De Finetti's bets on partially evaluated Kripke frames

Tommaso Flaminio[1] and Hykel Hosni[2]

[1] IIIA - CSIC
Campus de la Univ. Autònoma de Barcelona s/n 08193 Bellaterra, Spain.
Email: tommaso@iiia.csic.es
[2] Scuola Normale Superiore,
Piazza dei Cavalieri 7 Pisa, Italy.
Email: hykel.hosni@sns.it

Abstract. De Finetti's conception of events is one of the most distinctive aspects of his theory of probability, yet it appears to be somewhat elusive. The purpose of this note is to set up a formal framework in which a rigorous characterisation of this notion, and its cognate modelling assumptions, gives rise to a detailed formalisation of the betting problem which underlies the celebrated Dutch Book Argument. In particular, we introduce *partially evaluated Kripke frames*, relational structures which put the implicit modal semantics of events on a rigorous footing and allow us to refine the notion of coherence originally put forward by de Finetti. As our main result shows, this refinement captures an intuitive condition which de Finetti imposed on the betting problem, namely that it is irrational to bet on an event which may be true, but whose truth will never be ascertained by the players.

Keywords: Uncertain reasoning, Events; De Finetti's betting problem; Partial valuations; Kripke frames.

1 The problem

One of the most compelling justifications for the probabilistic representation of uncertainty builds on the assumption that an agent's degrees of belief can be revealed by their disposition to behave in suitably specified betting problems.[2, 12, 3, 7, 8, 13]. The gist of the ensuing *Dutch book argument* is that, *under certain circumstances*, it is clearly irrational to behave in such a way as to lose money no matter what, a kind of behaviour which the agent avoids just if their degrees of belief conform to the laws of probability. Mapping the intuitive concept of "irrationality" to the mathematically definable one of "incurring sure loss" (in suitably specified choice problems) has been one of the fundamental steps in the development of Bayesian theory, which has played a fundamental role in the foundations and applications of Artificial Intelligence and related fields for the past three decades. The seminal [11] and the recent [5] can be taken as ideal references to delimit the spectrum of such a development.

Let $E_1, \ldots, E_n$ be a set of events of interest. A *betting problem* is the choice that an idealised agent called *bookmaker* must make when publishing a *book*, i.e. when making the assignment $B = \{(E_i, \beta_i) : i = 1, \ldots, n\}$ in which each event of interest E_i is given value $\beta_i \in [0, 1]$. Once a book has been published, a *gambler* can place bets on an event E_i by paying $\alpha_i\beta_i$ to the bookmaker. In return for this payment, the gambler will receive the following payoff: α_i, if E_i obtains and nothing otherwise. Note that "betting on an event" effectively amounts, for the gambler, to choosing a real-valued α which determines the amoung payable to the bookmaker. In order to avoid potential distorsions arising from the diminishing value of money, de Finetti makes the "rigidity hypothesis" [3], requiring that α should be small.

De Finetti constructs the betting problem in such a way as to force the bookmaker to publish *fair betting odds* for book B. To this end, two modelling assumptions are built into the problem, namely that (i) the bookmaker is forced to accept any number of bets on B and (ii) when betting on E_i, gamblers can choose the sign of the stakes α_i, thereby unilateraly imposing a payoff swap to the bookmaker. Conditions (i-ii) force the bookmaker to publish books with zero-expectation, for doing otherwise may offer the gambler the possibility of making a sure profit, possibily by swapping payoffs. As the game is zero-sum, this is equivalent to forcing the bookmaker into sure loss. In this context, de Finetti proves that the axioms of probability are necessary and sufficient to secure the bookmaker against this possibility.

The crux of the Dutch book argument is therefore the identification of the agent's degrees of belief with the *price* they are willing to pay for an uncertain reward which depends on the *future* truth value of some *presently unknown* propositions – the *events* on which the agents are betting. This clearly suggests that the semantics of events, which bears directly on the definition of probability, is implicitly endowed with an *epistemic structure*. The purpose of this paper is to give this structure an explicit logical characterisation.

