Secure and Optimal Base Contraction in Graded Łukasiewicz Logics.

Pere PARDO [a,1], Pilar DELLUNDE [a,b] and Lluís GODO [a]
[a] *Institut d'Investigació en Intel·ligència Artificial (IIIA - CSIC)*
[b] *Universitat Autònoma de Barcelona (UAB)*

Abstract. The operation of base contraction was successfully characterized for a very general class of logics using the notion of remainder sets. Although, in the general case, this notion is inadequate for revision, where it is replaced by maximal consistent subsets. A natural question is wether this latter notion allows for a definition of contraction-like operators and, in case it does, what differences there exist w.r.t. standard contraction. We make some steps towards this direction for the case of graded logic **RPL**: we characterize contraction operators with a fixed security-threshold $\epsilon > 0$; we prove soundness of (an optimal) ω-contraction operation, and a collapse theorem from ω- to some ϵ-contraction for finite theories.

Keywords. Base contraction, monotonic logics, fuzzy logics, involutive negation

Introduction

The field of Belief Change studies how *can* a set of sentences -in some fixed logic $\mathcal{S}$- (i) adapt to make room for some new input in a consistent way (revision), or (ii) stop making some old claim provable anymore (contraction). This field originated in the work of Alchourrón, Gärdenfors and Makinson, theory change, for deductively closed sets of sentences. In this

A base is a simply a set of sentences T in the language of $\mathcal{S}$: that is, it is not required to be $\mathrm{Cn}_{\mathcal{S}}$-closed. (For convenience, in this paper we will also use *theory* to refer to bases; this convention does not apply to *theory change*, though.) Base belief change is an important area of research that has attained a high level of generality in its objects of study. It is known that for any logic satisfying rather weak conditions, base belief change operators (revision and contraction) can be characterized by the method of partial meet (from [1] in their study of theory change). There exist in the literature many other variants of change operations. We refer the reader to [9] for an overview.

Formal results for logics with the deduction property (e.g. classical propositional logic) are extremely simple and elegant. This simplicity is lost when we

[1]Corresponding author: Pere Pardo, Campus UAB, E-08193 Bellaterra, Catalonia (Spain); e-mail: pardo@iiia.csic.es

move to a maximally general class of logics. For instance, identity between change operators defined by (i) remainders, or (ii) maximal consistent subsets (*coherers*, henceforth), holds for logics with the deduction property but fails in general. In the general case, remainders are natural candidates to understand contraction [6], while coherers are the natural choice for the case of revision [6,2,7]. Nonetheless, researching (slightly different) change operators from the non-natural tool is interesting in itself. In this paper, we make some steps towards a consistency-based definition of contraction-like operators in Łukasiewicz t-norm based fuzzy logics $Ł(\mathcal{C})$ expanded with truth-constants $\overline{r}$ for each $r \in \mathcal{C} \subseteq [0,1]$, for a suitable (countable) set $\mathcal{C}$, see [3]. Rational Pavelka Logic **RPL**, introduced in [5] (simplifying previous work in [8] for $\mathbb{R} \cap [0,1]$), is the case $\mathcal{C} = [0,1] \cap \mathbb{Q}$. In graded logics, it makes sense to consider contraction-like operators with a security-threshold $\epsilon > 0$. We characterize first this class of ϵ-contraction operators, for $\epsilon > 0$. This is a first step towards an optimal contraction operation, ω-contraction, which is proved to be sound w.r.t. a list of axioms (slightly different to standard, Rem-based contraction from Hansson and Wasserman [6]). Finally, a finite-collapse result is proved showing that for each finite theory, any ω-contraction operator collapses to some ϵ-contraction operator.

1. Preliminaries: the (local) deduction property.

One of the main differences between the original results and the current, more general, point of view originates in the deduction properties assumed to hold in the class of logics where characterization is possible. With more detail, logics were assumed first to have the (classical) deduction property. This property consists in satisfying the (classical) deduction theorem[2]. Some logics do not have a contractive conjunction (e.g. Łukasiewicz or product logics): $\nvdash_{\mathcal{S}} \varphi \rightarrow \varphi^n$, where φ^n stands for $\varphi \& \cdots \& \varphi$ (n times). These logics can be characterized by a weaker property: satisfying the local deduction theorem, defined next.

Definition 1.1. A logic $\mathcal{S}$ satisfies the *deduction theorem* iff for all $T \subseteq \mathbf{Fm}$, and $\varphi, \psi \in \mathbf{Fm}$, we have: $T \cup \{\varphi\} \vdash_{\mathcal{S}} \psi \iff T \vdash_{\mathcal{S}} \varphi \rightarrow \psi$. Logic $\mathcal{S}$ satisfies the *local deduction theorem* iff for all T, φ, ψ as above, $T \cup \{\varphi\} \vdash_{\mathcal{S}} \psi \iff (\exists n \in \omega) T \vdash_{\mathcal{S}} \varphi^n \rightarrow \psi$.

