Canonical Extension of Possibility Measures to Boolean Algebras of Conditionals

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Abstract. In this paper we study conditional possibility measures within the algebraic setting of Boolean algebras of conditional events. More precisely, we focus on the possibilistic version of the Strong Conditional Event Problem, introduced for probabilities by Goodman and Nguyen, and solved in finitary terms in a recent paper by introducing the so-called Boolean algebras of conditionals. Our main result shows that every possibility measure on a finite Boolean algebra can be canonically extended to an unconditional possibility measure on the resulting Boolean algebra of conditionals, in such a way that the canonical extension and the conditional possibility, determined in usual terms by any continuous t-norm, coincide on every basic conditional expression.

Keywords: Conditional possibility · Boolean algebras of conditionals · Strong conditional event problem

1 Introduction

The issue of conditioning in the framework of possibility theory has been discussed at large in the literature, starting from the pioneering work by Hisdal [15] and then followed by Dubois, Prade and colleagues (see e.g. [1,9–11]), de Cooman and Walley [8,17] and Coletti and colleagues [2–6] among others.

Comparing to the case of probability, where there is a common agreement on taking the conditional probability $P(a \mid b)$ as the ratio $P(a \wedge b)/P(b)$ (in case $P(b) > 0$), a conditional possibility $\Pi(a \mid b)$ has mainly been defined in the literature in (at least) two different ways: in an ordinal setting, the min-based conditioning sets $\Pi_{\min}(a \mid b) = \Pi(a \wedge b)$ if $\Pi(b) > \Pi(a \wedge b)$ and 1 otherwise; and in a numerical setting, the product-based conditioning defines $\Pi_{prod}(a \mid b)$ as $\Pi(a \wedge b)/\Pi(b)$ if $\Pi(b) > 0$ and 1 otherwise. More in general, if $*$ is a continuous t-norm and $\Rightarrow_*$ is its residuum, one can define $\Pi^+_*(a \mid b) = \Pi(b) \Rightarrow_* \Pi(a \wedge b)$. Such $\Pi^+_*(a \mid b)$ is in fact the greatest solution x of the equation $\Pi(a \wedge b) = x * \Pi(b)$. Notice that the two definitions above correspond to the choice $* =$ minimum t-norm and $* =$ product t-norm. However, as also discussed in the literature, see e.g. [10], not every choice of $*$ is compatible with the satisfaction of some

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rationality postulates for conditional possibility that are counterparts to Cox's axioms for conditional probability, and moreover the definition of Π_*^+ itself has some problems as well, and has to be slightly modified to another expression that we will denote with Π_*.

In this paper we deal with conditional possibility measures within the algebraic setting of Boolean algebras of conditionals [13], a recently proposed setting for measure-free conditionals endowed with the structure of a Boolean algebra. More precisely, we focus on a possibilistic version of the Strong Conditional Event Problem, considered for probabilities by Goodman and Nguyen [14], and solved in finitary terms in [13] with the help of those structures. Indeed, Boolean algebras of conditional events are introduced with the aim of playing for conditional probability a similar role Boolean algebras of plain events play for probability theory. To do so, the authors try to identify which properties of conditional probability depend on the algebraic features of conditional events and which properties are intrinsic to the nature of the measure itself.

Clearly, a similar analysis can be done for the notion of conditional possibility, with the advantage that we can now study the suitability of Boolean algebras of conditionals as a common algebraic framework for conditional events also in relation to conditional possibility, in the sense of checking whether conditional possibilities can be regarded as plain possibility measures on algebras of conditional events. Our main result shows that this is actually the case for a very general notion of conditional possibility. Namely, if $\mathbf{A}$ is any finite Boolean algebra and Π is a possibility measure on $\mathbf{A}$, then we show that, for any continuous t-norm $*$, Π can be always extended to a (unconditional) possibility measure μ_Π on the Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$ built over $\mathbf{A}$, in such a way that, for every basic conditional object $(a \mid b) \in \mathcal{C}(\mathbf{A})$, $\mu_\Pi(`(a \mid b)')$ coincides with the conditional possibility $\Pi_*(a \mid b)$.

This paper is structured as follows. Next Sect. 2 is dedicated to recall basic notions and results on conditional possibility theory, while Boolean algebras of conditionals will be briefly recalled in Sect. 3. Our main result, namely the solution of the possibilistic version of the Strong Conditional Event Problem, will be proved in Sect. 4. Finally, in Sect. 5, we will conclude and present future work on this subject.

2 Possibility and Conditional Possibility Measures

In this section we will recall basic notions and results about (conditional) possibility theory [9,18]. We will assume the reader to be familiar with the algebraic setting of Boolean algebras. For every finite Boolean algebra $\mathbf{A}$, we will henceforth denote by $\mathfrak{at}(\mathbf{A})$ the finite set of its atoms.

A possibility measure on a Boolean algebra $\mathbf{A}$ is a mapping $\Pi : A \to [0, 1]$ satisfying the following properties:

(Π1) $\Pi(\top) = 1$,
(Π2) $\Pi(\bot) = 0$,
(Π3) $\Pi(\bigvee_{i\in I} a_i) = \sup\{\Pi(a_i) \mid i \in I\}$

If $\mathbf{A}$ is finite, which will be our setting in this paper, then ($\Pi 3$) can just be replaced by

$(\Pi 3)^*$ $\Pi(a \vee b) = \max(\Pi(a), \Pi(b))$

In the finite setting, possibility measures are completely determined by their corresponding (normalized) possibility distributions on the atoms of the algebra. Namely, $\Pi : A \to [0, 1]$ is a possibility measure iff there is a mapping $\pi : \mathfrak{at}(\mathbf{A}) \to [0, 1]$ with $\max\{\pi(\alpha) : \alpha \in \mathfrak{at}(\mathbf{A})\} = 1$ such that

$$\Pi(a) = \max\{\pi(\alpha) : \alpha \in \mathfrak{at}(\mathbf{A}), \alpha \leq a\}.$$

Clearly, in such a case $\pi(\alpha) = \Pi(\alpha)$ for each $\alpha \in \mathfrak{at}(\mathbf{A})$.

When we come to define conditional possibility measures there have been several proposals in the literature, see e.g. a summary of them in [17]. Nevertheless, $\Pi(a \mid b)$ has been traditionally chosen as the greatest solution of the equation [10,12]:

$$\Pi(a \wedge b) = x * \Pi(b) \tag{1}$$

where either $* = \min$ or $* =$ product. This leads to these two definitions:

$$\Pi_{\min}(a \mid b) = \begin{cases} \Pi(a \wedge b), \text{ if } \Pi(b) > \Pi(a \wedge b) \\ 1, \quad\quad\quad \text{otherwise} \end{cases}$$

$$\Pi_{prod}(a \mid b) = \begin{cases} \Pi(a \wedge b)/\Pi(b), \text{ if } \Pi(b) > 0 \\ 1, \quad\quad\quad\quad\quad \text{otherwise} \end{cases}$$

$\Pi_{\min}(\cdot \mid \cdot)$ has been known as the qualitative conditioning of Π while $\Pi_{prod}(\cdot \mid \cdot)$ has been known as the quantitative conditioning of Π.