2 Events, facts and propositions

De Finetti's conception of events is one of the most distinctive –and distinctively puzzling– aspects of his theory of probability. In [4], for instance, he puts forward the following crucial defining feature of this elusive notion:

> [T]he characteristic feature of what I refer to as an "event" is that the circumstances under which the event will turn out to be "verified" or "disproved" have been fixed in advance. [4]

The underlying intuition, which is also developed in various points of the "critical appendix" to [3] can be unravelled as follows. From the *logical* point of view, events are propositions, and a such can either be true or false. In this sense, de Finetti is a firm believer in classical, as opposed to non-classical logics. Yet from the *qualitative epistemic* point of view, an event can be either certainly known to be true (false) or "unknown". Finally, from the *quantitative epistemic* point of view, an agent can expresses their degrees of belief (uncertainty) on a particular event by assigning it with a real number in $[0,1]$. The only constraint that a rational agent must satisfy in this assignment is that it should not be incoherent, i.e. it should never lead a bookmaker to be open to the possibility of being forced into sure loss by a gambler.

Going back to the "characteristic feature" of events, if bookmaker and bettor *knew* the (logical) truth value of say event E at the time of betting, coherence determines that exactly one of the following alternatives must hold: Either they exchange money for nothing or the bookmaker faces the possibility sure loss (because any deviation from the logical truth value would result in sure win for the gambler). In either case, there is no proper "uncertainty" to be measured about E. This arises precisely when bookmaker and gamblers agree on which *facts* will determine the logical truth value of E and agree that those facts do not hold at the current (that is, to the act of signing the contract) state of the world.

The remainder of this section is devoted to formalising these notions.

2.1 Formal preliminaries

Let $L = \{q_1, \ldots, q_n\}$ be a classical propositional language. The set of sentences $SL = \{\varphi, \psi, \theta, \ldots\}$ is inductively built up from L through the propositional connectives $\wedge, \vee, \to$, and $\neg$, as usual. We use $\bot$ to denote *falsum*. *Valuations* are mappings ω form L into $\{0,1\}$ that naturally extend to SL by the truth-functionality of the propositional connectives (with the usual stipulation that $\omega(\bot) = 0$, for all valuations ω). We denote by $\Omega(L)$ (or simply Ω when the set of variables is clear by the context) the class of all valuations over L.

A *partial valuation* on L is a map $\nu : X \subseteq L \to \{0,1\}$. In other words a map ν is a partial valuation, provided that there is (at least) a valuation $\omega \in \Omega$ such that for every $q \in L$,

$$\nu(q) = \begin{cases} \omega(q) & \text{if } q \in X; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The class of all partial valuations over L is denoted by $\Omega^P(L)$ (or simply Ω^P when L is clear from the context). For $\nu : X \to \{0,1\}$ $\mu : Y \to \{0,1\}$ in $\Omega^P(L)$ we say that μ *extends* ν (and we write $\nu \subseteq \mu$) if $X \subseteq Y$, and for every $x \in X$, $\nu(x) = \mu(x)$.

Finally, for every formula φ we set $[\varphi] = \{\psi \in SL : \vdash \varphi \leftrightarrow \psi\}$, where as usual, $\vdash$ denotes the classical provability relation. We conform to the custom of referring to the equivalence classes $[\varphi]$ as to *the proposition* φ.

Note that de Finetti's notion of event is captured by propositions, rather than sentences. In fact, in the present logical framework, the "circumstances" under which the propositions turn out to be true or false are nothing but the valuations in $\Omega(L)$. As a consequence, de Finetti's notion of *book* is formally defined by (probability) assignments on a (finite) set of propositions. We will reserve the expression *propositional books* to refer to these particular assignments.

