A similarity-based three-valued modal logic approach to reason with prototypes and counterexamples

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Abstract In this paper we focus on the application of similarity relations to formalise different kinds of graded approximate reasoning with gradual concepts. In particular we extend a previous approach that studies properties of a kind of approximate consequence relations for gradual propositions based on the similarity between both prototypes and counterexamples of the antecedent and the consequent. Here we define a graded modal extension of Łukasiewicz's three-valued logic $Ł_3$ and we show how the above mentioned approximate consequences can be interpreted in this modal framework, while preserving both prototypes and counterexamples.

1 Introduction

Indistinguishability relations, also known as *fuzzy similarity relations*, are suitable graded generalisations of the classical notion of equivalence relation that go back to Zadeh, Trillas and colleagues [39, 32, 33, 34].

Definition 1 Let W be a universe and let $*$ be a t-norm. A binary fuzzy relation $S : W \times W \longrightarrow [0,1]$ is called an *indistinguishability relation* with respect to $*$ (or an $*$-indistinguashability relation, or $*$-similarity relation) if, for any $x, y, z \in W$, the following properties hold:

- Reflexivity: $S(x,x) = 1$,
- Symmetry: $S(x,y) = S(y,x)$,

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- $*$-Transitivity: $S(x,y) * S(y,z) \leqslant S(x,z)$.

Additionally, if the following property holds, then S is called a *strict* indistinguishability relation:

- Strictness: $S(x,y) = 1$ iff $x = y$,

Actually, this definition is a generalization of Zadeh's concept of fuzzy similarity relations introduced in [39], that corresponds to the particular case of min-indistinguishability relations, that is, when one takes $* = \min$ above. Actually, min-indistinguishability relations are a very particular class of fuzzy relations since all the cuts of a min-similarity relation turn out to be classical equivalence relations, and hence it defines a partition tree (a partition for each cut of the relation, and in which the partition in one level is a refinement of the partition in the level above it). This does not hold for a $*$-similarity with $* \neq \min$.

Fuzzy similarity relations have been extensively studied from a mathematical point of view, see for example [35, 27, 25, 28, 2, 11, 26, 15, 16] and its applications cover many different topics like classification, analogical reasoning, preferences, interpolation, morphology, etc. In this paper we are interested in the application of similarity relations to formalise different kinds of graded approximate reasoning with gradual concepts. This approach dates back to Ruspini [31] where he proposes a semantics for fuzzy sets based on the idea of prototypes and similarity relations. Along this line, there have been a number of contributions towards the logical formalisation of (graded) approximate inference patterns of the kind "if φ then *approximately* ψ", either as graded consequence relations and with unary or binary modalities [18, 21, 22, 37, 38], but always dealing with classical propositions φ and ψ. In a recent paper [20], this approach has been extended to cope with vague concepts (or propositions) based on the similarity between both prototypes and counterexamples of the antecedent and the consequent. This approach is a natural generalization for Łukasiewicz's three-valued logic $Ł_3$ of the notion of logical consequence that preserves truth-degrees ($\models^{\leqslant}$). In this paper, we define a graded modal extension of Łukasiewicz's three-valued logic $Ł_3$, where modalities $\Diamond_a \varphi$ stand for *approximately* φ *to the degree at least* a, where a is a value in a finite scale $G \subset [0,1]$, and we show how one can interpret in this modal framework the above mentioned approximate consequences preserving both prototypes and counterexamples.

The paper is structured as follows. After this short introduction, Sections 2 and Section 3 are devoted to briefly recall material from [20]. Namely, in Section 2 we overview a logical approach to reason with vague concepts represented by examples and counterexamples based on the three-valued Łukasiewicz logic $Ł_3$, while in Section 3 we characterize three similarity-based graded notions of approximate logical consequence among vague propositions. Finally, in Section 4 we formally define a multi-modal extension of $Ł_3$ to capture reasoning about the approximate consequences, and prove its completeness. We end up with some conclusions.

2 Prototypes, counter-examples and borderline cases: a simple 3-valued model for gradual properties

2.1 A 3-valued approach

A vague property, in the sense of gradual, is characterized by the existence of borderline cases; that is, objects or situations for which the property only partially applies.

In a recent paper [20], the authors investigate how a logic for vague concepts can be defined starting from the most basic description of a vague property or concept φ in terms of two subsets of the set Ω of situations: the set of *examples* –situations where φ definitely applies– and the set of *counterexamples* –situations where φ does not apply for sure–, denoted $[\varphi^+]$ and $[\varphi^-]$ respectively.