Under the deduction property, the original tool used to define partial meet, remainder sets, becomes dual to that of maximal consistent subsets of T (introduced in [10]; see also [2])

Definition 1.2. Let $\mathcal{S}$ be some finitary monotonic logic, and $T \subseteq \mathbf{Fm}_{\mathcal{S}}$ be some theory and $\varphi \in \mathbf{Fm}$ some input. A *remainder* set X for T and φ is a subset of T

[2]Following [4], we define a logic as a finitary and structural consequence relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathbf{Fm}) \times \mathbf{Fm}$, for some algebra of formulas $\mathbf{Fm}$. That is, $\mathcal{S}$ satisfies (1) If $\varphi \in \Gamma$ then $\Gamma \vdash_{\mathcal{S}} \varphi$, (2) If $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_{\mathcal{S}} \varphi$, (3) If $\Gamma \vdash_{\mathcal{S}} \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash_{\mathcal{S}} \psi$ then $\Delta \vdash_{\mathcal{S}} \varphi$ (*consequence relation*); (4) If $\Gamma \mathcal{S} \varphi$ then for some finite $\Gamma_0 \subseteq \Gamma$ we have $\Gamma_0 \vdash_{\mathcal{S}} \varphi$ (*finitarity*); (5) If $\Gamma \vdash_{\mathcal{S}} \varphi$ then $e[\Gamma] \vdash_{\mathcal{S}} e(\varphi)$ for all substitutions $e \in Hom(\mathbf{Fm}, \mathbf{Fm})$ (*structurality*). In addition, we assume that logics contain the following symbols in its language: conditional $\rightarrow$, and *falsum* $\overline{0}$.

$\subseteq$-maximal w.r.t. the property: $X \nvdash_{\mathcal{S}} \varphi$. A *coherer* set X for T and φ is a subset T $\subseteq$-maximal w.r.t. the property: $X \cup \{\varphi\} \nvdash_{\mathcal{S}} \overline{0}$. We denote by $\mathrm{Rem}(T, \varphi)$ and $\mathrm{Con}(T, \varphi)$ the set of remainders and coherers of T and φ.

Lemma 1.3. *Let $\mathcal{S}$ be some logic having the deduction property. Then* $\mathrm{Rem}(T, \neg\varphi) = \mathrm{Con}(T, \varphi)$.

Definition 1.4. For a given set $\mathbb{X}$ of families of sets, a *selection function* γ is a function selecting a non-empty subset of each family in $\mathbb{X}$.

$$\gamma : \mathbb{X} \longrightarrow \bigcup_{X \in \mathbb{X}} \mathcal{P}(X), \text{ with } \emptyset \neq \gamma(X) \subseteq X$$

A Rem-selection function for T takes $\mathbb{X} = \{\mathrm{Con}(T, \varphi) \mid \varphi \in \mathbf{Fm}\}$. The definition of Con-selection functions (for some T) is analogous.

Each Rem-selection function γ induces a contraction operator by means of the following

$$T \ominus_\gamma \varphi \quad = \quad \bigcap \gamma(\mathrm{Rem}(T, \varphi))$$

and each Con-selection function γ gives rise to a revision operator

$$T \circledast_\gamma \varphi \quad = \quad \bigcap \gamma(\mathrm{Con}(T, \varphi)) \cup \{\varphi\}$$

The operation of expansion is simply defined as $T \oplus \varphi = T \cup \{\varphi\}$. Observe this operation does not depend on any selection function; in contrast to the other operations, expansion cannot be guaranteed to output consistent theories (even if input φ is consistent).

Now, assume remainders (unprovability) to be the natural tool for partial meet contraction, while coherers naturally define partial meet revision. The following identities remain (provably) true under this assumption.

Theorem 1.5. *(Levi and Harper identities) If $\mathcal{S}$ has the deduction property, then:*

$$\begin{array}{llll} \textit{(Levy)} & T \circledast_\gamma \varphi & = & (T \ominus_\gamma \neg\varphi) \oplus \varphi \\ \textit{(Harper)} & T \ominus_\gamma \neg\varphi & = & (T \circledast \varphi) \cap T \end{array}$$

The following example shows this coincidence between Rem and Con need not hold for logics satisfying only the local deduction theorem (as Łukasiewicz):

Exemple 1.6. (In (Ł)) Let $T = \{p^2 \to_{\text{Ł}} \overline{0}\}$. We have $\mathrm{Rem}(T, p \to_{\text{Ł}} \overline{0}) = \{T\}$, but $\mathrm{Con}(T, p) = \{\emptyset\}$. Since there is only one remainder for T, namely T itself, applying Levy's RHS leads to $(T \ominus \neg p) \oplus p = T \cup \{p\}$ which is an inconsistent base[3].

The next result, whose proof depends upon Zorn's Lemma (i.e. the Axiom of Choice in **ZFC** set theory), is extensively used later on.

[3]The reason is simple: (1) since $p \in T \cup \{p\}$ we have $T \cup \{p\} \vdash_{\text{Ł}} p^n$; (2) since $T \cup \{p\} \vdash_{\text{Ł}} p^2 \to_{\text{Ł}} \overline{0}$, by modus ponens with (1) we derive $\overline{0}$.

Lemma 1.7. *(From [7]) Let $\mathcal{S}$ be some finitary logic and T_0 a theory. For any $X \subseteq T_0$, if $X \cup T_1$ is consistent, then X can be extended to some Y with $Y \in$ $\mathrm{Con}(T_0, T_1)$.*

In the remaining of the paper we make some steps into obtaining consistency-based contraction operators for a particularly suited logic, **RPL**. This is motivated by the possibility of defining an a Con-based operation of contraction which differs from the Rem-based operation as characterized by Hansson and Wasserman.

The idea, roughly, is as follows: we first characterize ϵ-contraction, which is a suboptimal operation of contraction, in the sense that a security-threshold $\epsilon > 0$ must be fixed beforehand. Then we try to recover optimality in contraction by defining another operation, ω-contraction, as the limit of an infinite series of ϵ-contraction operators.

2. Success-based contraction: ϵ-contraction

We introduce an involutive graded negation $\neg_\epsilon$ for Łukasiewicz t-norm based graded logics like **RPL**.