Definition 1. *Let $\varphi \in SL$, and let $\nu \in \Omega^P(L)$. (1) We say that ν* realizes *φ, written $\nu \Vdash \varphi$, if $\nu(\varphi)$ is defined.*

(2) We say that ν realizes *$[\varphi]$, written $\nu \Vdash [\varphi]$, if there exists at least a $\psi \in [\varphi]$ such that $\nu \Vdash \psi$. In this case we assign $\nu(\gamma) = \nu(\psi)$ for every $\gamma \in [\varphi]$.*

We now introduce a simple relational structure which, unlike the classical notion of a Kripke structure, admits of partial valuations. Let W be a finite set of nodes interpreted, as usual, as possible worlds. Let $e : W \to \Omega^P(L)$ such that for every $w \in W$, $e(w) = \nu_w : X_w \subseteq L \to \{0, 1\}$ is a partial valuation. Note that to avoid cumbersome notation we will write ν_w instead of $e(w)$ to denote the partial valuation associated to w, and similarly, we denote by X_w the subset of L for which ν_w is defined. Finally, let $R \subseteq W \times W$ be an *accessibility relation*. We call a triplet (W, e, R) a *partially evaluated Kripke frame* (**pekf** for short).

Let $w \in W$ and let $[\varphi]$ be a proposition. We say that that w *decides* $[\varphi]$, if $\nu_w \Vdash [\varphi]$.

Remark 1. In every **pekf** $K = (W, e, R)$, the role of e is to identify every world w with the partial valuation ν_w attached to it. Therefore we will henceforth identify two worlds w and w' whenever $\nu_w = \nu_{w'}$.

Definition 2 (Events and Facts). *Let (W, e, R) be a* **pekf***, $w \in W$ and $\varphi \in SL$. Then we say that a proposition $[\varphi]$ is:*

- *A w-*event *iff $\nu_w \nVdash [\varphi]$, and for every $\nu \in \Omega^P(L)$ such that $\nu \supseteq \nu_w$ and $\nu \Vdash [\varphi]$, there exists w' such that $R(w, w')$, and $\nu_{w'} = \nu$.*
- *A w-*fact *iff $\nu_w \Vdash [\varphi]$.*

For every $w \in W$ we denote by $\mathcal{E}(w)$ and $\mathcal{F}(w)$ the classes of w-events, and w-facts respectively.

The main point of Definition 2 is that in **pekf** events (and facts) are relativised to a specific state of the world. In analogy with the modal consequence relations [1], facts and events have distinct "local" and "global" properties.

A proposition $[\varphi]$ might be an event *with respect to a given world* w (i.e. $[\varphi]$ is a w-event), but not for some other w'.

Example 1. Let $L = \{q_1, \ldots, q_6\}$, and set $\psi_1 = q_1 \wedge q_2$, $\psi_2 = q_1 \vee q_2$, and $\varphi = q_1 \vee (q_2 \to q_4)$. Suppose (W, R) is as illustrated in Fig. 1 where at each node the following partial valuations are defined:

(w_1) $\nu_{w_1}(q_1) = 0$, $\nu_{w_1}(q_3) = 1$, with ν_{w_1} otherwise undefined on L (o.u);
(w_2) $\nu_{w_2}(q_5) = 0$, $\nu_{w_2}(q_6) = 1$, with ν_{w_2} o.u;
(w_3) $\nu_{w_3} \supseteq \nu_{w_1}$, $\nu_{w_3}(q_2) = 0$, $\nu_{w_3}(q_4) = 1$, with ν_{w_3} o.u;
(w_4) $\nu_{w_4} \supseteq \nu_{w_1}$, $\nu_{w_4}(q_2) = 1$, $\nu_{w_4}(q_4) = 1$, with ν_{w_4} o.u;
(w_5) $\nu_{w_5} \supseteq \nu_{w_1}$, $\nu_{w_5}(q_2) = 0$, $\nu_{w_5}(q_5) = 0$, with ν_{w_5} o.u;
(w_6) $\nu_{w_6} \supseteq \nu_{w_1}$, $\nu_{w_6}(q_2) = 1$, $\nu_{w_6}(q_4) = 1$, $\nu_{w_6}(q_6) = 0$, with $\nu_{w_6}(q_5)$ undefined;
(w_7) $\nu_{w_7}(q_2) = 1$, $\nu_{w_7}(q_3) = 0$ with ν_{w_7} o.u.;
(w_8) $\nu_{w_8}(q) = 1$ for all $q \in L$;
(w_9) $\nu_{w_9} = \nu_{w_6}$