The consistency condition $[\varphi^+] \cap [\varphi^-] = \emptyset$ is assumed to hold. Further, the remaining set of situations $[\varphi^\sim] = \Omega \setminus ([\varphi^+] \cup [\varphi^-])$ are assumed to be those where φ only partially applies, that is, the set of borderline cases. In such a scenario, one is led to a three-valued framework, where for each situation $w \in \Omega$, either $w \in [\varphi^+]$, $w \in [\varphi^-]$ or $w \in [\varphi^\sim]$.[1]

To define a logic to deal with these 3-valued concepts, a first question is how to define the prototypes and counter-examples of compound concepts from their basic constituents. Let us consider a language with four connectives: conjunction ($\wedge$), disjunction ($\vee$), negation ($\neg$) and implication ($\rightarrow$). The rules for $\wedge$, $\vee$ and $\neg$ seem clear to be given as follows:

$$
\begin{array}{ll}
[(\varphi \wedge \psi)^+] = [\varphi^+] \cap [\psi^+], & [(\varphi \wedge \psi)^-] = [\varphi^-] \cup [\psi^-], \\
[(\varphi \vee \psi)^+] = [\varphi^+] \cup [\psi^+], & [(\varphi \vee \psi)^-] = [\varphi^-] \cap [\psi^-], \\
[(\neg\varphi)^+] = [\varphi^-], & [(\neg\varphi)^-] = [\varphi^+].
\end{array}
$$

The case for $\rightarrow$ is not as straightforward as for the previous connectives since several choices can be considered. For instance, if w is a borderline case for both φ and ψ then it can be considered as an example for $\varphi \rightarrow \psi$ rather than a borderline case of $\varphi \rightarrow \psi$. The former choice leads to the well-known three-valued Łukasiewicz logic, while the latter would lead to Kleene's three-valued logic (see e.g. [14] for a relevant discussion on three-valued logical representations of imperfect information). In this paper, we follow [20] and will use the three-valued Łukasiewicz logic Ł$_3$ as base logic to reason with vague concepts, we will thus stick to the following rule for $\rightarrow$:

$$
[(\varphi \rightarrow \psi)^+] = [\varphi^-] \cup [\psi^+] \cup ([\varphi^\sim] \cap [\psi^\sim]), \quad [(\varphi \rightarrow \psi)^-] = [\varphi^+] \cap [\psi^-].
$$

[1] It is worth noticing that in this 3-valued model, the set $[\varphi^\sim]$ is not meant to represent the situations the agent does not know whether φ applies or not; rather it is meant to represent the situations where the concept only partially applies, or equivalently, the situation that are borderline cases for the concept φ (see [17] for a discussion on this topic).

2.2 A refresher on 3-valued Łukasiewicz logic Ł$_3$

Let us briefly recall the formal logical framework of the 3-valued Łukasiewicz logic Ł$_3$, see e.g. [13, 24]. Let *Var* denote a (finite) set of atomic concepts, or propositional variables, from which compound concepts (or formulas) are built using the connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$. We will denote the set of formulas by $\mathscr{L}_3(Var)$, or simply by $\mathscr{L}_3$ in case of no doubt. Further, we identify the set of all possible situations Ω with the set of all evaluations w of atomic concepts *Var* into the truth set $\{0,1/2,1\}$, that is, $\Omega = \{0,1/2,1\}^{Var}$, with the following intended meaning: for every $\alpha \in Var$, $w(\alpha) = 1$ means that w is an example of α (resp. w is a model of α in logical terms), $w(\alpha) = 0$ means that w is a counterexample of α (resp. w is a counter-model of α), and $w(\alpha) = 1/2$ means that w is borderline situation for α, i.e. it is neither an example nor a counterexample. According to the previous discussion, truth-evaluations w will be extended to compound concepts according to the semantics of Ł$_3$, defined by the following truth-functions:[2] for all $x, y \in \{0,1/2,1\}$,

$$x \wedge y = \min(x,y), \quad x \vee y = \max(x,y), \quad x \rightarrow y = \min(1, 1-x+y), \quad \neg x = 1-x.$$

In Ł$_3$, strong conjunction and disjunction connectives can be defined from $\rightarrow$ and $\neg$ as follows: for all $\varphi, \psi \in \mathscr{L}_3$, $\varphi \otimes \psi := \neg(\varphi \rightarrow \neg\psi)$ and $\varphi \oplus \psi := \neg\varphi \rightarrow \psi$.[3] Actually, for each concept $\varphi \in \mathscr{L}_3$, three related *Boolean* concepts can be defined using the connective $\otimes$:

$$\varphi^+ := \varphi \otimes \varphi, \quad \varphi^- := (\neg\varphi) \otimes (\neg\varphi) = (\neg\varphi)^+, \quad \varphi^\sim := \neg\varphi^+ \wedge \neg\varphi^-,$$

with the following semantics:

$$\begin{array}{llll} w(\varphi^+) = 1 & \text{if} \quad w(\varphi) = 1, & w(\varphi^+) = 0 & \text{otherwise;} \\ w(\varphi^\sim) = 1 & \text{if} \quad w(\varphi) = 1/2, & w(\varphi^\sim) = 0 & \text{otherwise;} \\ w(\varphi^-) = 1 & \text{if} \quad w(\varphi) = 0, & w(\varphi^-) = 0 & \text{otherwise.} \end{array}$$

Therefore, if for $\star \in \{+,-,\sim\}$ we let $[\varphi^\star] = \{w \in \Omega \mid w(\varphi^\star) = 1\}$, then $[\varphi^+], [\varphi^-], [\varphi^\sim]$ capture respectively the (classical) sets of examples, counterexamples and borderline cases of φ.