Actually, in a finite setting, both expressions satisfy very intuitive properties a conditional possibility $\Pi(. \mid .)$ should enjoy. For instance, for any $a, b \in A$ and any $\alpha, \gamma \in \mathfrak{at}(\mathbf{A})$ we have:

P1. $\Pi(\cdot \mid b)$ is a possibility measure on $\mathbf{A}$, if $\Pi(b) > 0$
P2. $\Pi(a \mid b) = \Pi(a \wedge b \mid b)$, and hence $\Pi(a \mid b) = 0$ if $a \wedge b = \bot$ and $\Pi(b) > 0$
P3. If $\Pi(\alpha) < \Pi(\gamma)$ then $\Pi(\alpha \mid b) < \Pi(\gamma \mid b)$, if $\alpha, \gamma \leq b$
P4. If $\Pi(b) = 1$ then $\Pi(a \mid b) = \Pi(a \wedge b)$
P5. If $\Pi(a) = 0$ and $\Pi(b) > 0$ then $\Pi(a \mid b) = 0$

Notice however that in case the Boolean algebra $\mathbf{A}$ is infinite, $\Pi_{\min}$ may fail to satisfy P1, that is, $\Pi_{\min}(\cdot \mid b)$ might fail to be a possibility measure in a strict sense, even if $\Pi(b) > 0$, see [7].

Actually, if $*$ is a continuous t-norm, properties P2-P4 keep holding for the conditional possibility measure $\Pi^+_*(\cdot \mid \cdot)$ defined as the maximal solution for x in Eq. (1), in other words, by the conditional possibility defined as

$$\Pi^+_*(a \mid b) = \Pi(b) \Rightarrow_* \Pi(a \wedge b), \tag{2}$$

where $\Rightarrow_*$ is the residuum of $*$, that is the binary operation defined as $x \Rightarrow_* y = \max\{z \in [0,1] \mid x * z \leq y\}$. If $*$ is without zero-divisors,[1] then $\Pi^+_*(\cdot \mid \cdot)$ further satisfies P1 and P5 as well. For every continuous t-norm $*$, the pair $(*, \Rightarrow_*)$ is known as a *residuated pair*. When $* = \min$ or $* =$ product, we obtain the above definitions of $\Pi_{\min}$ and Π_{prod} as particular cases. Note that if $*$ has zero divisors, P1 fails because $\Pi^+_*(\bot \mid b) = \Pi(b) \Rightarrow_* 0$ is not guaranteed to be 0 when $\Pi(b) > 0$ [7,10]. For the same reason P5 can fail as well. For instance, if $*$ is Łukasiewicz t-norm, $\Pi(b) \Rightarrow_* 0 = 1 - \Pi(b) > 0$ when $\Pi(b) < 1$. To avoid this situation and other issues related to the conditioning by an event of measure zero, in [4] the authors considered a modified expression for Π^+_*, called T_{DP}-conditional possibility, on which we base the following definition.

Definition 1. *Given a possibility measure Π on a Boolean algebra* $\mathbf{A}$ *and a continuous t-norm $*$, we define the mapping $\Pi_* : A \times A' \to [0,1]$, where $A' = A \setminus \{\bot\}$, as follows:*

$$\Pi_*(a \mid b) = \begin{cases} \Pi(b) \Rightarrow_* \Pi(a \wedge b), & \text{if } a \wedge b \neq \bot \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

In this way, $\Pi_*(a \mid b) = 0$ whenever $a \wedge b = \bot$, even if $\Pi(b) = 0$ (while $\Pi^+_*(a \mid b) = 1$ in that case). For any continuous t-norm $*$, Π_* keeps satisfying P2-P4, and if $*$ has no zero-divisors then P1 and P5 as well. Moreover, for any continuous t-norm $*$, Π_* also satisfies both P1 and P2 when $\Pi(b) = 0$.

However, if $*$ is a left-continuous t-norm, to start with, (1) might not have a greatest solution. Indeed, if we define Π_* as in (3) for a left-continuous t-norm $*$, we obtain the following characterisation.

Proposition 1. *If $*$ is a left-continuous t-norm, the following are equivalent:*

(i) *$*$ is continuous.*
(ii) *$\Pi_*(a \mid d) * \Pi_*(b \mid a \wedge d) = \Pi_*(a \wedge b \mid d)$, for all possibility measures Π on any Boolean algebra* $\mathbf{A}$, *and for all $a, b, c \in$* $\mathbf{A}$.
(iii) *$\Pi(a \wedge b) = \Pi(a) * \Pi_*(b \mid a)$, for all possibility measures Π on any Boolean algebra* $\mathbf{A}$, *and for all $a, b, c \in$* $\mathbf{A}$.

Proof. That (i) implies (ii) is proved in [4, Prop. 1]. Nevertheless we show here a more compact proof by using the fact that $*$ is continuous iff, together with its residuum $\Rightarrow_*$, it satisfies the divisibility equation $x * (x \Rightarrow_* y) = \min(x, y)$, see e.g. [16]. Notice first that if $a \wedge b \wedge d = \bot$, then $\Pi_*(a \mid d) * \Pi_*(b \mid a \wedge d) = \Pi_*(a \mid d) * 0 = 0 = \Pi_*(a \wedge b \mid d)$, by definition of Π_*. Suppose now that $a \wedge b \wedge d \neq \bot$ (and thus also $a \wedge d \neq \bot$), then

[1] A t-norm $*$ has no zero-divisors when $x * y = 0$ implies either $x = 0$ or $y = 0$, see e.g. [16].

$$\begin{aligned}
\Pi_*(a \mid d) * \Pi_*(b \mid a \wedge d) &= \Pi_*(a \mid d) * (\Pi(a \wedge d) \Rightarrow_* \Pi(a \wedge b \wedge d)) \\
&= \Pi_*(a \mid d) * ((\Pi(d) * \Pi_*(a \mid d)) \Rightarrow_* \Pi(a \wedge b \wedge d)) \\
&= \Pi_*(a \mid d) * (\Pi_*(a \mid d) \Rightarrow_* (\Pi(d) \Rightarrow_* \Pi(a \wedge b \wedge d))) \\
&= \min(\Pi_*(a \mid d), \Pi(d) \Rightarrow_* \Pi(a \wedge b \wedge d)) \\
&= \min(\Pi_*(a \mid d), \Pi_*(a \wedge b \mid d)) \\
&= \Pi_*(a \wedge b \mid d).
\end{aligned}$$

That (ii) implies (iii) can be easily seen by setting $d = \top$. We now show that (iii) implies (i) by contraposition. Suppose that $*$ is not continuous, this means that divisibility does not hold and then there exist $x, y \in [0, 1]$ such that $y < x$ and $x * (x \Rightarrow_* y) < y$. Then we can find a Boolean algebra $\mathbf{A}$ with a possibility measure Π where $\Pi(a) * \Pi_*(b \mid a) < \Pi(a \wedge b)$. Take $\mathbf{A}$ to be the Boolean algebra generated by two elements a and b. The atoms of $\mathbf{A}$ are conjunctions of literals of the generators. We can then define a measure Π by setting $\Pi(a \wedge b) = y$, $\Pi(a \wedge \neg b) = x$, $\Pi(\neg a \wedge \neg b) = \Pi(\neg a \wedge b) = 1$. Then $\Pi(a) = \max(x, y) = x$. Thus, $\Pi(a) * \Pi_*(b \mid a) = \Pi(a) * (\Pi(a) \Rightarrow_* \Pi(a \wedge b)) = x * (x \Rightarrow_* y) < y = \Pi(a \wedge b)$, so (iii) would not hold and the proof is complete. □

Thus, continuity of the t-norm $*$ is equivalent to requiring both usual Bayes' Theorem (iii) and the generalized version of it (ii).

