LP formulation for regional-optimal bounds Technical Report TR-IIIA-2011-01

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Abstract

This technical report is written as a support material for the reader in [1], detailing the transformations to simplify an initial program to compute a tight bound for a $\mathcal{C}$-optimal assignment into a linear program (LP).

In this technical report we show how the initial program to compute a tight bound for a $\mathcal{C}$-optimal assignment, can be transformed into a linear program. Our departure point is the following program.

Find $\mathcal{R}$, $x^{\mathcal{C}}$ and x^* that
minimize $\frac{R(x^{\mathcal{C}})}{R(x^*)}$
subject to $x^{\mathcal{C}}$ being a $\mathcal{C}$-optimal for $\mathcal{R}$

We start by analyzing what exactly means saying that $x^{\mathcal{C}}$ is $\mathcal{C}$-optimal. The condition can be expressed as: for each x inside region $\mathcal{C}$ of $x^{\mathcal{C}}$ we have that $R(x^C) \geq R(x)$. However, instead of considering all the assignments for which x^C is guaranteed to be optimal, we consider only the subset of assignments such that the set of variables that deviate with respect to x^C take the same value than in the optimal assignment. If we restrict to this subset of assignments, then each neighborhood covers a $2^{|C^\alpha|}$ assignments, one for each subset of variables in the neighborhood. Let 2^{C^α} stand for the set of all subsets of the neighborhood C^α. Then for each $A^k \in 2^{C^\alpha}$ we can define an assignment x^{α_k} such that for every variable x_i in a relation completely covered by A^k we have that $x_i^{\alpha_k} = x_i^*$, and for every variable x_i that is not covered at all by A^k we have that $x_i^{\alpha_k} = x_i^C$. Then, we can write the value of $x_i^{\alpha_k}$ as

$$R(x_k^\alpha) = \sum_{S \in T(A^k)} S(x_k^\alpha) + \sum_{S \in P(A^k)} S(x_k^\alpha) + \sum_{S \in N(A^k)} S(x_k^\alpha) \tag{1}$$

Now, the definition of $\mathcal{C}$-optimal can be expressed as $A^k \in \{2^{C^{\alpha_k}} | C^{\alpha_k} \in \mathcal{C}\}$:

$$R(x^C) \geq R(x_k^\alpha) = \sum_{S \in T(A^k)} S(x_k^\alpha) + \sum_{S \in P(A^k)} S(x_k^\alpha) + \sum_{S \in N(A^k)} S(x_k^\alpha) \tag{2}$$

that, by setting partially covered relations to the minimum possible reward (0 assuming non-negative rewards), results in:

$$R(x^C) \geq \sum_{S \in T(A^k)} S(x_k^\alpha) + \sum_{S \in N(A^k)} S(x_k^\alpha) \quad \forall A^k \in \{2^{C^{\alpha_k}} | C^{\alpha_k} \in \mathcal{C}\} \tag{3}$$

where $T(A^k)$ is the set of completely covered relations, $P(A^k)$ the set of partially covered relations and $N(A^k)$ the set of relations not covered at all.

Given the definition of $\mathcal{C}$-optimality of equation 3, we can proceed on specifying the linear programming formulation of the initial problem. First, we assume that $x_-^{\mathcal{C}} = \langle 0, \ldots, 0 \rangle$ and $x^* = \langle 1, \ldots, 1 \rangle$ where 0 and 1 stand for the first and second value in each variable domain. This assumption can be made without loosing generality. Second, we create two real positive variables for each relation $S \in \mathcal{R}$, one representing $S(x^{\mathcal{C}})$, noted as x_S, and another one representing $S(x^*)$, noted as y_S. x$^{\mathcal{C}}$ in $\mathcal{C}^\alpha$ $R(x^{\mathcal{C}}) \geq R(x^\alpha)$ using a single equation, concretely the one that sets every variable in $\mathcal{C}^\alpha$ to 1.

Third, to obtain the LP we can normalize the rewards of our optimal to add up to one ($\sum_{S \in \mathcal{R}} y_S = 1$). This is a common procedure for turning a linear fractional program into a linear program.

Fourth, we add all the constraints from equation 3, to guarantee the optimality of x^C.

The linear program resulting from these is as follows:

minimize $\sum_{S \in \mathcal{R}} x_S$
subject to
$\sum_{S \in \mathcal{R}} y_S = 1$
and for each $A^k \in \{2^{C^{\alpha_k}} | C^{\alpha_k} \in \mathcal{C}\}$ also subject to
$\sum_{S \in \mathcal{R}} x_S \geq \sum_{S \in T(A^k)} y_S + \sum_{S \in N(A^k)} x_S$

where

- x is a vector of positive real numbers representing the values for each relation of the $\mathcal{C}$-optimal
- y is a vector of real numbers representing the values for each relation of the optimal of the problem
- $T(A^k)$ contains the relations completely covered by A^k, and
- $N(A^k)$ contains the relations that are not covered by C^α at all.

References

[1] M. Vinyals, E. Shieh, J. Cerquides, J. A. Rodriguez-Aguilar, Z. Yin, M. Tambe, and E. Bowring, *Quality guarantees for region optimal DCOP algorithms*, In: Proceedings of 10th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2011) (to appear).