The usual notion of logical consequence in 3-valued Łukasiewicz logic is defined as follows: for any set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \models \varphi \quad \text{if,} \quad \text{for any evaluation } w,\ w(\psi) = 1 \text{ for all } \psi \in \Gamma, \text{ then } w(\varphi) = 1.$$

It is well known that this consequence relation can be axiomatized by the following axioms and rule (see e.g. [13]):

(Ł1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
(Ł2) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$

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[2] We use the same symbols of connectives to denote their corresponding truth-functions.

[3] One could take $\rightarrow$ and $\neg$ as the only primitive connectives since $\wedge$ and $\vee$ can be defined from $\rightarrow$ and $\neg$ as well: $\varphi \wedge \psi = \varphi \otimes (\varphi \rightarrow \psi)$ and $\varphi \vee \psi = (\varphi \rightarrow \psi) \rightarrow \psi$.

(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
(Ł4) $(\varphi \vee \psi) \to (\psi \vee \varphi)$
(Ł5) $\varphi \oplus \varphi \leftrightarrow \varphi \oplus \varphi \oplus \varphi$
(MP) The rule of modus ponens: $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$

This axiomatic system, denoted $Ł_3$, is strongly complete with respect to the above semantics; that is, for a set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$, where $\vdash$, the notion of proof for $Ł_3$, is defined from the above axioms and rule in the usual way.

Remark: In the sequel we will restrict ourselves on considerations about logical consequences from a *finite* set of premises. In such a case, if $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ then $\Gamma \models \psi$ iff $\varphi_1 \wedge \ldots \wedge \varphi_n \models \psi$, and hence it will be enough to consider premises consisting of a single formula.

2.3 Dealing with both prototypes and counter-examples: the logic $Ł_3^{\leqslant}$

It is evident that, for any formulas φ, ψ, $\varphi \models \psi$ can be equivalently expressed as $[\varphi^+] \subseteq [\psi^+]$. This makes clear that $\models$ is indeed the consequence relation that preserves the examples of concepts. Similarly one can also consider the consequence relation that preserves counterexamples [20]. Namely, one can contrapositively define a falsity-preserving consequence as:

$$\begin{array}{lll} \varphi \models^C \psi & \text{if} & \neg\psi \models \neg\varphi, \text{ that is,} \\ & \text{if} & \text{for any evaluation } w,\ w(\psi) = 0 \text{ implies } w(\varphi) = 0. \end{array}$$

Unlike classical logic, in 3-valued Łukasiewicz logic it is not the case that $\varphi \models \psi$ iff $\neg\psi \models \neg\varphi$. As we have seen that the former amounts to require $[\varphi^+] \subseteq [\psi^+]$, while the latter, as shown next, amounts to require $[\psi^-] \subseteq [\varphi^-]$. Clearly these conditions, in general, are not equivalent, except when φ and ψ do not have borderline cases, that is, when $[\varphi^+] \cup [\varphi^-] = [\psi^+] \cup [\psi^-] = \Omega$.

Equivalently, $\varphi \models^C \psi$ holds iff $[\psi^-] \subseteq [\varphi^-]$, and iff for any evaluation $v \in \Omega$, $v(\varphi) \geqslant 1/2$ implies $v(\psi) \geqslant 1/2$, or in other words, $[\varphi^+] \cup [\varphi^\sim] \subseteq [\psi^+] \cup [\psi^\sim]$. Now we define the consequence relation that preserves both examples and counterexamples in the natural way.

Definition 2 $\varphi \models^{\leqslant} \psi$ if $\varphi \models \psi$ and $\varphi \models^C \psi$, that is, if $[\varphi^+] \subseteq [\psi^+]$ and $[\psi^-] \subseteq [\varphi^-]$.

Note that $\varphi \models^{\leqslant} \psi$ iff, for any $w \in \Omega$, $w(\varphi) \leqslant w(\psi)$, that justifies the use of the superscript $\leqslant$ in the symbol of consequence relation. Indeed, the consequence relation $\models^{\leqslant}$ is known in the literature as the *degree-preserving* companion of $\models$, as opposed to the *truth-preserving* consequence $\models$, that preserves the truth-value '1' , see e.g. [3].

$\models^{\leqslant}$ can also be axiomatized by taking as axioms those of $Ł_3$ and the following two inference rules:

$$(Adj): \frac{\varphi, \quad \psi}{\varphi \wedge \psi} \qquad (rMP): \frac{\varphi, \quad \vdash \varphi \to \psi}{\psi}$$

The resulting logic is denoted by $Ł_3^{\leqslant}$, and its notion of proof is denoted by $\vdash^{\leqslant}$. Notice that (rMP) is a weakened version of modus ponens, called restricted modus ponens, since $\varphi \to \psi$ has to be a theorem of $Ł_3$ for the rule to be applicable.

Therefore, $Ł_3^{\leqslant}$ (and its semantical counterpart $\models^{\leqslant}$) appears as a more natural logical framework to reason about concepts described by examples and counterexamples than the usual three-valued Łukasiewicz logic $Ł_3$.