The problem of conditioning by an event with zero measure (already mentioned regarding Definition 1) and other related problems led Coletti and colleagues, see e.g. [2,5,6], to put forward an axiomatic approach to the definition of conditional possibility, similar to the case of conditional probability, where the notion of conditional possibility is primitive and not derived from a (unconditional) possibility. The following definition is basically from [2].

Definition 2. *Given a t-norm $*$, a $*$-conditional possibility*[2] *measure on* $\mathbf{A}$ *is a binary mapping* $\overline{\Pi}(\cdot \mid \cdot) : A \times A' \to [0, 1]$, *where* $A' = A \setminus \{\bot\}$, *satisfying the following conditions:*

($C\Pi1$) $\overline{\Pi}(a \mid b) = \overline{\Pi}(a \wedge b \mid b)$, *for all* $a \in A, b \in A'$
($C\Pi2$) $\overline{\Pi}(\cdot \mid b)$ *is a possibility measure for each* $b \in A'$
($C\Pi3$) $\overline{\Pi}(a \wedge b \mid c) = \overline{\Pi}(b \mid a \wedge c) * \overline{\Pi}(a \mid c)$, *for all* $a, b, c \in A$ *such that* $a \wedge c \in A'$.

Regarding ($C\Pi2$), note that if $\overline{\Pi}(b \mid \top) = 0$ then the possibility measure $\Pi_b(\cdot) = \overline{\Pi}(\cdot \mid b)$ is such that $\Pi_b(a) = 0$ if $a \wedge b = \bot$ and $\Pi_b(a) = 1$ otherwise.

The requirement of the operation $*$ in ($C\Pi3$) to be a t-norm is very natural and has been discussed and justified by several authors, specially in relation to the simpler, particular case $\overline{\Pi}(a \wedge b \mid \top) = \overline{\Pi}(b \mid a) * \overline{\Pi}(a \mid \top)$, see e.g. [2,5,10]. Moreover, although they might be a bit weaker, arguments different from the ones discussed above to further require $*$ to be continuous and without zero-divisors can also be put forward here:

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[2] Called T-conditional possibility in [5,6].

- no zero-divisors: in the case $b \leq a \leq c$ and $(C\Pi 3)$ yields $\overline{\Pi}(b \mid c) = \overline{\Pi}(b \mid a) * \overline{\Pi}(a \mid c)$, then if $*$ has zero-divisors, it can happen that $\overline{\Pi}(b \mid c) = 0$ (which can be read as "$(b \mid c)$ is not possible"), while $\overline{\Pi}(b \mid a)$ and $\overline{\Pi}(a \mid c)$ are both positive, which is unintuitive.
- continuity: roughly speaking, if we require that small changes in $\overline{\Pi}(b \mid a \wedge c)$ and $\overline{\Pi}(a \mid c)$ imply small changes in $\overline{\Pi}(a \wedge b \mid c)$.

Note that when $c = \top$, $(C\Pi 3)$ simplifies to:

$$\overline{\Pi}(a \wedge b \mid \top) = \overline{\Pi}(b \mid a) * \overline{\Pi}(a \mid \top),$$

so that $\overline{\Pi}(a \mid b)$ is a solution of Eq. (1) once we identify $\Pi(.)$ with $\overline{\Pi}(. \mid \top)$. Moreover, for every (unconditional) Π on $\mathbf{A}$, it turns out that Π_* (as defined in Definition 1) is the greatest $*$-conditional possibility $\overline{\Pi}$ on $A \times A'$ agreeing with Π on $\mathbf{A}$, that is, such that $\overline{\Pi}(. \mid \top) = \Pi(.)$.

Proposition 2 (cf. [4]). *For every continuous t-norm $*$ and for every possibility measure Π on $\mathbf{A}$, let $C_*(\Pi)$ denote the set of $*$-conditional possibilities agreeing with Π on $\mathbf{A}$. Then $\Pi_* = \max\{C_*(\Pi)\}$, i.e. $\Pi_* \in C_*(\Pi)$ and $\Pi_* \geq \overline{\Pi}$ for every $\overline{\Pi} \in C_*(\Pi)$.*

Proof. That Π_* is a $*$-conditional possibility is shown in [4, Prop.1] and that $\Pi_*(. \mid \top) = \Pi(.)$ is clear by definition. Suppose $\overline{\Pi}$ is a $*$-conditional possibility agreeing with Π, and let $a, b \in A$ such that $a \wedge b \neq \bot$ (otherwise $\overline{\Pi}(a \mid b) = \Pi_*(a \mid b) = 0$). Then we have $\Pi(a \wedge b) = \overline{\Pi}(a \wedge b \mid \top) = \overline{\Pi}(b \mid a) * \overline{\Pi}(a \mid \top) = \overline{\Pi}(b \mid a) * \Pi(a)$, and hence $\overline{\Pi}(b \mid a) \leq (\Pi(a) \Rightarrow_* \Pi(a \wedge b)) = \Pi_*(b \mid a)$.

3 A Brief Recap on Boolean Algebras of Conditionals

In this section we recall basic notions and results from [13] where, for any Boolean algebra $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top)$, a Boolean algebra of conditionals, denoted $\mathcal{C}(\mathbf{A})$, is built. Intuitively, a Boolean algebra of conditionals over $\mathbf{A}$ allows *basic conditionals*, i.e. objects of the form $(a \mid b)$ for $a \in A$ and $b \in A' = A \setminus \{\bot\}$, to be freely combined with the usual Boolean operations up to certain extent, but always satisfying the following requirements:

(R1) For every $b \in A'$, the conditional $(b \mid b)$ will be the top element of $\mathcal{C}(\mathbf{A})$, while $(\neg b \mid b)$ will be the bottom;

(R2) Given $b \in A'$, the set of conditionals $A \mid b = \{(a \mid b) : a \in A\}$ will be the domain of a Boolean subalgebra of $\mathcal{C}(\mathbf{A})$, and in particular when $b = \top$, this subalgebra will be isomorphic to $\mathbf{A}$;

(R3) In a conditional $(a \mid b)$ we can replace the consequent a by $a \wedge b$, that is, the conditionals $(a \mid b)$ and $(a \wedge b \mid b)$ represent the same element of $\mathcal{C}(\mathbf{A})$;

(R4) For all $a \in A$ and all $b, c \in A'$, if $a \leq b \leq c$, then the result of conjunctively combining the conditionals $(a \mid b)$ and $(b \mid c)$ must yield the conditional $(a \mid c)$.

