\mathrm{Fi}(X, a \to b) \cap \mathrm{Fi}(X, b \to a) = \mathrm{Fi}(X)$ *for each $\mathcal{L}$-algebra $\boldsymbol{A}$ and each set $X \cup \{a, b\} \subseteq A$.*
4. *Linear filters coincide with finitely $\cap$-irreducible ones in each $\mathcal{L}$-algebra.*
5. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^\ell(\mathrm{L})$.
6. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}} \subseteq \mathbf{MOD}^\ell(\mathrm{L})$.

Proof. If $T \nvdash_{\mathrm{L}} \chi$, then there is a $\mathbf{B} = \langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^\ell(\mathrm{L})$ and a $\mathbf{B}$-evaluation e such that $e[T] \subseteq F$ and $e(\chi) \notin F$. We define $T' = e^{-1}[F]$. Obviously T' is a theory, $T \subseteq T'$ and $T' \nvdash_{\mathrm{L}} \chi$. Since $\leq_{\mathbf{B}}$ is a linear order, $e(\varphi) \leq_{\mathbf{B}} e(\psi)$ or $e(\psi) \leq_{\mathbf{B}} e(\varphi)$ for each φ and ψ. Thus either $e(\varphi \to \psi) \in F$ or $e(\psi \to \varphi) \in F$, i.e. $\varphi \to \psi \in T'$ or $\psi \to \varphi \in T'$.

[13]Notice that this set of variables is not countable, so it does not satisfy the cardinality restriction that we have assumed from the beginning of the chapter for the sake of simplicity. However, an inspection of the relevant parts of the general theory of logical calculi that we have introduced so far shows that everything needed for this proof would work as well without that restriction. Therefore, in this proof we can violate our general assumption without problems.

The second claim: if $T \nvdash_{\mathrm{L}} \chi$, then (using the LEP) there is a linear theory $T' \supseteq T$, such that $T' \nvdash_{\mathrm{L}} \chi$. Assume (without a loss of generality) that $T' \vdash_{\mathrm{L}} \varphi \to \psi$, then obviously $T, \varphi \to \psi \nvdash_{\mathrm{L}} \chi$.

The third claim: follows from the SLP using the transfer theorem proved above.

The fourth claim: let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra. One direction is Proposition 3.1.4. The other one obviously holds for $F = A$ and otherwise follows from the previous claim counterpositively: assume that there are $a, b \in A$ such that $a \to b \notin F$ and $b \to a \notin F$, i.e. $F \subsetneq \mathrm{Fi}(F, a \to b)$ and $F \subsetneq \mathrm{Fi}(F, b \to a)$. Therefore we obtain $F = \mathrm{Fi}(F) = \mathrm{Fi}(F, a \to b) \cap \mathrm{Fi}(F, b \to a)$ and so F is finitely $\cap$-reducible.

The fifth claim: it is an easy corollary of the previous claim and Theorem 2.3.14.

The final claim is a trivial consequence of the previous one. □

Observe that, in fact, only the LEP has been proved directly from semilinearity; all the remaining claims of the theorem above have been shown from their direct predecessors. Now, an obvious question is when these claims are equivalent. First notice that claim 5 tells us that semilinear logics are complete w.r.t. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$, a known property of *finitary* logics established in Theorem 2.3.16, where even more is shown: completeness w.r.t. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$, which is exactly the property needed for the final implication in the proof (namely, that claim 6 implies semilinearity). However, this property is rather obscure, and hence we choose to formulate the following characterization theorem in terms of finitarity.

THEOREM 3.1.8 (Characterization of semilinear logics). *Let* L *be a weakly implicative logic. The following are equivalent:*

1. L *is semilinear.*
2. L *has the* LEP.

Furthermore, if L *is finitary the list of equivalences can be expanded with:*

3. L *has the* SLP.
4. L *has the transferred* SLP.
5. *Linear filters coincide with finitely* $\cap$*-irreducible ones in each* $\mathcal{L}$*-algebra.*
6. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^{\ell}(\mathrm{L})$.
7. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}} \subseteq \mathbf{MOD}^{\ell}(\mathrm{L})$.

Proof. All we have to do is to prove two implications. One of them, *7 implies 1*, is an immediate consequence of Theorem 2.3.16; we show the final one: *2 implies 1*. Assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$, let T be the theory generated by Γ and $T' \supseteq T$ a linear theory such that $T' \nvdash_{\mathrm{L}} \varphi$. From part 3 of Lemma 2.2.9 we know $\mathbf{LindT}_{T'} \in \mathbf{MOD}^*(\mathrm{L})$ and its second part trivially entails that $[T']$ is a linear filter, i.e. $\mathbf{LindT}_{T'} \in \mathbf{MOD}^{\ell}(\mathrm{L})$. The rest of the proof is the same as the proof of the completeness theorem 2.2.13. □

The previous theorems have several interesting and important corollaries. The first one uses the trivial observation that $\varphi, \psi \to \varphi \vdash \psi \to \varphi$ and for regular implications also $\varphi, \varphi \to \psi \vdash \psi \to \varphi$. Thus, by the SLP, we derive $\varphi \vdash \psi \to \varphi$.

COROLLARY 3.1.9. *Every regularly implicative semilinear logic is also Rasiowa-implicative.*

Another interesting corollary is obtained by observing that the LEP of a logic is preserved in all its axiomatic extensions (it is based on the fact that any theory of a logic L which contains a set of axioms $\mathcal{A}$ is a theory in $\mathrm{L} + \mathcal{A}$ as well).

COROLLARY 3.1.10. *All axiomatic extensions of a semilinear logic are semilinear too.*

This corollary is particularly useful when presenting a large class of weakly implicative logics which are not semilinear no matter which weak implication we might take as principal.

It is quite easy to show that a given logic is not semilinear for a fixed principal implication. Consider e.g. intuitionistic logic with its usual implication: the well-known fact that the linear Heyting algebras do not generate the variety of Heyting algebras does the job. The next example uses again our characterization theorem (together with Lemma 3.1.3) to show much more:

EXAMPLE 3.1.11. Intuitionistic logic is not semilinear w.r.t. any principal implication.

Proof. We provide two alternative proofs of this fact. First a very simple *ad hoc* one, and then a more sophisticated proof using the machinery introduced in the present chapter which has the advantage of providing a general method to show undefinability of semilinear implications in many logics.
(1) Let IPC be the intuitionistic propositional logic. Assume that $\to'$ is a semilinear implication in IPC and we show that $p \to' q \dashv\vdash_{\mathrm{IPC}} p \to q$ (where $\to$ is the usual implication of intuitionistic logic), which entails that $\to$ is a semilinear implication—a contradiction. One direction is simple: from $p, p \to' q \vdash_{\mathrm{IPC}} q$ we obtain (using Deduction Theorem) $p \to' q \vdash_{\mathrm{IPC}} p \to q$. The reverse direction: using the first direction we obtain $q \to' p, p \to q \vdash_{\mathrm{IPC}} q \to p$. Since, trivially, $q \to' p, p \to q \vdash_{\mathrm{IPC}} p \to q$ and the symmetrizations of any pair of weak implications are interderivable (Corollary 2.2.4), we obtain $q \to' p, p \to q \vdash_{\mathrm{IPC}} p \to' q$. Now, using the following trivial fact $p \to' q, p \to q \vdash_{\mathrm{IPC}} p \to' q$ and the SLP, we conclude that $p \to q \vdash_{\mathrm{IPC}} p \to' q$.
(2) IPC can be presented as $\mathrm{FL_{ewc}}$, and hence it is a substructural logic in the scope of Theorem 2.5.11. In fact, it is a Rasiowa-implicative logic and $\mathbf{MOD}^*(\mathrm{IPC}) = \{\langle \boldsymbol{A}, \{\overline{1}^{\boldsymbol{A}}\}\rangle \mid \boldsymbol{A} \in \mathbb{HA}\}$, where $\mathbb{HA}$ is the variety of Heyting algebras. The isomorphism (see Theorem 2.4.5) between filters and congruences in any Heyting algebra $\boldsymbol{A}$ tells us that $\{\overline{1}^{\boldsymbol{A}}\}$ is $\cap$-irreducible in $\mathcal{F}i_{\mathrm{IPC}}(\boldsymbol{A})$ if, and only if, the identity relation is $\cap$-irreducible in $Con(\boldsymbol{A})$, i.e. $\boldsymbol{A}$ is subdirectly irreducible. Thus, $\mathbf{MOD}^*(\mathrm{IPC})_{\mathrm{RSI}} = \{\langle \boldsymbol{A}, \{\overline{1}^{\boldsymbol{A}}\}\rangle \mid \boldsymbol{A} \in \mathbb{HA}_{\mathrm{SI}}\}$. Assume now, in search of a contradiction, that $\to'$ is a weak implication in IPC, $\mathbf{MOD}^\ell(\mathrm{IPC})$ is the class of its linear models and IPC is complete w.r.t. $\mathbf{MOD}^\ell(\mathrm{IPC})$. By Theorem 3.1.8, we have $\{\langle \boldsymbol{A}, \{\overline{1}^{\boldsymbol{A}}\}\rangle \mid \boldsymbol{A} \in \mathbb{HA}_{\mathrm{SI}}\} \subseteq \mathbf{MOD}^\ell(\mathrm{IPC})$. Now, it is sufficient to consider a subdirectly irreducible Heyting algebra where the natural lattice order is not linear (it is well-known that these algebras exist) and it will have two incomparable filters (IPC-filters are known to be the same as lattice filters over the Heyting algebra). Then Lemma 3.1.3 gives the contradiction. □

Combining this example and the previous corollary we obtain:

PROPOSITION 3.1.12. *If* L *is a substructural logic that can be axiomatically extended to* IPC*, then it is not semilinear w.r.t. any principal implication.*

Prominent logics falling under the scope of the previous proposition are the following: SL_X and FL_X for any $X \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. On the other hand, observe that the second proof in Example 3.1.11 can be used in many other weakly implicative finitary logics L: all one needs is to find a member of $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ whose algebra admits two incomparable logical filters. For instance, consider now the variety $\mathbb{V}$ of pointed residuated lattices generated by the symmetric rotation (see this construction e.g. in [46, 63]) of all Heyting algebras. Clearly, its corresponding logic has an involutive negation, that is, it proves $\neg\neg\varphi \to \varphi$. Reasoning exactly in the same way as before, we can show that this logic is not semilinear w.r.t. any principal implication and thus the same holds for any substructural logic whose algebraic semantics contains $\mathbb{V}$. A particular case of that is Girard's linear logic (without exponentials).

At the end of this subsection we present another corollary of Theorem 3.1.8 that shows that semilinearity of implications is preserved under intersections of logics and discuss some of its consequences.

COROLLARY 3.1.13. *The intersection of a family of semilinear logics in the same language is a semilinear logic.*

Proof. Let $\mathcal{I}$ be a family of semilinear logics and $\hat{\mathrm{L}}$ its intersection. We show that $\hat{\mathrm{L}}$ has the LEP. Let T be a theory in $\hat{\mathrm{L}}$ and $\varphi \notin T$, i.e. $T \nvdash_{\hat{\mathrm{L}}} \varphi$. Thus there has to be a logic $\mathrm{L} \in \mathcal{I}$ such that $T \nvdash_{\mathrm{L}} \varphi$, i.e. $\varphi \notin \mathrm{Th}_{\mathrm{L}}(T)$. Thus by the LEP of L there is a linear theory T' in L such that $T' \supseteq \mathrm{Th}_{\mathrm{L}}(T) \supseteq T$ and $\varphi \notin T'$. Since T' is also a theory in $\hat{\mathrm{L}}$, the proof is done. □

On the other hand, the inconsistent logic is trivially semilinear. Thus, the following definition is sound:

DEFINITION 3.1.14 (Logic L^ℓ). *Given a weakly implicative logic* L*, we denote by* L^ℓ *the least semilinear logic extending* L*.*

In the next subsection we will give a method to axiomatize L^ℓ. However, it is very simple to determine a complete semantics for this logic:

PROPOSITION 3.1.15. *Let* L *be a weakly implicative logic. Then* $\mathrm{L}^\ell = \models_{\mathbf{MOD}^\ell(\mathrm{L})}$ *and* $\mathbf{MOD}^\ell(\mathrm{L}^\ell) = \mathbf{MOD}^\ell(\mathrm{L})$*.*

Moreover, finitarity is preserved when taking the least semilinear extension:

PROPOSITION 3.1.16. *If* L *is a finitary weakly implicative logic, then so is* L^ℓ.

Proof. Recall the notion of finitary companion of a logic S, denoted as $\mathcal{FC}(\mathrm{S})$, which is the largest finitary logic contained in S. Thus, since L is finitary, we know that $\mathrm{L} \subseteq \mathcal{FC}(\mathrm{L}^\ell) \subseteq \mathrm{L}^\ell$. If we show that $\mathcal{FC}(\mathrm{L}^\ell)$ is semilinear, we obtain $\mathcal{FC}(\mathrm{L}^\ell) = \mathrm{L}^\ell$ and hence L^ℓ is finitary. Actually, one can easily show, by checking that the SLP is preserved, that semilinearity is preserved in general when taking the finitary companion of a logic. □

Note that the proof of the previous theorem also says that if L is finitary, then L^ℓ is the intersection of all its *finitary* semilinear extensions.

L	Axiom(s) needed to axiomatize L^ℓ
FL	$\{\gamma_1(\varphi \to \psi) \vee \gamma_2(\psi \to \varphi) \mid \gamma_1, \gamma_2$ iterated conjugates$\}$
$\mathrm{FL_e}$	$((\varphi \to \psi) \wedge \overline{1}) \vee ((\psi \to \varphi) \wedge \overline{1})$
$\mathrm{FL_{ew}}$	$(\varphi \to \psi) \vee (\psi \to \varphi)$
$\mathrm{IPC}_\to$	$\{[(\varphi \to \psi) \to (\psi \to \varphi)] \to (\psi \to \varphi), [(\psi \to \varphi) \to (\varphi \to \psi)] \to (\varphi \to \psi)\}$

Table 6. Axiomatization of notable substructural semilinear logics

3.2 Disjunction and semilinearity

In this subsection we consider the relationships between p-disjunctions, semilinear implications, and their related properties. In particular, provided that a couple of simple syntactic conditions are satisfied, we will see that a logic is p-disjunctional iff it is semilinear, we will obtain axiomatizations for L^ℓ logics, and new characterizations of p-disjunctions.

We start by demonstrating that for a p-disjunctional logic we can very easily axiomatize its least semilinear extension (in particular, we show that it is an *axiomatic* extension).

THEOREM 3.2.1 (Axiomatization of L^ℓ). *Let* L *be a finitary p-disjunctional weakly implicative logic. Then* L^ℓ *is the extension of* L *with the axiom(s):*

$$(\mathrm{P}_\nabla) \quad \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla (\psi \to \varphi).$$

Proof. Using Proposition 3.1.15 we know that $\mathrm{L}^\ell = \models_{\mathbf{MOD}^\ell(\mathrm{L})}$. The proof is completed by Theorem 2.7.29; we only need to observe that a matrix $\mathbf{A} \in \mathbf{MOD}^\ell(\mathrm{L})$ iff $\mathbf{A} \models P$, where P is the positive clause $F(\varphi \to \psi) \vee F(\psi \to \varphi)$. □

The axiom(s) (P_∇) is (are) called the *prelinearity axiom(s)*. Observe that these axioms holds in a semilinear logic for an arbitrary p-protodisjunction ∇ (using SLP to $\varphi \to \psi \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla (\psi \to \varphi)$ and $\psi \to \varphi \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla (\psi \to \varphi)$). As we have seen in Subsection 2.7, substructural logics over FL are typical examples of finitary weakly implicative p-disjunctional logics. This allows us to achieve some well-known results (collected in Table 6) as corollaries of the theorem above.

Later in this subsection (Example 3.2.7) we will see alternative axiomatizations of $\mathrm{FL}^\ell_{\mathrm{e}}$ and in Theorem 3.6.5 of Chapter IV we will see a substantially simplified axiomatic system for FL^ℓ. A natural question is how to axiomatize the least semilinear extensions of logics which are not p-disjunctional or where the p-disjunction is unknown. Of course, a first idea is to choose a *suitable* p-protodisjunction ∇ and extend this logic into L^∇ and then proceed by the previous theorem. But it is not so simple: how would we know that $\mathrm{L}^\nabla \subseteq \mathrm{L}^\ell$? In order to overcome this problem, we introduce a pair of consecutions which play a kind of dual rôle to (P_∇):

$$(\mathrm{MP}_\nabla) \quad \varphi \to \psi, \varphi \nabla \psi \vdash_{\mathrm{L}} \psi \qquad \text{and} \qquad \varphi \to \psi, \psi \nabla \varphi \vdash_{\mathrm{L}} \psi.$$

PROPOSITION 3.2.2. (MP_∇) *is satisfied in:*

- *any logic for any* ∇ *satisfying the* PCP*,*
- *any substructural (not necessarily lattice-disjunctive!) logic for* $\nabla = \vee$*.*

Proof. The first claim is simple (from $\varphi, \varphi \to \psi \vdash \psi$ and $\psi, \varphi \to \psi \vdash \psi$). To prove the second one observe that any substructural logic proves: $\varphi \to \psi \vdash \varphi \vee \psi \to \psi$. □

The introduced consecutions (P_∇) and (MP_∇) are indeed natural binding conditions for implication and disjunction, as shown by the next lemma and theorem.

LEMMA 3.2.3. *Let* L *be a weakly implicative logic in* $\mathcal{L}$*,* ∇ *a p-protodisjunction, and* $\boldsymbol{A}$ *an* $\mathcal{L}$*-algebra.*

- *If* L *fulfils* (MP_∇)*, then each linear filter in* $\boldsymbol{A}$ *is* ∇*-prime.*
- *If* L *fulfils* (P_∇)*, then each* ∇*-prime filter in* $\boldsymbol{A}$ *is linear.*

Proof. The first claim: assume that F is linear ($a \to^{\boldsymbol{A}} b \in F$ or $b \to^{\boldsymbol{A}} a \in F$) and $a \nabla^{\boldsymbol{A}} b \subseteq F$. Thus from (MP_∇) we obtain that $b \in F$ or $a \in F$.

The second claim: assume that F is not linear, i.e. there are elements a, b such that $x = a \to^{\boldsymbol{A}} b \notin F$ and $y = b \to^{\boldsymbol{A}} a \notin F$. $x \nabla^{\boldsymbol{A}} y = (a \to^{\boldsymbol{A}} b) \nabla^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} a) \subseteq F$ because L satisfies (P_∇), thus F is not ∇-prime. □

THEOREM 3.2.4 (Interplay of p-disjunctions and semilinearity). *Let* L *be a finitary weakly implicative logic. The following are equivalent:*

- L *is p-disjunctional and satisfies* (P_∇)*.*
- L *is semilinear and satisfies* (MP_∇)*.*

Thus in particular:

1. *Let* L *be a weakly implicative logic satisfying* (P_∇) *and* (MP_∇)*. Then,* L *is semilinear iff* L *is p-disjunctional.*
2. *Let* L *be a p-disjunctional weakly implicative logic. Then,* L *is semilinear iff* L *satisfies* (P_∇)*.*
3. *Let* L *be a semilinear logic and* ∇ *a p-protodisjunction. Then,* L *is p-disjunctional iff* L *satisfies* (MP_∇)*.*

Proof. Top-to-bottom implication: we know that each finitary p-disjunctional logic satisfies (MP_∇) and has the PEP (Theorem 2.7.23) and, since from (P_∇) we know that ∇-prime theories are linear, we obtain the LEP and the proof is done. The second direction is analogous. □

Notice that the first claim of this theorem provides many additional characterizations of semilinearity by means of Theorems 2.7.15 and 2.7.23 in a broad class of finitary logics satisfying (P_∇) and (MP_∇). Similarly, notice that the other two claims reduce, in huge classes of logics, the validity of a meta-rule, SLP or PCP, to the validity of a simple rule, (P_∇) or (MP_∇).

This theorem has two interesting corollaries. First, we can use Corollary 2.7.17 to extend Corollary 3.1.10 from axiomatic *extensions* to axiomatic *expansions*.

COROLLARY 3.2.5. *Let* L_1 *be a p-disjunctional weakly implicative logic and let* L_2 *be an axiomatic expansion which is weakly implicative with the same principal implication. If* L_1 *is semilinear, then so is* L_2.

Second, we can return to our original goal of providing an axiomatization for L^ℓ. Recall that by L^∇ we denote the weakest logic extending L where ∇ is a p-disjunction (see Theorem 2.7.27 for an axiomatization of this logic).

COROLLARY 3.2.6. *Let* L *be a finitary weakly implicative logic and* ∇ *a p-proto-disjunction* satisfying (MP_∇). *Then* L^ℓ *is the extension of* L^∇ *by* (P_∇).

Proof. Since $\mathrm{L}^\nabla + (\mathrm{P}_\nabla)$ is an axiomatic extension of L^∇, ∇ remains a p-disjunction there (by Corollary 2.7.17). Thus, by Theorem 3.2.4, it is a semilinear logic.

Let L' be a finitary semilinear extension of L. Clearly L' satisfies (MP_∇) as well and thus by Theorem 3.2.4 it is a p-disjunctional logic. Thus $\mathrm{L}^\nabla \subseteq \mathrm{L}'$ and, since the Theorem 3.2.4 also tells that L' satisfies (P_∇), the proof is done. □

The restriction of this corollary to p-disjunctional logics (which, of course, satisfy (MP_∇) and $\mathrm{L}^\nabla = \mathrm{L}$) gives an alternative proof of Theorem 3.2.1. As we have seen, substructural logics with $\vee$ in the language form a big class of logics satisfying $(\mathrm{MP}_\vee)$. Let us summarize some consequences of the previous claims for these logics and add some more interesting ones. In the beginning of this subsection we have seen one way to axiomatize L^ℓ: identify a good p-disjunction and add the prelinearity axioms for this p-disjunction. The previous corollary provides an alternative way for substructural logics with $\vee$ in the language which produces less elegant axiomatizations, but can be seen as more robust, because it does not require to identify a p-disjunction in L. We simply extend L into $\mathrm{L}^\vee$ (just by adding the $\vee$-forms of all rules, see Theorem 2.7.29) and then we add prelinearity written using $\vee$.

EXAMPLE 3.2.7. $\mathrm{FL}_\mathrm{e}^\ell$ is the extension of FL_e by the rule $\varphi \vee \psi \vdash (\varphi \wedge \bar{1}) \vee \psi$ and the axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$. The $\vee$-form of *modus ponens*, $\varphi \vee \chi, (\varphi \to \psi) \vee \chi \vdash \psi \vee \chi$, already holds in FL_e, which is not difficult to prove.

LEMMA 3.2.8. *Let* L *be a lattice-disjunctive substructural logic. Then the following are equivalent:*

$(\mathrm{P}_\vee)$ $\vdash_\mathrm{L} (\varphi \to \psi) \vee (\psi \to \varphi)$

$(\mathrm{lin}_\wedge)$ $\vdash_\mathrm{L} (\varphi \wedge \psi \to \chi) \to (\varphi \to \chi) \vee (\psi \to \chi)$

$(\mathrm{lin}_\vee)$ $\vdash_\mathrm{L} (\chi \to \varphi \vee \psi) \to (\chi \to \varphi) \vee (\chi \to \psi)$.

Proof. We prove the equivalence of the first two claims; the equivalence of the first and the third ones is prove analogously. First recall that $\varphi \to \psi \vdash_\mathrm{L} \varphi \to \varphi \wedge \psi$ and so $\varphi \to \psi \vdash_\mathrm{L} (\varphi \wedge \psi \to \chi) \to (\varphi \to \chi)$. The proof is completed by $(\vee 1)$ and the PCP. The other direction: from $\varphi \wedge \psi \to \varphi \wedge \psi$ we obtain $(\varphi \to \varphi \wedge \psi) \vee (\psi \to \varphi \wedge \psi)$. The rest is simple. □

PROPOSITION 3.2.9. *Let* L *be a finitary substructural logic with* $\vee$ *in its language. Then:*

- L *is semilinear iff it is lattice-disjunctive and satisfies* $(\mathrm{P}_\vee)$.
- L^ℓ *is the extension of* $\mathrm{L}^\vee$ *by any of these axioms:* $(\mathrm{P}_\vee)$, $(\mathrm{lin}_\wedge)$, *or* $(\mathrm{lin}_\vee)$.

The next proposition is a generalization of Example 2.7.10.

PROPOSITION 3.2.10. *Let* L *be a finitary Rasiowa-implicative substructural semilinear logic. Then the set* $\nabla = \{(p \to q) \to q, (q \to p) \to p\}$ *is a disjunction.*

Proof. We can easily show that ∇ is a protodisjunction because $p \vdash_{\mathrm{L}} (p \to q) \to q$ ($(\mathrm{P_{SL}}4)$ of Proposition 2.5.5) and $q \vdash_{\mathrm{L}} (p \to q) \to q$ (W).