3 A similarity-based refined framework

In the previous section we have discussed a logic for reasoning about vague concepts described in fact as 3-valued fuzzy sets. A more fine grained representation, moving from 3-valued to $[0,1]$-valued fuzzy sets, can be introduced by assuming the availability of a (fuzzy) similarity relation $S : \Omega \times \Omega \to [0,1]$ among situations. Indeed, for instance, assume that all examples of φ are examples of ψ, but some counterexamples of ψ are not counterexamples of φ. Hence, we cannot derive that ψ follows from φ according to $\models^{\leqslant}$. However, if these counterexamples of ψ greatly resemble to counterexamples of φ, it seems reasonable to claim that ψ follows *approximately* from φ.

Actually, starting from Ruspini's seminal work [31], a similar approach has already been investigated in the literature in order to extend the notion of entailment in classical logic in different frameworks and using formalisms, see e.g. [22]. Following this line, here we recall from [20] a graded generalization of the $\models^{\leqslant}$ in the presence of similarity relation S on the set of 3-valued Łukasiewicz interpretations Ω, that allows to draw approximate conclusions.

Since, by definition $\varphi \models^{\leqslant} \psi$ if both $\varphi \models \psi$ and $\varphi \models^{C} \psi$, that is, if $[\varphi^+] \subseteq [\psi^+]$ and $[\psi^-] \subseteq [\varphi^-]$, it seems natural to define that ψ is an approximate consequence of φ to some degree $a \in [0,1]$ when every example of φ is similar (at least to the degree a) to some example of ψ, as well as every counterexample of ψ is similar (to at least to the degree a) to some counterexample of φ. In other words, this means that to relax $\models^{\leqslant}$ we propose to relax both $\models$ and $\models^{C}$. This idea is formalized next, where we assume that a $*$-similarity relation $S : \Omega \times \Omega \to [0,1]$ is given. Moreover, for any subset $A \subseteq \Omega$ and value $a \in [0,1]$ we define its *a-neighborhood* as

$$A^a = \{w \in \Omega \mid \text{there exists } w' \in A \text{ such that } S(w,w') \geqslant a\}.$$

Definition 3 For any pair of formulas φ, ψ and for each degree $a \in [0,1]$, we define the graded consequence relations $\models_a, \models_a^{C}$ and $\models_a^{\leqslant}$ as follows:

(i) $\varphi \models_a \psi$ if $[\varphi^+] \subseteq [\psi^+]^a$
(ii) $\varphi \models_a^{C} \psi$ if $[\psi^-] \subseteq [\varphi^-]^a$
(iii) $\varphi \models_a^{\leqslant} \psi$ if both $[\varphi^+] \subseteq [\psi^+]^a$ and $[\psi^-] \subseteq [\varphi^-]^a$.

Taking into account that for any formula χ we have $[(\neg\chi)^+] = [\chi^-]$, it is clear that both $\models^C_a$ and $\models^{\leqslant}_a$ can be expressed in terms of $\models^a$. Namely, $\varphi \models^C_a \psi$ iff $\neg\psi \models_a \neg\varphi$ and $\varphi \models^{\leqslant}_a \psi$ iff $\varphi \models_a \psi$ and $\neg\psi \models_a \neg\varphi$.

The consequence relations $\models_a$'s are very similar to the so-called approximate graded entailment relations defined in [18] and further studied in [22]. The main difference is that in [18] the authors consider classical propositions while here we consider three-valued Łukasiewicz propositions. Nevertheless, as shown in [20], one can prove very similar characterizing properties for the $\models^a$'s. In the following theorem we assume the language is built from a *finite* set of propositional variables *Var*, and for each evaluation $w \in \Omega$, $\overline{w}$ denotes the following proposition:

$$\overline{w} = \Big(\bigwedge_{p\in Var:\ w(p)=1} p^+\Big) \wedge \Big(\bigwedge_{p\in Var:\ w(p)=1/2} p^\sim\Big) \wedge \Big(\bigwedge_{p\in Var:\ w(p)=0} p^-\Big).$$

So, $\overline{w}$ is a (Boolean) formula which encapsulates the complete description provided by w. Moreover, for every $w' \in \Omega$, $w'(\overline{w}) = 1$ if $w' = w$ and $w'(\overline{w}) = 0$ otherwise.

Theorem 1 *([20]) The following properties hold for the family $\{\models_a\}_{a\in[0,1]}$ of graded entailment relations on $\mathscr{L}_3$ induced by a $*$-similarity relation S on Ω:*

(i) Nestedness: if $\varphi \models_a \psi$ and $b \leqslant a$, then $\varphi \models_b \psi$
(ii) $\models_1$ coincides with $\models$, while $\models \subsetneq \models_a$ if $a < 1$. Moreover, if $\psi \not\models \bot$, then $\varphi \models_0 \psi$ for any φ.
(iii) Positive-preservation: $\varphi \models_a \psi$ iff $\varphi^+ \models_a \psi^+$
(iv) $*$-Transitivity: if $\varphi \models_a \psi$ and $\psi \models_b \chi$ then $\varphi \models_{a*b} \chi$
(v) Left-OR: $\varphi \vee \psi \models_a \chi$ iff $\varphi \models_a \chi$ and $\psi \models_a \chi$
(vi) Restricted Right-OR: for all $w \in \Omega$, $\overline{w} \models_a \varphi \vee \psi$ iff $\overline{w} \models_a \varphi$ or $\overline{w} \models_a \psi$
(vii) Restricted symmetry: for all $w, w' \in \Omega$, $\overline{w} \models_a \overline{w'}$ iff $\overline{w'} \models_a \overline{w}$
(viii) Consistency preservation: if $\varphi \not\models \bot$ then $\varphi \models_a \bot$ only if $a = 0$
(ix) Continuity from below: If $\varphi \models_a \psi$ for all $a < b$, then $\varphi \models_b \psi$