R4 encodes a sort of restricted chaining of conditionals and it is inspired by the chain rule of conditional probabilities: $P(a \mid b) \cdot P(b \mid c) = P(a \mid c)$ whenever $a \leq b \leq c$.

In mathematical terms, the formal construction of the algebra of conditionals $\mathcal{C}(\mathbf{A})$ is done as follows. One first considers the free Boolean algebra $\mathbf{Free}(A \mid A) = (Free(A \mid A), \sqcap, \sqcup, \sim, \perp^*, \top^*)$ generated by the set $A \mid A = \{(a \mid b) : a \in A, b \in A'\}$. Then, in order to accommodate the requirements R1-R4 above, one considers the smallest congruence relation $\equiv_{\mathfrak{C}}$ on $\mathbf{Free}(A \mid A)$ satisfying:

(C1) $(b \mid b) \equiv_{\mathfrak{C}} \top^*$, for all $b \in A'$;
(C2) $(a_1 \mid b) \sqcap (a_2 \mid b) \equiv_{\mathfrak{C}} (a_1 \wedge a_2 \mid b)$, for all $a_1, a_2 \in A$, $b \in A'$;
(C3) $\sim(a \mid b) \equiv_{\mathfrak{C}} (\neg a \mid b)$, for all $a \in A$, $b \in A'$;
(C4) $(a \wedge b \mid b) \equiv_{\mathfrak{C}} (a \mid b)$, for all $a \in A$, $b \in A'$;
(C5) $(a \mid b) \sqcap (b \mid c) \equiv_{\mathfrak{C}} (a \mid c)$, for all $a \in A$, $b, c \in A'$ such that $a \leq b \leq c$.

Note that (C1)–(C5) faithfully account for the requirements R1-R4 where, in particular, (C2) and (C3) account for R2. Finally, the algebra $\mathcal{C}(\mathbf{A})$ is defined as follows.

Definition 3. *For every Boolean algebra* $\mathbf{A}$*, the* Boolean algebra of conditionals *of* $\mathbf{A}$ *is the quotient structure* $\mathcal{C}(\mathbf{A}) = \mathbf{Free}(A \mid A)/_{\equiv_{\mathfrak{C}}}$.

To distinguish the operations of $\mathbf{A}$ from those of $\mathcal{C}(\mathbf{A})$, the following signature is adopted:

$$\mathcal{C}(\mathbf{A}) = (\mathcal{C}(A), \sqcap, \sqcup, \sim, \perp_{\mathfrak{C}}, \top_{\mathfrak{C}}).$$

Since $\mathcal{C}(\mathbf{A})$ is a *quotient* of $\mathbf{Free}(A \mid A)$, elements of $\mathcal{C}(\mathbf{A})$ are equivalence classes, but without danger of confusion, one can henceforth identify classes $[t]_{\equiv_{\mathfrak{C}}}$ with one of its representative elements, in particular, by t itself. For instance, using this notation convention, the following equalities, which correspond to (C1)–(C5) above, hold in any Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$, for all $a, a' \in A$ and $b, c \in A'$:

1. $(b \mid b) = \top_{\mathfrak{C}}$;
2. $(a \mid b) \sqcap (c \mid b) = (a \wedge c \mid b)$;
3. $\sim(a \mid b) = (\neg a \mid b)$;
4. $(a \wedge b \mid b) = (a \mid b)$;
5. if $a \leq b \leq c$, then $(a \mid b) \sqcap (b \mid c) = (a \mid c)$.

A basic observation is that if $\mathbf{A}$ is finite, $\mathcal{C}(\mathbf{A})$ is finite as well, and hence atomic. Therefore, in the finite case, it is of crucial importance, for uncertainty measures on $\mathcal{C}(\mathbf{A})$ like probabilities or possibilities, to identify the atoms of $\mathcal{C}(\mathbf{A})$, since these measures are fully determined by their values on the atoms of the algebra they are defined upon.

If $\mathbf{A}$ is a Boolean algebra with n atoms, i.e. $|\mathbb{at}(\mathbf{A})| = n$, it is shown in [13] that the atoms of $\mathcal{C}(\mathbf{A})$ are in one-to-one correspondence with sequences $\overline{\alpha} = \langle \alpha_1, \ldots, \alpha_{n-1} \rangle$ of $n-1$ pairwise different atoms of $\mathbf{A}$, each of these sequences giving rise to an atom $\omega_{\overline{\alpha}}$ defined as the following conjunction of $n-1$ basic conditionals:

$$\omega_{\overline{\alpha}} = (\alpha_1 \mid \top) \sqcap (\alpha_2 \mid \neg\alpha_1) \sqcap \ldots \sqcap (\alpha_{n-1} \mid \neg\alpha_1 \wedge \ldots \wedge \neg\alpha_{n-2}), \qquad (4)$$

It is then clear that $|\mathbb{at}(\mathcal{C}(\mathbf{A}))| = n!$.

Now, assume $\mathbb{at}(\mathbf{A}) = \{\alpha_1, \ldots, \alpha_n\}$ and let us consider a basic conditional of the form $(\alpha_1 \mid b)$ for $\alpha_1 \leq b$. Let us denote the set of atoms below $(\alpha_1 \mid b)$ by $\mathbb{at}_{\leq}(\alpha_1 \mid b) = \{\omega_{\overline{\gamma}} \in \mathbb{at}(\mathcal{C}(\mathbf{A})) \mid \omega_{\overline{\gamma}} \leq (\alpha_1 \mid b)\}$. For every $j = 1, \ldots, n$, define

$$\mathbb{S}_j = \{\omega_{\overline{\gamma}} \in \mathbb{at}_{\leq}(\alpha_1 \mid b) \mid \overline{\gamma} = \langle \gamma_1, \ldots, \gamma_{n-1} \rangle \text{ with } \gamma_1 = \alpha_j\}.$$

In other words, for all j, $\mathbb{S}_j$ denotes the set of all those atoms below $(\alpha_1 \mid b)$ of the form $\omega_{\overline{\gamma}}$ where $\overline{\gamma}$ is the sequence $\langle \gamma_1, \ldots, \gamma_{n-1} \rangle$ and $\gamma_1 = \alpha_j$. In [13, Lemma 5.1] it is proved the following result.

Lemma 1. *The set* $\{\mathbb{S}_j\}_{j=1\ldots,n}$ *is a partition of* $\mathbb{at}_{\leq}(\alpha_1 \mid b)$*, i.e.* $\mathbb{at}_{\leq}(\alpha_1 \mid b) = \cup_{j=1,n} \mathbb{S}_j$*, and* $\mathbb{S}_i \cap \mathbb{S}_i = \emptyset$ *whenever* $i \neq j$*. Moreover, for* $j \geq 2$*,* $\mathbb{S}_j = \emptyset$ *if* $\alpha_j \leq b$*.*

Now, let us describe the sets $\mathbb{S}_j$'s. For every sequence $\langle \alpha_1, \ldots, \alpha_i \rangle$ of pairwise different atoms of $\mathbf{A}$ with $i \leq n-1$, let us consider the following set of atoms:

$$[\![\alpha_1, \ldots, \alpha_i]\!] = \{\omega_{\overline{\gamma}} \in \mathbb{at}(\mathbf{A}) \mid \overline{\gamma} = \langle \alpha_1, \ldots, \alpha_i, \sigma_{i+1}, \ldots, \sigma_{n-1}\}.$$

In other words, $[\![\alpha_1, \ldots, \alpha_i]\!]$ stands for the subset of $\mathbb{at}(\mathcal{C}(\mathbf{A}))$ whose elements are in one-one correspondence with those sequences $\overline{\gamma}$ having $\langle \alpha_1, \ldots, \alpha_i \rangle$ as initial segment. The next proposition recalls how the sets $\mathbb{S}_j$ are obtained from the sets $[\![\alpha_1, \ldots, \alpha_i]\!]$.