Next we show that ∇ satisfies the PCP. Assume that $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash \chi$. Thus clearly $\Gamma, \varphi \to \psi, \varphi \nabla \psi \vdash_{\mathrm{L}} \psi$ and so $\Gamma, \varphi \to \psi, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$; analogously for $\psi \to \varphi$. The SLP completes the proof. $\square$

Notice that if the logic from the previous proposition would contain $\vee$ in the language, we would obtain $(p \to q) \to q, (q \to p) \to p \dashv\vdash p \vee q$. A question is whether we could internalize this equivalence. First observe that $\nabla' = \{(p \rightsquigarrow q) \to q, (q \rightsquigarrow p) \to p\}$ would be a disjunction as well. Then we can prove:

PROPOSITION 3.2.11. *Let* L *be a Rasiowa-implicative substructural semilinear logic. Then:*

$$\vdash_{\mathrm{L}} \varphi \vee \psi \leftrightarrow [(\varphi \rightsquigarrow \psi) \to \psi] \wedge [(\psi \rightsquigarrow \varphi) \to \varphi]$$
$$\vdash_{\mathrm{L}} \varphi \wedge \psi \leftrightarrow [\varphi \mathbin{\&} (\varphi \to \psi)] \vee [\psi \mathbin{\&} (\psi \to \varphi)].$$

Furthermore the logic L *extends* $\mathrm{SL_e}$ *iff*

$$\vdash_{\mathrm{L}} \varphi \vee \psi \leftrightarrow [(\varphi \to \psi) \to \psi] \wedge [(\psi \to \varphi) \to \varphi].$$

Proof. The first claim: Left-to-right direction is simple. The converse one is based on a simple observation: $\varphi \to \psi \vdash_{\mathrm{L}} [(\varphi \rightsquigarrow \psi) \to \psi] \to \psi$, a consequence of (symm) and $(\mathrm{P_{SL}}4)$ of Proposition 2.5.5.

The second claim: clearly $\varphi \mathbin{\&} (\varphi \to \psi) \to \psi$ and $\psi \mathbin{\&} (\psi \to \varphi) \to \psi$, thus $[\varphi \& (\varphi \to \psi)] \vee [\psi \& (\psi \to \varphi)] \to \psi$; the rest is simple. The converse direction: assume that $\varphi \to \psi$. Thus $\varphi \to (\varphi \& (\varphi \to \psi))$ and so $\varphi \wedge \psi \leftrightarrow (\varphi \& (\varphi \to \psi)) \vee (\psi \& (\psi \to \varphi))$. The rest easily follows from the SLP.

One direction of the third claim trivially follows from the first claim. To prove the converse one, observe that the assumption $\chi \vee \psi \leftrightarrow [(\chi \to \psi) \to \psi] \wedge [(\psi \to \chi) \to \chi]$ entails: $\psi \to ((\psi \to \chi) \to \chi)$. Then we obtain $[((\psi \to \chi) \to \chi) \to (\varphi \to \chi)] \to [\psi \to (\varphi \to \chi)]$ (using (Sf)). The proof is completed by another instance of (Sf), namely: $\varphi \to (\psi \to \chi) \vdash ((\psi \to \chi) \to \chi) \to (\varphi \to \chi)$. $\square$

The following proposition is a straightforward corollary of the fact that semilinear logics are complete with respect to linearly ordered matrices, whose algebraic reducts are clearly distributive lattices.

PROPOSITION 3.2.12. *Let* L *be a substructural semilinear logic with* $\wedge$ *and* $\vee$ *in its language and* $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$. *Then the* $\{\wedge, \vee\}$*-reduct of* $\boldsymbol{A}$ *is a distributive lattice.*

We end this subsection by using the characterization theorems 2.7.15 and 2.7.23, and Remark 2.7.24, to demonstrate many equivalent conditions for a connective ∇ to be a disjunction. These theorems were formulated for finitary logics; now we show that in the case of semilinear logics we can drop this condition.

THEOREM 3.2.13 (Characterizations of (p-)disjunctions in (possibly infinitary) semilinear logics). *Let* L *be a* semilinear *logic with a presentation* $\mathcal{AS}$ *and p-protodisjunction* ∇. *Then the following are equivalent:*

1. L *has the* PCP *w.r.t.* ∇.
2. L *proves* (MP_∇).
3. L *has the* PEP *w.r.t.* ∇.
4. L *has the* sPCP *w.r.t.* ∇.
5. ∇ *satisfies* (C_∇), (I_∇), *and* $R^\nabla \subseteq \mathrm{L}$ *for each* $R \in \mathrm{L}$.
6. ∇ *satisfies* (C_∇), (I_∇), *and* $R^\nabla \subseteq \mathrm{L}$ *for each* $R \in \mathcal{AS}$.
7. $\mathrm{Fi}(Y) \cap \mathrm{Fi}(Z) = \mathrm{Fi}(Y \nabla^{\boldsymbol{A}} Z)$ *for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and each* $Y, Z \subseteq A$.
8. $\mathrm{Fi}(X, x) \cap \mathrm{Fi}(X, y) = \mathrm{Fi}(X, x \nabla^{\boldsymbol{A}} y)$ *for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and each* $X \cup \{x, y\} \subseteq A$.

Moreover, if ∇ *is parameter-free, we can add:*

9. *For every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and every* $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, *F is finitely* $\cap$*-irreducible iff it is* ∇*-prime.*
10. *For every* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$, $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ *iff F is* ∇*-prime.*
11. *For every* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ *if, and only if,* $\Gamma \models_{\mathbf{MOD}^*_p(\mathrm{L})} \varphi$.

Proof. Recall that L satisfies (P_∇). The implications 1→2 and 2→3 are trivial (recall that each semilinear logic has the LEP).

3→4: assume that $\Gamma, \Phi \nabla \Psi \nvdash_{\mathrm{L}} \chi$, then using the PEP there is a ∇-prime theory $T \supseteq \Gamma \cup (\Phi \nabla \Psi)$ such that $T \nvdash_{\mathrm{L}} \chi$. If $\Phi \subseteq T$ we know that $\Gamma, \Phi \nvdash_{\mathrm{L}} \chi$. Assume otherwise, i.e. there is $\psi \in \Phi \setminus T$. Then, since $\{\psi\} \nabla \Psi \subseteq T$ and T is ∇-prime, we obtain $\Psi \subseteq T$ and so $\Gamma, \Psi \nvdash_{\mathrm{L}} \chi$.

From Remarks 2.7.16 and 2.7.19 we know that conditions $4, 5, 6$, and 7 are equivalent and because implications 7→8 and 8→1 are trivial, we have established the equivalence of the first eight claims.

1→9: on one hand, we know that linear and ∇-prime filters coincide; on the other hand, from Proposition 3.1.4, we know that linear filters are finitely $\cap$-irreducible. From 8 we have that $\mathrm{Fi}(F, x \to^{\boldsymbol{A}} y) \cap \mathrm{Fi}(F, y \to^{\boldsymbol{A}} x) = \mathrm{Fi}(F, (x \to^{\boldsymbol{A}} y) \nabla^{\boldsymbol{A}} (y \to^{\boldsymbol{A}} x)) = F$ and so we obtain that finitely $\cap$-irreducible filters are linear.

9→10: direct consequence of Theorem 2.3.14.

$10{\to}11$: we know that L is complete w.r.t. $\mathbf{MOD}^{\ell}(\mathrm{L})$, thus by Proposition 3.1.4 it is also complete w.r.t. $\mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}}$, and the claim follows from 10.

The final implication $11{\to}1$ is proved in the same way as the corresponding implication in Theorem 2.7.23. □

COROLLARY 3.2.14. *Let* L_1 *be a semilinear logic with a p-protodisjunction which satisfies* (MP_∇) *and* L_2 *a* finitary *weakly implicative expansion of* L_1 *by a set of consecutions* $\mathcal{C}$*. Then,* L_2 *is semilinear if, and only if,* $R^\nabla \subseteq \mathrm{L}_2$ *for each* $R \in \mathcal{C}$*.*

Proof. Observe from the assumption we obtain that L_i satisfies (MP_∇) and (P_∇) for $i = 1, 2$ and $R^\nabla \subseteq \mathrm{L}_1$ for each $R \in \mathrm{L}_1$.

Because L_2 is semilinear we can use Theorem 3.2.13 to complete the proof of one direction. The converse direction: we know that L_2 has an axiomatic system closed under the formation of ∇-forms of its rules. Therefore, since it is finitary, we know that it is p-disjunctional (Theorem 2.7.15) and so Theorem 3.2.4 completes the proof. □

EXAMPLE 3.2.15. We show that the logic $\mathrm{MTL}_\triangle$ (introduced in Chapter I) is semilinear. Recall that this logic is an expansion of MTL (i.e. the logic $\mathrm{FL}^{\ell}_{\mathrm{ew}}$) by a new unary connective $\triangle$, by adding the deduction rule $\varphi \vdash_{\mathrm{MTL}_\triangle} \triangle\varphi$ and some additional axioms (see Chapter I) which ensure that it remains weakly implicative with the same principal implication. Clearly the logics MTL and $\mathrm{MTL}_\triangle$ satisfy the conditions of the previous corollary, so if we show that $\varphi \vee p \vdash_{\mathrm{MTL}_\triangle} \triangle\varphi \vee p$ the proof is done.

Using the deduction rule we obtain $\varphi \vee p \vdash_{\mathrm{MTL}_\triangle} \triangle(\varphi \vee p)$ and then it is enough to recall that $\triangle(\varphi \vee \psi) \to \triangle\varphi \vee \triangle\psi$ and $\triangle\psi \to \psi$ are among the axioms of $\mathrm{MTL}_\triangle$.

3.3 Strengthening completeness: densely ordered chains

We have proposed semilinear logics as a useful mathematical notion to encompass and study most examples of fuzzy logics by characterizing them as the logics of linearly ordered matrices. However, a genuine item in the agenda of fuzzy logics is that of looking for finer complete semantics based on some particular kind of linearly ordered models such as standard models based on the real unit interval, or models defined on the rational unit interval, or models over finite chains. In this section we focus on the semantics of densely ordered linear matrices, a common feature of both real- and rational-valued ones. We will characterize completeness with respect to this semantics by means of a special kind of filter and some meta-rules, in a similar fashion to what we have already done for disjunctions and semilinear implications.

DEFINITION 3.3.1 (Dense filter). *Let* L *be a weakly implicative logic and* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ *an* L*-matrix. Then* F *is called a* dense *filter if* F *is linear and for every* $a, b \in A$ *such that*[14] $a <_{\mathbf{A}} b$ *there is* $z \in A$ *such that* $a <_{\mathbf{A}} z$ *and* $z <_{\mathbf{A}} b$*.*

A matrix $\mathbf{A}$ *is called a* dense linear matrix *if it is reduced and* F *is dense (equivalently: if* $\leq_{\mathbf{A}}$ *is a dense order). The class of all dense linear* L*-models is denoted as* $\mathbf{MOD}^{\delta}(\mathrm{L})$*.*

[14]By $a <_{\mathbf{A}} b$ we understand $a \leq_{\mathbf{A}} b$ and $b \not\leq_{\mathbf{A}} a$; note that we are not assuming that $\mathbf{A}$ is reduced, so this convention is not trivial.

The *Dense Extension Property* DEP will be defined analogously as the LEP and the PEP but with some non-trivial changes. Like in the case of disjunctions, where we characterize (in finitary logics) the defining meta-rule PCP by some suitable filter extension principle, we start with the meta-rule DP which was already introduced in the literature. Thus, our goal is to provide a corresponding filter extension principle. The problem is that DP is not *structural* because it refers to an *unused* propositional variable. That is the reason why we will be forced to formulate the DEP only in Lindenbaum matrices, and not for theories but for some particular sets of formulae. These definitions will still allow us to obtain a nice interplay between a filter extension principle, a completeness property, and a logical meta-rule, as in the previous cases.

DEFINITION 3.3.2 (Density Property). *Let* L *be a weakly implicative logic and* ∇ *a p-protodisjunction. We say that* L *has the* Density Property DP *with respect to* ∇ *if for any set of formulae* $\Gamma \cup \{\varphi, \psi, \chi\}$ *and any variable* p *not occurring in* $\Gamma \cup \{\varphi, \psi, \chi\}$ *the following holds: if* $\Gamma \vdash_{\mathrm{L}} (\varphi \to p) \nabla (p \to \psi) \nabla \chi$*, then* $\Gamma \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla \chi$*.*

DEFINITION 3.3.3 (Dense Extension Property). *Let* L *be a weakly implicative logic. We say that* L *has the* Dense Extension Property DEP *if every set of formulae* Γ *such that* $\Gamma \nvdash_{\mathrm{L}} \varphi$ *and there are infinitely many variables not occurring in* Γ *can be extended into a dense theory* $T \supseteq \Gamma$ *such that* $T \nvdash_{\mathrm{L}} \varphi$*.*

In order to prove the characterization of dense completeness in terms of the DEP, we need the following technical lemma.

LEMMA 3.3.4. *Let* L *be a weakly implicative logic,* $\mathbf{A} \in \mathbf{MOD}^{\delta}(\mathrm{L})$*,* T *a theory, and* φ *a formula. If* $T \not\models_{\mathbf{A}} \varphi$*, then there is a countable submatrix* $\mathbf{A}'$ *of* $\mathbf{A}$ *such that* $\mathbf{A}' \in \mathbf{MOD}^{\delta}(\mathrm{L})$ *and* $T \not\models_{\mathbf{A}'} \varphi$*.*

Proof. Clearly $\mathbf{A}$ is non-trivial and so it is infinite (because its matrix order $\leq$ is dense). Let e be any evaluation witnessing $T \not\models_{\mathbf{A}} \varphi$; we define K as the subset of A containing valuations assigned to all subformulae of formulae from $T \cup \{\varphi\}$ by e. Clearly K is countable. We define two sequences K_i of countable subsets of A and $\mathbf{A}_i$ of submatrices of $\mathbf{A}$ as: $K_1 = K$ and for $i > 0$:

- $\mathbf{A}_i$ is the submatrix generated by K_i. (Clearly each $\mathbf{A}_i$ is countable.)
- K_{i+1} is any countable dense subset of A containing A_i.[15]

Clearly $\mathbf{A}_i$ is a directed family of reduced matrices and so their union $\mathbf{A}'$ is in $\mathbf{MOD}^{*}(\mathrm{L})$ (see [24, Theorem 0.7.2]). Obviously $\mathbf{A}'$ is a countable submatrix of $\mathbf{A}$ such that $\mathbf{A}' \in \mathbf{MOD}^{\delta}(\mathrm{L})$ (because of the construction) and $T \not\models_{\mathbf{A}'} \varphi$ (the evaluation e does the job because $K \subseteq A'$). □

THEOREM 3.3.5 (Characterization of dense completeness). *Let* L *be a weakly implicative logic. Then,* $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\delta}(\mathrm{L})}$ *if, and only if,* L *has the* DEP.

[15]Consider any two elements $a, b \in A_i$ such that $a < b$ and there is no element of A_i between a and b. There has to be a set $X_{a,b} \subseteq A \cap [a, b]$, $X_{a,b}$ isomorphic to Q. Then we construct A_{i+1} with the desired properties simply by adding all such sets to A_i.

Proof. Right-to-left: we repeat the usual completeness proof via constructing the appropriate Lindenbaum–Tarski matrix with an interesting twist to overcome the restrictions of the DEP.

Let us consider a set of formulae $\Gamma \cup \{\varphi\}$ such that $\Gamma \nvdash_{\mathrm{L}} \varphi$. Let us enumerate the propositional variables and define substitutions σ and σ' by setting: $\sigma(v_i) = v_{2i}$, $\sigma'(v_{2i}) = v_i$, and $\sigma'(v_{2i+1}) = v_i$ for each $i \geq 0$. Observe that $\sigma'\sigma\psi = \psi$ for any formula ψ. Thus also $\sigma[\Gamma] \nvdash_{\mathrm{L}} \sigma\varphi$ (otherwise, by structurality, $\sigma'\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma'\sigma\varphi$, i.e. $\Gamma \vdash_{\mathrm{L}} \varphi$—a contradiction). Notice that there are infinitely many variables not occurring in $\sigma[\Gamma]$ and so we can use the DEP to obtain a dense theory T such that $T \supseteq \sigma[\Gamma]$ and $T \nvdash_{\mathrm{L}} \sigma\varphi$. Take the matrix $\mathbf{A} = \mathbf{LindT}_T = \langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega_{\mathrm{L}}(T), [T]\rangle$, observe that $\mathbf{A} \in \mathbf{MOD}^{\delta}(\mathrm{L})$, and consider the $\mathbf{A}$-evaluation $e(\psi) = [\psi]_T$. We know that $e[T] \subseteq [T]$ and $e(\sigma\varphi) \notin [T]$.

Let us now consider the $\mathbf{A}$-evaluation $e'(\psi) = e(\sigma\psi)$ and observe that $e'(\varphi) = e(\sigma\varphi) \notin [T]$. As $\sigma[\Gamma] \subseteq T$, we obtain that $e'[\Gamma] = e[\sigma[\Gamma]] \subseteq e[T] \subseteq [T]$. Thus, we obtain $T \not\models_{\mathbf{A}} \varphi$.

Left-to-right: consider a set Γ of formulae with infinitely many unused variables and a formula δ such that $\Gamma \nvdash_{\mathrm{L}} \delta$. We can use our assumption to obtain a dense linear L-matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ and an $\mathbf{A}$-evaluation e such that $e[\Gamma] \subseteq F$ and $e(\delta) \notin F$. Without loss of generality we can assume that $\mathbf{A}$ is countable (due to previous lemma). Let us consider, for any $a \in A$, a variable v_a not occurring in $\Gamma \cup \{\delta\}$ (such variables exist). Further consider an $\mathbf{A}$-evaluation e' such that $e'(p) = e(p)$ for variables in $\Gamma \cup \{\delta\}$ and $e'(v_a) = a$ for $a \in A$.

Consider the set of formulae $T = \{\varphi \mid e'(\varphi) \in F\}$. Clearly T is a theory, $T \supseteq \Gamma$, and $\delta \notin T$; it remains to be shown that T is dense in L. Linearity is simple (for each φ and ψ, clearly $e'(\varphi) \to^{\mathbf{A}} e'(\psi) \in F$ or $e'(\psi) \to^{\mathbf{A}} e'(\varphi) \in F$). Observe that $\varphi <_{\langle \mathrm{Fm},T\rangle} \psi$ iff $e'(\varphi) <_{\mathbf{A}} e'(\psi)$. In this case, since $\mathbf{A}$ is dense, there is $a \in A$ such that $e'(\varphi) <_{\mathbf{A}} a = e'(v_a) <_{\mathbf{A}} e'(\psi)$. Thus $\varphi <_{\langle \mathrm{Fm},T\rangle} v_a$ and $v_a <_{\langle \mathrm{Fm},T\rangle} \psi$. □

Combining this result with the characterization of semilinearity in terms of LEP in Theorem 3.1.8 we obtain:

COROLLARY 3.3.6. *Let* L *be a weakly implicative logic. Then, if* L *has the* DEP *then it has the* LEP.

We consider now the relation of the DEP with the Density Property DP.

LEMMA 3.3.7. *Let* L *be a weakly implicative logic with a p-protodisjunction* ∇ *satisfying the* (MP_{∇}). *Then, the* DEP *implies the* DP.

Proof. Assume that $\Gamma \nvdash_{\mathrm{L}} (\varphi \to \psi) \nabla \chi$ and $\Gamma \vdash_{\mathrm{L}} (\varphi \to p) \nabla (p \to \psi) \nabla \chi$ for some variable p not occurring in $\Gamma, \varphi, \psi, \chi$. From the first assumption we know that there is a formula $\delta \in ((\varphi \to \psi) \nabla \chi)$ such that $\Gamma \nvdash_{\mathrm{L}} \delta$. Thus there is a dense linear L-matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ and an $\mathbf{A}$-evaluation e such that $e[\Gamma] \subseteq F$ and $e(\delta) \notin F$. Clearly $e(\varphi) \not\leq e(\psi)$ (otherwise $e(\varphi \to \psi) \in F$ and so $e(\delta) \in F$) and so $e(\psi) < e(\varphi)$ (because $\mathbf{A}$ is linear). Since $\mathbf{A}$ is dense, there is an element $e(\psi) < a < e(\varphi)$. Take evaluation $e'(v) = a$ for $v = p$ and $e(v)$ otherwise (clearly $e'[\Gamma] \subseteq F$). Then for the elements $\nu_1 = e'(\varphi \to p)$ and $\nu_2 = e'(p \to \psi)$ we have $e'(\nu_i) \notin F$. Notice that also $e'(\chi) \notin F$

(otherwise $e(\delta) \in F$). Using Lemma 3.2.3 we know that F is also a ∇-prime filter and so $e'(\nu_1) \nabla e'(\nu_2) \nabla e'(\chi) \not\subseteq F$. Therefore, $\Gamma \nvdash_{\mathrm{L}} (\varphi \to p) \nabla (p \to \psi) \nabla \chi$—a contradiction. □

For finitary logics, and in the absence of parameters in ∇, we can prove the equivalence of the DEP and the DP, and thus give the main result of this subsection: the syntactical characterization of completeness with respect to dense linear models by means of a meta-rule.

THEOREM 3.3.8 (Characterization of dense completeness). *Let* L *be a finitary semilinear disjunctional logic. Then the following are equivalent:*

1. $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\delta}(\mathrm{L})}$.
2. L *has the* DEP.
3. L *has the* DP.

Proof. It is enough to prove that 3 implies 2. Consider a set of formulae Γ such that $\Gamma \nvdash_{\mathrm{L}} \varphi$ and there are infinitely many variables not occurring in Γ. Let us enumerate all pairs of formulae. We introduce two sequences of sets of formulae Γ_i and A_i such that $\Gamma_i \nvdash_{\mathrm{L}} A_i$. We start with $\Gamma_0 = \Gamma$ and $A_0 = \{\varphi\}$. Given any $i > 0$ consider the following cases:

- If $\Gamma_i, \varphi_i \to \psi_i \nvdash_{\mathrm{L}} A_i$, then we define $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_i \to \psi_i\}$ and $A_{i+1} = A_i$.
- If $\Gamma_i, \varphi_i \to \psi_i \vdash_{\mathrm{L}} A_i$, then we define $\Gamma_{i+1} = \Gamma_i \cup \{\psi_i \to \varphi_i\}$ and $A_{i+1} = A_i \nabla (\varphi_i \to p) \nabla (p \to \psi_i)$ for some variable p not occurring in $\Gamma_i \cup A_i \cup \{\varphi_i, \psi_i\}$ (since there are infinitely many variables not occurring in Γ, we can find in each step some unused one; notice that this would be no longer true if ∇ would be parameterized).

Now we prove by induction that $\Gamma_i \nvdash_{\mathrm{L}} A_i$ for every i. When $i = 0$ it is true by assumption. For the induction step, if we have proceeded by the first case, clearly $\Gamma_{i+1} \nvdash_{\mathrm{L}} A_{i+1}$. Otherwise, assume, by the way of contradiction, that $\Gamma_{i+1} \vdash_{\mathrm{L}} A_{i+1}$, i.e. that $\Gamma_i, \psi_i \to \varphi_i \vdash_{\mathrm{L}} A_{i+1}$. As clearly also $\Gamma_i, \varphi_i \to \psi_i \vdash_{\mathrm{L}} A_{i+1}$ (using the assumption of the second case and properties of protodisjunction) we can use the SLP to obtain $\Gamma_i \vdash_{\mathrm{L}} A_{i+1}$. Thus we also have $\Gamma_i \vdash_{\mathrm{L}} A_i \nabla (\varphi_i \to \psi_i)$ using DP. Finally observe that from $\Gamma_i, A_i \vdash_{\mathrm{L}} A_i$ and $\Gamma_i, \varphi_i \to \psi_i \vdash_{\mathrm{L}} A_i$ we can obtain $\Gamma_i, A_i \nabla (\varphi_i \to \psi_i) \vdash_{\mathrm{L}} A_i$ via the syntactical characterization of p-disjunctions in Theorem 2.7.15. Putting this together we obtain $\Gamma_i \vdash_{\mathrm{L}} A_i$—a contradiction with the induction hypotheses.

Define T as the L-theory generated by the union of all Γ_i's. First observe that $T \nvdash_{\mathrm{L}} A_i$ for each i (otherwise by finitarity there would be some j such that $\Gamma_j \vdash_{\mathrm{L}} A_i$ and so clearly $\Gamma_{\max\{i,j\}} \vdash_{\mathrm{L}} A_{\max\{i,j\}}$—a contradiction).

Thus $T \nvdash_{\mathrm{L}} \varphi$ and from the construction it follows that T is linear. Now assume that $T \nvdash_{\mathrm{L}} \varphi_i \to \psi_i$, then we had to proceed via the second case in the construction (otherwise already $\Gamma_{i+1} \vdash_{\mathrm{L}} \varphi_i \to \psi_i$) thus also $T \nvdash_{\mathrm{L}} \varphi_i \to p$ and $T \nvdash_{\mathrm{L}} p \to \psi_i$ (because otherwise $T \vdash_{\mathrm{L}} A_{i+1}$). □

Note that the premises of the previous theorem are fulfilled by any substructural semilinear logic with $\vee$ in its language. Analogously to the case of L^{ℓ} and L^{∇}, we consider now the problem of finding the weakest extension of a logic enjoying completeness with respect to dense linear models.