*Conversely, for any family of graded entailment relations $\{\vdash_a\}_{a\in[0,1]}$ on $\mathscr{L}_3$ satisfying the above properties, there exists a *-similarity relation S such that $\vdash_a$ coincides with $\models_a$ for each $a \in [0,1]$.*

Actually, the above properties also indirectly characterize $\models^{\leqslant}_a$ since, in the finite setting, $\models_a$ (and thus $\models^C_a$ as well) can be derived from $\models^{\leqslant}_a$ in the following sense: $\varphi \models_a \psi$ holds iff for every $w \in \Omega$ such that $w(\varphi) = 1$ there exists $w' \in \Omega$ such that $w(\psi) = 1$ and $\overline{w} \models^{\leqslant}_a \overline{w'}$.

However, a nicer characterization of $\models^{\leqslant}_a$ can be obtained if we extend the language of Ł$_3$ with the truth-constant $\overline{\frac{1}{2}}$.

Lemma 1 *([20]) For any formulas in the expanded language, the following conditions hold:*

- $\varphi \models_a \psi$ *iff* $\varphi \models^{\leqslant}_a \psi \vee \overline{\frac{1}{2}}$
- $\varphi \models^C_a \psi$ *iff* $\varphi \wedge \overline{\frac{1}{2}} \models^{\leqslant}_a \psi$

As a consequence, we have that $\varphi \models_a^{\leqslant} \psi$ *iff* $\varphi \models_a^{\leqslant} \psi \vee \overline{\frac{1}{2}}$ *and* $\varphi \wedge \overline{\frac{1}{2}} \models_a^{\leqslant} \psi$.

From this, one can get the following representation for the $\models_a^{\leqslant}$ consequence relations.

Theorem 2 *([20]) Let* $\{\vdash_a^{\leqslant}\}_{a \in G}$ *be a set of consequence relations on the expanded language satisfying the following conditions:*

- $\varphi \vdash_a^{\leqslant} \psi$ *iff* $\neg\psi \vdash_a^{\leqslant} \neg\varphi$
- $\varphi \vdash_a^{\leqslant} \psi$ *iff* $\varphi \vdash_a^{\leqslant} \psi \vee \overline{\frac{1}{2}}$ *and* $\varphi \wedge \overline{\frac{1}{2}} \vdash_a^{\leqslant} \psi$
- *The set of relations* $\{\vdash_a\}_{a \in G}$*, where* $\varphi \vdash_a \psi$ *is defined as* $\varphi \vdash_a^{\leqslant} \psi \vee \overline{\frac{1}{2}}$*, satisfy the conditions (i)-(ix) of Theorem 1.*

Then, there exists a similarity relation $S : \Omega \times \Omega \to G$ *such that* $\vdash_a^{\leqslant} = \models_a^{\leqslant}$*, for any* $a \in G$*.*

4 A multi-modal approach to reason about the similarity-based graded entailments

As we have seen in Section 3, the three kinds of similarity-based graded entailments $\models_a$, $\models_a^C$ and $\models_a^{\leqslant}$ are based on an idea of *neighbourhood* of a set of interpretations (see Def. 3) . Indeed, for instance, given a $*$-similarity relation S on the set of interpretations Ω, the idea of a graded entailment $\varphi \models_a \psi$ is to replace the classical constraint that the set $[\varphi^+]$ of models (prototypes) of φ have to be included in the set $[\psi^+]$ of models (prototypes of ψ, to a more relaxed condition in the sense that $[\varphi^+]$ needs only to be included in the neighbourhood of $[\psi^+]$ of radius a (i.e. each model of φ has to be at least a-similar to some model of ψ). Similarly with the other graded entailments $\models_a^C$ and $\models_a^{\leqslant}$.

In the framework of relational semantics for modal logic, such neighbourhoods can be nicely captured by a certain class of generalized Kriple frames of the form (W,S), where W is a set of possible worlds and S a $*$-similarity relation on W. Then S induces a nested set of binary accessibility relations on W: for each value a, we can define the accessibility relation S_a among interpretations (or worlds in the modal logic terminology) in the natural way: $(w,w') \in S_a$ if $S(w,w') \geqslant a$, that is, S_a is the a-cut of S. In fact, for each value a, (W,S_a) is a classical Kripke frame, and the semantics of a corresponding possibility modal formula $\Diamond_a \varphi$ exactly corresponds to the notion of a-neighbourhood of (the set of models of) φ (see for instance [21] for the case of dealing with classical propositions.)[4]

[4] Note that this is not to be confused with the so-called *neighbourhood semantics* (also known as Scott-Montague semantics) for modal logics (see e.g. [12]), a more general semantics that, instead of using relational frames (W,R) consisting of a set W of worlds and an accessibility relation R, it is based on neighborhood frames (W,N), where N is a neighborhood function $N : W \to 2^{2^W}$ assigning to each possible world of W a set of subsets of W.