Proposition 3 ([13, Proposition 5.3])**.** *Let* $\mathbf{A}$ *be a finite Boolean algebra with atoms* $\alpha_1, \ldots, \alpha_n$ *and let* $b \in A$ *such that* $\neg b = \beta_1 \vee \ldots \vee \beta_k$ *and let* $\beta_k = \alpha_j$*. For every* $t = 2, \ldots, k-1$*, denote by* $\mathbb{P}_t$ *the set of permutations* $p : \{1, \ldots, t\} \to \{1, \ldots, t\}$*. Then,*

$$\mathbb{S}_j = [\![\alpha_j, \alpha_1]\!] \cup \bigcup_{i=1}^{k} [\![\alpha_j, \beta_i, \alpha_1]\!] \cup \bigcup_{t=2}^{k-1} \bigcup_{p \in \mathbb{P}_t} [\![\alpha_j, \beta_{p(1)} \ldots, \beta_{p(t)} \alpha_1]\!].$$

4 The Strong Conditional Event Problem for Possibility Measures

Our aim in this section is to solve the following *possibilistic* version of the Goodman and Nguyen's *strong conditional event problem*:

> **Possibilistic Strong Conditional Event Problem**: Given a possibility measure Π on a finite Boolean algebra $\mathbf{A}$, find an extension of Π to a possibility measure μ_Π on the Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$ such that, for any $(a \mid b) \in \mathcal{C}(\mathbf{A})$, $\mu_\Pi(a \mid b)$ coincides with the $*$-conditional possibility $\Pi_*(a \mid b)$ for some suitable continuous t-norm $*$.

Since the main results we need to prove next in order to provide a solution for this problem are quite general, in what follows, unless stated otherwise, $*$ will denote an arbitrary t-norm and $\Rightarrow$ a residuated implication, so that we will not assume, in general, that $\Rightarrow$ is the residuum of $*$. A pair $(*, \Rightarrow)$ of this kind will be simply called a *t-norm implication-pair*, or simply a ti-*pair*.

Given a possibility measure Π on a finite Boolean algebra $\mathbf{A}$ with n atoms and a ti-pair $(*, \Rightarrow)$, we start by defining a mapping $\mu_\Pi : \mathbb{at}(\mathcal{C}(\mathbf{A})) \to [0,1]$ as follows: for every atom of the Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$ as in Eq. (4), we define

$$\mu_\Pi(\omega_{\overline{\alpha}}) = \Pi(\alpha_1) * (\Pi(\neg\alpha_1) \Rightarrow \Pi(\alpha_2)) * \ldots * (\Pi(\bigwedge_{j=1}^{n-2} \neg\alpha_i) \Rightarrow \Pi(\alpha_{n-1})).$$

For the sake of a lighter and shorter notation, we will henceforth write $\frac{\Pi(a)}{\Pi(b)}\!\!\succ$ in place of $\Pi(b) \Rightarrow \Pi(a)$, so that the above expression becomes

$$\mu_\Pi(\omega_{\overline{\alpha}}) = \Pi(\alpha_1) * \frac{\Pi(\alpha_2)}{\Pi(\neg\alpha_1)}\!\!\succ * \ldots * \frac{\Pi(\alpha_{n-1})}{\Pi(\bigwedge_{j=1}^{n-2} \neg\alpha_i)}\!\!\succ. \tag{5}$$

Lemma 2. *For any* ti-*pair* $(*, \Rightarrow)$, μ_Π *is a normalized possibility distribution on* $\mathbb{at}(\mathcal{C}(\mathbf{A}))$.

Proof. We have to show that μ_Π takes value 1 for some atom of $\mathcal{C}(\mathbf{A})$. First of all note that (5) can be equivalently expressed as

$$\mu_\Pi(\omega_{\overline{\alpha}}) = \Pi(\alpha_1) * \frac{\Pi(\alpha_2)}{\Pi(\alpha_2 \vee \alpha_3 \vee \ldots \vee \alpha_n)}\!\!\succ * \ldots * \frac{\Pi(\alpha_{n-1})}{\Pi(\alpha_{n-1} \vee \alpha_n)}\!\!\succ.$$

Since Π is a possibility measure on $\mathbf{A}$, we can rank its atoms $\mathbb{at}(\mathbf{A})$ according to the values taken by Π, so that we can assume $\mathbb{at}(\mathbf{A}) = \{\beta_1, \beta_2, \ldots, \beta_n\}$ with $1 = \Pi(\beta_1) \geq \Pi(\beta_2) \geq \ldots \geq \Pi(\beta_n)$. In this way, for $1 \leq i \leq n-1$, $\Pi(\beta_i) = \max_{j=i}^{n} \Pi(\beta_j) = \Pi(\beta_i \vee \beta_{i+1} \vee \ldots \vee \beta_n)$, and hence $\frac{\Pi(\beta_i)}{\Pi(\beta_i \vee \beta_{i+1} \vee \ldots \vee \beta_n)}\!\!\succ = 1$. Let $\overline{\beta}$ to be the sequence $\overline{\beta} = \langle \beta_1, \beta_2, \ldots, \beta_{n-1} \rangle$. Then, the corresponding atom $\omega_{\overline{\beta}}$ of $\mathcal{C}(\mathbf{A})$ is such that $\mu_\Pi(\omega_{\overline{\beta}}) = \Pi(\beta_1) * \frac{\Pi(\beta_2)}{\Pi(\beta_2 \vee \beta_3 \vee \ldots)}\!\!\succ * \ldots * \frac{\Pi(\beta_{n-1})}{\Pi(\beta_{n-1} \vee \beta_n)}\!\!\succ = 1$. $\square$

In light of the above lemma, we extend μ_Π to a possibility measure on $\mathcal{C}(\mathbf{A})$, denoted by the same symbol $\mu_\Pi : \mathcal{C}(\mathbf{A}) \to [0,1]$, by defining, for every $t \in \mathcal{C}(\mathbf{A})$,

$$\mu_\Pi(t) = \max\{\mu_\Pi(\omega_{\overline{\alpha}}) \mid \omega_{\overline{\alpha}} \in \mathbb{at}(\mathcal{C}(\mathbf{A})) \text{ and } \omega_{\overline{\alpha}} \leq t\}.$$

Then, so defined, μ_Π is indeed a possibility measure on $\mathcal{C}(\mathbf{A})$.