LEMMA 3.3.9. *Let $\mathcal{I}$ be a family of weakly implicative logics in the same language and $\hat{\mathrm{L}}$ its intersection. If every logic of $\mathcal{I}$ has the* DEP*, then so has $\hat{\mathrm{L}}$.*

Proof. Let Γ be a set of formulae with infinitely many variables not occurring in it and φ a formula such that $\Gamma \nvdash_{\hat{\mathrm{L}}} \varphi$. Thus there has to be a logic $\mathrm{L} \in \mathcal{I}$ such that $\Gamma \nvdash_{\mathrm{L}} \varphi$. Thus by the DEP of L there is a dense L-theory $T \supseteq \Gamma$ and $\varphi \notin T$. Since clearly T is also an $\hat{\mathrm{L}}$-theory, the proof is done. □

This, together with the fact that any weakly implicative logic has at least one extension which is complete with respect to its dense linear models (namely the inconsistent logic), gives the following result:

THEOREM 3.3.10 (The logic L^{δ} and its semantics). *Let* L *be a weakly implicative logic. Then there is the weakest logic extending* L *which is complete w.r.t. its dense linear models. Let us denote this logic as L^{δ}.*

The proofs of the following two results run parallel to those of their analogues in previous sections.

PROPOSITION 3.3.11. *Let* L *be a weakly implicative logic. Then* $\vdash_{\mathrm{L}^{\delta}} = \models_{\mathbf{MOD}^{\delta}(\mathrm{L})}$ *and* $\mathbf{MOD}^{\delta}(\mathrm{L}^{\delta}) = \mathbf{MOD}^{\delta}(\mathrm{L})$.

PROPOSITION 3.3.12. *Let* L *be a finitary weakly implicative logic. Then L^{δ} is finitary.*

THEOREM 3.3.13 (L^{δ} in finitary logics). *Let* L *be a finitary semilinear disjunctional logic. Then, L^{δ} is equal to the intersection of all its extensions satisfying the* DP *iff this intersection is finitary and semilinear.*

Proof. Let us denote that intersection as $\hat{\mathrm{L}}$. One direction is a simple consequence of Corollary 3.3.6. The converse direction: clearly $\hat{\mathrm{L}}$ enjoys the DP and is disjunctional (due to Theorem 3.2.4). Thus by Theorem 3.3.8 it has the DEP and so $\mathrm{L}^{\delta} \subseteq \hat{\mathrm{L}}$. Lemma 3.3.7 tells us that each extension of L with the DEP has also the DP, thus $\hat{\mathrm{L}} \subseteq \mathrm{L}^{\delta}$. □

These results simplify and give a new insight into an approach used in the fuzzy logic literature to prove dense completeness (presented in Section 4.2 of Chapter III). Indeed, this approach starts from a suitable proof-theoretic description of a logic L, which then is extended into a proof-system for the intersection of all extensions of L satisfying the DP just by adding DP as a rule (in the proof-theoretic terms, not as we understand rules here). This rule is then shown to be eliminable (using analogs of the well-known cut-elimination techniques). Thus we can conclude by the previous theorem that $\mathrm{L} = \mathrm{L}^{\delta}$ (or equality of their finite derivations, or equality of their sets of theorems, depending on how the elimination of DP was done), and hence the original logic is complete w.r.t. its dense linear models (of course, our general theory is not helpful in this last step, because here one needs to use specific properties of the logic in question).

3.4 Strengthening completeness: arbitrary classes of chains

We move now to arbitrary semantics of linearly ordered matrices. So far we have only considered completeness as the equality of two logics, i.e. two structural consequence relations, one syntactically presented, the other semantically defined. However, it is usual in the literature to study weaker notions of completeness: equality of the sets of theorems and equality of finite sequents. We formalize this by introducing three types of completeness properties according to the cardinality of the sets of premises in the derivations. Later we will obtain characterizations for these properties and relations between them.

DEFINITION 3.4.1 (Completeness properties). *Let* L *be a weakly implicative semilinear logic and* $\mathbb{K} \subseteq \mathbf{MOD}^{\ell}(\mathrm{L})$. *We say that* L *has the property of:*

- Strong $\mathbb{K}$-completeness, $\mathrm{S}\mathbb{K}\mathrm{C}$ *for short, when for every set of formulae* $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_{\mathrm{L}} \varphi$ *if, and only if,* $\Gamma \models_{\mathbb{K}} \varphi$.
- Finite strong $\mathbb{K}$-completeness, $\mathrm{FS}\mathbb{K}\mathrm{C}$ *for short, when for every* finite *set of formulae* $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_{\mathrm{L}} \varphi$ *if, and only if,* $\Gamma \models_{\mathbb{K}} \varphi$.
- $\mathbb{K}$-completeness, $\mathbb{K}\mathrm{C}$ *for short, when for every formula* φ: $\vdash_{\mathrm{L}} \varphi$ *if, and only if,* $\models_{\mathbb{K}} \varphi$.

Of course, the $\mathrm{S}\mathbb{K}\mathrm{C}$ implies the $\mathrm{FS}\mathbb{K}\mathrm{C}$, and the $\mathrm{FS}\mathbb{K}\mathrm{C}$ implies the $\mathbb{K}\mathrm{C}$. Our next aim is to prove characterizations of these properties that will allow, for particular choices of semilinear logics and classes of linearly ordered models, either to show or to falsify the corresponding completeness properties. Although a more general approach is possible, here such characterizations will be obtained by using the algebraizability condition of the logics.

In reduced matrices of algebraically implicative logics, as have seen, the filters are equationally definable, and so each reduced matrix is uniquely determined by its algebraic reduct. Thus, by a slight abuse of language, we will use the symbol $\models_{\mathbb{K}}$ (for a class of L-chains $\mathbb{K}$) not only for the equational consequence but also for the semantical consequence on the set of formulae given by the corresponding class of matrices. The confusion cannot happen as it is always clear from the context whether we speak about formulae or equations (see, e.g. the next theorem).

CONVENTION 3.4.2. *From now on, until the end of this subsection, we assume that* L *is an algebraically implicative semilinear logic with a principal implication* $\to$ *and* $\mathbb{K}$ *a class of* L*-chains.*

First, we can obtain the following equivalent algebraic properties for each type of completeness:

THEOREM 3.4.3 (Algebraic characterization of completeness properties).

1. L *has the* $\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{V}(\mathbf{ALG}^*(\mathrm{L})) = \mathbf{V}(\mathbb{K})$.
2. L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{Q}(\mathbf{ALG}^*(\mathrm{L})) = \mathbf{Q}(\mathbb{K})$.
3. L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{ALG}^*(\mathrm{L}) = \mathbf{ISP}_{\sigma\text{-}f}(\mathbb{K})$.

Proof. Let us prove the first claim. For the left-to-right direction take an arbitrary equation $\varphi \approx \psi$. Then: $\models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ iff $\vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$ iff $\models_{\mathbb{K}} \varphi \leftrightarrow \psi$ iff $\models_{\mathbb{K}} \varphi \approx \psi$. Therefore $\mathbf{ALG}^*(\mathrm{L})$ and $\mathbb{K}$ satisfy the same equations and hence they generate the same variety. The other direction is straightforward.

The remaining points are proved analogously using that quasivarieties are characterized by quasiequations, and the classes closed under the operator $\mathbf{ISP}_{\sigma\text{-}f}$ are characterized by generalized quasiequations with countably many premises (we can omit this operator on the left side of the equation because that $\mathbf{ALG}^*(\mathrm{L})$ is closed under $\mathbf{ISP}_{\sigma\text{-}f}$, see Proposition 2.3.13). □

Observe that in the first claim of the previous theorem we could have written only $\mathbf{H}(\mathbf{ALG}^*(\mathrm{L}))$ instead of $\mathbf{V}(\mathbf{ALG}^*(\mathrm{L}))$ and if $\mathbf{ALG}^*(\mathrm{L})$ is a quasivariety (e.g. if L is finitary) we could write just $\mathbf{ALG}^*(\mathrm{L})$ instead of $\mathbf{Q}(\mathbf{ALG}^*(\mathrm{L}))$ in the second claim.

Other useful characterizations of completeness properties are obtained in terms of embeddability. To present them, we need first one definition and one lemma.

DEFINITION 3.4.4 (Directed set of formulae). *A set of formulae Ψ is* directed *if for each $\varphi, \psi \in \Psi$ there is $\chi \in \Psi$ such that both $\varphi \to \chi$ and $\psi \to \chi$ are provable in* L *(we call χ an* upper bound *of φ and ψ).*

LEMMA 3.4.5. *Assume that* L *is finitary and has the* S$\mathbb{K}$C. *Then for every set of formulae Γ and every directed set of formulae Ψ the following are equivalent:*

- $\Gamma \nvdash_{\mathrm{L}} \psi$ *for each* $\psi \in \Psi$.
- *There is a matrix $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^{\ell}(\mathrm{L})$ with $\boldsymbol{A} \in \mathbb{K}$ and an $\boldsymbol{A}$-evaluation e such that $e[\Gamma] \subseteq F$ and $e[\Psi] \cap F = \emptyset$.*

Proof. One direction is obvious. For the other one, first assume that there exists a propositional variable v not appearing in $\Gamma \cup \Psi$. Define the set $\Gamma' = \Gamma \cup \{\psi \to v \mid \psi \in \Psi\}$. We show that $\Gamma' \nvdash_{\mathrm{L}} v$ by the way of contradiction. Assume that $\Gamma' \vdash_{\mathrm{L}} v$. Thus there are finite sets $\hat{\Gamma} \subseteq \Gamma$ and $\hat{\Psi} \subseteq \Psi$ such that $\hat{\Gamma} \cup \{\psi \to v \mid \psi \in \hat{\Psi}\} \vdash_{\mathrm{L}} v$. Let $\delta \in \Psi$ denote an upper bound of the formulae in $\hat{\Psi}$. Since $\Gamma \nvdash_{\mathrm{L}} \delta$, we know that there is a matrix $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ and an evaluation e such that $e[\Gamma] \subseteq F$ and $e(\delta) \notin F$. We define the evaluation e' as $e'(p) = e(p)$ for each $p \neq v$ and $e'(v) = e(\delta)$. Clearly, $e'[\hat{\Gamma} \cup \{\psi \to v \mid \psi \in \hat{\Psi}\}] \subseteq F$ and $e'(v) \notin F$—a contradiction.

Now, by the S$\mathbb{K}$C, there are $\langle \boldsymbol{B}, G\rangle \in \mathbf{MOD}^{\ell}(\mathrm{L})$ with $\boldsymbol{B} \in \mathbb{K}$ and e such that $e[\Gamma'] \subseteq G$ and $e(v) \notin G$. Thus $e[\Psi] \cap G = \emptyset$ (if $e(\psi) \in G$ for some $\psi \in \Psi$ then, since $e[\Gamma'] \subseteq G$, we would obtain $e(v) \in G$—a contradiction).

Assume now that $\Gamma \cup \Psi$ uses all propositional variables. In this case we consider an enumeration of all propositional variables $\{v_n \mid n \in \mathsf{N}\}$ and take the substitutions σ, σ' defined by $\sigma(v_n) = v_{n+1}$ and $\sigma'(v_{n+1}) = v_n$ (and $\sigma'(v_0) = v_0$); note that $\sigma' \circ \sigma$ is the identity. Observe that $\sigma[\Gamma] \nvdash_{\mathrm{L}} \sigma(\psi)$ for each $\psi \in \Psi$ (indeed, if $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\psi)$, by structurality we would have $\sigma'[\sigma[\Gamma]] \vdash_{\mathrm{L}} \sigma'\sigma(\psi)$, i.e. $\Gamma \vdash_{\mathrm{L}} \psi$). The variable v_0 does not appear in $\sigma[\Gamma] \cup \sigma[\Psi]$, so we can apply the previous reasoning to these sets an obtain a model and an evaluation $e \circ \sigma$ as desired. □

THEOREM 3.4.6 (Characterization of strong completeness). *Assume that* L *is finitary and lattice-disjunctive. Then the following are equivalent:*

1. L *has the* S$\mathbb{K}$C.

2. *Every non-trivial countable member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$ *is embeddable into some member of* $\mathbb{K}$.

3. *Every countable member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RSI}}$ *is embeddable into some member of* $\mathbb{K}$.

Proof. 1→2: Take a countable $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$ and let F be its filter. Consider a set of pairwise different variables $\{v_a \mid a \in A\}$ (we can do it because $\boldsymbol{A}$ is countable) and the following sets of formulae:

$$\Gamma = \{c(v_{a_1}, \ldots, v_{a_n}) \leftrightarrow v_{c^{\boldsymbol{A}}(a_1,\ldots,a_n)} \mid \langle c, n\rangle \in \mathcal{L} \text{ and } a_1, \ldots, a_n \in A\},$$

$$\Psi = \{v_{a_1} \vee \ldots \vee v_{a_n} \mid n \in \mathsf{N} \text{ and } a_1, \ldots, a_n \in A \setminus F\}.$$

Clearly Ψ is directed and $\Gamma \nvdash_{\mathrm{L}} \psi$ for each $\psi \in \Psi$. Indeed, take the $\boldsymbol{A}$-evaluation $e(v_a) = a$; we have $e[\Gamma] \subseteq F$ but $a_1 \vee \ldots \vee a_n \notin F$ (otherwise, since F is prime, $a_i \in F$ for some i—a contradiction).

Now we use the S$\mathbb{K}$C and Lemma 3.4.5 to obtain $\langle \boldsymbol{B}, G\rangle \in \mathbf{MOD}^\ell(\mathrm{L})$ with $\boldsymbol{B} \in \mathbb{K}$ and a $\boldsymbol{B}$-evaluation e such that $e[\Gamma] \subseteq G$ and $e(\psi) \notin G$ for each $\psi \in \Psi$. Consider the mapping $f\colon A \to B$ defined as $f(a) = e(v_a)$. It is clear that f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. We show that it is one-one: take $a, b \in A$ such that $a \neq b$ and assume, without loss of generality, $a \to^{\boldsymbol{A}} b \notin F$. Therefore, $f(a) \to^{\boldsymbol{B}} f(b) = e(v_a) \to^{\boldsymbol{B}} e(v_b) = e(v_{a \to^{\boldsymbol{A}} b}) \notin G$ and thus $f(a) \neq f(b)$.

2→3: Obvious.

3→1: Suppose that for some Γ and φ we have $\Gamma \nvdash_{\mathrm{L}} \varphi$. Then, since L is finitary, by Theorem 2.3.16, there are $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ and e such that $e[\Gamma] \subseteq F$ and $e(\varphi) \notin F$. Let $\boldsymbol{B}$ be the countable subalgebra of $\boldsymbol{A}$ generated by $e[Fm_{\mathcal{L}}]$. Consider the submatrix $\langle \boldsymbol{B}, B \cap F\rangle \in \mathbf{MOD}^\ell(\mathrm{L})$. $\boldsymbol{B}$ is not necessarily subdirectly irreducible but it is representable as a subdirect product of a family of $\{\boldsymbol{C}_i \mid i \in I\} \subseteq \mathbf{ALG}^*(\mathrm{L})_{\mathrm{RSI}}$; let G_i be their corresponding filters and let α be the representation homomorphism. It is clear that $e[\Gamma] \subseteq B \cap F$ and $e(\varphi) \notin B \cap F$. There is some $j \in I$ such that $(\pi_j \circ \alpha)(e(\varphi)) \notin G_j$. $\boldsymbol{C}_j$ is a countable member of $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RSI}}$, so by the assumption there is a matrix $\langle \boldsymbol{C}, G\rangle \in \mathbf{MOD}^\ell(\mathrm{L})$ with $\boldsymbol{C} \in \mathbb{K}$ and an embedding $f\colon \boldsymbol{C}_j \hookrightarrow \boldsymbol{C}$, and hence, using this model and the evaluation $f \circ \pi_j \circ \alpha \circ e$, we obtain $\Gamma \not\models_{\mathbb{K}} \varphi$. □

DEFINITION 3.4.7 (Partial embeddability). *Given two algebras* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *of the same language* $\mathcal{L}$*, we say a finite subset* X *of* $\boldsymbol{A}$ *is partially embeddable into* $\boldsymbol{B}$ *if there is a one-to-one mapping* $f\colon X \to B$ *such that for each* $\langle c, n\rangle \in \mathcal{L}$ *and each* $a_1, \ldots, a_n \in X$ *satisfying* $c^{\boldsymbol{A}}(a_1, \ldots, a_n) \in X$*,* $f(c^{\boldsymbol{A}}(a_1, \ldots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \ldots, f(a_n))$*.*

A class $\mathbb{K}$ *of algebras is* partially embeddable into *a class* $\mathbb{K}'$ *if every finite subset of every member of* $\mathbb{K}$ *is partially embeddable into a member of* $\mathbb{K}'$*.*

THEOREM 3.4.8 (Characterization of finite strong completeness). *Let* L *be a finitary lattice-disjunctive logic with a* finite *language* $\mathcal{L}$. *Then the following are equivalent:*

1. L *has the* FS$\mathbb{K}$C.
2. *Every countable non-trivial member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$ *is partially embeddable into* $\mathbb{K}$.
3. *Every non-trivial member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$ *is partially embeddable into* $\mathbb{K}$.
4. *Every member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RSI}}$ *is partially embeddable into* $\mathbb{K}$.
5. *Every countable member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RSI}}$ *is partially embeddable into* $\mathbb{K}$.

Proof. The implications $3\to4$ and $4\to5$ are trivial; $5\to1$ is proved analogously to the implication $3\to1$ of Theorem 3.4.6.

$1\to2$: Take a countable $\boldsymbol{A}\in\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$, with filter F, and a finite set $B\subseteq A$. Define the set $B'=B\cup\{a\to^{\boldsymbol{A}}b\mid a,b\in B\}$. Consider a set of pairwise different variables $\{v_a\mid a\in A\}$, a formula $\varphi=\bigvee_{a\in B'\setminus F}v_a$, and the following set of formulae (notice a difference between this set and the set Γ from the proof of Theorem 3.4.6):

$$\Gamma=\{c(v_{a_1},\dots,v_{a_n})\leftrightarrow v_{c^{\boldsymbol{A}}(a_1,\dots,a_n)}\mid\langle c,n\rangle\in\mathcal{L}\text{ and }a_1,\dots,a_n,c^{\boldsymbol{A}}(a_1,\dots,a_n)\in B'\}.$$

Observe that Γ is finite and $\Gamma\nvdash_{\mathrm{L}}\varphi$ (use the $\boldsymbol{A}$-evaluation $e(v_a)=a$). Thus, by the FS$\mathbb{K}$C, there is $\boldsymbol{C}\in\mathbb{K}$, with filter G, and a $\boldsymbol{C}$-evaluation e such that $e[\Gamma]\subseteq G$ and $e(\varphi)\notin G$. Define a mapping $f\colon B\to C$ as $f(a)=e(v_a)$. Obviously f is a partial homomorphism. We show that f is one-to-one. Take $a,b\in B$ such that $a\neq b$. We know that $a\to^{\boldsymbol{A}}b\in B$; also, without loss of generality, we can assume $a\to^{\boldsymbol{A}}b\notin F$. Thus, $f(a)\to^{\boldsymbol{C}}f(b)=e(v_a)\to^{\boldsymbol{C}}e(v_b)=e(v_{a\to^{\boldsymbol{A}}b})\notin G$ (because $e(\varphi)\notin G$) and so $f(a)\neq f(b)$.

$2\to3$: Take any $\boldsymbol{A}\in\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$, a finite $X\subseteq A$ and consider the countable subalgebra $\boldsymbol{B}\subseteq\boldsymbol{A}$ generated by X. It is enough to prove that $\boldsymbol{B}$ is finitely subdirectly irreducible relative to $\mathbf{ALG}^*(\mathrm{L})$. Let F,G be the filters such that $\langle\boldsymbol{A},F\rangle,\langle\boldsymbol{B},G\rangle\in\mathbf{MOD}^*(\mathrm{L})$ (we know that $G=B\cap F$ and, because the logic is algebraically implicative, $\mathrm{Fi}^{\boldsymbol{A}}(G)=F$). Suppose, by the way of contradiction, that $G=G_1\cap G_2$ for some $G_1,G_2\in\mathcal{F}i_{\mathrm{L}}(\boldsymbol{B})$ such that $G\subsetneq G_1,G_2$. Take $b_i\in G_i\setminus G$. Observe that $b_1,b_2\notin F$ and thus $F\subsetneq\mathrm{Fi}^{\boldsymbol{A}}(G_i)$. By Theorem 2.7.15 we have: $G_1\cap G_2=\mathrm{Fi}^{\boldsymbol{B}}(G_1\vee G_2)=G\subseteq F$. Finally we obtain: $\mathrm{Fi}^{\boldsymbol{A}}(G_1)\cap\mathrm{Fi}^{\boldsymbol{A}}(G_2)=\mathrm{Fi}^{\boldsymbol{A}}(G_1\vee G_2)\subseteq F$, which implies that F is not finitely $\cap$-irreducible—a contradiction. $\square$

REMARK 3.4.9. *Notice that the implications from* 2, 3, 4, *or* 5 *to* 1 *hold also for infinite languages, whereas the converse ones do not (as shown by the following example).*

EXAMPLE 3.4.10. Consider the language $\mathcal{L}$ resulting from $\mathcal{L}_0=\{\&,\to,\wedge,\vee,\overline{0},\overline{1}\}$ by adding a denumerable set $C=\{c_n\mid n\in\mathsf{N}\}$ of new 0-ary connectives. Let G_C be the conservative expansion of Gödel–Dummett logic in this language with no additional axioms or rules. It is a semilinear Rasiowa-implicative logic (in fact, it is a core fuzzy logic; see Chapter I). Let $\boldsymbol{G}_C$ denote its equivalent algebraic semantics, which in fact is the variety of Gödel algebras with infinitely many constants arbitrarily interpreted. Now consider the subclass $\mathcal{R}_1$ of algebras from $\boldsymbol{G}_C$ defined over $[0,1]$ in which all constants, except for a finite number, are interpreted as 1.

Consider any finite set $\Gamma \cup \{\varphi\}$ such that $\Gamma \nvdash_{\mathrm{G}_C} \varphi$. Then also $\Gamma \nvdash_{\mathrm{G}} \varphi$, where we understand the new constants just as propositional variables. Thus by the strong standard completeness of Gödel–Dummett logic, there is a $[0,1]_{\mathrm{G}}$-evaluation e such that $e[\Gamma] \subseteq \{1\}$ and also $e(\varphi) < 1$. We construct a G_C-algebra $\boldsymbol{A}$ resulting from $[0,1]_{\mathrm{G}}$ by setting $c_n^{\boldsymbol{A}} = e(c_n)$ for every c_n occurring in $\Gamma \cup \{\varphi\}$ and $c_n^{\boldsymbol{A}} = 1$ otherwise. Notice that e can be viewed as an $\boldsymbol{A}$-evaluation and, since $\boldsymbol{A} \in \mathcal{R}_1$ (because $\Gamma \cup \{\varphi\}$ contains only finitely many constants) we obtain, $\Gamma \not\models_{\mathcal{R}_1} \varphi$. Thus we have shown that the FS$\mathcal{R}_1$C holds for G_C.

On the other hand, let us by $[0,1]_0$ denote the Gödel algebra on $[0,1]$ with all new constants interpreted as 0. Clearly, any partial subalgebra of $[0,1]_0$ containing 0 does not partially embed into any algebra in $\mathcal{R}_1$.

Nevertheless, we can give the following characterization for the FS$\mathbb{K}$C that holds even for logics in infinite languages without disjunction.

THEOREM 3.4.11 (Characterization of finite strong completeness). *If* L *is finitary, then the following are equivalent:*

1. L *satisfies the* FS$\mathbb{K}$C.
2. *Every* L*-chain belongs to* $\mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$.

Proof. 1→2: if L satisfies the FS$\mathbb{K}$C then, by Theorem 3.4.3, $\mathbf{ALG}^*(\mathrm{L}) = \mathbf{Q}(\mathbb{K})$. It follows from [25, Lemma 1.5] that every relative finitely subdirectly irreducible member of $\mathbf{Q}(\mathbb{K})$ (i.e. each L-chain) belongs to $\mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$.

2→1: if every L-chain belongs to $\mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$, since every L-algebra is representable as subdirect product of L-chains we have that

$$\mathbf{ALG}^*(\mathrm{L}) \subseteq \mathbf{P}_{\mathrm{SD}}(\mathbf{ISP}_{\mathrm{U}}(\mathbb{K})) \subseteq \mathbf{Q}(\mathbb{K}) \subseteq \mathbf{ALG}^*(\mathrm{L}).$$

Therefore by Theorem 3.4.3, L has the FS$\mathbb{K}$C. □

We know that the S$\mathbb{K}$C means that L and $\models_{\mathbb{K}}$ coincide; we can also formulate the FS$\mathbb{K}$C in a similar manner:

PROPOSITION 3.4.12. *Assume that* L *is finitary. Then* L *has the* FS$\mathbb{K}$C *if, and only if,* L *is the finitary companion of* $\models_{\mathbb{K}}$.