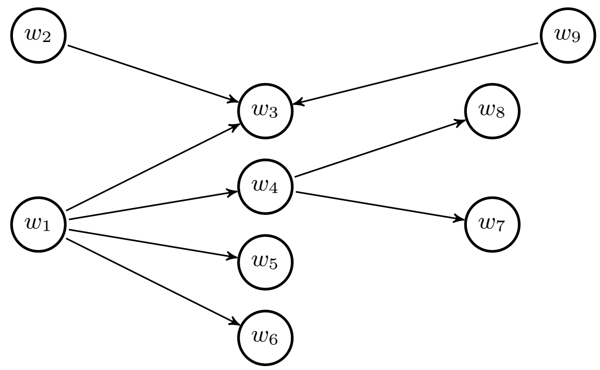

Fig. 1. The partially evaluated Kripke frame of Example 1.

To see that propositions $[\psi_1]$ and $[\psi_2]$ are w_1-events note that for every $\nu \in \Omega^P$ such that $\nu \supseteq \nu_{w_1}$ and $\nu \Vdash [\varphi_i]$ (for $i = 1, 2$), there exists $w \in W$ such that $\nu = \nu_w$. In addition, say the partial valuation ν such that $\nu(p_1) = 0$ and $\nu(p_2) = 0$ and is o.u, coincides with ν_{w_3} the partial valuation ν' such that $\nu'(q_1) = 0$, and $\nu'(q_2) = 1$ and is o.u, is, say ν_{w_4}, and so forth. On the other hand, $[\varphi]$ is not a w_1-event. To see this note that there is no ν'' which, for every $w \in W$, is defined over q_1, q_2, q_3 only and o.u.

Note that $[\psi_1]$ in Example 1 is a w_1-event whose truth value is determined by the partial valuation ν_{w_1}. Indeed, every ν which extends ν_{w_1} (namely $\nu_{w_3}, \nu_{w_4}, \nu_{w_5}$ and ν_{w_6}) assigns 0 to ψ_1. This motivates the following definition:

Definition 3. *Let (W, e, R) be a* **pekf** *let $w \in W$, and let $[\psi]$ be a w-event. Then we say that $[\psi]$ is* surely true *(resp.* false*) if for every $\nu \in \Omega^P(L)$ such that $\nu_w \subseteq \nu$ and $\nu \Vdash [\psi]$, $\nu(\psi) = 1$ (resp. $\nu(\psi) = 0$).*

In Example 1 above, $[\psi_1]$ is a surely false w_1-event, and $[q_2 \vee q_3]$ is a surely true w_4-event.

Notice in a **pekf**, two worlds may be identical but unrelated by R, as w_6 and w_9 in Example 1 (i.e. R is not reflexive). This is certainly incompatible with the underlying epistemic interpetation of the concept of events that we intend to formalise. Thus, in the remainder of this paper we will be concerned with adding suitable structure to **pekf**s.

3 Models with memory, and complete worlds

A partially evaluated Kripke frame $K = (W, e, R)$ is said to be *monotonic* if satisfies the following property:

(M) for every $w, w' \in W$, if $R(w, w')$, then $\nu_w \subseteq \nu_{w'}$.