Definition 2.1. Given a logic Ł$(\mathcal{C})$ for some suitable $\mathcal{C}$, we denote by (φ, r) the sentence $\overline{r} \rightarrow_{\text{Ł}} \varphi$. Now, given some graded sentece (φ, r), we define

$$\neg_\epsilon(\varphi, r) = (\neg\varphi, 1 - r + \epsilon) \text{ for } \epsilon \in [0, r]$$

In the remaining of the paper, we denote by *selection function* a Con-selection function γ for T, taking $\mathbb{X} = \{\mathrm{Con}(T, \varphi) \mid \varphi \in \mathbf{Fm}\}$ and defining a base ϵ-contraction operator for T:

$$T \ominus_\gamma^\epsilon (\varphi, r) = \bigcap \gamma(\mathrm{Con}(T, \neg_\epsilon(\varphi, r)))$$

Definition 2.2. Given Ł$(\mathcal{C})$, let T be a base. We define a (partial meet) *base ϵ-contraction operator* for T as a function $\ominus^\epsilon : \mathbf{Fm} \to \mathcal{P}(\mathbf{Fm})$ such that there is some selection function γ for T with

$$T \ominus^\epsilon (\varphi, r) \quad = \quad T \ominus_\gamma^\epsilon (\varphi, r), \qquad \text{for any } (\varphi, r) \in \mathbf{Fm}.$$

In general, it is important that the ϵ-parameter is positive, $\epsilon > 0$. This enforces the property of (Success) to hold. For the case $\epsilon = 0$, consider the next example:

Example 2.3. Let $\epsilon = 0$ and $T = \{\varphi\}$ with $\varphi = (p, r)$. Then $\mathrm{Con}(T, (\neg\varphi, 1 - r)) = \{T\}$. Hence, γ is unique and contraction becomes $\bigcap\{T\} = T$. But then, the axiom of (Success) is not fulfilled, since $T \ominus^0 (p, r) \vdash_{\mathbf{RPL}} (p, r)$.

Consider the next axiom list, defined for each $\epsilon \in [0, r]$.

(G1$^\epsilon$)	$T \ominus^\epsilon (\varphi, r) \cup \{\neg_\epsilon(\varphi, r)\}$ is consistent, if $\neg_\epsilon(\varphi, r)$ is	(ϵ-Consistency)
(G2$^\epsilon$)	$T \ominus^\epsilon (\varphi, r) \subseteq T$	(Inclusion)
(G3$^\epsilon$)	For all $(\psi, s) \in \mathbf{Fm}$, if $(\psi, s) \in T - T \ominus^\epsilon (\varphi, r)$, then there exists $T'(T \ominus^\epsilon (\varphi, r) \subseteq T' \subseteq T$ with $T' \cup \{\neg_\epsilon(\varphi, r)\}$ consistent and $T' \cup \{\neg_\epsilon(\varphi, r), (\psi, s)\}$ inconsistent	(ϵ-Relevance)
(G4$^\epsilon$)	If, for all $T' \subseteq T$ we have $(T' \cup \{\neg_\epsilon(\varphi, r)\} \nvdash_{\mathcal{S}} \overline{0}$ iff $T' \cup \neg_\epsilon(\varphi', r') \nvdash_{\mathcal{S}} \overline{0})$ then $T \ominus^\epsilon (\varphi, r) = T \ominus^\epsilon (\varphi', r')$	(ϵ-Uniformity)

Theorem 2.4. *For any operator (for T) $\ominus^\epsilon : \mathbf{Fm} \to \mathcal{P}(\mathbf{Fm})$ we have the following:*

$$\ominus^\epsilon \textit{ satisfies } (\mathrm{G1}^\epsilon) - (\mathrm{G4}^\epsilon) \quad \textit{iff} \quad \ominus^\epsilon = \ominus^\epsilon_\gamma \textit{ for some selection function } \gamma \textit{ for } T$$

Proof. (Soundness) (G1$^\epsilon$) Since for all $X \in \mathrm{Con}(T, \neg_\epsilon(\varphi, r))$, we have $X \cup \{\neg_\epsilon(\varphi, r)\}$ is consistent, the same holds for $X \in \gamma(\mathrm{Con}(T, \neg_\epsilon(\varphi, r)))$. Hence, $\bigcap \gamma(\mathrm{Con}(T, \neg_\epsilon(\varphi, r))) \cup \{\neg_\epsilon(\varphi, r)\}$ is consistent. (G2$^\epsilon$) Trivial, by definition of $\ominus^\epsilon_\gamma$. (G3$^\epsilon$) Let $(\psi, s) \in T - T \ominus^\epsilon_\gamma (\varphi, r)$. Then, $(\psi, s) \notin X$, for some $X \in \gamma(Con(T, \neg_\epsilon(\varphi, r)))$. In consequence, $X \cup \{\neg_\epsilon(\varphi, r)\}$ is consistent. On the other side, since $(\psi, s) \in T$, from $(\psi, s) \notin X$ we infer $X \cup \{\neg_\epsilon(\varphi, r), (\psi, s)\}$ is inconsistent. Such X is what we are looking for, so we put $T' = X$. (G4$^\epsilon$) Assume for all $T' \subseteq T$ we have T' is consistent with $\neg_\epsilon(\varphi, r)$ iff it is consistent with $\neg_\epsilon(\varphi', r')$. Then by Lemma [7, 6], we have $\mathrm{Con}(T, \neg_\epsilon(\varphi, r)) = \mathrm{Con}(T, \neg_\epsilon(\varphi', r'))$. Since γ is a function, applying γ to both sets preserves the identity. Finally, applying $\bigcap$ to each side of the corresponding equation preserves identity too, so $T \ominus^\epsilon_\gamma (\varphi, r) = T \ominus^\epsilon_\gamma (\varphi', r')$.