Proof. The direction from right to left is obvious. Assume that L has the FS$\mathbb{K}$C and take $\mathrm{L}' = \mathcal{FC}(\models_{\mathbb{K}})$, the finitary companion of $\models_{\mathbb{K}}$. Then we have: $\Gamma \vdash_{\mathrm{L}'} \varphi$ iff there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models_{\mathbb{K}} \varphi$ iff there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}} \varphi$ (by the FS$\mathbb{K}$C) iff $\Gamma \vdash_{\mathrm{L}} \varphi$ (by finitarity of L). □

COROLLARY 3.4.13. *Assume that* L *is finitary and* $\models_{\mathbb{K}}$ *is finitary too (e.g. whenever* $\mathbf{P}_{\mathrm{U}}\mathbf{I}(\mathbb{K}) \subseteq \mathbf{I}(\mathbb{K})$*). Then* L *has the* S$\mathbb{K}$C *if, and only if,* L *has the* FS$\mathbb{K}$C.

COROLLARY 3.4.14. *Assume that* L *is finitary and enjoys the* FS$\mathbb{K}$C. *Then* L *has the* $\mathbf{SP}_{\mathrm{U}}(\mathbb{K})$C.

Now we show that the failure of completeness properties is inherited by conservative expansions.

PROPOSITION 3.4.15. *Let* L' *be a conservative expansion of* L, $\mathbb{K}'$ *a class of* L'*-chains and* $\mathbb{K}$ *the class of their* L*-reducts.*

- *If* L' *enjoys the* $\mathbb{K}'\mathrm{C}$*, then* L *enjoys the* $\mathbb{K}\mathrm{C}$.
- *If* L' *enjoys the* $\mathrm{FS}\mathbb{K}'\mathrm{C}$*, then* L *enjoys the* $\mathrm{FS}\mathbb{K}\mathrm{C}$.
- *If* L' *enjoys the* $\mathrm{S}\mathbb{K}'\mathrm{C}$*, then* L *enjoys the* $\mathrm{S}\mathbb{K}\mathrm{C}$.

Proof. All the implications are proved in a similar way. Let us prove as an example the first one. We want to show that L has the $\mathbb{K}\mathrm{C}$ and we do it contrapositively: assume $\nvdash_{\mathrm{L}} \varphi$. Since L' is a conservative expansion of L, we also have $\nvdash_{\mathrm{L}'} \varphi$ and so by the $\mathbb{K}'\mathrm{C}$ we obtain $\not\models_{\boldsymbol{A}'} \varphi$ for some $\boldsymbol{A}' \in \mathbb{K}'$. Thus also $\not\models_{\boldsymbol{A}} \varphi$ for the reduct $\boldsymbol{A}$ of $\boldsymbol{A}'$. Because $\boldsymbol{A} \in \mathbb{K}$ we obtain $\not\models_{\mathbb{K}} \varphi$. □

An interesting semantics for which we can apply the characterization of strong completeness is that formed by finite chains:

PROPOSITION 3.4.16. *Assume that* L *is finitary and lattice-disjunctive and let us by* $\mathcal{F}$ *denote the class of all finite* L*-chains. Then the following are equivalent:*

1. L *enjoys the* $\mathrm{S}\mathcal{F}\mathrm{C}$.
2. *All* L*-chains are finite.*
3. *There exists* $n \in \mathsf{N}$ *such each* L*-chain has at most* n *elements.*
4. *There exists* $n \in \mathsf{N}$ *such that* $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

Proof. 1→2: From Theorem 3.4.6 we know that every countable L-chain is embeddable into some member of $\mathcal{F}$, thus there are not infinite countable L-chains and so by the downward Löwenheim–Skolem Theorem there are no infinite chains.

2→3: If all the algebras in $\mathbf{ALG}^*(\mathrm{L})$ are finite then there must a bound for their length, because otherwise by means of an ultraproduct we could build an infinite one.

3→1: Trivial.

3→4: Take an arbitrary L-chain $\boldsymbol{A}$, with filter F, and elements $a_0, \dots, a_n \in A$. Since A has at most n elements it is impossible that $a_0 > a_1 > \cdots > a_n$, thus there is some k such that $a_k \leq a_{k+1}$, i.e. $a_k \to^{\boldsymbol{A}} a_{k+1} \in F$, and hence it satisfies the formula. Since the logic is complete w.r.t. chains, the proof is done.

4→2: Suppose that there is an L-chain $\boldsymbol{A}$, with filter F and elements $a_0, \dots, a_n \in A$ such that $a_0 > a_1 > \cdots > a_n$. Then $a_i \to^{\boldsymbol{A}} a_{i+1} \notin F$, for every $i < n$, and as F is $\vee$-prime we know that $\not\models_{\boldsymbol{A}} \bigvee_{i<n}(x_i \to x_{i+1})$. □

COROLLARY 3.4.17. *For a finitary lattice-disjunctive logic* L *and a natural number* n*, the axiomatic extension* $\mathrm{L}_{\leq n}$ *obtained by adding the schema* $\bigvee_{i<n}(x_i \to x_{i+1})$*, is a semilinear logic which is strongly complete with respect the* L*-chains of length less than or equal to* n.

Finally, as other examples of meaningful semantics based on some particular class of chains, we consider chains defined over intervals of real or rational numbers. Completeness with respect these kind of semantics has been a traditional item in the agenda of fuzzy logics (giving rise to some of the so-called *standard completeness theorems*).

DEFINITION 3.4.18 (Real and rational semantics). *The class* $\mathcal{R} \subseteq \mathbf{MOD}^{\ell}(\mathrm{L})$ *is defined as:* $\boldsymbol{A} \in \mathcal{R}$ *if the domain of* $\boldsymbol{A}$ *is the closed, open, or semi-open real unit interval and* $\leq_{\boldsymbol{A}}$ *is the usual order on reals. The class* $\mathcal{Q} \subseteq \mathbf{MOD}^{\ell}(\mathrm{L})$ *is analogously defined requiring rational unit intervals as domains.*

THEOREM 3.4.19 (Relations between real and rational completeness properties). *Let* L *be a finitary logic.*

1. L *has the* $\mathrm{FS}\mathcal{Q}\mathrm{C}$ *iff it has the* $\mathrm{S}\mathcal{Q}\mathrm{C}$.
2. *If* L *has the* $\mathcal{R}\mathrm{C}$*, then it has the* $\mathcal{Q}\mathrm{C}$.
3. *If* L *has the* $\mathrm{FS}\mathcal{R}\mathrm{C}$*, then it has the* $\mathrm{S}\mathcal{Q}\mathrm{C}$.

Proof. The three claims are proved in a similar way by using the downward Löwenheim–Skolem Theorem. Let us show the first one as an example. If L has the $\mathrm{FS}\mathcal{Q}\mathrm{C}$, then it also has the $\mathrm{S}\mathbf{P}_{\mathrm{U}}(\mathcal{Q})\mathrm{C}$ by Corollary 3.4.14. Assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$. Then there is $\boldsymbol{A} \in \mathbf{P}_{\mathrm{U}}(\mathcal{Q})$, with filter F, and an evaluation e such that $e[\Gamma] \subseteq F$ and $e(\varphi) \notin F$. It is clear that $\boldsymbol{A}$ is a densely ordered chain. Consider the countable set $S = \{e(p) \mid p$ propositional variable in $\Gamma \cup \{\varphi\}\}$. By the downward Löwenheim–Skolem Theorem (considering algebras as first-order structures), there exists a countable elementary substructure $\boldsymbol{B} \subseteq \boldsymbol{A}$, such that $S \subseteq B$. Therefore, $\boldsymbol{B}$ is also densely ordered and hence isomorphic to an element of $\mathcal{Q}$,[16] and so we have a counterexample showing the $\mathrm{S}\mathcal{Q}\mathrm{C}$. □

Observe that, if we restrict ourselves to finitary logics, the completeness properties with respect to $\mathcal{Q}$ are, in fact, equivalent to completeness properties with respect to the whole class of densely ordered linear models. Indeed, when we have an evaluation over a densely ordered linear model providing a counterexample to some derivation, we can apply the downward Löwenheim–Skolem Theorem to the (countable) subalgebra generated by the image of all formulae by the evaluation and obtain a rational countermodel. In particular, the $\mathrm{S}\mathcal{Q}\mathrm{C}$ turns out to be the completeness property characterized in Theorem 3.3.8.

4 First-order predicate semilinear logics

In this section we introduce the basics of the theory of first-order predicate semilinear logics. We will see that for each semilinear logic L one can define *two* natural predicate logics: $\mathrm{L}\forall^{\mathrm{m}}$, the minimal one (because it is complete w.r.t. *all* matrices), and its strengthening $\mathrm{L}\forall$ (complete w.r.t. *linearly ordered* matrices). Interestingly enough, and unlike what happens in the propositional case, these two logics need not coincide (see Example 4.1.18).

[16] We use the well-known fact that any two countable (bounded) dense orders are isomorphic.

Our goal is to find axiomatizations for both logics. We will show that $\mathrm{L}\forall^{\mathrm{m}}$ can be nicely axiomatized for nearly all weakly implicative logics (in fact we could axiomatize it for all of them but at price of increased complexity). However, to axiomatize $\mathrm{L}\forall$ we need to restrict to logics with a reasonable disjunction connective. Thus we make the following convention:

CONVENTION 4.0.1. *In order to simplify the formulation of upcoming definitions and results, let us assume from now on that each logic* L *has the following properties:*

- L *is a weakly implicative* semilinear *logic with principal implication* $\to$.
- L *has a p-protodisjunction* ∇ *satisfying* (MP_∇) *(and thus all the properties from Theorem 3.2.13, like the* PCP *and the* PEP*).*
- *The language of* L *contains a truth constant* $\overline{1}$ *satisfying the consecutions* $\varphi \dashv\vdash \overline{1} \to \varphi$.

Typical (though not exhaustive) examples of logics satisfying the above conditions are substructural semilinear logics which either have both $\overline{1}$ *and* $\vee$ *in their language or are Rasiowa-implicative.*

Notice that for any L satisfying this convention and any L-matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ it is the case that $\overline{1}^{\boldsymbol{A}} = \min_{\leq_{\mathbf{A}}} F$. It is important to add that all the results about the minimal predicate logic $\mathrm{L}\forall^{\mathrm{m}}$ proved in this section would also hold for logics satisfying only the third condition of the above convention; however to minimize the complexity of this section, we prefer to keep all the assumptions from the beginning.

In the first subsection we deal with basic syntactic and semantic notions. Notice that our restriction to the class of logics above (in particular the third condition) is used for the first time in Example 4.1.15 to demonstrate the soundness of generalization rule. The other two assumptions are used in Example 4.1.17 to show the soundness of the ∇-form of this rule in the semantics based on chains. The second subsection presents axiomatizations of both predicate logics and proves their soundness. The third subsection shows alternative simpler axiomatizations in predicate substructural logics and other syntactical properties of these logics like the Local Deduction Theorem. The fourth subsection contains the proof of completeness of both kinds of predicate logics with respect to the presented axiomatizations by means of a generalization of the Henkin-style proof used for classical and some non-classical first-order logics. Finally, the fifth subsection studies a notion of Skolemization for our first-order logics and considers the semantics of witnessed models.

4.1 Basic syntactic and semantic notions

The following definitions are absolutely standard; we present them for the reader's convenience. Let us fix a propositional language $\mathcal{L}$ and a logic L.

DEFINITION 4.1.1 (Predicate language). *A* predicate language $\mathcal{P}$ *is a triple* $\langle \mathbf{P}, \mathbf{F}, \mathbf{ar}\rangle$, *where* $\mathbf{P}$ *is a non-empty set of predicate symbols,* $\mathbf{F}$ *is a set of function symbols, and* $\mathbf{ar}$ *is a function assigning to each predicate and function symbol a natural number called the* arity *of the symbol. The functions* f *for which* $\mathbf{ar}(f) = 0$ *are called* object constants. *The predicates* P *for which* $\mathbf{ar}(P) = 0$ *are called* truth constants.

Let us further fix a predicate language $\mathcal{P} = \langle \mathbf{P},\ \mathbf{F},\ \mathbf{ar}\rangle$ and a denumerable set V whose elements are called *object variables*.

DEFINITION 4.1.2 (Term). *The set of* $\mathcal{P}$-terms *is the minimum set* X *such that:*

- $V \subseteq X$*, and*
- *if* $t_1, \dots, t_n \in X$ *and* f *is an* n*-ary function symbol, then* $f(t_1, \dots, t_n) \in X$.

DEFINITION 4.1.3 (Formulae). *An* atomic $\mathcal{P}$-formula *in any expression* $P(t_1, \dots, t_n)$ *where* P *is an* n*-ary predicate symbol and* $t_1, \dots, t_n$ *are* $\mathcal{P}$*-terms. Atomic* $\mathcal{P}$*-formulae and nullary logical connectives of* $\mathcal{L}$ *are called* atomic $\langle\mathcal{L}, \mathcal{P}\rangle$-formulae. *The set of* $\langle\mathcal{L}, \mathcal{P}\rangle$-formulae *is the minimum set* X *such that:*

- X *contains the atomic* $\langle\mathcal{L}, \mathcal{P}\rangle$*-formulae,*
- X *is closed under logical connectives of* $\mathcal{L}$*, and*
- *if* $\varphi \in X$ *and* x *is an object variable, then* $(\forall x)\varphi, (\exists x)\varphi \in X$.

CONVENTION 4.1.4. *We speak about* $\mathcal{P}$*-formulae if the propositional language is clear from the context and we speak about terms and formulae if both the propositional and the predicate languages are clear from the context. The same convention will be used for any other notion defined in this section parameterized by propositional or predicate languages.*

Given a set of formulae Γ*, we denote by* $\mathcal{P}_\Gamma$ *the predicate language containing exactly the predicate and function symbols that occur in formulae of* Γ.

DEFINITION 4.1.5 (Bound and free variables, closed terms and formulae). *An* occurrence *of a variable* x *in a formula* φ *is* bound *if it is in the scope of some quantifier over* x*; otherwise it is called a* free *occurrence.*

A variable is free in a formula φ *if it has a free occurrence in* φ*. A term is* closed *if it contains no variables. A formula is* closed *if it has no free variables; closed formulae are also known as* sentences.

CONVENTION 4.1.6. *Instead of* $\xi_1, \dots, \xi_n$ *(where* ξ_i*'s are terms or formulae and* n *is arbitrary or fixed by the context) we shall sometimes write just* $\vec{\xi}$.

Unless stated otherwise, by the notation $\varphi(\vec{z})$ *we signify that* all *free variables of* φ *are among those in the vector of pairwise different object variables* $\vec{z}$.

If $\varphi(x_1, \dots, x_n, \vec{z})$ *is a formula and we replace all free occurrences of* x_i*'s in* φ *by terms* t_i*, we denote the resulting formula in the context simply by* $\varphi(t_1, \dots, t_n, \vec{z})$.

DEFINITION 4.1.7 (Substitutability). *A term* t *is* substitutable *for the object variable* x *in a formula* $\varphi(x, \vec{z})$ *iff no occurrence of any variable occurring in* t *is bound in* $\varphi(t, \vec{z})$ *unless it was already bound in* $\varphi(x, \vec{z})$.

DEFINITION 4.1.8 (Theory). *A theory* T *is a pair* $\langle\mathcal{P}, \Gamma\rangle$*, where* $\mathcal{P}$ *is a predicate language and* Γ *is a set of* $\mathcal{P}$*-formulae. A theory is called* closed *if all its formulae are sentences.*

For convenience we sometimes identify the theory T and its set of formulae Γ and say that T is a $\mathcal{P}$-theory to indicate that its language is $\mathcal{P}$ (see e.g. Definition 4.1.12).

DEFINITION 4.1.9 (Structure). *A* $\mathcal{P}$-structure $\mathfrak{S}$ *is a pair* $\langle \mathbf{A}, \mathbf{S} \rangle$ *where* $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ *and* $\mathbf{S}$ *has the form* $\langle S, \langle P_{\mathbf{S}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{S}} \rangle_{f \in \mathbf{F}} \rangle$*, where* S *is a non-empty domain;* $P_{\mathbf{S}}$ *is an* n*-ary fuzzy relation, i.e. a function* $S^n \to A$*, for each* n*-ary predicate symbol* $P \in \mathbf{P}$ *with* $n \geq 1$ *and an element of* A *if* P *is a truth constant;* $f_{\mathbf{S}}$ *is a function* $S^n \to S$ *for each* n*-ary function symbol* $f \in \mathbf{F}$ *with* $n \geq 1$ *and an element of* S *if* f *is an object constant.*

Sometimes, $\mathbf{S}$ is called an $\mathbf{A}$-*structure for* $\mathcal{P}$ and we write $P_{\mathfrak{S}}$ instead of $P_{\mathbf{S}}$.

DEFINITION 4.1.10 (Evaluation). *Let* $\mathfrak{S}$ *be a* $\mathcal{P}$*-structure. An* $\mathfrak{S}$-evaluation *of the object variables is a mapping* v *which assigns to each variable an element from* S*.*

Let v *be an* $\mathfrak{S}$*-evaluation,* x *a variable, and* $a \in S$*. Then* $\mathrm{v}[x{\to}a]$ *is an* $\mathfrak{S}$*-evaluation such that* $\mathrm{v}[x{\to}a](x) = a$ *and* $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ *for each object variable* $y \neq x$*.*

DEFINITION 4.1.11 (Truth definition). *Let* $\mathfrak{S} = \langle \mathbf{A}, \mathbf{S} \rangle$ *be a* $\mathcal{P}$*-structure and* v *an* $\mathfrak{S}$*-evaluation. We define* values *of the terms and* truth values *of the formulae in* $\mathfrak{S}$ *for an evaluation* v *as:*

$$
\begin{array}{rcll}
\|x\|^{\mathfrak{S}}_{\mathrm{v}} & = & \mathrm{v}(x), & \\
\|f(t_1,\dots,t_n)\|^{\mathfrak{S}}_{\mathrm{v}} & = & f_{\mathbf{S}}(\|t_1\|^{\mathfrak{S}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{S}}_{\mathrm{v}}), & \textit{for } f \in \mathbf{F} \\
\|P(t_1,\dots,t_n)\|^{\mathfrak{S}}_{\mathrm{v}} & = & P_{\mathbf{S}}(\|t_1\|^{\mathfrak{S}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{S}}_{\mathrm{v}}), & \textit{for } P \in \mathbf{P} \\
\|c(\varphi_1,\dots,\varphi_n)\|^{\mathfrak{S}}_{\mathrm{v}} & = & c^{\mathbf{A}}(\|\varphi_1\|^{\mathfrak{S}}_{\mathrm{v}}, \dots, \|\varphi_n\|^{\mathfrak{S}}_{\mathrm{v}}), & \textit{for } c \in \mathcal{L} \\
\|(\forall x)\varphi\|^{\mathfrak{S}}_{\mathrm{v}} & = & \inf_{\leq^{\mathbf{A}}}\{\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}[x\to a]} \mid a \in S\}, & \\
\|(\exists x)\varphi\|^{\mathfrak{S}}_{\mathrm{v}} & = & \sup_{\leq^{\mathbf{A}}}\{\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}[x\to a]} \mid a \in S\}. &
\end{array}
$$

If the infimum or supremum does not exist, we take its value as undefined. We say that $\mathfrak{S}$ *is* safe *iff* $\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}}$ *is defined for each* $\mathcal{P}$*-formula* φ *and each* $\mathfrak{S}$*-evaluation* v*.*

We set the following useful denotations for a structure $\mathfrak{S} = \langle\langle \boldsymbol{A}, F\rangle, \mathbf{S}\rangle$. We write

- $\|\varphi(a_1,\dots,a_n)\|^{\mathfrak{S}}$ instead of $\|\varphi(x_1,\dots,x_n)\|^{\mathfrak{S}}_{\mathrm{v}}$ for $\mathrm{v}(x_i) = a_i$,
- $\mathfrak{S} \models \varphi[\mathrm{v}]$ if $\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}} \in F$,
- $\mathfrak{S} \models \varphi$ if $\mathfrak{S} \models \varphi[\mathrm{v}]$ for each $\mathfrak{S}$-evaluation v,
- $\mathfrak{S} \models \Gamma$ if $\mathfrak{S} \models \varphi$ for each $\varphi \in \Gamma$.

To keep the traditional notation from the literature we could also write $\|\varphi\|^{\mathbf{A}}_{\mathbf{S},\mathrm{v}}$ instead of $\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}}$ (see e.g. Chapter XI).

DEFINITION 4.1.12 (Model). *Let* T *be a* $\mathcal{P}$*-theory and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$*. A* $\mathcal{P}$*-structure* $\mathfrak{M} = \langle \mathbf{A}, \mathbf{M} \rangle$ *is called a* $\mathbb{K}$*-model of* T *if it is* safe*,* $\mathbf{A} \in \mathbb{K}$*, and* $\mathfrak{M} \models T$*.*

We use just the term '$\mathbf{A}$-model' instead of '$\{\mathbf{A}\}$-model' and we also use this term for its safe $\mathbf{A}$-structure. When the logic L is known from the context, we just use the terms 'model' and 'ℓ-model' instead of '$\mathbf{MOD}^*(\mathrm{L})$-model' and '$\mathbf{MOD}^\ell(\mathrm{L})$-model'. Notice that as each theory comes with its fixed predicate language we do not need to specify the language of $\mathfrak{M}$ when we say that it is a model of the theory T. By a slight abuse of language we will use the term 'model' instead of 'safe $\mathcal{P}$-structure', when the language is clear from the context.

DEFINITION 4.1.13 (Consequence relation). *Let* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. *A $\mathcal{P}$-formula φ is a semantical (sentential) consequence of a $\mathcal{P}$-theory T w.r.t. the class $\mathbb{K}$, in symbols $T \models_{\mathbb{K}} \varphi$, if for each $\mathbb{K}$-model $\mathfrak{M}$ of T we have $\mathfrak{M} \models \varphi$.*

Note that both in the definition of model and semantical consequence, the language of the theory T plays a minor rôle; basically they could be formulated just for sets of formulae. Indeed we can prove the following:

PROPOSITION 4.1.14. *Let* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$ *and* $\Gamma \cup \{\varphi\}$ *a set of $\mathcal{P}$-formulae. Then the following are equivalent:*

1. $\langle \mathcal{P}', \Gamma\rangle \models_{\mathbb{K}} \varphi$ *for all* $\mathcal{P}' \supseteq \mathcal{P}$.
2. $\langle \mathcal{P}', \Gamma\rangle \models_{\mathbb{K}} \varphi$ *for some* $\mathcal{P}' \supseteq \mathcal{P}$.
3. $\langle \mathcal{P}, \Gamma\rangle \models_{\mathbb{K}} \varphi$.

Proof. If we show that for each $\mathcal{P}' \supseteq \mathcal{P}$ we have: $\langle \mathcal{P}, \Gamma\rangle \models_{\mathbb{K}} \varphi$ iff $\langle \mathcal{P}', \Gamma\rangle \models_{\mathbb{K}} \varphi$ the proof is done. One direction is simple as any $\mathcal{P}$-reduct of a $\mathbb{K}$-model of $\langle \mathcal{P}', \Gamma\rangle$ is clearly a $\mathbb{K}$-model of $\langle \mathcal{P}, \Gamma\rangle$.

To prove the second one we need to show that for any safe $\mathcal{P}$-structure $\mathfrak{M} = \langle \mathbf{A}, \mathbf{M}\rangle$ we can construct a safe $\mathcal{P}'$-structure $\mathfrak{M}' = \langle \mathbf{A}, \mathbf{M}'\rangle$ such that $\mathfrak{M}$ is its $\mathcal{P}$-reduct. The idea is clear: take $\mathfrak{M}$, define the interpretations of additional function and predicate symbols in an arbitrary way and check that the resulting structure is still safe. Let s be an arbitrary element from M, ψ an arbitrary $\mathcal{P}$-sentence and $a = \|\psi\|^{\mathfrak{M}}$. For any functional symbol f from $\mathcal{P}' \setminus \mathcal{P}$ we define $f_{\mathfrak{M}'}(\vec{x}) = s$ and for any predicate symbol P from $\mathcal{P}' \setminus \mathcal{P}$ we set $P_{\mathfrak{M}'}(\vec{x}) = a$. Clearly $\mathfrak{M}$ is the $\mathcal{P}$-reduct of $\mathfrak{M}'$ and if we show that $\mathfrak{M}'$ is safe the proof is done.