Definition 4. *For every possibility measure* Π *on a Boolean algebra* $\mathbf{A}$ *and for every* ti-*pair* $(*, \Rightarrow)$, *the possibility measure* $\mu_\Pi : \mathcal{C}(\mathbf{A}) \to [0,1]$ *is called the* $(*, \Rightarrow)$-canonical extension *of* Π.

Example 1. Let $\mathbf{A}$ be a Boolean algebra with 3 atoms (say $\alpha_1, \alpha_2, \alpha_3$) and 8 elements, and let Π be the possibility distribution $\Pi(\alpha_1) = 0.7, \Pi(\alpha_2) = 1, \Pi(\alpha_3) = 0.2$. The corresponding algebra of conditionals $\mathcal{C}(\mathbf{A})$ has $3! = 6$ atoms and 64 elements. In particular, the atoms are the following compound conditionals:

$$\omega_1 = (\alpha_1 \mid \top) \sqcap (\alpha_2 \mid \neg\alpha_1),\ \omega_2 = (\alpha_1 \mid \top) \sqcap (\alpha_3 \mid \neg\alpha_1),\ \omega_3 = (\alpha_2 \mid \top) \sqcap (\alpha_1 \mid \neg\alpha_2)$$
$$\omega_4 = (\alpha_2 \mid \top) \sqcap (\alpha_3 \mid \neg\alpha_2),\ \omega_5 = (\alpha_3 \mid \top) \sqcap (\alpha_1 \mid \neg\alpha_3),\ \omega_6 = (\alpha_3 \mid \top) \sqcap (\alpha_2 \mid \neg\alpha_3)$$

Now, let us consider the basic conditionals $t_1 = (\alpha_1 \mid \neg\alpha_3) = (\alpha_1 \mid \neg\alpha_1 \vee \alpha_2)$ and $t_2 = (\alpha_2 \vee \alpha_3 \mid \top)$. It is not hard to check that $t_1 = \omega_1 \sqcup \omega_2 \sqcup \omega_5; t_2 = \omega_3 \sqcup \omega_4 \sqcup \omega_5 \sqcup \omega_6; t_1 \sqcap t_2 = \omega_5; \neg t_1 = \omega_3 \sqcup \omega_4 \sqcup \omega_6$.

Taking $*$ = product and $\Rightarrow$ its residuum, the extension μ_Π of Π to $\mathcal{C}(\mathbf{A})$ is determined by the following distribution on its atoms given by (5):

$$\mu_\Pi(\omega_1) = 0.7 \cdot 1 = 0.7,\ \mu_P(\omega_2) = 0.7 \cdot (0.2/1) = 0.14,$$
$$\mu_P(\omega_3) = 1 \cdot (0.7/0.7) = 1,\ \mu_P(\omega_4) = 1 \cdot (0.2/0.7) = 2/7,$$
$$\mu_P(\omega_5) = 0.2 \cdot 0.7 = 0.14,\ \mu_P(\omega_6) = 0.2 \cdot 1 = 0.2.$$

Then we can compute the possibility degree of any other (basic or compound) conditional. For instance:

$$\mu_\Pi(t_1) = \mu_\Pi(\alpha_1 \mid \neg\alpha_1 \vee \alpha_2) = \max\{\mu_\Pi(\omega_1), \mu_\Pi(\omega_2), \mu_\Pi(\omega_5)\} = 0.7$$
$$\mu_\Pi(t_2) = \mu_\Pi(\alpha_2 \vee \alpha_3 \mid \top) = \max\{\mu_\Pi(\omega_3), \mu_\Pi(\omega_4), \mu_\Pi(\omega_5), \mu_\Pi(\omega_6)\} = 1$$
$$\mu_\Pi(t_1 \sqcap t_2) = \mu_\Pi(\omega_5) = 0.2$$
$$\mu_\Pi(\neg t_1) = \max\{\mu_\Pi(\omega_3), \mu_\Pi(\omega_4), \mu_\Pi(\omega_6)\} = 1.$$ □

For what follows, recall the sets $\mathbb{S}_j$ and $[\![\alpha_1, \ldots, \alpha_i]\!]$ as defined in Sect. 3.

Lemma 3. *Let $\alpha_{i_1}, \ldots, \alpha_{i_t} \in \mathfrak{at}(\mathbf{A})$. Then*

$$\mu_\Pi([\![\alpha_{i_1}, \ldots, \alpha_{i_t}]\!]) = \Pi(\alpha_{i_1}) * \frac{\Pi(\alpha_{i_2})}{\Pi(\neg\alpha_{i_1})}\!\succ * \cdots * \frac{\Pi(\alpha_{i_t})}{\Pi(\neg\alpha_{i_1} \wedge \neg\alpha_{i_2} \wedge \ldots \wedge \neg\alpha_{i_{t-1}})}\!\succ.$$

Proof. Let $\gamma_1, \ldots, \gamma_l$ be the atoms of $\mathbf{A}$ different from $\alpha_{i_1}, \ldots, \alpha_{i_t}$. Thus, $\omega_{\overline{\delta}} \in [\![\alpha_{i_1}, \ldots, \alpha_{i_t}]\!]$ iff $\overline{\delta} = \langle \alpha_{i_1}, \ldots, \alpha_{i_t}, \sigma_1, \ldots, \sigma_{l-1}\rangle$ where $\overline{\sigma} = \langle \sigma_1, \ldots, \sigma_{l-1}\rangle$ is any string which is obtained by permuting $l-1$ elements in $\{\gamma_1, \ldots, \gamma_l\}$.

Therefore, if $\Psi = \neg\alpha_{i_1} \wedge \ldots \wedge \neg\alpha_{i_t} (= \gamma_1 \vee \ldots \vee \gamma_l)$, by letting

- $K = \Pi(\alpha_{i_1}) * \frac{\Pi(\alpha_{i_2})}{\Pi(\neg\alpha_{i_1})}\!\succ * \ldots * \frac{\Pi(\alpha_{i_t})}{\Pi(\neg\alpha_{i_1} \wedge \neg\alpha_{i_2} \wedge \ldots \wedge \neg\alpha_{i_{t-1}})}\!\succ$,
- $H = \bigvee_{\overline{\sigma}} \frac{\Pi(\sigma_1)}{\Pi(\Psi)}\!\succ * \frac{\Pi(\sigma_2)}{\Pi(\Psi \wedge \neg\sigma_1)}\!\succ * \ldots * \frac{\Pi(\sigma_{l-1})}{\Pi(\Psi \wedge \neg\sigma_1 \wedge \ldots \wedge \neg\sigma_{n-2})}\!\succ$,

it is clear that $\mu_\Pi([\![\alpha_{i_1}, \ldots, \alpha_{i_t}]\!]) = K * H$. We now prove $H = 1$ by induction on the number l of atoms different from $\alpha_{i_1}, \ldots, \alpha_{i_t}$, i.e. $l = n - t$:

(Case 0) The basic case is for $l = 2$. In this case $H = \frac{\Pi(\gamma_1)}{\Pi(\Psi)}\!\succ \vee \frac{\Pi(\gamma_2)}{\Pi(\Psi)}\!\succ = \frac{\Pi(\gamma_1 \vee \gamma_2)}{\Pi(\Psi)}\!\succ$. Thus, the claim trivially follows because $\Psi = \gamma_1 \vee \gamma_2$.