(Completeness) Let $\ominus^\epsilon$ be an operator (for T) $\ominus^\epsilon : \mathbf{Fm} \to \mathcal{P}(\mathbf{Fm})$ satisfying (G1$^\epsilon$) – (G4$^\epsilon$). We define a function γ

$$\gamma(\mathrm{Con}(T, \neg_\epsilon(\varphi, r))) = \{X \in \mathrm{Con}(T, \neg_\epsilon(\varphi, r)) : X \supseteq T \ominus^\epsilon (\varphi, r)\}$$

We have to show that for γ for T, γ is both (1) well-defined (for $\neg_\epsilon$-sentences), (2) a $\neg_\epsilon$-selection function for T, and (3) that for all $(\varphi, r) \in \mathbf{Fm}$, we have $T \ominus^\epsilon (\varphi, r) = T \ominus^\epsilon_\gamma (\varphi, r)$.
(**1**) γ is well-defined (for $\neg_\epsilon$-sentences). Let $\mathrm{Con}(T, \neg_\epsilon(\varphi, r)) = \mathrm{Con}(T, \neg_\epsilon(\varphi', r'))$. By Lemma 6 in [7], since $\ominus^\epsilon$ satisfies (G4$^\epsilon$), we infer $T \ominus^\epsilon (\varphi, r) = T \ominus^\epsilon (\varphi', r')$. From this and the initial assumption we obtain, using the above definition of γ, that $\gamma(\mathrm{Con}(T, \neg_\epsilon(\varphi, r))) = \gamma(\mathrm{Con}(T, \neg_\epsilon(\varphi', r')))$. (**2**) γ is a $\neg_\epsilon$-selection function for T (for $\neg_\epsilon$-sentences). Let $\mathrm{Con}(T, \neg_\epsilon(\varphi, r))$ be non-empty. Since, by (G1$^\epsilon$), $T \ominus^\epsilon (\varphi, r) \cup \{\neg_\epsilon(\varphi, r)\} \nvdash_{\mathcal{S}} \overline{0}$, we obtain from Lemma 1.7 that $T \ominus^\epsilon (\varphi, r) \subseteq X \in \mathrm{Con}(T, \neg_\epsilon(\varphi, r))$, for some $X \in \mathrm{Con}(T, \neg_\epsilon(\varphi, r))$. By the above definition of γ, this implies $X \in \gamma(T, \neg_\epsilon(\varphi, r))$, so this set is not empty.(**3**) We check the $T \ominus^\epsilon_\gamma (\varphi, r) \subseteq T \ominus^\epsilon (\varphi, r)$ first. Let $(\psi, s) \in T \ominus^\epsilon_\gamma (\varphi, r)$. Hence, (ψ, s) in T. Now, assume the contrary of what we want to prove: $(\psi, s) \notin T \ominus^\epsilon (\varphi, r)$. Then, by (G3$^\epsilon$), there is T' with $T \ominus^\epsilon (\varphi, r) \subseteq T' \subseteq T$. Then we have (i) $T' \cup \{\neg_\epsilon(\varphi, r)\}$ consistent while (ii) $T' \cup \{\neg_\epsilon(\varphi, r), (\psi, s)\}$ is inconsistent. We have that (i) plus $T \ominus^\epsilon (\varphi, r) \subseteq T$ (which holds by (G2$^\epsilon$)) jointly imply $T \ominus^\epsilon (\varphi, r) \subseteq X \in \mathrm{Con}(T, \neg_\epsilon(\varphi, r))$. We also have (ii) implies $X \cup \{\neg_\epsilon(\varphi, r), (\psi, s)\}$ is inconsistent, so $(\psi, s) \notin X$. Since $X \supseteq T \ominus^\epsilon_\gamma (\varphi, r)$

we have that $X \in \gamma(\text{Con}(T, \neg_\epsilon(\varphi, r)))$. But then $(\psi, s) \notin \bigcap \gamma(\text{Con}(T, \neg_\epsilon(\varphi, r)))$, thus contradicting the initial assumption $(\psi, s) \in T \ominus^\epsilon_\gamma (\varphi, r)$. In consequence, $(\psi, s) \in T \ominus^\epsilon (\varphi, r)$. For the other direction $T \ominus^\epsilon (\varphi, r) \subseteq T \ominus^\epsilon_\gamma (\varphi, r)$ consider: $(\psi, s) \in T \ominus^\epsilon (\varphi, r)$. Then, let $X \in \gamma(\text{Con}(T, \neg_\epsilon(\varphi, r)))$. Since $X \subseteq T \ominus^\epsilon (\varphi, r)$, we have $(\psi, s) \in X$. In consequence $(\psi, s) \in \bigcap \gamma(\text{Con}(T, \neg_\epsilon(\varphi, r)))$, so that $(\psi, s) \in T \ominus^\epsilon_\gamma (\varphi, r)$. □

Let (Success) be the (G1)-like axiom $(\varphi, r) \notin T \ominus (\varphi, r)$.