Assume (without loss of generality) that there is a $\mathcal{P}'$-formula $\chi(x, \vec{y})$ and a sequence of elements $\vec{r}$ from M such that $\inf\{\|\chi(r, \vec{r})\|^{\mathfrak{M}'} \mid r \in M\}$ does not exist. Let z be a variable different from x and $\vec{y}$. Then for any $\mathcal{P}'$-term $t(x, \vec{y})$ we construct a $\mathcal{P}$-term $\hat{t}(x, \vec{y}, z)$ by replacing each subterm of $t(x, \vec{y})$ given by some $f \in \mathcal{P}' \setminus \mathcal{P}$ by z. Clearly for each $u, \vec{u}$ from M we have: $\|t(u, \vec{u})\|^{\mathfrak{M}'} = \left\|\hat{t}(u, \vec{u}, s)\right\|^{\mathfrak{M}}$. Let further $\hat{\chi}(x, \vec{y}, z)$ be the $\mathcal{P}$-formula resulting from $\chi(x, \vec{y})$ by replacing each:

- atomic subformula given by a predicate symbol $P \in \mathcal{P}' \setminus \mathcal{P}$ by the sentence ψ,
- term t by the term $\hat{t}$.

Notice that $\|\chi(r, \vec{r})\|^{\mathfrak{M}'} = \|\hat{\chi}(r, \vec{r}, s)\|^{\mathfrak{M}}$ for each $r, \vec{r} \in S$. Therefore, the infimum of $\{\|\hat{\chi}(r, \vec{r}, s)\|^{\mathfrak{M}} \mid r \in M\}$ does not exist, i.e. $\mathfrak{M}$ is not safe—a contradiction. $\square$

Now we give a series of examples. The first two demonstrate that we need to have unit in the language for the validity of the well-known generalization rule; the other two show that in first-order semilinear logics, unlike in the propositional case, the consequence relations $\models_{\mathbf{MOD}^{\ell}(\mathrm{L})}$ and $\models_{\mathbf{MOD}^{*}(\mathrm{L})}$ need not coincide.

EXAMPLE 4.1.15. We show that for any L: $\varphi \models_{\mathbf{MOD}^{*}(\mathrm{L})} (\forall x)\varphi$. Consider a model $\mathfrak{M} = \langle\langle \boldsymbol{A}, F\rangle, \mathbf{M}\rangle$ of φ and an $\mathfrak{M}$-evaluation e. We know that $\|\varphi\|^{\mathfrak{M}}_{e[x\to a]} \in F$ for each $a \in M$. Because L satisfies Convention 4.0.1, we know from Proposition 2.5.10 that $\inf^{\mathbf{A}}\{\|\varphi\|^{\mathfrak{M}}_{e[x\to a]} \mid a \in M\} \geq \inf^{\mathbf{A}} F = \overline{1} \in F$ (the first infimum exists as $\mathfrak{M}$ is safe).

EXAMPLE 4.1.16. Consider the logic L given by the three-valued reduced matrix with domain $\{a, b, \bot\}$, filter $\{a, b\}$ and $\to$ defined as:

$\to$	$\bot$	a	b
$\bot$	a	a	a
a	$\bot$	a	$\bot$
b	$\bot$	$\bot$	b

Clearly, L is a weakly implicative logic, though not satisfying Convention 4.0.1, and it is not difficult to build a model showing that $\varphi \not\models_{\mathbf{MOD}^{*}(\mathrm{L})} (\forall x)\varphi$.

EXAMPLE 4.1.17. We show that for any lattice-disjunctive logic L: $\varphi \vee \psi \models_{\mathbf{MOD}^{\ell}(\mathrm{L})} ((\forall x)\varphi) \vee \psi$ whenever x is not free in ψ. Consider an ℓ-model $\mathfrak{M}$ of $\varphi \vee \psi$ and an $\mathfrak{M}$-evaluation e. If $\mathfrak{M} \models \psi[e]$ we are done. Assume that $\mathfrak{M} \not\models \psi[e]$, then also $\mathfrak{M} \not\models \psi[e[x\to a]]$ for each $a \in M$ (because x is not free in ψ!). Since the filter in the matrix is $\vee$-prime, we know that $\mathfrak{M} \models \varphi[e[x\to a]]$; the rest of the proof is the same as in Example 4.1.15.

EXAMPLE 4.1.18. $\models_{\mathbf{MOD}^{*}(\mathrm{G})}$ is different from $\models_{\mathbf{MOD}^{\ell}(\mathrm{G})}$ for the Gödel–Dummett logic G and so they also differ for any logic weaker than G. Indeed, from the previous example we know $\varphi \vee \psi \models_{\mathbf{MOD}^{\ell}(\mathrm{G})} ((\forall x)\varphi) \vee \psi$ and we show that $\varphi \vee \psi \not\models_{\mathbf{MOD}^{*}(\mathrm{G})} ((\forall x)\varphi) \vee \psi$. Consider $\varphi = P(x)$ for a unary predicate P and $\psi = c$ a nullary predicate. Take the G-algebra $\boldsymbol{A}$ whose domain is $\{0, \alpha\} \cup \{1/n \mid n \in \mathsf{N}\}$; the lattice operations are given by the lattice order: elements different from α are ordered as usual, $0 \leq \alpha \leq 1$, α is incomparable with all the other elements; residual conjunction $\&^{\boldsymbol{A}} = \wedge^{\boldsymbol{A}}$; and implication $\to^{\boldsymbol{A}}$ is its residuum ($c \to^{\boldsymbol{A}} 1 = 0 \to^{\boldsymbol{A}} c = c \to^{\boldsymbol{A}} c = 1$, $c \to^{\boldsymbol{A}} 0 = c \to^{\boldsymbol{A}} 1/n = 1/2$, and $1 \to^{\boldsymbol{A}} c = 1/n \to^{\boldsymbol{A}} c = c$, and defined as usual for the remaining values). It is easy to check that the equation $(x \to y) \vee (y \to x) \approx \overline{1}$ is valid in $\boldsymbol{A}$, so we have a non-linearly ordered G-algebra. Now take N as the domain of a first-order structure $\mathbf{S}$ and interpret $c_{\mathbf{S}} = \alpha$ and $P_{\mathbf{S}}(n) = 1/n$, and we have the desired counterexample.

At the end of this subsection we introduce two special kinds of models, exhaustive and fully-named ones.

DEFINITION 4.1.19 (Exhaustive model). *Let $\mathfrak{M} = \langle \mathbf{A}, \mathbf{M}\rangle$ be a model for $\mathcal{P}$. We define the set*

$$A^{\mathrm{exh}\mathfrak{M}} = \{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \mid \varphi \textit{ a } \mathcal{P}\textit{-formula and } \mathrm{v} \textit{ an } \mathfrak{M}\textit{-evaluation}\}$$

and say that $\mathfrak{M}$ is exhaustive *if $A = A^{\mathrm{exh}\mathfrak{M}}$.*

Intuitively, $\mathfrak{M}$ is exhaustive if A only contains the necessary values to interpret first-order formulae of the language. The following straightforward proposition shows that for any model we can always restrict to its exhaustive submodel.

PROPOSITION 4.1.20. *Let* $\mathfrak{M} = \langle\langle \boldsymbol{A}, F\rangle, \mathbf{M}\rangle$ *be a model. Then:*

- *There is a subalgebra* $\boldsymbol{A}^{\mathrm{exh}\mathfrak{M}}$ *of* $\boldsymbol{A}$ *with domain* $A^{\mathrm{exh}\mathfrak{M}}$.
- *There is a submatrix* $\mathbf{A}^{\mathrm{exh}\mathfrak{M}} = \langle \boldsymbol{A}^{\mathrm{exh}\mathfrak{M}}, F \cap A^{\mathrm{exh}\mathfrak{M}}\rangle$ *of* $\langle \boldsymbol{A}, F\rangle$.
- *If we denote by* $\mathfrak{M}^{\mathrm{exh}}$ *the model* $\langle \mathbf{A}^{\mathrm{exh}\mathfrak{M}}, \mathbf{M}'\rangle$, *where* $P_{\mathbf{M}'} = P_{\mathbf{M}}$ *and* $F_{\mathbf{M}'} = F_{\mathbf{M}}$ *for each predicate symbol* P *and functional symbol* F, *then* $\mathfrak{M}^{\mathrm{exh}}$ *is exhaustive and for each formula* φ *and each* $\mathfrak{M}$*-evaluation* v *holds:*

$$\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} = \|\varphi\|^{\mathfrak{M}^{\mathrm{exh}}}_{\mathrm{v}}.$$

DEFINITION 4.1.21 (Fully named model). *Let* $\mathfrak{M}$ *be a model. We say that* $\mathfrak{M}$ *is* fully named *if for each* $m \in M$ *there is a closed term* t*, such that* $t_{\mathfrak{M}} = m$.

4.2 Axiomatic systems and soundness

As we have seen above, the two natural semantical consequence relations we have introduced for first-order semilinear logics may be different in general. The goal of this subsection is to propose axiomatizations for both of them and show their basic properties including their soundness, their completeness will be proved later, in Subsection 4.4.

DEFINITION 4.2.1 (Predicate logics $\mathrm{L}\forall^{\mathrm{m}}$ and $\mathrm{L}\forall$). *Let* L *be a logic in* $\mathcal{L}$. *The* minimal predicate logic over L *(in a predicate language* $\mathcal{P}$*), denoted as* $\mathrm{L}\forall^{\mathrm{m}}$*, is the logic defined by the following axiomatic system:*

(P)	*the rules resulting from the consecutions of* L *by substituting propositional variables by* $\langle\mathcal{L},\mathcal{P}\rangle$*-formulae*
$(\forall 1)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi(x,\vec{z}) \to \varphi(t,\vec{z})$, *where* t *is substitutable for* x *in* φ
$(\exists 1)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi(t,\vec{z}) \to (\exists x)\varphi(x,\vec{z})$, *where* t *is substitutable for* x *in* φ
$(\forall 2)$	$\chi \to \varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \chi \to (\forall x)\varphi$, *where* x *is not free in* χ
$(\exists 2)$	$\varphi \to \chi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)\varphi \to \chi$, *where* x *is not free in* χ.

Further, we define the predicate logic over L *(in a predicate language* $\mathcal{P}$*), denoted as* $\mathrm{L}\forall$*, as the extension of* $\mathrm{L}\forall^{\mathrm{m}}$ *by the following two rules:*

$(\forall 2)^{\nabla}$	$(\chi \to \varphi) \nabla \psi \vdash_{\mathrm{L}\forall} (\chi \to (\forall x)\varphi) \nabla \psi$, *where* x *is neither free in* χ *nor in* ψ
$(\exists 2)^{\nabla}$	$(\varphi \to \chi) \nabla \psi \vdash_{\mathrm{L}\forall} ((\exists x)\varphi \to \chi) \nabla \psi$, *where* x *is neither free in* χ *nor in* ψ.

CONVENTION 4.2.2. *Many results and definitions in this section are valid for both logics* $\mathrm{L}\forall^{\mathrm{m}}$ *and* $\mathrm{L}\forall$. *To simplify matters, when a definition or a theorem does not specifically mention a particular predicate logic we mean that it holds for both of them.*

Notice that we have omitted the propositional language $\mathcal{L}$ in the symbol for the predicate logics over L for it is always that of L. Omitting the symbol for the predicate language could be more confusing. In order to avoid possible problems, we first define the notion of proof from a $\mathcal{P}$-theory T in the (minimal) predicate logic over L in a predicate language $\mathcal{P}$ in the same way we did it in the propositional case, denoting it by means of $\vdash$. We can obtain the analog of Proposition 4.1.14 either as a consequence of the completeness theorem or by a direct syntactical proof (as in classical logic).[17] Finally, observe that there is no need to mention the used p-disjunction in the symbol for L$\forall$, because we know that all p-disjunctions are mutually derivable.

PROPOSITION 4.2.3. *Let and* $\Gamma \cup \{\varphi\}$ *a set of* $\mathcal{P}$*-formulae. Then the following are equivalent:*

1. $\langle \mathcal{P}', \Gamma \rangle \vdash \varphi$ *for all* $\mathcal{P}' \supseteq \mathcal{P}$.
2. $\langle \mathcal{P}', \Gamma \rangle \vdash \varphi$ *for some* $\mathcal{P}' \supseteq \mathcal{P}$.
3. $\langle \mathcal{P}, \Gamma \rangle \vdash \varphi$.

PROPOSITION 4.2.4. *Let* $\mathcal{AS}$ *be a presentation of* L*, then the group of rules* (P) *in the axiomatization of* L$\forall^{\mathrm{m}}$ *can be equivalently replaced by*

$(P_{\mathcal{AS}})$ *the rules resulting from the rules of* $\mathcal{AS}$ *by the substitution of the propositional variables by* $\langle \mathcal{L}, \mathcal{P} \rangle$*-formulae.*

Furthermore, axioms $(\forall 2)$ *and* $(\exists 2)$ *are redundant in the axiomatization of* L$\forall$.

Proof. The proof of the first claim is straightforward. We prove the second one: using (PD) we know that $\chi \to \varphi \vdash (\chi \to \varphi) \nabla (\chi \to (\forall x)\varphi)$, thus from $(\forall 2)^{\nabla}$ we obtain $\chi \to \varphi \vdash (\chi \to (\forall x)\varphi) \nabla (\chi \to (\forall x)\varphi)$. The idempotency of ∇ completes the proof of $(\forall 2)$. The proof of $(\exists 2)$ is analogous. □

Later we will see simpler axiomatizations of L$\forall^{\mathrm{m}}$ and L$\forall$ for particular choices of a logic L.

PROPOSITION 4.2.5. *The following consecutions are provable:*

$(\forall 0)$	$\varphi \vdash (\forall x)\varphi$	
$(T1)$	$\varphi \to \psi \vdash (\forall x)\varphi \to (\forall x)\psi$	
$(T2)$	$\varphi \to \psi \vdash (\exists x)\varphi \to (\exists x)\psi$	
$(T3)$	$\vdash \varphi \leftrightarrow (\forall x)\varphi$	*if* x *is not free in* φ
$(T4)$	$\vdash (\exists x)\varphi \leftrightarrow \varphi$	*if* x *is not free in* φ
$(T5)$	$\vdash (\forall x)\varphi(x, \vec{z}) \leftrightarrow (\forall x')\varphi(x', \vec{z})$	*if* x' *does not occur in* $\varphi(x, \vec{z})$
$(T6)$	$\vdash (\exists x)\varphi(x, \vec{z}) \leftrightarrow (\exists x')\varphi(x', \vec{z})$	*if* x' *does not occur in* $\varphi(x, \vec{z})$
$(T7)$	$\vdash (\forall x)(\forall y)\varphi \leftrightarrow (\forall y)(\forall x)\varphi$	
$(T8)$	$\vdash (\exists x)(\exists y)\varphi \leftrightarrow (\exists y)(\exists x)\varphi$.	

[17] A hint of the proof in the finitary case: consider a proof of φ from $\langle \mathcal{P}', \Gamma \rangle$, i.e. a sequence of $\mathcal{P}'$-formulae. We can transform any element of this proof by the following process: replace any term $f(\vec{s})$, where $f \notin \mathcal{P}$, by an unused variable, and replace any atomic subformula $Q(\vec{s})$, where Q is an n-ary symbol not in $\mathcal{P}$, by an arbitrary $\mathcal{P}$-formula $\chi(\vec{x})$ with n free variables. It can be seen that the resulting sequence of formulae *is* a proof of φ from $\langle \mathcal{P}, \Gamma \rangle$.

Proof. The proof of generalization $(\forall 0)$ is a simple corollary of rule $(\forall 2)$ used for $\chi = \overline{1}$. We show the proof of odd claims (for $\forall$); the proofs for $\exists$ are analogous.

(T1) Using $(\forall 1)$ and (T) we obtain $\varphi \to \psi \vdash (\forall x)\varphi \to \psi$. Rule $(\forall 2)$ completes the proof.

(T3) One implication is axiom $(\forall 1)$. To prove the second one starts from (R) in the form $\vdash \varphi \to \varphi$ and the rule $(\forall 2)$ completes the proof.

(T5) Clearly $\vdash (\forall x)\varphi(x, \vec{z}) \to \varphi(x', \vec{z})$ by $(\forall 1)$ (x' is clearly substitutable for x in φ). Rule $(\forall 2)$ completes the proof of one implication (x' is clearly not free in $(\forall x)\varphi(x, \vec{z})$). The proof of the second implication is symmetric.

(T7) From $(\forall 1)$ and $(T1)$ we obtain $(\forall x)(\forall y)\varphi(x, y, \vec{z}) \to (\forall x)\varphi(x, y, \vec{z})$. Rule $(\forall 2)$ completes the proof of one implication. The proof of the second one is symmetric. □

Observe that the condition "x' does not occur in $\varphi(x, \vec{z})$" is unnecessarily strong and could be replaced by "x' is both substitutable for x and not free in $\varphi(x, \vec{z})$ and x is both substitutable for x' and not free in $\varphi(x', \vec{z})$".

From the group of the rules (P) and rules $(T1)$, $(T2)$ we obtain (by induction on the complexity of the formula χ):

THEOREM 4.2.6 (Congruence Property). *Let φ, ψ, δ be sentences. Then:*

- $\vdash \varphi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \psi \vdash \psi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash \varphi \leftrightarrow \psi$.

Further assume that χ is a formula and $\hat{\chi}$ is obtained from χ by replacing some occurrences φ by ψ. Then

$$\varphi \leftrightarrow \psi \vdash \chi \leftrightarrow \hat{\chi}.$$

The next straightforward proposition shows that we can restrict our attention to sentences, as usual in first-order logics.

PROPOSITION 4.2.7. *Let $\Gamma \cup \{\varphi\}$ be arbitrary formulae. We denote by $\forall\varphi$ the universal closure of φ (i.e. if $x_1, \ldots, x_n$ are the free variables in φ, then $\forall\varphi = (\forall x_1)\ldots(\forall x_n)\varphi$), and by $\forall\,\Gamma$ the set of universal closures of all formulae in Γ. Then: $\Gamma \vdash \varphi$ iff $\forall\,\Gamma \vdash \forall\varphi$.*

The following theorem shows that free variables behave as constants naming arbitrary elements.

THEOREM 4.2.8 (Constants Theorem). *Let $\Sigma \cup \{\varphi(x, \vec{z})\}$ set of formulae and c a constant not occurring in $\Sigma \cup \{\varphi(x, \vec{z})\}$. Then $\Sigma \vdash \varphi(c, \vec{z})$ iff $\Sigma \vdash \varphi(x, \vec{z})$.*

Proof. The right-to-left direction follows easily from $(\forall 0)$ and $(\forall 1)$. Assume that $\Sigma \vdash \varphi(c, \vec{z})$. Let the sequence $\langle \alpha_1, \ldots, \alpha_n \rangle$ be a proof of $\varphi(c, \vec{z})$ from Σ (assuming for simplicity that the logic is finitary; for the general case the proof is analogous). Let y

be a variable different from x and not occurring in the formulae $\alpha_1, \ldots, \alpha_n$. We denote by $S^y_c(\alpha_i)$ the substitution in α_i of each occurrence of c by y. We will show that $\langle S^y_c(\alpha_0), \ldots, S^y_c(\alpha_n)\rangle$ is a proof of $S^y_c(\alpha_n) = \varphi(y, \vec{z})$ from $\Sigma_0 = \Sigma \cap \{\alpha_1, \ldots, \alpha_n\}$. This will end the proof because then $\Sigma_0 \vdash (\forall y)\varphi(y, \vec{z})$ (by $(\forall 0)$), hence $\Sigma_0 \vdash \varphi(x, \vec{z})$ (by $(\forall 1)$), so finally $\Sigma \vdash \varphi(x, \vec{z})$.

If $\alpha_i \in \Sigma$, then $S^y_c(\alpha_i) = \alpha_i \in \Sigma_0$, because y does not occur in Σ_0. If α_i results from a rule in (P), then the same holds for $S^y_c(\alpha_i)$ because the substitution preserves the propositional structure of formulae. Assume that $\alpha_i = (\forall v)\psi(v, \vec{z}) \to \psi(t, \vec{z})$, where t is substitutable for v in ψ. Then $S^y_c(\alpha_i) = (\forall v)S^y_c(\psi(v, \vec{z})) \to S^y_c(\psi(t, \vec{z}))$, where t is substitutable for v in $S^y_c(\psi)$, is still an instance of $(\forall 1)$. The case of $(\exists 1)$ is analogous. Assume that α_i results from an application of the rule $(\forall 2)$, i.e. $\alpha_j = \chi \to \psi$, for some $j < i$, and $\alpha_i = \chi \to (\forall v)\psi$, where v is not free in χ. Then $S^y_c(\alpha_j) = S^y_c(\chi) \to S^y_c(\psi)$ and $S^y_c(\alpha_i) = S^y_c(\chi) \to (\forall v)S^y_c(\psi)$, where v is not free in $S^y_c(\chi)$, so the formula still results from an application of $(\forall 2)$. The remaining rules are analogously checked. □

The next lemma and its two corollaries show that the p-disjunction retains some good properties in the first-order logic L∀, such as closure under ∇-forms and the PCP.

LEMMA 4.2.9. *For each set* Γ *of formulae and formula* φ *such that* $\Gamma \vdash_{\mathrm{L}\forall} \varphi$ *we have* $\Gamma \nabla \psi \vdash_{\mathrm{L}\forall} \varphi \nabla \psi$ *for each sentence* ψ.

Proof. We show $\Gamma \nabla \psi \vdash_{\mathrm{L}\forall} \delta \nabla \psi$ for each δ appearing in the proof of φ from Γ. If $\delta \in \Gamma$ or it is an axiom, the proof is trivial. Now assume that $\Gamma' \vdash_{\mathrm{L}\forall} \delta$ is the inference rule we use to obtain δ. From the induction assumption we have $\Gamma \nabla \psi \vdash_{\mathrm{L}\forall} \Gamma' \nabla \psi$. Since $\Gamma' \nabla \psi \vdash_{\mathrm{L}\forall} \delta \nabla \psi$ (for propositional rules due the PCP of L, for first-order rules due to our definition of the axiomatic system for L∀), the proof of this claim is done. □

COROLLARY 4.2.10. *The following consecution is provable in* L∀*:*

$(\forall 0)^{\nabla}$ $\quad \varphi \nabla \psi \vdash_{\mathrm{L}\forall} (\forall x)\varphi \nabla \psi$*, where* x *is not free in* ψ.

Proof. Let $\varphi(x, \vec{y}), \psi(\vec{y})$ be formulae, x a variable not among $\vec{y}$, and $\vec{c}$ constants not occurring in those formulae. By the previous lemma we obtain: $\varphi(x, \vec{c}) \nabla \psi(\vec{c}) \vdash_{\mathrm{L}\forall} (\forall x)(\varphi(x, \vec{c})) \nabla \psi(\vec{c})$. Since $\varphi(x, \vec{y}) \nabla \psi(\vec{y}) \vdash_{\mathrm{L}\forall} \varphi(x, \vec{c}) \nabla \psi(\vec{c})$ (using $(\forall 0)$ and $(\forall 1)$), we obtain $\varphi(x, \vec{y}) \nabla \psi(\vec{y}) \vdash_{\mathrm{L}\forall} (\forall x)(\varphi(x, \vec{c})) \nabla \psi(\vec{c})$. The Constants Theorem completes the proof. □

COROLLARY 4.2.11. L∀ *enjoys the* sPCP *(and therefore also* PCP*) and the* SLP*, i.e. for each* $\mathcal{P}$*-theory* T *and* $\mathcal{P}$*-sentences* φ, ψ, *and* χ *holds:*

$$\frac{T \vdash_{\mathrm{L}\forall} \chi \qquad S \vdash_{\mathrm{L}\forall} \chi}{T \nabla S \vdash_{\mathrm{L}\forall} \chi} \qquad\qquad \frac{T, \varphi \to \psi \vdash_{\mathrm{L}\forall} \chi \qquad T, \psi \to \varphi \vdash_{\mathrm{L}\forall} \chi}{T \vdash_{\mathrm{L}\forall} \chi}.$$

Proof. Assume that $T \vdash_{\mathrm{L}\forall} \chi$ and $S \vdash_{\mathrm{L}\forall} \chi$. Using the Lemma 4.2.9 we obtain that $S \nabla \chi \vdash_{\mathrm{L}\forall} \chi \nabla \chi$ and $T \nabla \psi \vdash_{\mathrm{L}\forall} \chi \nabla \psi$ for each $\psi \in S$ and so $T \nabla S \vdash_{\mathrm{L}\forall} \chi \nabla S$. Using (I_{∇}) and (C_{∇}) we obtain $T \nabla S \vdash_{\mathrm{L}\forall} \chi$.

To prove the SLP we start from $T, \varphi \to \psi \vdash_{\mathrm{L}\forall} \chi$ and $T, \psi \to \varphi \vdash_{\mathrm{L}\forall} \chi$ and by the PCP we obtain $T, (\varphi \to \psi) \nabla (\psi \to \varphi) \vdash_{\mathrm{L}\forall} \chi$. Knowing that L satisfies the (P_{∇}) we obtain $T \vdash_{\mathrm{L}\forall} \chi$. □

We leave the proof of the soundness of both our logics with respect to their intended semantics as an exercise for the reader. Recall that in Example 4.1.18 we have seen that $\vdash_{\mathrm{L}\forall} \subseteq \models_{\mathbf{MOD}^*(\mathrm{L})}$ need not be true in general.