In this section we will define a multi-modal logic over 3-valued Łukasiewicz logic $Ł_3$ expanded with the truth-constant $\overline{\frac{1}{2}}$, and we will see how the above notions of graded entailment can be faithfully captured in this logic. Actually, this 3-valued modal logic is a richer framework with a more expressive power. To avoid unnecessary complications, we will make the following assumptions: all $*$-similarity relations S will take values in a finite set $G \subset [0,1]$, containing 0 and 1, and $*$ will be a *finite* t-norm operation on G, that is, $(G,*,\leqslant,1,0)$ will be a finite totally ordered semi-group. In this way, we keep our language finitary and will avoid the use of an infinitary inference rule to cope with Property (ix) of Theorem 1. Then, we will expand the propositional language of $Ł_3$ with the (finite) set of modal operators $\{\Box_a : a \in G\}$ ($\Diamond_a$ will be used as abbreviations of $\neg\Box_a\neg$) by means of the usual rules. We will denote the modal language by $\mathscr{L}_\Box$.

For a given finite t-norm $(G,*,1,0)$ as above, the semantics will be given by $*$-similarity Kripke frames (W,S), where W is a set of possible worlds and $S : W \times W \to G$ is a *strict* $*$-similarity relation. A 3-valued $*$-similarity Kripke model is a structure $M = (W,S,e)$, where (W,S) is as above and $e : W \times Var \to \{0,1/2,1\}$ is a 3-valued evaluation of propositional variables for each possible world. An evaluation $e(w,\cdot)$ is extended to arbitrary formulas using the truth-functions of $Ł_3$ with the following special stipulations:

- $e(w,\overline{1/2}) = 1/2$
- $e(w,\Box_a\varphi) = \min\{e(w',\varphi) \mid (w,w') \in S_a\}$
- $e(w,\Diamond_a\varphi) = \max\{e(w',\varphi) \mid (w,w') \in S_a\}$

where, as already mentioned, $S_a = \{(w,w') \mid w,w' \in W, S(w,w') \geqslant a\}$. The corresponding notions of satisfiability, validity and consequence are respectively as follows:

- $(M,w) \models \varphi$ if $e(w,\varphi) = 1$.
- $M \models \varphi$ if $(M,w) \models \varphi$ for every $w \in W$.

Given a similarity scale $(G,*,1,0)$, we will denote by $\mathsf{SK}(G)$ the class of similarity Kripke models $M = (W,S,e)$ where S is a G-valued *-strict similarity relation on W. Then, we can finally define the notion of consequence relative to $\mathsf{SK}(G)$: for any set of formulas $\Gamma \cup \{\varphi\}$,

- $\Gamma \models_{\mathsf{SK}(G)} \varphi$ if, for every model $M = (W,S,e) \in \mathsf{SK}(G)$ and world $w \in W$, $(M,w) \models \psi$ for every $\psi \in \Gamma$ implies $(M,w) \models \varphi$.

Next, we aim at providing a complete aximatization for this 3-valued modal logic. We start by observing that each modality $\Box_a$ is interpreted in a Kripke model $M = (W,S,e)$ by the a-cut S_a of the similarity relation, i.e. each graded modality has associated a crisp accessibility relation S_a. Thus, we have to look at what properties these crisp relations S_a have. It is clear that S_a is a reflexive and symmetric relation for every $a \in G$. Moreover, due to the $*$-transitivity property of S, the following transitivity-like inclusions hold for any $a,b \in G$: $S_a \circ S_b \subseteq S_{a*b}$, where $\circ$ denotes usual composition of relations. Hence, if $a * a = a$, the relation S_a is also transitive. Therefore, each operator $\Box_a$ is a sort of 3-valued KTB modality, as the following well-known axioms are valid in every Kripke model for all $a \in G$:

(K_a) $\Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$
(T_a) $\Box_a\varphi \to \varphi$
(B_a) $\varphi \to \Box_a\Diamond_a\varphi$

while the following generalized form of Axiom 4 is also valid for every $a, b \in G$:

($4_{a,b}$) $\Box_{a*b}\varphi \to \Box_b(\Box_a\varphi)$, or equivalently $\Diamond_a(\Diamond_b\varphi) \to \Diamond_{a*b}\varphi$.

Thus, if $a * a = a$ (e.g. for $a = 0$ or $a = 1$ at least), then $\Box_a$ can be considered as 3-valued S5 modality as it satisfies many-valued versions of Axioms K, T, B and 4 axioms.

Moreover, since the S_a's relations form a nested set, in the sense that $S_a \subseteq S_b$ if $a \geqslant b$, then any Kripke model validates the following axiom:

($N_{a,b}$) $\Box_b\varphi \to \Box_a\varphi$, if $a \geqslant b$.

Finally, let us observe the boundary properties of the relations S_a (when $a = 0$ or $a = 1$), namely: $S_0 = W \times W$ and $S_1 = \{(w, w) \mid w \in W\}$.

These properties and existing results for many-valued modal logics in general [4] and for some modal extensions of finite-valued Łukasiewicz logics $Ł_n$ [1] lead to the definition of the following axiomatic system.