Lemma 2.5. *For each logic $\mathcal{S} = \text{Ł}(\mathcal{C})$, any $\epsilon > 0$ and any contraction operator $\ominus$ the following hold:*

(i) *if $\ominus$ satisfies* (G1$^\epsilon$)*, then it also satisfies* (Success)
(ii) *if $\ominus$ satisfies* (Success)*, then it also satisfies* (G1^0)
(iii) *if $\ominus$ satisfies* (G1$^\epsilon$)*, then it also satisfies* (G1$^{\epsilon'}$) *for any $\epsilon' < \epsilon$.*

Proof. *(i)* Let $\ominus$ satisfy (G1$^\epsilon$) for some $\epsilon > 0$. Then $T \ominus (\varphi, r)$ is consistent with $\neg_\epsilon(\varphi, r)$. This implies $(\varphi, r) \notin \text{Cn}_\mathcal{S}(T \ominus (\varphi, r))$. *(ii)* Assume $(\varphi, r) \notin \text{Cn}_\mathcal{S}(T \ominus (\varphi, r))$. We use Lemma 3.3.7 from [5] to obtain $T \ominus (\varphi, r) \cup \{\varphi \to_{\text{L}} \overline{r}\}$ is consistent. Since internal negation $\neg$ is involutive $(\varphi \to_{\text{L}} \overline{r})$ is equivalent to $(\neg\varphi, 1 - r) = \neg_0(\varphi, r)$. But then $T \ominus (\varphi, r)$ is consistent with $\neg_0(\varphi, r)$. Hence, $\ominus \models$ (G1^0). (iii) This is obvious. □

Levi and Harper identities hold between ϵ-contraction of (φ, r) and revision by $\neg_\epsilon(\varphi, r)$. The former $T \ominus^\epsilon_\gamma (\varphi, r) \cup \{\neg_\epsilon(\varphi, r)\} = T \circledast_\gamma \neg_\epsilon(\varphi, r)$ follows from the definitions. For the latter, we have the following result.

Proposition 2.6. *For any theory T, $\epsilon > 0$ and selection function γ, we have that*

$$T \ominus^\epsilon_\gamma (\varphi, r) = T \circledast_\gamma \neg_\epsilon(\varphi, r) \cap T$$

Proof. We have $(\psi, s) \in (\bigcap \gamma(\text{Con}(T, \neg_\epsilon(\varphi, r))) \cup \{\neg_\epsilon(\varphi, r)\}) \cap T$ iff $(\psi, s) \in (\bigcap \gamma(\text{Con}(T, \neg_\epsilon(\varphi, r))) \cap T) \cup (\{\neg_\epsilon(\varphi, r)\} \cap T)$. Now consider the next possibilities: Case $\neg_\epsilon(\varphi, r) \in T$. Then, by maximality, for each $X \in \text{Con}(T, \neg_\epsilon(\varphi, r))$ we have $\neg_\epsilon(\varphi, r) \in X$, so that $\neg_\epsilon(\varphi, r) \in \bigcap \gamma(\text{Con}(T, \neg_\epsilon(\varphi, r)))$; and $T \circledast_\gamma \neg_\epsilon(\varphi, r) = T \ominus^\epsilon_\gamma (\varphi, r)$. Case $\neg_\epsilon(\varphi, r) \notin T$. Then, trivially the last identity is also the case since $\{\neg_\epsilon(\varphi, r)\} \cap T = \emptyset$. In both cases the identity $T \ominus^\epsilon_\gamma (\varphi, r) = T \circledast_\gamma \neg_\epsilon(\varphi, r)$ holds. □

We saw in Example 2.3 the need for a fixed ϵ to be > 0. Doing so, though, a positive value for ϵ may result in unnecessary contractions:

Example 2.7. Let $T = \{(p, 0.9)\}$ and $\varphi = (p, 0.93)$. In case $\epsilon > 0.3$, we have $T \ominus^\epsilon \varphi = \emptyset$; otherwise, for $\epsilon \leq 0.3$, the output is again T. Intuitively the former case output $\emptyset$, where $\epsilon > 0.3$, is wrong due to ϵ being too high. By setting $\epsilon \leq 0.3$ we have $T \cup \{\neg_\epsilon \varphi\}$ is consistent. Observe in this example, one can safely set $\epsilon = 0$, which looks optimal[4].

[4] This suggests the possibility of defining, alternatively,

For comparison purposes between the mentioned Rem-based contraction and our Con-based proposal, consider the notion of *deg*-closed bases. This condition requires bases to be closed in the following sense: if $(\varphi, r) \in T$ and $s \leq r$, then $(\varphi, s) \in T$. We denote by $\mathrm{Cn}_{\mathbb{Q}}(T)$ the *deg*-closure of T; then, a base T is *deg*-closed if $T = \mathrm{Cn}_{\mathbb{Q}}(T)$. This notion was introduced in [2] as an essential component in the framework adopted therein (see [7] for a comparison with pure bases in the case of revision). The condition of being *deg*-closed makes change operations less drastic, since (possibly) lower but maximally consistent degrees are kept for formulas being targeted by the operation in question. Normal bases, as Example 2.7 shows, suffer major changes. Now, under the condition of *deg*-closure, Con-based contraction may be preferred over Rem-based contraction, since only the former operation preserves the property of being *finitely axiomatizable* (for *deg*-closed bases):

Example 2.8. Let $T = \{(p, s) : s \leq 0.9\}$ and $\varphi = (p, 0.9)$. Then, $\mathrm{Cn}_{\mathbb{Q}}(T) \ominus^{\epsilon} \varphi = \mathrm{Cn}_{\mathbb{Q}}(T \smallsetminus \{(p, s) : s > 0.9 - \epsilon\})$. Observe this is just $\mathrm{Cn}_{\mathbb{Q}}(\{(p, 0.9 - \epsilon)\})$. On the other side, for Rem-based contraction, we would have $T \ominus \varphi = \{(p, s) : s < 0.9\}$ which cannot be finitely axiomatized.

Finally, an example showing the convenience of an alternative contraction which does not commit oneself to any given $\epsilon > 0$.