THEOREM 4.2.12 (Soundness of first-order logics). *Let* L *be a logic. Then:*

$$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \subseteq \models_{\mathbf{MOD}^*(\mathrm{L})} \qquad\qquad \vdash_{\mathrm{L}\forall} \subseteq \models_{\mathbf{MOD}^{\ell}(\mathrm{L})}.$$

4.3 Predicate substructural logics

In this subsection we focus on the predicate logics over *substructural* logics. We will see that the axiomatic systems corresponding to their predicate logics can be presented in a simpler way. Recall that, according to Convention 2.5.9, a propositional weakly implicative logic L in a language $\mathcal{L}$ is a substructural logic if L is an expansion of the $\mathcal{L} \cap \mathcal{L}_{\mathrm{SL}}$-fragment of SL. Thus in particular all $\mathcal{L}$-consecutions provable in SL are provable in L too. Unfortunately the situation is more complicated in the predicate case. For instance, if L is the $\to$-fragment of SL, we do not know whether $\mathrm{L}\forall^{\mathrm{m}}$ is the $\to$-fragment of $\mathrm{SL}\forall^{\mathrm{m}}$. Therefore e.g. from the fact that the upcoming formula $(\forall 2')$ is a theorem of $\mathrm{SL}\forall^{\mathrm{m}}$ we cannot infer that it is a theorem of $\mathrm{L}\forall^{\mathrm{m}}$ even though its only propositional connective is $\to$.

Therefore we are not going to prove our claims just for SL and assume that they will transfer to the proper fragments, but we formulate the forthcoming theorems for the smallest fragments in which we can express their proofs.[18] For simplicity, we will however tacitly assume that whenever we formulate some claim in relation with some logic, this logics has at least the necessary connectives to express the claim. We start with the first-order version of the Duality Theorem 2.5.8 (the definition of mirror image for predicate formulae is the natural extension of Definition 2.5.2). Its proof is straightforward: all new predicate axioms and rules have a principal implication $\to$ which can be easily replaced by $\rightsquigarrow$ using the rule (symm).

THEOREM 4.3.1 (Duality Theorem). *Let* $\{\to, \rightsquigarrow\} \subseteq \mathcal{L} \subseteq \mathcal{L}_{\mathrm{SL}}$ *and* L *the* $\mathcal{L}$*-fragment of* SL_X *for* $X \subseteq \{\mathrm{a}, \mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. *For any* $\mathcal{P}$*-theory* T *and any* $\mathcal{P}$*-formula* φ*:*

$$T \vdash \varphi \qquad \textit{iff} \qquad T' \vdash \varphi'.$$

PROPOSITION 4.3.2. *Let* L *be a substructural logic. Let* φ*,* ψ*,* χ *be formulae and* x *a variable not free in* χ*. The following hold:*

- $(T9)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\chi \to (\forall x)\varphi) \to (\forall x)(\chi \to \varphi)$
- $(T10)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} ((\exists x)\varphi \to \chi) \to (\forall x)(\varphi \to \chi)$
- $(T11)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\chi \to \varphi) \to (\chi \to (\exists x)\varphi)$
- $(T12)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\varphi \to \chi) \to ((\forall x)\varphi \to \chi)$
- $(T13)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi \wedge (\forall x)\psi \leftrightarrow (\forall x)(\varphi \wedge \psi)$
- $(T14)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\varphi \vee \psi) \leftrightarrow (\exists x)\varphi \vee (\exists x)\psi$
- $(T15)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi \vee \chi \to (\forall x)(\varphi \vee \chi)$
- $(T16)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\varphi \wedge \chi) \to (\exists x)\varphi \wedge \chi.$

[18] Of course, the resulting language restriction can be seen only as an upper bound, because one might always expect to find a proof in smaller language, ideally using only the language of the claim itself.

If $\&$ is in the language, then we also have:

$(\forall 2')$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$

$(T1')$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$.

If $\rightsquigarrow$ is in the language, then we also have:

$(\exists 2')$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$

$(T2')$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$

$(T17)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\varphi \,\&\, \chi) \leftrightarrow (\exists x)\varphi \,\&\, \chi$.

If $\vee$ is in the language, then we also have:

$(\forall 3)$ $\vdash_{\mathrm{L}\forall} (\forall x)(\varphi \vee \chi) \to (\forall x)\varphi \vee \chi$

$(\exists 3)$ $\vdash_{\mathrm{L}\forall} (\exists x)\varphi \wedge \chi \to (\exists x)(\varphi \wedge \chi)$.

Proof. The proofs of the first four statements are simple: use $(\forall 1)$ or $(\exists 1)$, then prefixing or suffixing, and then $(\forall 2)$ or $(\exists 2)$. The proofs of the left-to-right directions in the second four statements are also simple: use $(\forall 1)$ or $(\exists 1)$, monotonicity of $\vee$ or $\wedge$, and then $(\forall 2)$ or $(\exists 2)$. Let us show the proof of right-to-left direction of $(T13)$ ($(T14)$ is fully analogous): from $\varphi \wedge \psi \to \varphi$ we get $(\forall x)(\varphi \wedge \psi) \to \varphi$ using $(\forall 1)$ and so by $(\forall 2)$ also $(\forall x)(\varphi \wedge \psi) \to (\forall x)\varphi$, analogously for ψ and then rule $\wedge 3$ completes the proof.

$(\forall 2')$ From $(\forall 1)$ we get $(\forall x)(\chi \to \varphi) \to (\chi \to \varphi)$ and so $\chi \,\&\, (\forall x)(\chi \to \varphi) \to \varphi$. Using the rule $(\forall 2)$ and residuation again completes the proof.

$(T1')$ From $(\forall 1)$, suffixing, and prefixing we get $(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to \psi)$, $(\forall 2)$ and $(\forall 2')$ complete the proof.

$(\exists 2')$ From $(\forall 1)$ we get $(\forall x)(\varphi \to \chi) \to (\varphi \to \chi)$ and so $\varphi \to ((\forall x)(\varphi \to \chi) \rightsquigarrow \chi)$. Using the rule $(\exists 2)$ and $(\mathrm{E})_{\rightsquigarrow}$ again completes the proof.

$(T2')$ The proof is analogous to the proof of $(T1')$.

$(T17)$ From $(\exists 1)$ we get $\varphi \,\&\, \chi \to (\exists x)\varphi \,\&\, \chi$. $(\exists 2)$ completes the proof of one direction. The reverse one: we start with $\chi \to (\varphi \to \varphi \,\&\, \chi)$ then apply $(\forall 2)$, and then $(T2')$ completes the proof.

$(\forall 3)$ From $(\forall x)(\varphi \vee \chi) \to \varphi \vee \chi$ we get $((\forall x)(\varphi \vee \chi) \to \varphi) \vee ((\forall x)(\varphi \vee \chi) \to \chi)$ (by $(\mathrm{lin}_{\vee})$). Using $(\forall 0)^{\vee}$ we obtain $(\forall x)((\forall x)(\varphi \vee \chi) \to \varphi) \vee ((\forall x)(\varphi \vee \chi) \to \chi)$, and so by $(\forall 2')$ we obtain $((\forall x)(\varphi \vee \chi) \to (\forall x)\varphi) \vee ((\forall x)(\varphi \vee \chi) \to \chi)$. The rest is simple.

$(\exists 3)$ From $\varphi \wedge \chi \to (\exists x)(\varphi \wedge \chi)$ we get $[\varphi \to (\exists x)(\varphi \wedge \chi)] \vee [\chi \to (\exists x)(\varphi \wedge \chi)]$ (by $(\mathrm{lin}_{\wedge})$). Using $(\forall 0)^{\vee}$ we obtain $(\forall x)[\varphi \to (\exists x)(\varphi \wedge \chi)] \vee [\chi \to (\exists x)(\varphi \wedge \chi)]$, and so by $(\exists 2')$ we obtain $[(\exists x)\varphi \to (\exists x)(\varphi \wedge \chi)] \vee [\chi \to (\exists x)(\varphi \wedge \chi)]$. Therefore: $(\exists x)\varphi \wedge \chi \to (\exists x)(\varphi \wedge \chi)$. □

Notice that, as corollaries of $(T13)$ and $(T14)$, we easily obtain the provability of $(\forall x)\varphi \wedge \chi \leftrightarrow (\forall x)(\varphi \wedge \chi)$ and $(\exists x)(\varphi \vee \chi) \leftrightarrow (\exists x)\varphi \vee \chi$, for x not free in χ.

REMARK 4.3.3. *Notice that the quantification theory in first-order substructural logics is almost classical. In fact, it is much closer to the intuitionistic one as the only two unprovable quantifier shifts for implication are those which are also unprovable in intuitionistic logic. More formally, let* L *be any logic that can be expanded to intuitionistic logic; then due to the soundness we know that:*

$(T11^r)$ $\nvdash_{\mathrm{L}\forall^{\mathrm{m}}} (\chi \to (\exists x)\varphi) \to (\exists x)(\chi \to \varphi)$

$(T12^r)$ $\nvdash_{\mathrm{L}\forall^{\mathrm{m}}} ((\forall x)\varphi \to \chi) \to (\exists x)(\varphi \to \chi)$.

Next we show how to simplify the axiomatic systems of our predicate logics. The previous proposition showed us a wide class of logics satisfying the precondition of the following theorem.

THEOREM 4.3.4 (Simpler axiomatization of $\mathrm{L}\forall^{\mathrm{m}}$). *Let us assume that the logic* $\mathrm{L}\forall^{\mathrm{m}}$ *proves* $(\forall 2')$ *and* $(\exists 2')$. *Then* $\mathrm{L}\forall^{\mathrm{m}}$ *can be alternatively axiomatized by* (P) *and the following:*

$(\forall 1)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi(x, \vec{z}) \to \varphi(t, \vec{z})$, *where* t *is substitutable for* x *in* φ

$(\exists 1)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi(t, \vec{z}) \to (\exists x)\varphi(x, \vec{z})$, *where* t *is substitutable for* x *in* φ

$(\forall 2')$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, *where* x *is not free in* ψ

$(\exists 2')$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, *where* x *is not free in* ψ

$(\forall 0)$ $\varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi$.

Furthermore, an alternative axiomatization of $\mathrm{L}\forall$ *can be obtained extending this system with:*

$(\forall 0)^{\nabla}$ $\varphi \nabla \psi \vdash_{\mathrm{L}\forall} ((\forall x)\varphi) \nabla \psi$, *where* x *is not free in* ψ.

Proof. The first part of the claim is trivial. One direction of the second claim follows from Corollary 4.2.10. To prove the second direction observe that the logic just defined satisfies an analog of Lemma 4.2.9 and Constants Theorem, and thus we can prove $(\forall 2)^{\nabla}$ and $(\exists 2)^{\nabla}$ in the same way we have proved $(\forall 0)^{\nabla}$ in Corollary 4.2.10. □

As a consequence we obtain the next theorem, for which we need a rather rich language. We formulate it for expansions of SL but, in fact, the presence of $\rightsquigarrow, \&, \vee$, and $\overline{1}$ would suffice. Recall that we are restricted to semilinear logics in this whole section.

THEOREM 4.3.5 (Axiomatization of first-order substructural logics). *Let* L *be an expansion of* SL. *Then* $\mathrm{L}\forall$ *can be alternatively axiomatized by* (P) *and the following:*

$(\forall 1)$ $\vdash_{\mathrm{L}\forall} (\forall x)\varphi(x, \vec{z}) \to \varphi(t, \vec{z})$, *where* t *is substitutable for* x *in* φ

$(\exists 1)$ $\vdash_{\mathrm{L}\forall} \varphi(t, \vec{z}) \to (\exists x)\varphi(x, \vec{z})$, *where* t *is substitutable for* x *in* φ

$(\forall 2')$ $\vdash_{\mathrm{L}\forall} (\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, *where* x *is not free in* ψ

$(\exists 2')$ $\vdash_{\mathrm{L}\forall} (\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, *where* x *is not free in* ψ

$(\forall 3)$ $\vdash_{\mathrm{L}\forall} (\forall x)(\varphi \vee \psi) \to (\forall x)\varphi \vee \psi$, *where* x *is not free in* ψ

$(\forall 0)$ $\varphi \vdash_{\mathrm{L}\forall} (\forall x)\varphi$.

Of course in both previous theorems in the propositional part of the axiomatic system of $\mathrm{L}\forall^{\mathrm{m}}$ and $\mathrm{L}\forall$ we can replace (P) by $(P_{\mathcal{AS}})$ where $\mathcal{AS}$ is an arbitrary axiomatic system for L (see Proposition 4.2.4). Let us show that in the case of Łukasiewicz logic both predicate logics coincide:

COROLLARY 4.3.6. $Ł\forall = Ł\forall^m$.

Proof. It is enough to show that $Ł\forall^m$ proves $(\forall 3)$. From $(\alpha \vee \beta) \leftrightarrow ((\alpha \to \beta) \to \beta)$ and $(T1')$ we obtain $(\forall x)(\varphi \vee \psi) \to (\forall x)((\psi \to \varphi) \to \varphi)$. Now, again by $(T1')$, we have $(\forall x)((\psi \to \varphi) \to \varphi) \to ((\forall x)(\psi \to \varphi) \to (\forall x)\varphi)$. By $(T9)$ and suffixing, $((\forall x)(\psi \to \varphi) \to (\forall x)\varphi) \to ((\psi \to (\forall x)\varphi) \to (\forall x)\varphi)$, and so finally we obtain $((\psi \to (\forall x)\varphi) \to (\forall x)\varphi) \to (\forall x)\varphi \vee \psi$. Transitivity ends the proof. □

The next proposition, which can be seen as quantifier shift of $\exists$ over the defined unary connectives ^{n}, is crucial in the proof of Corollary 4.3.10 which will play an important rôle in Subsection 4.5.

PROPOSITION 4.3.7. *Let* L *be an associative substructural logic with* $\vee$ *in its language. Then:*

$(T18)$ $\quad \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\varphi^n) \leftrightarrow ((\exists x)\varphi)^n$.

Proof. The left-to-right direction is simple: using $(\exists 1)$ n-times and monotonicity of $\&$ we obtain $\varphi^n \to ((\exists x)\varphi)^n$; $(\exists 2)$ completes the proof.[19] We prove the converse direction for $n = 2$; the proof for $n > 2$ is analogous. First observe the provability of the propositional formula

$$\alpha \,\&\, \beta \to \alpha^2 \vee \beta^2$$

(from $\alpha \to \beta$ we obtain $\alpha \,\&\, \beta \to \beta \,\&\, \beta$ and so $\alpha \,\&\, \beta \to \alpha^2 \vee \beta^2$; we obtain the same from $\beta \to \alpha$ and hence the SLP completes the proof). Assume that x is free in φ (otherwise the proof is trivial) and no other variables are free in φ (this assumption only simplifies the notation, the proof for a formula with more free variables would essentially be the same). Choose a variable y which does not occur in φ, then clearly $\varphi(x) \,\&\, \varphi(y) \to \varphi^2(x) \vee \varphi^2(y)$ and so by $(\exists 1)$, the properties of $\vee$ and $(T6)$ (in the form: $(\exists x)\varphi^2 \leftrightarrow (\exists y)\varphi^2(y)$) we get $\varphi(x) \,\&\, \varphi(y) \to (\exists x)\varphi^2$. Thus by $(\exists 2)$ we obtain $(\exists y)(\varphi(x) \,\&\, \varphi(y)) \to (\exists x)\varphi^2$ and so by $(T17)$ and $(T6)$ we have $\varphi(x) \,\&\, (\exists x)\varphi \to (\exists x)\varphi^2$. We just repeat the last three steps to complete the proof. □

Our next aim is to prove a form of local deduction theorem for predicate substructural logics. Let $\mathcal{P}$ be a predicate language and DT be a set of $\star$-formulae (i.e. propositional formulae in language $\mathcal{L}$ built from the normal set of variables enhanced with a new distinguished variable $\star$; see the beginning of Subsection 2.6). By $\mathrm{DT}_{\mathcal{P}}$ we denote the set of formulae resulting from any $\star$-formula from DT by replacing all its propositional variables other than $\star$ by arbitrary $\mathcal{P}$-sentences. Note that elements of $\mathrm{DT}_{\mathcal{P}}$ are not $\mathcal{P}$-formulae, but if we substitute all occurrences of $\star$ by a $\mathcal{P}$-sentence we get another $\mathcal{P}$-sentence.

[19] Notice that the proof of the left-to-right direction does not use the associativity assumption.

THEOREM 4.3.8 (Local Deduction Theorem for $\mathrm{L}\forall^{\mathrm{m}}$). *Let* L *be a substructural logic with* $\&, \rightsquigarrow$ *and* $\overline{1}$ *in its language. Let* $\mathcal{P}$ *be a predicate language. Assume that* L *is* almost (MP)*-based with a set of basic deductive terms* bDT. *Then for each* $\mathcal{P}$*-theory* T*,* $\mathcal{P}$*-formula* ψ *and* $\mathcal{P}$*-sentence* φ*, we have:*

$$T, \varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi \qquad \textit{iff} \qquad T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \delta(\varphi) \to \psi \quad \textit{for some } \delta \in \Pi(\mathrm{bDT}^*)_{\mathcal{P}}.$$

Proof. The observation that $\varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \delta(\varphi)$ for each $\delta \in \Pi(\mathrm{bDT}^*)_{\mathcal{P}}$ completes the proof of the right-to-left direction. To prove the converse one we first observe (using Theorem 4.3.4, Proposition 4.3.2 and the comments after Theorem 4.3.5) that $\mathrm{L}\forall^{\mathrm{m}}$ can be axiomatized using *modus ponens*, rules of the form $\varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \delta(\varphi)$ for $\delta \in \Pi(\mathrm{bDT}^*)_{\mathcal{P}}$, and $\varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi$. The proof runs along the lines of the proof of Theorem 2.6.3. The induction base and the induction steps for all the rules except $\varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi$ are done in the same way as in the propositional case. Let us deal with the remaining one, i.e. assume that $\chi = (\forall x)\psi$. From the induction assumption there has to be $\delta_\psi \in \Pi(\mathrm{bDT}^*)_{\mathcal{P}}$ such that $T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \delta_\psi(\varphi) \to \psi$. Using $(\forall 2)$ we obtain $T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \delta_\psi(\varphi) \to (\forall x)\psi$ and so setting $\delta_\chi = \delta_\psi$ completes the proof. □

Using Theorem 4.3.5 we can easily prove the analog of the theorem above for $\mathrm{L}\forall$.

THEOREM 4.3.9 (Local Deduction Theorem for $\mathrm{L}\forall$). *Let* L *be a substructural logic expanding* SL. *Let* $\mathcal{P}$ *be a predicate language. Assume that* L *is* almost (MP)*-based with the set of basic deductive terms* bDT. *Then for each* $\mathcal{P}$*-theory* T*,* $\mathcal{P}$*-formula* ψ *and* $\mathcal{P}$*-sentence* φ*, we have:*

$$T, \varphi \vdash_{\mathrm{L}\forall} \psi \qquad \textit{iff} \qquad T \vdash_{\mathrm{L}\forall} \delta(\varphi) \to \psi \quad \textit{for some } \delta \in \Pi(\mathrm{bDT}^*)_{\mathcal{P}}.$$

As a corollary we obtain the following claim for $\mathrm{L}\forall$ which will be useful in Subsection 4.5.

COROLLARY 4.3.10. *Let* L *be an axiomatic expansion of* $\mathrm{FL}_{\mathrm{e}}^{\ell}$*. Then for each predicate language* $\mathcal{P}$*, each* $\mathcal{P}$*-theory* T*, each* $\mathcal{P}$*-formula* $\varphi(x)$*, and any constant* $c \notin \mathcal{P}$ *holds that* $T \cup \{\varphi(c)\}$ *is a conservative expansion (in the logic* $\mathrm{L}\forall$*) of* $T \cup \{(\exists x)\varphi(x)\}$*.*

Proof. Assume that $T \cup \{\varphi(c)\} \vdash_{\mathrm{L}\forall} \psi$. Then, by the Local Deduction Theorem, there is n such that $T \vdash_{\mathrm{L}\forall} (\varphi(c) \wedge \overline{1})^n \to \psi$. Thus by the Constants Theorem and $(\exists 2)$ we obtain $T \vdash_{\mathrm{L}\forall} (\exists x)(\varphi(x) \wedge \overline{1})^n \to \psi$. Using $(T18)$ and $(\exists 3)$ we obtain $T \vdash_{\mathrm{L}\forall} ((\exists x)(\varphi(x) \wedge \overline{1}))^n \to \psi$ and $T \vdash_{\mathrm{L}\forall} ((\exists x)\varphi(x) \wedge \overline{1})^n \to \psi$. Local Deduction Theorem completes the proof. □

4.4 Completeness theorem

In this subsection we show that the axiomatic systems $\mathrm{L}\forall^{\mathrm{m}}$ and $\mathrm{L}\forall$ are respectively presentations of the semantically defined first-order logics $\models_{\mathbf{MOD}^*(\mathrm{L})}$ and $\models_{\mathbf{MOD}^{\ell}(\mathrm{L})}$, i.e. we prove two completeness theorems by showing that the reverse inclusions in Theorem 4.2.12 hold as well. To this end we need the notions of linear and $\forall$-Henkin theory. Again we proceed for both logics at once.

DEFINITION 4.4.1 (Linear and $\forall$-Henkin theories). *Let $\mathcal{P}$ be a predicate language. A $\mathcal{P}$-theory T is*

- Linear *if for each pair of $\mathcal{P}$-sentences φ, ψ we have $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$.*
- $\forall$-Henkin *if for each $\mathcal{P}$-formula ψ such that $T \nvdash (\forall x)\psi(x)$ there is an object constant c in $\mathcal{P}$ such that $T \nvdash \psi(c)$.*

Note that the quantifier $(\forall x)$ could be omitted from the definition. Next we introduce the notions of Lindenbaum–Tarski algebra and canonical model of a theory T.

DEFINITION 4.4.2 (Lindenbaum–Tarski algebra). *Let φ be a $\mathcal{P}$-sentence and T a $\mathcal{P}$-theory. We define*

$$[\varphi]_T = \{\psi \mid \psi \textit{ a } \mathcal{P}\textit{-sentence and } T \vdash \varphi \leftrightarrow \psi\}.$$

The Lindenbaum–Tarski matrix of T, denoted by $\mathbf{LindT}_T$, has the domain $L_T = \{[\varphi]_T \mid \varphi$ a $\mathcal{P}$-sentence$\}$, operations:

$$c^{\mathbf{LindT}_T}([\varphi_1]_T, \ldots, [\varphi_n]_T) = [c(\varphi_1, \ldots, \varphi_n)]_T$$

(for each n-ary connective c of L *and each $\mathcal{P}$-sentences $\varphi_1, \ldots, \varphi_n$), and the filter*

$$[T] = \{[\varphi]_T \mid \varphi \textit{ a } \mathcal{P}\textit{-sentence and } T \vdash \varphi\}.$$

The definition is sound due to the congruence property of $\leftrightarrow$ proved in Theorem 4.2.6. The Lindenbaum–Tarski matrix of a theory T will allow us to define a model of T, the so-called *canonical model*, where the formulae not provable from T are not valid. The following proposition shows that the matrix belongs to the corresponding classes with respect to which we want to prove completeness. Next proposition (and to a large extent its proof) is analogous to Lemma 2.2.9 for propositional logics.

PROPOSITION 4.4.3. *Let T a $\mathcal{P}$-theory. Then:*

1. $[\varphi]_T \leq_{\mathbf{LindT}_T} [\psi]_T$ *iff* $T \vdash \varphi \to \psi$.
2. $\mathbf{LindT}_T \in \mathbf{MOD}^*(\mathrm{L})$.
3. $\mathbf{LindT}_T \in \mathbf{MOD}^\ell(\mathrm{L})$ *if, and only if, T is linear.*

Proof. We show these three claims for the minimal logic; the proofs for L$\forall$ are completely analogous. Part 1 is proved by the following chain of simple equivalencies: $[\varphi]_T \leq_{\mathbf{LindT}_T} [\psi]_T$ iff $[\varphi]_T \to^{\mathbf{LindT}_T} [\psi]_T \in [T]$ iff $[\varphi \to \psi]_T \in [T]$ iff $T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi \to \psi$.