Definition 4 Given a finite similarity scale $(G, *, 1, 0)$, the axiomatic system $Ł_3^{\Box}(G)$ consists of the following groups of axioms and rules, where the subindices a, b run over G:

($Ł_3$) axioms of $Ł_3$,
(bk) $\neg(\overline{1/2}) \leftrightarrow \overline{1/2}$
(K_a) $\Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$,
($A_a^{\wedge}$) $(\Box_a\varphi \wedge \Box_a\psi) \to \Box_a(\varphi \wedge \psi)$,
(A_a^c) $\Box_a(\overline{1/2} \to \varphi) \leftrightarrow (\overline{1/2} \to \Box_a\varphi)$,
($A_a^{\oplus}$) $(\Box_a\varphi \oplus \Box_a\varphi) \leftrightarrow \Box_a(\varphi \oplus \varphi)$,

(T_a) $\Box_a\varphi \to \varphi$
(B_a) $\varphi \to \Box_a\Diamond_a\varphi$,
($4_{a,b}$) $\Box_{a*b}\varphi \to \Box_b(\Box_a\varphi)$

($N_{a,b}$) $\Box_b\varphi \to \Box_a\varphi$, if $b \leqslant a$
(C_1) $\varphi \to \Box_1\varphi$

(MP) from φ and $\varphi \to \psi$ derive ψ
(Nec) from φ derive $\Box_a\varphi$

We will denote by $\vdash_{Ł_3^{\Box}(G)}$ the notion of proof in $Ł_3^{\Box}(G)$ defined as usual from the above axioms and rules.

The first group of axioms ($Ł_3$)-($A_a^{\oplus}$) corresponds to the axiomatization of each $\Box_a$ for the minimal modal logic over $Ł_3$ with the truth constant $\overline{1/2}$ with respect to the semantics with crisp accessibility relations, see [4]. The second group aims at capturing the three characteristic properties of the relations S_a above mentioned, namely reflexivity, symmetry and $*$-transitivity, see e.g. [1]. Finally, $(N_{a,b})$ captures the nestedness of the graded operators, while (C_1) aims at capturing the particular behaviour of the modal operator for the extremal value $a=1$: $\Box_1\varphi$ collapses with φ itself since for any strict similarity $S(w,w')=1$ iff $w=w'$.

Now we can prove that the axiomatic system $Ł_3^{\Box}(G)$ is indeed complete with respect to the intended semantics, that is, the class of similarity Kripke models $\mathsf{SK}(G)$.

Theorem 3 *$Ł_3^{\Box}(G)$ is complete w.r.t. the class of models $\mathsf{SK}(G)$, that is, for any set of formulas $\Gamma\cup\{\varphi\}$, $\Gamma\models_{\mathsf{SK}(G)}\varphi$ iff $\Gamma\vdash_{Ł_3^{\Box}(G)}\varphi$.*

Proof (Sketch) We have to prove that if $\Gamma\not\vdash_{Ł_3^{\Box}(G)}\varphi$ there is a model $M\in\mathsf{SK}(G)$ and a world $w\in W$ such that $(M,w)\models\psi$ for every $\psi\in\Gamma$, but $(M,w)\not\models\varphi$.

Actually, we will make the proof in two steps. In the first step we will build a canonical model $M^c=(W^c,S^c,e^c)$, where S^c is a (non-necessarily strict) $*$-similarity relation on W^c, and a world $v\in W^c$ such that $(M^c,v)\models\psi$ for every $\psi\in\Gamma$, but $(M^c,v)\not\models\varphi$. For this, we basically use the results and proofs of [1] where the authors define KD45 and S5-like modal logics over a k-valued Łukasiewicz logic $Ł_k$.

Consider for each modal formula φ its propositional counterpart φ^* by treating subformulas of the form $\Box_a\psi$ for some $a\in G$ as new propositional variables. So formally, we can consider a propositional language $\mathscr{L}_3(Var^*)$ built from the extended set of variables $Var^*=Var\cup\{(\Box_a\psi)^*\mid\psi\in\mathscr{L}_3(Var),a\in G\}$, and let Ω^* the set of $Ł_3$-evaluations over the fomulas from $\mathscr{L}_3(Var^*)$. Then, the canonical model $M^c=(W^c,S^c,e^c)$ is defined as follows:

- $W^c=\{w\in\Omega^*\mid\forall\varphi^*\in\Lambda: w(\varphi^*)=1\}$ with $\Lambda=\{\varphi^*\mid\vdash_{Ł_3}\varphi\}$;
- $S^c_a=\{(w_1,w_2)\in\Omega^*\times\Omega^*\mid\forall\varphi\in\mathscr{L}_3(Var):\text{ if } w_1((\Box_a\varphi)^*)=1\text{ then } w_2(\varphi^*)=1\}$;
- $S^c(w_1,w_2)=\max\{a\in G\mid(w_1,w_2)\in S^a_c\}$, for all $w_1,w_2\in W^c$;
- $e^c(w,p)=w(p)$ for each variable $p\in Var$.