Example 2.9. Let $T = \{(p_n, \frac{1}{2^n}) \mid n \in \omega, n > 0\}$ and $\varphi_n = (p_n, \frac{1}{2^n} + \delta_n)$. We have

$$T \ominus^{\epsilon} \varphi_n = \begin{cases} T \smallsetminus \{(p_n, \frac{1}{2^n})\}, & \text{for } \epsilon > \delta_n, \text{ and} \\ T & \text{otherwise} \end{cases}$$

Now, if we let $\delta_n = \frac{1}{2^n}$, then for each $\epsilon > 0$ there is an $n \in \omega$ such that for input φ_n: $T \ominus^{\epsilon} \varphi_n = T \smallsetminus \{(p_n, \frac{1}{2^n})\}$, this not being necessary, since we have: $T \nvdash_{\mathbf{RPL}} \varphi_n$.

3. Towards Con-based (optimal) ω-contraction.

A problem posed by ϵ-contraction operators is their lack of flexibility: if we are to choose the value of ϵ beforehand, we may find that a given problem -say, ϵ-contraction of T by (φ, r)- is not optimally solved (ϵ being unnecessarily high, hence violating the principle of minimizing information loss. (Observe this problem does not affect Rem-based contraction operators from [6].) To solve this inconvenience we will define a contraction operation which is not committed to any particular ϵ.

Definition 3.1. A selection function for T is *conservative* iff for all $\epsilon' < \epsilon$ we have

$$T \ominus^{\epsilon}_{\gamma} (\varphi, r) \subseteq T \ominus^{\epsilon'}_{\gamma} (\varphi, r)$$

$$T \ominus \varphi = \begin{cases} T & \text{if } T \cup \{\neg_0 \varphi\} \text{ is consistent, (e.g. } T \nvdash_{\mathbf{RPL}} \varphi) \\ T \ominus^{\epsilon} \varphi & \text{otherwise} \end{cases}$$

We do not pursue this line here, but note it may also be applied to ω-contraction studied below.

Notice first that condition (iii) from the previous Lemma is weaker than conservativeness: a selection function -defined for both families $\mathrm{Con}(T, \neg_{\epsilon}(\varphi, r))$ and $\mathrm{Con}(T, \neg_{\epsilon'}(\varphi, r))$- can satisfy (G3$^{\epsilon}$), hence (G3$^{\epsilon'}$), with $T \ominus^{\epsilon}_{\gamma} (\varphi, r) \subsetneq T \ominus^{\epsilon'}_{\gamma} (\varphi, r)$. (For example, consider γ to be maxichoice in both cases but with selected elements X_{ϵ} and X'_{ϵ} not forming a $\subseteq$-chain.) Conservativeness can be enforced by imposing a further condition upon selection functions, which is defined next.

Definition 3.2. For $\mathbb{X}, \mathbb{Y}$ families of sets, we define:

$$\begin{array}{lll} \mathbb{X} \underset{\rightarrow}{\subseteq} \mathbb{Y} & \text{iff} & \text{for each } X \in \mathbb{X} \text{ there is some } Y \in \mathbb{Y} \text{ such that } X \subseteq Y \\ \mathbb{X} \underset{\leftarrow}{\subseteq} \mathbb{Y} & \text{iff} & \text{for each } Y \in \mathbb{Y} \text{ there is some } X \in \mathbb{X} \text{ such that } X \subseteq Y \end{array}$$

A selection function defined on $\mathcal{P}(\mathcal{P}(\mathbf{Fm})) - \{\emptyset\}$ is *family-conservative* iff $\mathbb{X} \underset{\rightarrow}{\subseteq} \mathbb{Y}$ implies $\gamma(\mathbb{X}) \underset{\leftarrow}{\subseteq} \gamma(\mathbb{Y})$.

We call such arrow-reversing selection functions γ *family-conservative* because they only select sets in the greater family $\mathbb{Y}$ which extend some set in the lesser family $\mathbb{X}$. Observe that (1) for any $\mathbb{X}, \mathbb{Y}$, $\mathbb{X} \underset{\leftarrow}{\subseteq} \mathbb{Y}$ implies $\bigcap \mathbb{X} \subseteq \bigcap \mathbb{Y}$; and (2) (by Lemma 1.7) for each theory T, sentence (φ, r) and degrees $\epsilon > 0$ we have: $\mathrm{Con}(T, (\varphi, r+\epsilon)) \underset{\rightarrow}{\subseteq} \mathrm{Con}(T, (\varphi, r))$. Facts (1) and (2) jointly imply that any family-conservative selection function is conservative.

By fixing some $\epsilon > 0$ in the definition of contraction, one enforces contraction to satisfy (Success), at the cost of committing ourselves to possibly suboptimal contraction operations for some problems. To overcome this problem, it would be desirable that parameter ϵ in $\ominus^{\epsilon}$-contraction automatically adjusted itself to find the optimal solution for the problem at hand -contracting T by (φ, r)-, while granting that (Success) still holds. With this idea in mind, we define partial meet base ω-contraction as follows, and prove its soundness after some auxiliary results, while completeness is left as an open problem.

$$T \ominus^{\omega}_{\gamma} (\varphi, r) = \bigcup_{\epsilon > 0} T \ominus^{\epsilon}_{\gamma} (\varphi, r)$$

In comparison to ϵ-contraction, we have that Levi does not hold for ω-contraction; but Harper remains valid:

Corollary 3.3. *For each theory T, selection function γ and (φ, r) we have that*

$$T \ominus^{\omega}_{\gamma} (\varphi, r) = (\bigcup_{\epsilon > 0} T \circledast_{\gamma} \neg_{\epsilon}(\varphi, r)) \cap T$$