If we show that $\mathbf{LindT}_T \in \mathbf{MOD}(\mathrm{L})$, the proofs of parts 2 and 3 trivially follow. Assume that $\Gamma \vdash_{\mathrm{L}} \psi$ and let us fix a $\mathbf{LindT}_T$-evaluation e such that $e[\Gamma] \subseteq [T]$ and we need to show that $e(\psi) \in [T]$. Let us define a mapping σ from propositional formulae to $\langle \mathcal{L}, \mathcal{P}\rangle$-sentences by induction over the complexity of formulae: $\sigma(v) \in e(v)$ (arbitrarily for each propositional variable v) and $\sigma(c(\varphi_1, \ldots, \varphi_n)) = c(\sigma\varphi_1, \ldots, \sigma\varphi_n)$ for each n-ary connective c and propositional formulae $\varphi_1, \ldots, \varphi_n$. Further we show that for each propositional formula φ we obtain $[\sigma\varphi]_T = e(\varphi)$ by induction: for variables it is clear, now assume that $\varphi = c(\varphi_1, \dots, \varphi_n)$, we obtain $[\sigma c(\varphi_1, \dots, \varphi_n)]_T = [c(\sigma\varphi_1, \dots, \sigma\varphi_n)]_T = c^{\mathbf{LindT}_T}([\sigma\varphi_1]_T, \dots, [\sigma\varphi_n]_T) = c^{\mathbf{LindT}_T}(e(\varphi_1), \dots, e(\varphi_n)) = e(c(\varphi_1, \dots, \varphi_n))$.

Since $e[\Gamma] \subseteq [T]$, we have $T \vdash \sigma[\Gamma]$. From $\Gamma \vdash_{\mathrm{L}} \psi$ we obtain $\sigma[\Gamma] \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \sigma\psi$ (due to the group (P) of rules in the axiomatization of $\mathrm{L}\forall^{\mathrm{m}}$). Taken together, we have $T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \sigma\psi$ and so $e(\psi) = [\sigma(\psi)]_T \in [T]$. □

LEMMA 4.4.4. *Let T be a $\forall$-Henkin $\mathcal{P}$-theory. Then for any $\mathcal{P}$-formula φ with only one free variable x holds:*

- $[(\forall x)\varphi]_T = \inf_{\leq_{\mathbf{LindT}_T}} \{[\varphi(c)]_T \mid c \in \mathbf{C}\}$,
- $[(\exists x)\varphi]_T = \sup_{\leq_{\mathbf{LindT}_T}} \{[\varphi(c)]_T \mid c \in \mathbf{C}\}$,

where $\mathbf{C}$ is the set of all closed $\mathcal{P}$-terms.

Proof. We prove only the first claim for the proof of the second one is completely analogous. It is simple to see that $[(\forall x)\varphi]_T$ is a lower bound: from axiom $(\forall 1)$ and part 1 of the previous proposition we obtain $[(\forall x)\varphi]_T \leq_{\mathbf{LindT}_T} [\varphi(c)]_T$ for all terms $c \in \mathbf{C}$.

Assume that $[\chi]_T \not\leq_{\mathbf{LindT}_T} [(\forall x)\varphi]_T$. Without loss of generality we assume that x is not free in χ (because by (T5) we know that $[(\forall x)\varphi]_T = [(\forall y)\varphi]_T$ if y does not occur in $\varphi(x)$). Thus $T \nvdash \chi \to (\forall x)\varphi$ and so $T \nvdash \chi \to \varphi(x)$ (by rule $(\forall 2)$) and $T \nvdash (\forall x)(\chi \to \varphi(x))$ (by rule $(\forall 0)$). By the $\forall$-Henkin property of T we obtain a constant $d \in \mathbf{C}$ such that $T \nvdash \chi \to \varphi(d)$. Thus finally $[\chi]_T \not\leq_{\mathbf{LindT}_T} [\varphi(d)]_T$, i.e. $[\chi]_T$ is not a lower bound of $\{[\varphi(c)]_T \mid c \in \mathbf{C}\}$. □

DEFINITION 4.4.5 (Canonical model). *Let T a $\forall$-Henkin $\mathcal{P}$-theory. The* canonical model *of T, denoted by $\mathfrak{CM}_T$, is the $\mathcal{P}$-structure $\langle \mathbf{LindT}_T, \mathbf{S}\rangle$ where the domain of $\mathbf{S}$ consists of the closed $\mathcal{P}$-terms,*

- $f_{\mathbf{S}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$ *for each n-ary function symbol $f \in \mathcal{P}$, and*
- $P_{\mathbf{S}}(t_1, \dots, t_n) = [P(t_1, \dots, t_n)]_T$ *for each n-ary predicate symbol $P \in \mathcal{P}$.*

Now we can easily prove the following proposition which shows that $\mathfrak{CM}_T$ is indeed a $\mathcal{P}$-model of T:

PROPOSITION 4.4.6. *Let T be a $\forall$-Henkin $\mathcal{P}$-theory. Then for each $\mathcal{P}$-sentence φ:*

1. $\|\varphi\|^{\mathfrak{CM}_T} = [\varphi]_T$.
2. $\mathfrak{CM}_T \models \varphi$ *if, and only if,* $T \vdash \varphi$.

Thus $\mathfrak{CM}_T$ is an exhaustive and fully-named model of T and furthermore T is linear if, and only if, $\mathfrak{CM}_T$ is an ℓ-model of T.

The following two theorems are crucial for the completeness proofs of our two logics. For now, we give the proof of the first one only; the second one is more involved and we postpone its proof right after the completeness theorem.

THEOREM 4.4.7. *Let $\mathcal{P}$ be a predicate language and $T \cup \{\varphi\}$ a $\mathcal{P}$-theory such that $T \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$. Then there is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$ and a $\forall$-Henkin $\mathcal{P}'$-theory (in* $\mathrm{L}\forall^{\mathrm{m}}$*) $T' \supseteq T$ such that $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$.*

Proof. Let $\mathcal{P}'$ be an expansion of $\mathcal{P}$ by countably many new object constants, and take $T' = \langle \mathcal{P}', T\rangle$. Take any $\mathcal{P}'$-formula $\psi(x)$, such that $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\psi(x)$. Thus $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi(x)$ and so $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi(c)$ for any c not occurring in $T' \cup \{\psi\}$ (because T' contains just $\mathcal{P}$-formulae and ψ is a finite object there always is such $c \in \mathcal{P}'$ and so we can use Constants Theorem). □

THEOREM 4.4.8. *Let* L *be a finitary logic, $\mathcal{P}$ be a predicate language, and $T \cup \{\varphi\}$ a $\mathcal{P}$-theory such that $T \nvdash_{\mathrm{L}\forall} \varphi$. Then there is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$ and a linear $\forall$-Henkin $\mathcal{P}'$-theory (in* $\mathrm{L}\forall$*) $T' \supseteq T$ such that $T' \nvdash_{\mathrm{L}\forall} \varphi$.*

The proof of the next two theorems is straightforward: soundness was already established and completeness is a corollary of Proposition 4.4.6 and Theorem 4.4.7 or Theorem 4.4.8 respectively.

THEOREM 4.4.9 (Completeness theorem for $\mathrm{L}\forall^{\mathrm{m}}$). *Let* L *be a logic and $T \cup \{\varphi\}$ a $\mathcal{P}$-theory. Then the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$.
- $T \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi$.
- *There is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$ such that $\mathfrak{M} \models \varphi$ for each exhaustive, fully named, model $\mathfrak{M}$ of $\langle \mathcal{P}', T\rangle$.*

THEOREM 4.4.10 (Completeness theorem for $\mathrm{L}\forall$). *Let* L *be a finitary logic and $T \cup \{\varphi\}$ a $\mathcal{P}$-theory. Then the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $T \models_{\mathbf{MOD}^\ell(\mathrm{L})} \varphi$.
- *There is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$ such that $\mathfrak{M} \models \varphi$ for each exhaustive, fully named, ℓ-model $\mathfrak{M}$ of $\langle \mathcal{P}', T\rangle$.*

The rest of this subsection is devoted to the promised proof of Theorem 4.4.8. To this end, we first need to prepare some notions and prove a crucial lemma. From now on we work in the logic $\mathrm{L}\forall$:

DEFINITION 4.4.11 (Restricted Henkin theory). *Let $\mathcal{P} \subseteq \mathcal{P}'$ be predicate languages. A $\mathcal{P}'$-theory T is* $\mathcal{P}$-$\forall$-Henkin *if for each $\mathcal{P}$-sentence $\varphi(x)$ such that $T \nvdash (\forall x)\varphi(x)$ there is a constant $c \in \mathcal{P}'$ such that $T \nvdash \varphi(c)$.*

Notice that when $\mathcal{P}' = \mathcal{P}$ we obtain the already defined (without the prefix '$\mathcal{P}$') notion of $\forall$-Henkin theory. Recall that $T \vdash S$ means that $T \vdash \psi$ for each $\psi \in S$ and so by $T \nvdash S$ we mean that there is $\psi \in S$ such that $T \nvdash \psi$.

CONVENTION 4.4.12. *Let Ψ be a set of $\mathcal{P}$-theories and T a $\mathcal{P}$-theory. We write $T \nVdash \Psi$ whenever $T \nvdash S$ for each $S \in \Psi$.*

DEFINITION 4.4.13 (Deductively directed set of theories). *A set of $\mathcal{P}$-theories Ψ is* deductively directed *if for each $T, S \in \Psi$ there is $R \in \Psi$ such that $T \vdash R$ and $S \vdash R$; we call R an* upper bound *of T and S.*

We are now ready to prove the Fundamental Lemma which will have Theorem 4.4.8 as a corollary. The level of generality of our result, dealing with logics with arbitrary p-disjunctions, forces us to use the technical complication of dealing with deductively directed sets of theories Ψ. Theorem 4.4.8 will be an application starting from the particular case when $\Psi = \{\{\varphi\}\}$.

LEMMA 4.4.14 (Fundamental Lemma). *Let* L *be a finitary logic, T a $\mathcal{P}$-theory and Ψ a deductively directed set of closed $\mathcal{P}$-theories such that $T \nVdash \Psi$. Then:*

1. *There is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$, a $\mathcal{P}'$-theory $T' \supseteq T$, and a deductively directed set of closed $\mathcal{P}'$-theories $\Psi' \supseteq \Psi$, such that*
   - *$T' \nVdash \Psi'$ and*
   - *each theory $S \supseteq T'$ in arbitrary language is $\mathcal{P}$-$\forall$-Henkin whenever $S \nVdash \Psi'$.*
2. *There is a linear $\mathcal{P}$-theory $T' \supseteq T$ such that $T' \nVdash \Psi$.*

Proof. 1. We construct the extensions by a transfinite recursion. The language $\mathcal{P}'$ is the expansion of $\mathcal{P}$ by new constants $\{c_\nu \mid \nu < ||\mathcal{P}||\}$. We also enumerate all $\mathcal{P}$-formulae with one free variable by ordinal numbers as χ_μ for $\mu < ||\mathcal{P}||$. We construct $\mathcal{P}'$-theories T_μ and sets of closed $\mathcal{P}'$-theories Ψ_μ such that $T_\mu \subseteq T_\nu$ and $\Psi_\mu \subseteq \Psi_\nu$ for each $\mu \leq \nu$, $T_\mu \nVdash \Psi_\mu$, and Ψ_μ is deductively directed. Let $T_0 = T$, $\Psi_0 = \Psi$, and observe that they fulfil our conditions.

For each $\mu \leq ||\mathcal{P}||$ we define: $T_{<\mu} = \bigcup_{\nu<\mu} T_\nu$ and $\Psi_{<\mu} = \bigcup_{\nu<\mu} \Psi_\nu$. Notice that from the induction assumption we obtain that $T_{<\mu} \nVdash \Psi_{<\mu}$ (due to the finitarity) and $\Psi_{<\mu}$ is deductively directed. We distinguish two possibilities:

(H1) If $T_{<\mu} \vdash R \mathbin{\nabla} (\forall x)\chi_\mu(x)$ for some $R \in \Psi_{<\mu}$, then we define $T_\mu = T_{<\mu} \cup \{(\forall x)\chi_\mu(x)\}$ and $\Psi_\mu = \Psi_{<\mu}$.

(H2) Otherwise we define $T_\mu = T_{<\mu}$ and $\Psi_\mu = \Psi_{<\mu} \cup \{R \mathbin{\nabla} \chi_\mu(c_\mu) \mid R \in \Psi_{<\mu}\}$.

We show that our conditions are met no matter which possibility occurred.

(H1) Ψ_μ is obviously deductively directed. Assume for contradiction that $T_\mu = T_{<\mu} \cup \{(\forall x)\chi_\mu(x)\} \vdash R'$ for some $R' \in \Psi_\mu$. We take an upper bound $\hat{R}$ of R and R' and notice that $T_{<\mu} \cup \{(\forall x)\chi_\mu(x)\} \vdash \hat{R}$ and $T_{<\mu} \cup R \vdash \hat{R}$. Thus using the sPCP (which can be proved from the PCP as in the propositional case) we obtain $T_{<\mu} \cup R \mathbin{\nabla} \{(\forall x)\chi_\mu(x)\} \vdash \hat{R}$ and so $T_{<\mu} \vdash \hat{R}$. Since $\hat{R} \in \Psi_{<\mu}$ we have a contradiction with $T_{<\mu} \nVdash \Psi_{<\mu}$.

(H2) Assume by the way of contradiction that $T_\mu = T_{<\mu} \vdash R$ for some $R \in \Psi_\mu$. From the induction assumption we know that $T_{<\mu} \nvdash R$ for each $R \in \Psi_{<\mu}$ and so R has to be of the form $R' \nabla \chi_\mu(c_\mu)$ for some $R' \in \Psi_{<\mu}$. Since c_μ does not appear in $T_{<\mu} \cup \Psi_{<\mu}$, we can use Constants Theorem to obtain $T_\mu \vdash R' \nabla \chi_\mu(x)$, and also $T_\mu \vdash R' \nabla (\forall x)\chi_\mu(x)$ (by $(\forall 0)^\nabla$), a contradiction with the fact that we are in the case (H2). To show that Ψ_μ is deductively directed we distinguish four cases: first if both $R, R' \in \Psi_{<\mu}$ then they have an upper bound already in $\Psi_{<\mu}$. Second assume that $R \in \Psi_{<\mu}$ and $R' = S \nabla \chi_\mu(c_\mu)$ for some $S \in \Psi_{<\mu}$. Let $\hat{R} \in \Psi_{<\mu}$ be the upper bound of R and S. Thus $\hat{R} \nabla \chi_\mu(c_\mu) \in \Psi_\mu$ is an upper bound of R (trivially) and R' (by the sPCP and the trivial fact that $\chi_\mu(c_\mu) \vdash S \nabla \chi_\mu(c_\mu)$). The final two cases are analogous.

Now take $T' = T_{<||\mathcal{P}||}$ and $\Psi' = \Psi_{<||\mathcal{P}||}$. Thus by the induction assumption $T' \not\Vdash \Psi'$. Let now S be any theory such that $T' \subseteq S$ and $S \not\Vdash \Psi'$. We show that S is $\mathcal{P}$-$\forall$-Henkin. Clearly for each $\mu < ||\mathcal{P}||$ if $S \nvdash (\forall x)\chi_\mu(x)$, then we must have used case (H2) (otherwise $T_\mu \vdash (\forall x)\chi_\mu(x)$ and so $S \vdash (\forall x)\chi_\mu(x)$). If $S \vdash \chi_\mu(c_\mu)$, then $S \vdash R \nabla \chi_\mu(c_\mu)$ for any $R \in \Psi_{<\mu}$. Since we have used case (H2), we know that $R \nabla \chi_\mu(c_\mu) \in \Psi_\mu$—a contradiction with $S \not\Vdash \Psi'$.

2. We say that T is maximally consistent w.r.t. Ψ if $T \not\Vdash \Psi$ and for each $\varphi \notin T$ there is $R \in \Psi$ such that $T, \varphi \vdash R$. By Zorn's Lemma be obtain a theory $T' \supseteq T$ which is maximally consistent w.r.t. Ψ. We only have to show that T' is linear. Assume that $\varphi \to \psi \notin T'$ and $\psi \to \varphi \notin T'$. Thus there are $R, S \in \Psi$ such that $T', \varphi \to \psi \vdash R$ and $T', \psi \to \varphi \vdash S$; consider an upper bound $\hat{R}$ of R and S and using the SLP we obtain that $T' \vdash \hat{R}$—a contradiction. □

Proof of Theorem 4.4.8. We construct our extension by induction over N. Take $T_0 = T$ and $\Psi_0 = \{\{\varphi\}\}$, $\mathcal{P}_0 = \mathcal{P}$. We construct predicate languages $\mathcal{P}_i$, $\mathcal{P}_i$-theories T_i, and deductively directed sets Ψ_i of closed $\mathcal{P}_i$-theories, such that $\mathcal{P}_{i-1} \subseteq \mathcal{P}_i$, $T_{i-1} \subseteq T_i$, $\Psi_{i-1} \subseteq \Psi_i$, and $T_i \not\Vdash \Psi_i$. Observe that the theory T_0, set Ψ_0, and language $\mathcal{P}_0$ satisfy $T_0 \not\Vdash \Psi_0$. The induction step: we use part 1 of Lemma 4.4.14 for $\mathcal{P}_i$, T_i, Ψ_i, and define their successors as $\mathcal{P}'_i$, T'_i, and Ψ'_i (the lemma assures us that our required conditions are fulfilled). Then we define $\mathcal{P}' = \bigcup\{\mathcal{P}_i \mid i \in \mathsf{N}\}$, the $\mathcal{P}'$-theory $\hat{T} = \bigcup\{T_i \mid i \in \mathsf{N}\}$, and $\Psi' = \bigcup\{\Psi_i \mid i \in \mathsf{N}\}$. Finally, we use part 2 of Lemma 4.4.14 for $\mathcal{P}'$, $\hat{T}$, and Ψ' and define T' as $\hat{T}'$.

Obviously T' is linear, $T_i \subseteq T'$, and $T' \not\Vdash \Psi_i$ for each i (thus in particular $T' \nvdash \varphi$). From part 1 of Lemma 4.4.14 and the definition of $\mathcal{P}'$ we obtain that T' is a $\mathcal{P}_i$-$\forall$-Henkin $\mathcal{P}'$-theory for each i, and so it is a $\forall$-Henkin $\mathcal{P}'$-theory. □

Notice that we have proved more: the maximal consistency of T with respect to φ.

4.5 ∃-Henkin theories, Skolemization, and witnessed semantics

In this subsection we will deal with first-order logics L∀ only. Our goal is twofold. First, we study a notion of Skolemization for these logics which, provided that the property in Corollary 4.3.10 is satisfied, allows to erase existential quantifiers in a formula by conservatively adding new functional symbols. Second, we deal with the particular stronger semantics of witnessed models, i.e. models where the truth value of each quantified formula coincides with the truth-value of some of its instances. We show that any logic L$\forall$ admitting Skolemization can be axiomatically extended to a logic enjoying completeness w.r.t. witnessed models.

We start by introducing the notion of $\exists$-Henkin theory, dual to the already introduced (and classically equivalent) notion of $\forall$-Henkin theory. It will be convenient to restrict its validity to a class Σ of formulae, which later will be determined by some particular syntactical property (e.g. those starting with the connective $\triangle$, or formulae satisfying excluded middle, or just all formulae). At the start, however, we need not assume anything. In the extreme case Σ could be just a single formula. Thus, let us fix a class Σ of formulae of arbitrary languages.

DEFINITION 4.5.1 ($\exists$-Henkin theory). *Let $\mathcal{P} \subseteq \mathcal{P}'$ be predicate languages. We say that a $\mathcal{P}'$-theory T is:*

- Σ-$\mathcal{P}$-$\exists$-Henkin *if for each $\mathcal{P}$-formula $\varphi(x) \in \Sigma$ such that $T \vdash (\exists x)\varphi(x)$ there is a constant $c \in \mathcal{P}'$ and $T \vdash \varphi(c)$.*
- Σ-Henkin *if it is $\forall$-Henkin and Σ-$\mathcal{P}'$-$\exists$-Henkin.*
- Henkin *if it is Σ-Henkin and Σ is the class of all formulae.*

DEFINITION 4.5.2 (preSkolem logic). *We say that* L$\forall$ *is Σ-preSkolem if $T \cup \{\varphi(c)\}$ is a conservative expansion of $T \cup \{(\exists x)\varphi(x)\}$ for each language $\mathcal{P}$, each $\mathcal{P}$-theory T, each $\mathcal{P}$-formula $\varphi(x) \in \Sigma$ and any constant $c \notin \mathcal{P}$.*

Again, if Σ is the class of all formulae we drop the prefix 'Σ-'.

EXAMPLE 4.5.3. In Corollary 4.3.10 we have seen that each predicate logic over an axiomatic expansion of $\mathrm{FL}_{\mathrm{e}}^{\ell}$ is preSkolem. This includes all core fuzzy logics introduced in Chapter I. We show additional examples of preSkolem logics based on $\triangle$-core fuzzy logics.

Let L be a $\triangle$-core fuzzy logic, and Σ be a class of all formulae of the form $\triangle\varphi$. We show that L$\forall$ is Σ-preSkolem. Let us first recall that L$\forall$ enjoys the Global Deduction Theorem: $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \triangle\varphi \to \psi$. Next assume that $T \cup \{\triangle\varphi(c)\} \vdash_{\mathrm{L}\forall} \psi$. Then by the Deduction Theorem $T \vdash_{\mathrm{L}\forall} \triangle\triangle\varphi(c) \to \psi$ and so $T \vdash_{\mathrm{L}\forall} \triangle\varphi(c) \to \psi$. Thus by the Constants Theorem and ($\exists$2) we obtain $T \vdash_{\mathrm{L}\forall} (\exists x)(\triangle\varphi(x)) \to \psi$ and so by *modus ponens*, we have: $T, (\exists x)(\triangle\varphi(x)) \vdash_{\mathrm{L}\forall} \psi$.

LEMMA 4.5.4 (Fundamental Lemma). *Let* L$\forall$ *a Σ-preSkolem predicate logic, T a $\mathcal{P}$-theory, and Ψ a deductively directed set of closed $\mathcal{P}$-theories such that $T \nVdash \Psi$. Then there is $\mathcal{P}' \supseteq \mathcal{P}$ and a $\mathcal{P}'$-theory $T' \supseteq T$ such that*

- *$T' \nVdash \Psi$ and*
- *each theory $S \supseteq T'$ in arbitrary language is Σ-$\mathcal{P}$-$\exists$-Henkin whenever $S \nVdash \Psi$.*

Proof. We construct our expansion by transfinite recursion as in the proof of the first part of Lemma 4.4.14. Let $\bar{\Sigma}$ be the set of all $\mathcal{P}$-formulae of the form $\varphi(x) \in \Sigma$. We expand our predicate language with new constants $\{c_\nu \mid \nu < ||\bar{\Sigma}||\}$ and enumerate all formulae from $\bar{\Sigma}$ by ordinal numbers as χ_μ.

We construct theories T_μ such that $T_\mu \subseteq T_\nu$ for $\mu \le \nu$ and $T_\mu \not\Vdash \Psi$. Let $T_0 = T$ and observe that it fulfils our condition. For each μ we define the set $T_{<\mu} = \bigcup_{\nu<\mu} T_\nu$. Notice that from the induction assumption and finitarity we obtain that $T_{<\mu} \not\Vdash \Psi$. We distinguish two possibilities:

(W1) If $T_{<\mu} \cup \{(\exists x)\chi_\mu(x)\} \not\Vdash \Psi$, we define $T_\mu = T_{<\mu} \cup \{\chi_\mu(c)\}$.

(W2) Otherwise we define $T_\mu = T_{<\mu}$.

In the case (W1) we use the fact that $T_{<\mu} \cup \{\chi_\mu(c_\mu)\}$ is a conservative expansion of $T_{<\mu} \cup \{(\exists x)\chi_\mu(x)\}$ (because L∀ is Σ-preSkolem) to obtain $T_\mu \not\Vdash \Psi$. In the case (W2) we obtain it trivially.

Take $T' = T_{<||\bar{\Sigma}||}$ and observe that clearly $T' \not\Vdash \Psi$. Let S be a theory in an arbitrary language such that $T' \subseteq S$ and $S \not\Vdash \Psi$. We show that S is Σ-$\mathcal{P}$-$\exists$-Henkin. If $S \vdash (\exists x)\chi_\mu(x)$ then we used case (W1) (from $T_{<\mu} \cup \{(\exists x)\chi_\mu(x)\} \vdash R$ for some $R \in \Psi$ we would obtain $S \vdash R$, a contradiction). Thus $T_\mu \vdash \chi_\mu(c_\mu)$ and so $S \vdash \chi_\mu(c_\mu)$. □

THEOREM 4.5.5. *The following are equivalent:*

1. L∀ *is* Σ*-preSkolem.*
2. *For each* $\mathcal{P}$*-theory* T, φ *such that* $T \nvdash \varphi$ *there is* $\mathcal{P}' \supseteq \mathcal{P}$ *and a linear* Σ*-Henkin* $\mathcal{P}'$*-theory* $T' \supseteq T$ *and* $T' \nvdash \varphi$.