Then, using the same techniques in [1], we can show that the fundamental *Truth Lemma* for the canonical model M^c, namely, for any modal formula φ and any $w\in W^c$, it holds that

$$e^c(w,\varphi)=w(\varphi^*).$$

Moreover, following [1] one can show that, so defined, S^c is reflexive, symmetric and $*$-transitive, and that $S^c(w,w')=1$ iff $w=w'$ thanks to Axiom $(C1)$.[5] However, even if S^c_0 turns out to be an equivalence relation, we cannot guarantee that $S^c_0=W^c\times W^c$ and hence we do not know whether $M^c\in\mathsf{SK}(G)$.

To remedy this problem, we need a further step. For this we need a previous lemma that shows that deductions in $Ł_3^{\Box}(G)$ can be reduced to propositional deductions in $Ł_3$.

[5] By definition, $(w,w')\in S^c_1$ iff for all φ, $w(\Box_1\varphi)=1$ implies $w'(\varphi)=1$, equivalent to the condition: for all φ, $w(\Box_1\varphi)\leqslant w'(\varphi)$. But by definition of W^c, $w(\Box_1\varphi)=w(\varphi)$, thus $(w,w')\in S^c_1$ iff for all φ, $w(\varphi)\leqslant w'(\varphi)$. Since S^c_1 is symmetric, then $(w,w')\in S^c_1$ iff for all φ, $w(\varphi)=w'(\varphi)$, therefore, iff $w=w'$.

Lemma. For any set of modal formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{Ł_3^\Box(G)} \varphi$ iff $\Gamma^* \cup \Lambda \vdash_{Ł_3} \varphi^*$

where Γ is defined as above. The proof is rather standard and it is omitted.

Continuing with the main proof, assume $\Gamma \not\vdash_{Ł_3^\Box(G)} \varphi$. Then by the above lemma and propositional completeness of $Ł_3$, there is a $Ł_3$-evaluation v such that $v(\Gamma^*) = v(\Lambda) = 1$ and $v(\varphi^*) < 1$. By definition, $v \in W^c$, and thus we can consider its equivalence class w.r.t. S_0^c, i.e. $W^0 = \{w \in W^c \mid (v,w) \in S_0^c\}$. Then, an easy computation shows that $M^0 = (W^0, S^0, e^0)$, where $S^0 = S^c$ and $e^0 = e^c$, belongs to the class $\mathsf{SK}(G)$, and by the Truth Lemma it preserves the evaluations. In particular, we have $e^0(v,\psi) = v(\psi^*) = 1$ for each $\psi \in \Gamma$, while $e^0(v,\varphi) = v(\varphi^*) < 1$. In other words, we have shown that $\Gamma \not\models_{\mathsf{SK}(G)} \varphi$. This finishes the proof. □

Finally, we can show that the graded entailments defined by similarity reasoning over $Ł_3$ can be faithfully captured in the multi-modal logic $Ł_3^\Box(G)$. Intuitively, consider an instance of a graded entailment like "$\varphi \models_a \psi$", whose intended meaning is "for all $w \in \Omega$, if $w \models \varphi$ then there exists w' such that $S(w,w) \geqslant a$ and $w' \models \psi$". To encode this condition in the modal framework, we use the universal modality $\Box_0$ to model the "for all $w \in \Omega$", while the rest of condition "if $w \models \varphi$ then there exists w' such that $S(w,w) \geqslant a$ and $w' \models \psi$" can be naturally encoded by the formula "$\varphi \to \Diamond_a \psi$". Therefore, we can encode "$\varphi \models_a \psi$" by the $Ł_3^\Box(G)$-formula $\Box_0(\varphi \to \Diamond_a \psi)$. In summary, the translations are as follows:

Graded consequences	$Ł_3^\Box(G)$-formulas
$\varphi \models_a \psi$	$\Box_0(\varphi \to \Diamond_a \psi)$
$\varphi \models_a^C \psi$	$\Box_0(\neg\varphi \to \Diamond_a \neg\psi)$, or $\Box_0(\Box_a \psi \to \varphi)$
$\varphi \models_a^{\leqslant} \psi$	$\Box_0((\varphi \to \Diamond_a \psi) \wedge (\Box_a \psi \to \varphi))$

5 Conclusions and dedication

In this paper we have first recalled an approach towards considering graded approximate entailments between vague concepts (or propositions) based on the similarity between both prototypes and counterexamples of the antecedent and the consequent, presented in [20]. This approach is a natural generalization for Łukasiewicz's three-valued logic $Ł_3$ of the notion consequence that preserves truth-degrees. Then, we have provided a modal logic formalisation, by defining a similarity-based graded modal extension of $Ł_3$, and have shown how this modal framework is expressive enough to accommodate reasoning about instances of the above approximate entailments.

This small paper on similarity-based reasoning is our humble tribute to Bernadette Bouchon-Meunier for her dedication to the field of approximate reasoning. Similarity relations have been one of the many nuclear research topics for Bernadette, as it is witnessed by her numerous and relevant contributions on this subject; here we cite only some representative works of hers relating together the notion of fuzzy similarity with analogical reasoning [10, 7, 6], similarity measures [5, 29, 30], and interpolation of fuzzy rules [9, 23, 8]. Her own remarkable studies and her relentless efforts to bring the computational intelligence and approximate reasoning community together have substantially deepened our knowledge and will continue to impact generations of computer scientists to come.

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