Proof. By Proposition 2.6, for each $\epsilon > 0$ we have $T \ominus^{\epsilon}_{\gamma} (\varphi, r) = T \circledast_{\gamma} \neg_{\epsilon}(\varphi, r) \cap T$. Hence $\bigcup_{\epsilon > 0} T \ominus^{\epsilon}_{\gamma} (\varphi, r) = \bigcup_{\epsilon > 0} (T \circledast_{\gamma} \neg_{\epsilon}(\varphi, r) \cap T) = (\bigcup_{\epsilon > 0} T \circledast_{\gamma} \neg_{\epsilon}(\varphi, r)) \cap T$. □

Next we show soundness of partial meet base ω-contraction operators in some logic $\mathcal{S} = \text{Ł}(\mathcal{C})$ with respect to the following axioms, where $\epsilon \in [0, 1-r]$.

(G1$^\omega$)	$(\varphi, r) \notin \mathrm{Cn}_{\mathcal{S}}(T \ominus (\varphi, r))$, if $\nvdash_{\mathcal{S}} (\varphi, r)$	(Success)
(G2$^\omega$)	$T \ominus (\varphi, r) \subseteq T$	(Inclusion)
(G3$^\omega$)	For all $(\psi, s) \in \mathbf{Fm}$, if $(\psi, s) \in T - T \ominus (\varphi, r)$, then for all $\epsilon > 0$ there exists T'_ϵ with $(T \circledast \neg_\epsilon(\varphi, r) \cap T) \subseteq$ $\subseteq T'_\epsilon \subseteq T$ and $T'_\epsilon \cup \{\neg_\epsilon(\varphi, r)\}$ consistent and $T'_\epsilon \cup \{\neg_\epsilon(\varphi, r), (\psi, s)\}$ inconsistent	(ω-Relevance)
(G4$^\omega$)	For any $(\varphi, r), (\varphi, r')$, if for all $T' \subseteq T$ we have that for all ϵ exists ϵ' (and viceversa for all ϵ' exists ϵ) such that $T' \cup \{\neg_\epsilon(\varphi, r)\} \nvdash_{\mathcal{S}} \overline{0}$ iff $T' \cup \neg_\epsilon(\varphi', r') \nvdash_{\mathcal{S}} \overline{0}$) then $T \ominus (\varphi, r) = T \ominus (\varphi', r')$	(ω-Uniformity)

Now we prove that ω-contraction is sound with respect to the axioms listed above. Notice the difference between (G3$^\omega$) and (G3) from Hansson and Wassermann: the former implies the latter only for logics with the deduction property.

Theorem 3.4. *Given some $Ł(\mathcal{C})$ and $\ominus$ an operator for T. Then, for any conservative selection function γ, $T \ominus_\gamma (\varphi, r)$ satisfies* (G1$^\omega$) – (G4$^\omega$).

Proof. (G1$^\omega$) Since $T \ominus^\epsilon_\gamma (\varphi, r)$ is consistent with $\neg_\epsilon(\varphi, r)$ we have that $(\varphi, r) \notin \mathrm{Cn}_{\mathcal{S}}(T \ominus^\epsilon_\gamma (\varphi, r))$. Since γ is conservative, $\{T \ominus^\epsilon_\gamma (\varphi, r)\}_{\epsilon>0}$ is a $\subseteq$-chain. Since $\mathcal{S}$ is finitary, these two facts imply that $(\varphi, r) \notin \mathrm{Cn}_{\mathcal{S}}(\bigcup_{\epsilon>0} T \ominus^\epsilon_\gamma (\varphi, r))$. (G2$^\omega$) By Corollary 3.3, $T \ominus_\gamma (\varphi, r) = (\bigcup_{\epsilon>0} T \circledast_\gamma \neg_\epsilon(\varphi, r)) \cap T$, which is a subset of T. (G3$^\omega$) Let $(\psi, s) \in T - T \ominus_\gamma (\varphi, r)$. This implies $(\psi, s) \notin T \circledast_\gamma \neg_\epsilon(\varphi, r)$, for any $\epsilon > 0$, so that for each $\epsilon > 0$ there is $X \in \gamma(\mathrm{Con}(T, (\varphi, r)))$ with $(\psi, s) \notin X$. Let $T'_\epsilon = X$. We have that $T \circledast_\gamma \neg_\epsilon(\varphi, r) \subseteq X \cup \{\neg_\epsilon(\varphi, r)\}$; since $X \subseteq T$, $T \circledast_\gamma \neg_\epsilon(\varphi, r) \cap T \subseteq X \cup (\{\neg_\epsilon(\varphi, r)\} \cap T)$. Wether $\neg_\epsilon(\varphi, r) \in T$ or not, maximality of X implies the latter set is X. Also, it is clear that $X \subseteq T$, that $X \cup \{\neg_\epsilon(\varphi, r)\}$ is consistent, and that $X \cup \{\neg_\epsilon(\varphi, r), (\psi, s)\}$ is not consistent, by definition of $\mathrm{Con}(T, \cdot)$. (G4$^\omega$) Assume for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $T' \subseteq T$, $T' \cup \{\neg_\epsilon(\varphi, r)\}$ is consistent iff $T' \cup \{\neg_\delta(\varphi', r')\}$ is consistent, and viceversa. Then, for each $\epsilon > 0$ there is $\delta > 0$ with $\mathrm{Con}(T, \neg_\epsilon(\varphi, r)) = \mathrm{Con}(T, \neg_\delta(\varphi', r'))$, hence with $\gamma(\mathrm{Con}(T, \neg_\epsilon(\varphi, r))) = \gamma(\mathrm{Con}(T, \neg_\delta(\varphi', r')))$. Let $(\psi, s) \in T \ominus_\gamma (\varphi, r)$. Then $(\psi, s) \in \bigcup_{\epsilon>0}(T \circledast_\gamma \neg_\epsilon(\varphi, r) \cap T)$, so that for some $\epsilon > 0$, $(\psi, s) \in \bigcap \gamma(\mathrm{Con}(T, \neg_\epsilon(\varphi, r))) \cap T$. By the assumption, there is $\delta > 0$ with $(\psi, s) \in \bigcap \gamma(\mathrm{Con}(T, \neg_\delta(\varphi', r')))$. Finally, $(\psi, s) \in \bigcup_{\delta>0} T \circledast_\gamma \neg_\delta(\varphi', r')$ so that $(\psi, s) \in T \ominus_\gamma (\varphi', r')$. The other direction is similar. □