Proof. Assume that L∀ is Σ-preSkolem and $T \nvdash \varphi$, some $\mathcal{P}$-formulae $T \cup \{\varphi\}$. We construct our extension by induction over N. Take $T_0 = T$ and $\Psi_0 = \{\{\varphi\}\}$, $\mathcal{P}_0 = \mathcal{P}$. We construct theories Ψ_i and T_i, and predicate languages $\mathcal{P}_i$ such that T_i is a $\mathcal{P}_i$-theory, Ψ_i is a directed set of $\mathcal{P}_i$-sentences, $T_i \not\Vdash \Psi_i$, and $\mathcal{P}_i \subseteq \mathcal{P}_j$, $T_i \subseteq T_j$ and $\Psi_i \subseteq \Psi_j$ for $i \le j$. Observe that the theory T_0, the set Ψ_0 and the language $\mathcal{P}_0$ fulfil these conditions. The induction step:

- If i is odd: use part 1 of Lemma 4.4.14 for $\mathcal{P}_i$, T_i, and Ψ_i; define their successors as $\mathcal{P}'_i$, T'_i, and Ψ'_i.
- If i is even: use Lemma 4.5.4 for $\mathcal{P}_i$, T_i, and Ψ_i; define their successors as $\mathcal{P}'_i$, T'_i, and Ψ_i.

Now we define $\mathcal{P}' = \bigcup\{\mathcal{P}_i \mid i \in \mathsf{N}\}$, $\hat{T} = \bigcup\{T_i \mid i \in \mathsf{N}\}$, and $\Psi' = \bigcup\{\Psi_i \mid i \in \mathsf{N}\}$. Finally, we use part 2 of Lemma 4.4.14 for $\mathcal{P}'$, $\hat{T}$, and Ψ' and define T' as $\hat{T}'$.

Obviously T' is linear, $T_i \subseteq T'$, and $T' \not\Vdash^{\mathcal{P}'} \Psi_i$ for each i. Thus from part 1 of Lemma 4.4.14 and part 2 of Lemma 4.5.4 and the definition of $\mathcal{P}'$ we obtain that T' is Σ-Henkin.

Let us now prove the converse direction. Take $T_1 = T \cup \{\varphi(c)\}$ and $T_2 = T \cup \{(\exists x)\varphi(x)\}$. We show that $T_2 \nvdash \chi$ implies $T_1 \nvdash \chi$ for each formula χ (assuming that c does not appear in $T \cup \{\varphi, \chi\}$). We know that there is $\mathcal{P}' \supseteq \mathcal{P}$ and a Σ-Henkin $\mathcal{P}'$-theory $T' \supseteq T_2$ such that $T' \nvdash \chi$. Since $T' \vdash (\exists x)\varphi(x)$, there is a $\mathcal{P}'$-constant c such that $T' \vdash \varphi(c)$. Thus in any model $\mathfrak{M}$ of T' holds: $\mathfrak{M} \models \varphi(c)$ and since $\mathfrak{CM}_{T'} \not\models \chi$ the proof is done. □

Now we are ready to prove that the preSkolem property allows (in fact, it is equivalent) to perform the usual process of Skolemization, i.e. introducing functional symbols to take care of existential quantifiers under the scope of universal quantifiers. To this end, we need a further technical restriction on the classes Σ.

DEFINITION 4.5.6 (Term-closed classes). *A class of formulae* Σ *is* term-closed *if for each formula* $\varphi(x, \vec{y}) \in \Sigma$*, each language* $\mathcal{P}$*, and each sequence of closed* $\mathcal{P}$*-terms* $\vec{t}$*, we have* $\varphi(x, \vec{t}) \in \Sigma$*.*

Typical examples of term-closed classes are the class of all formulae, the class of all formulae starting with $\triangle$ (see Example 4.5.3), or the class of all provably classical formulae (i.e. formulae such that $\vdash_{\mathrm{L}\forall} \varphi \vee \neg\varphi$, assuming that L expands $\mathrm{FL_{ew}}$).

THEOREM 4.5.7 (Skolemization). *Let* Σ *be a term-closed class of formulae. Then the following are equivalent:*

1. $\mathrm{L}\forall$ *is* Σ*-preSkolem.*
2. $T \cup \{(\forall\vec{y})\varphi(f_\varphi(\vec{y}), \vec{y})\}$ *is a conservative expansion of* $T \cup \{(\forall\vec{y})(\exists x)\varphi(x, \vec{y})\}$ *for each language* $\mathcal{P}$*, each* $\mathcal{P}$*-theory* T*, each* $\mathcal{P}$*-formula* $\varphi(x, \vec{y}) \in \Sigma$ *and any functional symbol* $f_\varphi \notin \mathcal{P}$ *of a proper arity.*

Proof. The proof that 2 implies 1 is trivial. The proof of the converse is analogous to the proof of the second part in Theorem 4.5.5. We denote $T \cup \{(\forall\vec{y})\varphi(f_\varphi(\vec{y}), \vec{y})\}$ as T_1 and $T \cup \{(\forall\vec{y})(\exists x)\varphi(x, \vec{y})\}$ as T_2. We show that $T_2 \nvdash \chi$ implies $T_1 \nvdash \chi$ for each formula χ. By Theorem 4.5.5 we know that there is $\mathcal{P}' \supseteq \mathcal{P}$ and a Σ-Henkin $\mathcal{P}'$-theory $T' \supseteq T_2$ such that $T' \nvdash \chi$, and hence $\mathfrak{CM}'_T \not\models \chi$. For each sequence $\vec{t}$ of closed $\mathcal{P}'$-terms $T' \vdash (\exists x)\varphi(x, \vec{t})$ (by ($\forall$1)) and hence there is a $\mathcal{P}'$-constant $c_{\vec{t}}$ such that $T' \vdash \varphi(c_{\vec{t}}, \vec{t})$ (we know that $\varphi(x, \vec{t}) \in \Sigma$ because Σ is term-closed). Since $c_{\vec{t}}$ is an element of the domain of $\mathfrak{CM}'_T$, we can define a model $\mathfrak{M}$ by expanding $\mathfrak{CM}'_T$ with one functional symbol defined as: $(f_\varphi)_{\mathfrak{M}}(\vec{t}) = c_{\vec{t}}$. Since, for each $\mathcal{P}'$-formula, obviously, $\mathfrak{M} \models \psi$ iff $\mathfrak{CM}'_T \models \psi$, we obtain: $\mathfrak{M} \models T$ and $\mathfrak{M} \not\models \chi$. Also clearly $\mathfrak{M} \models (\forall y)\varphi(f_\varphi(\vec{y}), \vec{y})$, and thus the proof is done. □

Our next aims are to consider witnessed models as a meaningful semantics for first-order semilinear logics and axiomatize the logic complete with respect to them. They are defined as those models where, resembling structures for first-order classical logic, every quantifier is realized by some particular element of the domain.

DEFINITION 4.5.8 (Witnessed model). *Let* Q *be either* $\forall$ *or* $\exists$*. We call a* $\mathcal{P}$*-formula* $\varphi(x, \vec{y})$ Q-witnessed *in an* ℓ*-model* $\mathfrak{M}$ *in language* $\mathcal{P}$ *if for each* $\vec{a} \in M$ *there in an element* $b \in M$ *such that*

$$\|(Qx)\varphi(x, \vec{a})\|^{\mathfrak{M}} = \|\varphi(b, \vec{a})\|^{\mathfrak{M}}.$$

Given a set Σ *of* $\mathcal{P}$*-formulae, we call a* ℓ*-model* $\mathfrak{M}$ *in language* $\mathcal{P}$ Σ-Q-witnessed *if each formula from* Σ *is* Q*-witnessed in* $\mathfrak{M}$*. Finally, we omit the prefix '*Q*-' if the formula (*ℓ*-model) is both* $\forall$*- and* $\exists$*-witnessed; we also omit the prefix '*Σ*-' if* Σ *is the set of all* $\mathcal{P}$*-formulae.*

DEFINITION 4.5.9 (Witnessing axioms). *Given a class* Σ *of formulae, we define the following classes*

$$W_\Sigma^\exists = \{(\exists x)((\exists y)\psi(y,\vec{z}) \to \psi(x,\vec{z})) \mid \psi \in \Sigma\}$$

$$W_\Sigma^\forall = \{(\exists x)(\psi(x,\vec{z}) \to (\forall y)\psi(y,\vec{z})) \mid \psi \in \Sigma\}.$$

Finally, we define $W_\Sigma = W_\Sigma^\forall \cup W_\Sigma^\exists$.

THEOREM 4.5.10 (Completeness w.r.t. witnessed models). *Let* Σ *be a term-closed class of formulae and* Q *be the symbol* $\forall$, $\exists$, *or the empty sequence. Let* $\mathrm{L}\forall$ *be* W_Σ^Q-*preSkolem. Then for each* T *and* φ *the following are equivalent:*

- $W_\Sigma^Q, T \vdash \varphi$.
- $\mathfrak{M} \models \varphi$ *for each* Σ-Q-*witnessed* ℓ-*model* $\mathfrak{M}$ *of* T.

Proof. One direction is simple, just observe that the axioms from W_Σ^Q are obviously tautologies in Σ-Q-witnessed models. To prove the converse one, assume that $W_\Sigma, T \nvdash \varphi$. Consider a W_Σ^Q-Henkin theory $T' \supseteq T$ which $T' \nvdash \varphi$ (such a theory exists due to Theorem 4.5.5). If we show that the canonical model $\mathfrak{CM}_{T'}$ is Σ-Q-witnessed, the proof is done. Let us assume that $Q = \exists$ and take $\psi(x,\vec{y}) \in \sigma$ and a sequence $\vec{t}$ of elements of the domain of the canonical model, i.e. closed terms. We know that $\psi(x,\vec{t}) \in \Sigma$ (since Σ is term-closed) and thus $T' \vdash (\exists x)((\exists y)\psi(y,\vec{t}) \to \psi(x,\vec{t}))$, thus (because T' is W_Σ^Q-Henkin) there has to be a constant c such that $T' \vdash (\exists y)\psi(y,\vec{t}) \to \psi(c,\vec{t})$ and so $[(\exists y)\psi(y,\vec{t})]_{T'} = [\psi(c,\vec{t})]_{T'}$. □

DEFINITION 4.5.11 (Witnessed extension). *Let* L *be a logic. We define the* witnessed *predicate logic over* L, *denoted as,* $\mathrm{L}\forall^{\mathrm{w}}$ *as the extension of* $\mathrm{L}\forall$ *by the following* witnessing axioms:

$$(\exists x)((\exists y)\psi(y,\vec{z}) \to \psi(x,\vec{z}))$$

$$(\exists x)(\psi(x,\vec{z}) \to (\forall y)\psi(y,\vec{z})).$$

COROLLARY 4.5.12. *Let* $\mathrm{L}\forall$ *be a preSkolem logic. Then for each theory* T *and formula* φ *the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall^{\mathrm{w}}} \varphi$.
- $\mathfrak{M} \models \varphi$ *for each witnessed* ℓ-*model* $\mathfrak{M}$ *of* T.

EXAMPLE 4.5.13. As prominent examples we check the validity of the witnessing axioms in the first-order versions of the three main logics based on continuous t-norms:

1. Łukasiewicz logic proves both witnessing axioms, i.e. $Ł\forall^w = Ł\forall$. Let us prove them. For the first one it is enough to prove $(\alpha \to (\exists x)\beta) \to (\exists x)(\alpha \to \beta)$ (for x being free in α); the axiom follows by taking $\alpha = (\exists y)\psi(y,\vec{z})$ and $\beta = \psi(x,\vec{z})$. We can easily show that Ł$\forall$ proves: $\neg(\exists x)(\alpha \to \beta) \to (\forall x)(\alpha \mathbin{\&} \neg\beta)$, $(\forall x)(\alpha \mathbin{\&} \neg\beta) \to \alpha \mathbin{\&} (\forall x)\neg\beta$, $\alpha \mathbin{\&} (\forall x)\neg\beta \to \neg(\alpha \to \neg(\forall x)\neg\beta)$, and $\neg(\alpha \to \neg(\forall x)\neg\beta) \to \neg(\alpha \to (\exists x)\beta)$. By transitivity and contraposition we are

done. Similarly, the other axiom follows from $((\forall x)\beta \to \alpha) \to (\exists x)(\beta \to \alpha)$, where x is not free in α, using theorems $((\forall x)\beta \to \alpha) \to (\neg\alpha \to \neg(\forall x)\beta)$, $(\neg\alpha \to \neg(\forall x)\beta) \to (\neg\alpha \to (\exists x)\neg\beta)$, $(\neg\alpha \to (\exists x)\neg\beta) \to (\exists x)(\neg\alpha \to \neg\beta)$, $(\exists x)(\neg\alpha \to \neg\beta) \to (\exists x)(\beta \to \alpha)$.

2. Product logic proves only one of the witnessing axioms. Indeed, it can be semantically shown that $(\alpha \to (\exists x)\beta) \to (\exists x)(\alpha \to \beta)$ is a tautology (an easy application of the fact that the implication in Product logic is left-continuous). For the second one, we can build a counterexample for $(\exists x)(P(x) \to (\forall y)P(y))$. Consider a model over the standard chain $[0,1]_\Pi$ such that N is its domain and $\{\|P(n)\| \mid n \in \mathsf{N}\}$ is a strictly decreasing sequence converging to 0.

3. In Gödel logic both witnessing axioms fail. Indeed, for the second one we can use the same counterexample as above (over the standard Gödel chain, of course). For the first one, we give a counterexample to the formula $(\exists x)((\exists y)P(y) \to P(x))$. Again, consider a model over the standard chain $[0,1]_{\mathrm{G}}$, such that N is its domain and $\{\|P(n)\| \mid n \in \mathsf{N}\}$ is any strictly increasing sequence converging to $r < 1$. Then $\|(\exists x)((\exists y)P(y) \to P(x))\| = \sup_{n\in\mathsf{N}}(\sup_{m\in\mathsf{N}} \|P(m)\| \to \|P(n)\|) = \sup_{n\in\mathsf{N}}(r \to \|P(n)\|) = \sup_{n\in\mathsf{N}} \|P(n)\| = r < 1$.

5 Historical remarks and further reading

Most of the basic notions used in this chapter come from the field of Algebraic Logic. This discipline was born in the XIXth century with the pioneering works of Boole, De Morgan, Pierce and others on classical logic, and has evolved into the study (with a heavy use of tools from Universal Algebra; see e.g. [10]) of classes of algebras providing semantics for non-classical logics. Typically, the connection between a propositional logic and a class of algebras is obtained by means of the Lindenbaum–Tarski method. The attempts to generalize this method have given rise to Abstract Algebraic Logic (AAL) as a natural evolution of the field aiming to understand the process by which a class of algebras can be associated to an arbitrary logic. Our presentation has strongly capitalized on the notions and methods from AAL too. Readers interested in this area and its history are referred to the survey [40] and the comprehensive monographs [24, 39, 88].

Section 2

The first four subsections of this section can be seen as a short introduction to AAL particularized, for didactic reasons, to the framework of weakly implicative logics (WIL). The notion of consequence operator was introduced by Tarski in [83] and the condition of structurality (invariance under substitutions) was added by Łoś and Suszko in [66]. Mainstream AAL has been extensively developed by the Polish school in the paradigm defined by this notion (see [24, 88]). Wójcicki introduced reduced matrices and reduced matrix models in [87] and implicitly obtained the corresponding completeness theorems. Schmidt Theorem (Theorem 2.3.3) is from [80]. The third bullet item of Theorem 2.2.7 motivates the name 'Leibniz congruence' (resembling Leibniz's principle of identity of indiscernibles, this result shows that a pair of formulae are congruent iff they are indistinguishable in the matrix model). The name appeared first in Blok and Pigozzi's paper [6]. The characterization in the third bullet item of Theorem 2.2.7 holds for arbitrary logics and can be deduced e.g. from Maltsev's Lemma (see [10, Lemma V.3.1]). Most results in Subsection 2.3 follow from Section 3.7 in [88], in particular, the subdirect decomposition theorem for reduced matrix models of a finitary logic and complete w.r.t. RSI reduced matrix models. Subsection 2.4 owes much to Blok and Pigozzi's 1989 memoir [7].

Weakly implicative logics were first introduced in [17] as a generalization of Rasiowa's notion of implicative logics (see [76]), which we have called here *Rasiowa-implicative logics*. In the terminology of [28], matrices for WIL coincide with the class of the *prestandard matrices* while ordered matrices coincide with *standard matrices*. Weakly implicative logics have been generalized in [21] to *weakly p-implicational logics*, in a pure AAL fashion, by considering a generalized notion of implication which, instead of being a binary connective, can be defined by sets of (possibly parameterized) formulae. This paper also studies the position of weakly implicative logics in the so-called *Leibniz hierarchy:* weakly implicative logics are a proper subclass of finitely equivalential logics (and therefore of protoalgebraic logics), our algebraically implicative logics are exactly those weakly implicative logics which are algebraizable and, if furthermore they are finitary, they are algebraizable in the sense of Blok and Pigozzi [7]. All the notions (except that of WIL) and results appearing in these four subsections can either be found in [24] or are particularizations of notions and results from this book to the context of weakly implicative logics. With a few exceptions,[20] all the results proved for (finitary) weakly implicative logics hold for arbitrary (finitary) logics.

Subsections 2.5 and 2.6 are dedicated to substructural logics. These logics can be roughly defined as those logical systems such that, when presented by means of Gentzen-style calculus, lack some of the so-called *structural rules:* exchange, weakening, contraction (see e.g. [75, 78, 81]). As such, this area covers a wide variety of systems independently developed since mid XXth century, including relevant logics [1], linear logic [48], Lambek calculus [65], fuzzy logics presented in the previous chapter, and other many-valued logics like monoidal logic [60]. In the last two decades, Algebraic Logic has developed a uniform approach to substructural logics as the logics of residuated lattices, i.e. propositional logics algebraizable in the sense of [7] whose equivalent algebraic semantics is a class of residuated lattices[21] (most results are collected in the recent monograph [43], where the weakest considered logic is FL). The notion of substructural logic we have proposed here extends this approach by considering the weaker base logic SL from [45] and allowing for well-behaved expansions and fragments. The definitions in these subsections are original, though most of the results (especially the algebraic variants of the proved rules) are folklore of the theory of residuated structures. The axiomatic systems for prominent existing substructural logics SL, FL, and $\mathrm{FL_e}$ have been taken from [45] and [43]. Also the deduction theorems for FL

[20] The exceptions are: the first two claims in Proposition 2.2.11, the second claim of Proposition 2.3.13, and Theorem 2.4.5.

[21] Then, classical and intuitionistic logics, although enjoying the structural rules, are included in the family of substructural logics as extreme cases.

and its main axiomatic extensions were already known (see e.g. [43, 44]), but our proofs of these theorems (using the notion of almost (MP)-based logic) and showing their relation with proof by cases properties are novelties of the present chapter.

The final subsection of Section 2 studies the notions of generalized disjunction defined using the proof by cases property. The different variants of generalized disjunctions had already been considered in the framework of Abstract Algebraic Logic in several works (see e.g. [22–24, 29, 39, 41, 84–86]). Our approach follows that of [24], where its wide generality is achieved by allowing a parameterized set of formulae instead of a single formula $p \vee q$, which gives rise to the notion of p-disjunction, in our terminology. We give a more systematic account, introducing classes of logics based on the properties of the disjunction they possess, including a new class of lattice-disjunctive logics (logics where $\vee$ is interpreted as the supremum of the order given by implication), show their separations (with one open problem, see Conjecture 2.7.12). The proofs of the Theorems 2.7.15, 2.7.20, and 2.7.23 are based on the proofs of corresponding theorems in [24, §2.5.1] (some of these results were known before, for detailed references see [24]), however or formulation of these theorems differs substantially from those in [24]. For a detailed exposition of the relation between the results in this subsection and previous works on disjunction see the paper [20], where we generalize the results for (p-)disjunctions in finitary logics (or semilinear logics in Subsection 3.2) presented here to arbitrary logics (non-necessarily finitary or semilinear). Finally, Theorem 2.7.29, where we show how to use (appropriate) disjunction to axiomatize positive universal classes of reduced matrices, was inspired by the paper [42] where the author proved (directly and without the notion of disjunction) a particular version of our result for the substructural logic FL.[22]

Section 3

The notions and results of the first subsection do not have many direct predecessors: the notion of weakly implicative semilinear logic was introduced by Cintula in [17], under the name 'weakly implicative *fuzzy* logic'. The present authors have generalized this class of logics to the context of weakly p-implicational (protoalgebraic) logics in [21] (where all the results of this subsection with exception of Theorem 3.1.6, which is new here, can be found). This paper introduced an important terminological change: the term 'fuzzy' (overloaded by other meanings) from [17] was replaced by (a more neutral) term 'semilinear'. This term was introduced by Olson and Raftery in [73] in the context of residuated lattices to describe the varieties whose subdirectly irreducible members are linear (following the tradition from Universal Algebra of calling a class of algebras 'semi$X$' whenever its subdirectly irreducible members have the property X). The spiritual predecessors of this work are far more numerous: from the beginning of Mathematical Fuzzy Logic it was clear that there are numerous logical systems deserving to be studied as *fuzzy logics*, thus already Hájek in his seminal monograph [53] considered not one, or even a few logics, but all axiomatic extensions of his Basic Fuzzy Logic. For the description of the way from extensions of BL to semilinear logics see the introduction of this chapter.

[22] Inspecting his proof, one can notice that its main ingredient can be seen as the demonstration that a particular set is a generalized disjunction in FL.

The second subsection studies the interplay between semilinearity and p-disjunctions. The importance of disjunction was well-known to the community of mathematical fuzzy logicians since the inception of fuzzy logics (viz. the axiom of prelinearity or its crucial rôle in first-order fuzzy logics). Several known important results involving disjunction (e.g. axiomatizations of important fuzzy logics) are obtained as corollaries of general theorems proved here. The first abstract study of this interplay was carried out in [86, Section 6] (where a less general version of Theorem 3.2.4 and its corollaries was proved) and will be generalized to weakly p-implicational logics in a forthcoming paper.

The final two subsections study refined completeness properties (w.r.t. distinguished classes of algebras), which has always been the central topic in fuzzy logic literature since the very beginning, taking in account the original motivation of fuzzy logics as many-valued systems taking truth values from the real unit interval. At first (in Subsection 3.3) we have concentrated on semantics given by dense chains. The crucial *density rule* originally appeared in [82] in a much more specific context, then was generalized to a wide class of fuzzy logics in [68], and finally it has been studied in [12] in a very general context of hypersequent calculi; however, the level of generality of this final study is clearly incomparable with ours (we subsume the first two). The final subsection generalizes the results of [18] (for ($\triangle$-)core fuzzy logics) to either algebraically implicative logic or, more specifically, lattice-disjunctive logics.

Section 4

Our approach to predicate logics follows that of Rasiowa from [76], where she generalizes the Rasiowa-Sikorski-style intuitionistic first-order predicate logic [77] to the class of logics we call *Rasiowa-implicative logics*. It starts from a propositional logic, which enjoys an equivalent algebraic semantics, and an implication connective defining an order relation on the algebras that allows to interpret the existential (resp. universal) quantification of a formula as the supremum (resp. infimum) of the values of its instances. The main result is the completeness w.r.t. all algebras of truth values. We can easily generalize this approach to weakly implicative logics, but in order to obtain completeness w.r.t. linearly ordered algebras for semilinear logics (as in the propositional case) some additional axioms are necessary. Historically speaking the first such example is Gödel–Dummett first-order predicate logic axiomatized in [61] (relative to the logic axiomatized in Rasiowa's way) by adding the axiom $(\forall 3)$: $(\forall x)(\varphi \vee \chi) \to (\forall x)\varphi \vee \chi$ (for x not free in χ).[23] Using this axiom many other first-order logics complete w.r.t. their linearly ordered algebras were axiomatized (see e.g. [15, 32, 53]). The next step forward was [56], where the proof of completeness was not only generalized to arbitrary languages (previous proofs were restricted to countable languages) but performed uniformly for all ($\triangle$-)core fuzzy logics (identifying the crucial rôle of disjunction in the process). Another noteworthy case is [19] where the first-order implicational fragment of MTL was axiomatized, using (as was later observed) axiom $(\forall 3)$ written for the generalized disjunction of MTL.

[23] In fact the *standard* Łukasiewicz first-order logic was axiomatized earlier in [59], but this is a peculiar case for two reasons: first in this logic the semantics given by all algebras and by chains coincide, and second the actually axiomatized logic in this paper is the first-order infinitary logic of the standard MV-algebra.

All subsections of this section, except for the last one, are dedicated to formalizing these ideas. The last subsection studies the so-called Σ-preSkolem logics, where we can prove a general form of Skolemization (already studied before in fuzzy logics, e.g. in [3]) and completeness w.r.t. witnessed models (i.e. models where the truth value of each quantified formula coincides with the truth-value of some of its instances). These models were first considered in [55] in the context of Łukasiewicz logic (see also [11] for a weaker notion useful in product logics). Our completeness theorem generalizes the one proved in [56] for core fuzzy logics to all preSkolem logics (which include all semilinear axiomatic expansions of the uninorm logic UL).

Other current trends in the research on first-order fuzzy logics that have not been covered here include the study of particular first-order Gödel logics (see e.g. [4]), the development of a model theory for fuzzy logics (see [18, 26, 56]) and works on description fuzzy logics (see [47, 55]).

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