(Case l) For any $j = 1, \ldots, l$ consider the strings $\overline{\sigma}$ such that $\sigma_1 = \gamma_j$. Therefore,

$$H = \left[\frac{\Pi(\gamma_1)}{\Pi(\Psi)}\!\succ * \bigvee_{\overline{\sigma}:\sigma_1=\gamma_1}\left(\frac{\Pi(\sigma_2)}{\Pi(\Psi \wedge \neg\gamma_1)}\!\succ * \ldots * \frac{\Pi(\sigma_{l-1})}{\Pi(\Psi \wedge \neg\sigma_1 \ldots \wedge \neg\sigma_{l-2})}\!\succ\right)\right] \vee \ldots$$
$$\ldots \vee \left[\frac{\Pi(\gamma_l)}{\Pi(\Psi)}\!\succ * \bigvee_{\overline{\sigma}:\sigma_1=\gamma_l}\left(\frac{\Pi(\sigma_2)}{\Pi(\Psi \wedge \neg\gamma_1)}\!\succ * \ldots * \frac{\Pi(\sigma_{l-1})}{\Pi(\Psi \wedge \neg\sigma_1 \wedge \ldots \wedge \neg\sigma_{l-2})}\!\succ\right)\right].$$

By inductive hypothesis, each term $\bigvee_{\overline{\sigma}:\sigma_1=\gamma_j}\left(\frac{\Pi(\sigma_2)}{\Pi(\Psi \wedge \neg\gamma_1)}\!\succ * \ldots * \frac{\Pi(\sigma_{l-1})}{\Pi(\Psi \wedge \neg\sigma_1 \wedge \ldots \wedge \neg\sigma_{l-2})}\!\succ\right)$ equals 1. Thus, $H = \bigvee_{j=1}^{l} \frac{\Pi(\gamma_j)}{\Pi(\Psi)}\!\succ = \frac{\Pi(\bigvee_{j=1}^{l} \gamma_j)}{\Pi(\Psi)}\!\succ = 1$ since $\Psi = \bigvee_{j=1}^{l} \gamma_j$. □

As we did in the last part of Sect. 3, let us focus on a basic conditional of the form $(\alpha_1 \mid b)$ for $b \geq \alpha_1$. Then, the next result immediately follows from Lemma 3 above together with the observation that $\mathbb{S}_1 = [\![\alpha_1]\!]$.

Lemma 4. *Let* $\mathbf{A}$ *be a finite Boolean algebra with atoms* $\alpha_1, \ldots, \alpha_n$*, let* $b \in A$ *be such that* $b \geq \alpha_1$*. Then, for every* ti*-pair* $(*, \Rightarrow)$*, the* $(*, \Rightarrow)$*-canonical extension* μ_Π *of a possibility measure* Π *of* $\mathbf{A}$ *satisfies,* $\mu_\Pi(\mathbb{S}_1) = \bigvee_{\omega_{\overline{\gamma}} \in \mathbb{S}_1} \mu_\Pi(\omega_{\overline{\gamma}}) = \Pi(\alpha_1)$.

Now we prove our main result that will directly lead to a finitary solution for the Possibilistic Strong Conditional Event Problem stated at the beginning of this section. In what follows, for every basic conditional $(a \mid b)$, we write $\mu_\Pi(a \mid b)$ instead of $\mu_\Pi(\text{‘}(a \mid b)\text{’})$.

Theorem 1. *For every possibility measure* $\Pi : \mathbf{A} \to [0,1]$ *and for every* ti*-pair* $(*, \Rightarrow)$*, the* $(*, \Rightarrow)$*-canonical extension* $\mu_\Pi : \mathcal{C}(\mathbf{A}) \to [0,1]$ *of* Π *to* $\mathcal{C}(\mathbf{A})$ *is such that, for every basic conditional* $(a \mid b)$*,*

$$\mu_\Pi(a \mid b) = \begin{cases} 0, & \text{if } a \wedge b = \bot \\ \Pi(b) \Rightarrow \Pi(a \wedge b), & \text{otherwise.} \end{cases}$$

Proof. Let Π be given as in the hypothesis, and let μ_Π be defined on $\mathcal{C}(\mathbf{A})$ as in (5). By definition and Lemma 2, μ_Π is a normalised possibility measure. Moreover, recall that $(a \mid b) = \bot_\mathcal{C}$ if $a \wedge b = \bot$, whence, by definition, $\mu_\Pi(a \mid b) = 0$. Thus, it is left to prove the claim for those $(a \mid b)$ with $a \wedge b > \bot$.

To this end, notice that each basic conditional $(a \mid b)$ equals $(\bigvee_{\alpha_i \leq a} \alpha_i \mid b) = \bigsqcup_{\alpha_i \leq a} (\alpha_i \mid b)$. Since μ_P is maxitive, let us prove the claim for conditionals of the form $(\alpha \mid b)$ for $\alpha \in \mathbb{at}(\mathbf{A})$ and $\alpha \leq b$. Without loss of generality we will assume $\alpha = \alpha_1$. By Lemma 1, $\{\mathbb{S}_j\}_{j=1,\ldots,n}$ is a partition of $\mathbb{at}_\leq(\alpha_1 \mid b)$. Thus,

$$\mu_\Pi(\alpha_1 \mid b) = \bigvee\nolimits_{j=1}^{n} \mu_\Pi(\mathbb{S}_j) = \mu_\Pi(\mathbb{S}_1) \vee \bigvee\nolimits_{j=2}^{n} \mu_\Pi(\mathbb{S}_j).$$

Also by Lemma 1, $\mathbb{S}_j = \emptyset$, and thus $\mu_\Pi(\mathbb{S}_j) = 0$, for all those j such that $\alpha_j \leq b$. Therefore, by Lemma 4 one has,

$$\mu_\Pi(\alpha_1 \mid b) = \bigvee\nolimits_{j=1}^{n} \mu_\Pi(\mathbb{S}_j) = \Pi(\alpha_1) \vee \bigvee\nolimits_{j : \alpha_j \leq \neg b} \mu_\Pi(\mathbb{S}_j).$$

From this equation it already follows $\mu_\Pi(\alpha_1 \mid b) \geq \Pi(\alpha_1)$. We have to prove that $\mu_\Pi(\alpha_1 \mid b) = \frac{\Pi(\alpha_1)}{\Pi(b)}\succ$.