Theorem 3.5. *(In $Ł(\mathcal{C})$) Let $\ominus_\gamma$ be an ω-contraction operator defined from a conservative selection function γ for T. If T is finite, then there is some $\epsilon > 0$ with*

$$T \ominus_\gamma (\varphi, r) = T \ominus^\epsilon_\gamma (\varphi, r)$$

Proof. Assume γ is conservative. Since T is finite, so are $T \ominus_\gamma (\varphi, r)$ and $\bigcup_{\epsilon>0} T \circledast_\gamma \neg_\epsilon(\varphi, r)$ (otherwise, these sets could not be subsets of the former; recall contraction is standard, not $\mathcal{C}$-closed). By being $\bigcup_{\epsilon>0} \bigcap \gamma(\mathrm{Con}(T, \neg_\epsilon(\varphi, r)))$ finite, there must be some $n \in \omega$ and $\epsilon_0 > \cdots > \epsilon_n > 0$ with $\bigcup_{\epsilon>0}(T \circledast_\gamma \neg_\epsilon(\varphi, r) \cap T) = \bigcup_{i\leq n}(T \circledast_\gamma \neg_{\epsilon_i}(\varphi, r) \cap T)$. Since γ is conservative and $\epsilon_n \leq \epsilon_i$ (for $i \leq n$), we have $T \ominus^\epsilon_\gamma \neg_{\epsilon_i}(\varphi, r) \subseteq T \ominus_\gamma \neg_{\epsilon_n}(\varphi, r)$. Hence, the join of all such i's is also a subset of $T \circledast_\gamma \neg_{\epsilon_n}(\varphi, r)$. In consequence, $T \ominus_\gamma (\varphi, r) \subseteq (T \circledast_\gamma \neg_{\epsilon_n}(\varphi, r) \cap T)$.

Since the other direction $\supseteq$ is obvious from the definition of $\ominus_\gamma$, we finally obtain $T \ominus_\gamma (\varphi, r) = T \ominus_\gamma^\epsilon (\varphi, r)$. □

This result shows that for finite bases in Łukasiewicz logic, ω-contraction operator $\ominus_\gamma$, with γ conservative, will produce no new outputs which cannot also be obtained by means of some ϵ-contraction. Still, this operation is interesting since it grants optimality of solutions, so we need not choose an ϵ-value from advance.

4. Conclusions and Future Work

We motivated the convenience of obtaining an alternative (consistency-based) operation of base contraction in graded expansions of Łukasiewicz logic and made some steps towards this result for (optimal) ω-contraction, including characterization of (secure) ϵ-contraction, soundness of ω-contraction, and a finite-collapse theorem between these base change operations. In the future, we hope to completely characterize the latter operation, as well as to obtain complexity results, whenever possible, for the operations studied in the present contribution.

Acknowledgements

The authors wish to thank Consolider project CSD2007-022 from the Spanish Ministry of Science and Innovation for financial support; P. Dellunde and Ll. Godo also thank MICINN project FFI2008-03126-E/FILO related to the LOMOREVI Eurocores Project FP006.

References

[1] Alchourrón, C., P. Gärdenfors, D. Makinson *On the Logic of Theory Change: Partial Meet Contraction and Revision Functions*, The Journal of Symbolic Logic, 50: 510-530 (1985)

[2] Booth, R. and E. Richter, *On Revising Fuzzy Belief Bases*, Studia Logica, 80:29-61 (2005).

[3] Esteva, F., J. Gispert, L. Godo, C. Noguera *Adding truth-constants to logics of continuous t-norms: axiomatization and completeness results* Fuzzy Sets and Systems, 185: 597-618 (2007)

[4] Font, J.M., R. Jansana A General Algebraic Semantics for Sentential Logics *Lecture Notes in Logic*, 7, Springer-Verlag (1996)

[5] Hájek, P., Metamatematics of Fuzzy Logic, *Trends in Logic*, 4, Kluwer Academic Publishers (1998).

[6] Hansson, S.o, R. Wasserman *Local change* Studia Logica, 70: 49-76 (2002)

[7] Pardo P. Base Belief Revision for Finitary Monotonic logics. *Proc. of ESSLLI 2009 Student Session*

[8] Pavelka, J. On fuzzy logic I, II, III. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 25, (1979) 45–52, 119–134, 447–464.

[9] Peppas P. Belief Revision, in: van Harmelen, F., V. Lifschitz, B. Porter (eds.) *Handbook of Knowledge Representation* Elsevier (2007)

[10] Zhang, D., N. Foo *Infnitary Belief Revision*, Journal of Philosophical Logic, 30, (2001) 525-570