In what follows we assume $\neg b = \beta_1 \vee \ldots \vee \beta_k$, where $\Pi(\beta_1) \leq \ldots \leq \Pi(\beta_k)$. The proof is divided in three steps:

(1) $\mu_\Pi(\alpha_1 \mid b) \leq \frac{\Pi(\alpha_1)}{\Pi(b)}\succ$. If $\overline{\omega} \leq (\alpha_1 \mid \beta)$, then it must be of the form

$$\overline{\omega} = (\beta^1 \mid \top) \wedge (\beta^2 \mid \neg\beta^1) \wedge \ldots \wedge (\beta^i \mid \neg\beta^1 \wedge \ldots \wedge \neg\beta^{i-1}) \wedge (\alpha_1 \mid \neg\beta^1 \wedge \ldots \wedge \neg\beta^i) \wedge \ldots$$

where $\beta^1, \ldots, \beta^i \leq \neg b$, or equivalently, $\neg\beta^1 \ldots \neg\beta^i \geq b$. Then we have $\overline{\omega} \leq (\alpha_1 \mid \neg\beta^1 \wedge \ldots \wedge \neg\beta^i)$, and hence

$$\mu_\Pi(\overline{\omega}) \leq \frac{\Pi(\alpha_1)}{\Pi(\neg\beta^1 \wedge \ldots \wedge \neg\beta^i)}\succ \leq \frac{\Pi(\alpha_1)}{\Pi(b)}\succ.$$

Therefore, $\mu_\Pi(\alpha_1 \mid b) = \bigvee_{j:\alpha_j \leq \neg b} \mu_\Pi(\mathbb{S}_j) \leq \frac{\Pi(\alpha_1)}{\Pi(b)}\!\succ$.
(2) $\Pi(b) < 1$ implies $\mu_\Pi(\alpha_1 \mid b) \geq \frac{\Pi(\alpha_1)}{\Pi(b)}\!\succ$. In this case, $\Pi(\neg b) = 1$, and thus $\Pi(\beta_k) = 1$. Let $i_b = \max\{j \in \{1, ..., k\} \mid \Pi(\beta_{i_b}) < \Pi(b)\}$, and consider the subset $S = [\![\beta_k, \beta_{k-1}, \ldots, \beta_{i_b+1}, \alpha_1]\!]$ of atoms of $\mathcal{C}(\mathbf{A})$. Proposition 3 shows that $S \subseteq \mathbb{S}_k$, whence, in particular, $S \subseteq \bigcup_{j:\alpha_j \leq \neg b} \mathbb{S}_j$. Then, by Lemma 3, we have:

$$\mu_\Pi(S) = \Pi(\beta_k) * \frac{\Pi(\beta_{k-1})}{\Pi(\neg\beta_k)}\!\succ * \ldots * \frac{\Pi(\beta_{i_b+1})}{\Pi(\neg\beta_k \wedge \ldots \wedge \neg\beta_{i_b+2})}\!\succ * \frac{\Pi(\alpha_1)}{\Pi(\neg\beta_k \wedge \ldots \wedge \neg\beta_{i_b+1})}\!\succ.$$

Notice that, for every $i \in \{i_b + 1, \ldots, k\}$, $\Pi(\neg\beta_k \wedge \neg\beta_{k-1} \ldots \wedge \neg\beta_i) = \Pi(\beta_{i-1} \vee \ldots \vee \beta_1 \vee b) = \Pi(\beta_{i-1}) \vee \Pi(b) = \Pi(\beta_{i-1})$, and moreover $\Pi(\neg\beta_k \wedge \ldots \wedge \neg\beta_{i_b+1}) = \Pi(\beta_{i_b} \vee \ldots \vee \beta_1 \vee b) = \Pi(b)$. Therefore we have:

$$\mu_\Pi(S) = \Pi(\beta_k) * \frac{\Pi(\beta_{k-1})}{\Pi(\beta_{k-1})}\!\succ * \ldots * \frac{\Pi(\beta_{i_b+1})}{\Pi(\beta_{i_b+1})}\!\succ * \frac{\Pi(\alpha_1)}{\Pi(b)}\!\succ = 1 * \ldots * 1 * \frac{\Pi(\alpha_1)}{\Pi(b)}\!\succ$$

and hence, $\mu_\Pi(\alpha_1 \mid b) \geq \mu_\Pi(S) = \frac{\Pi(\alpha_1)}{\Pi(b)}\!\succ$.
(3) $\Pi(b) = 1$ implies $\mu_\Pi(\alpha_1 \mid b) \geq \frac{\Pi(\alpha_1)}{\Pi(b)}\!\succ$. As above observed, $\mu_\Pi(\alpha_1 \mid b) \geq \Pi(\alpha_1) = \frac{\Pi(\alpha_1)}{\Pi(b)}\!\succ$.

Therefore, (1), (2) and (3) imply $\mu_\Pi(\alpha_1 \mid b) = \frac{\Pi(\alpha_1)}{\Pi(b)}\!\succ$. □

Let us observe that the above result holds under very general assumptions about the operations $*$ and $\Rightarrow$ under which μ_Π may fail to be a $*$-conditional possibility in the sense of Definition 2. However, when we restrict to the case where $*$ is a continuous t-norm and $(*, \Rightarrow_*)$ is a residuated pair, by Proposition 2 μ_Π does provide a solution to the possibilistic strong conditional event problem.

Corollary 1. *If Π is a possibility measure on a finite Boolean algebra $\mathbf{A}$, $*$ is a continuous t-norm and $\Rightarrow_*$ its residuum, then the $(*, \Rightarrow_*)$-canonical extension μ_Π of Π to $\mathcal{C}(\mathbf{A})$ restricted to basic conditionals yields a $*$-conditional possibility, in the sense that $\mu_\Pi(a \mid b) = \Pi_*(a \mid b)$ for every basic conditional $(a \mid b) \in \mathcal{C}(\mathbf{A})$.*

5 Conclusions and Future Work

In this paper we have shown that, in a finite setting, Boolean algebras of conditionals are suitable structures to accommodate conditional possibility measures, in a similar (but not completely analogous) way to that of conditional probabilities in [13]. We back up this claim by proving the Strong Conditional Event Problem for conditional possibility: for $*$ being a continuous t-norm without zero-divisors, any possibility measure Π in a Boolean algebra of events $\mathbf{A}$ extends to a possibility measure μ_Π on the full algebra of conditional events $\mathcal{C}(\mathbf{A})$ such that its restriction to basic conditionals is the $*$-conditional possibility $\Pi_*(. \mid .)$. This result indeed holds under very general assumptions. This suggests that the structure of a Boolean Algebra of Conditionals leaves room to accommodate notions of conditional possibility other than $\Pi_*(. \mid .)$. In particular, we plan to study when and how a $*$-conditional possibility on $A \times A'$, regarded as a partially specified (unconditional) possibility on $\mathcal{C}(\mathbf{A})$, can be extended to any compound conditional in the full algebra $\mathcal{C}(\mathbf{A})$.

Following the lines of [13], we will also investigate under which conditions a (plain) possibility measure Π on $\mathcal{C}(\mathbf{A})$ satisfies the axioms of a $*$-conditional possibility as in Definition 2 when restricted to basic conditionals. More precisely we will consider those possibilities satisfying $\Pi((a \mid b) \wedge (b \mid c)) = \Pi(a \mid b) * \Pi(b \mid c)$, for $a \leq b \leq c$ and $*$ being a continuous t-norm. These measures, in analogy with [13], can be called $*$-*separable*. Furthermore, we want to explore to what extent the results can be generalised when replacing the real unit interval $[0, 1]$ by more general scales for possibility measures, like Gödel or strict BL-algebras. Finally, we also plan to study the use of Boolean algebras of conditionals in relation to conditioning more general uncertainty models such as belief functions.

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