In the frame $K = (W, e, R)$ of Example 1, not all the information contained in a world w would necessarily be preserved when reaching an accessible w':

- at w_2 and w_3, the partial valuations are defined on subsets $X_{w_2}, X_{w_3} \subseteq L$ with $X_{w_2} \not\subseteq X_{w_3}$. As a consequence some variables are evaluated by ν_{w_2} and not by ν_{w_3}.
- moreover, q_3 is inconsistently evaluated at w_4 and at w_7

This sort of dynamical inconsistency across accessible worlds is clearly unsuitable to model the highly idealised bookmaker and gamblers of de Finetti's betting problem. Recall that in de Finetti's intuitive characterisation, at the time (i.e. world in W) at which the contract is signed, bookmaker and gambler agree on which conditions will realise the events in the book. This clearly presupposes some form of monotonic persistence of the underlying structure, which Property **(M)** guarantees. In particular, in every monotonic **pekf**, w-facts are w'-facts in each w' which is accessible from w. In addition their truth value once determined, is fixed throughout the frame.

Definition 4. *Let (W, e, R) be any monotonic* **pekf***, and let $w \in W$. A w-book is a propositional book $B = \{[\varphi_i] = \beta_i : i = 1, \ldots, n\}$ where the propositions $[\varphi_i]$ are w-events.*

Finally, we say that a **pekf** (W, e, R) is *complete* if the following property is satisfied:

(C) for every $\varphi \in SL$ and for every $\nu \in \Omega^P$ such that $\nu \Vdash [\varphi]$, there exists a $w_{[\varphi]} \in W$ such that $\nu = \nu_{w_{[\varphi]}} \Vdash [\varphi]$, i.e. $[\varphi]$ is a $w_{[\varphi]}$-fact.

4 No bets on inaccessible propositions

Suppose $w \in W$ is such that the proposition $[\varphi]$ is not realised in w, nor in any w' accessible from w. According to Definition 2 $[\varphi]$ is not a w-event. The main result of this note is that rational agents cannot bet on such propositions.

Definition 5 (Inaccessible propositions). *Let (W, e, R) be a* **pekf** *and $w \in W$. A proposition $[\varphi]$ is said to be w-*inaccessible *if $\nu_w \nVdash [\varphi]$ and for every w' such that $\nu_{w'} \Vdash [\varphi]$, $\neg R(w, w')$.*

Inaccessible propositions relativise the betting problem to the specific information available to the gambler and the bookmaker, thus being faithful to de Finetti's strict subjectivist perspective. According to this perspective all that matters is that the agents agree on which events are realised. Everything else is irrelevant to the determination of the payoff of the bet. [1]

The following example of w-inaccessible proposition also illustrates the expressive power of monotonic and complete **pekf**.

[1] Inaccessible propositions are, in this sense like Stephen Hawkins' dice, which God throws where we can't see them.

Example 2 ([6]). Consider an electron ϵ, and a world w. We are interested the position and the energy of ϵ at w. Let $[\varphi]$ and $[\psi]$ be the propositions expressing those measurements, respectively. Moreover let us assume that both $[\varphi]$ and $[\psi]$ are w-events. Indeed if at w we are uncertain about the position and the energy of ϵ, we can certainly perform experiments to determine them. But, what about $[\varphi] \wedge [\psi]$? Position and energy are represented by non-commuting operators in quantum theory, and we can assign an electron a definite position and a definite energy, but not both. This fact can be easily modelled in complete and monotonic **pekf** K, by forcing $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$ to be w-inaccessible .

The above example points out two important properties of w-events which are defined in a complete and monotonic **pekf**. The first, is that the class $\mathcal{E}(w)$ of w-events is not, in general, closed under the propositional connectives for every w. We will come back to this in next Section. The second interesting observation about Example 2 is that it implicitly defines an example of a coherent propositional book, which nonetheless is not a w-book for every w. To see this, let φ and ψ as above, and consider the propositional book $B = \{[\varphi] = 1/3, [\psi] = 3/7, [\varphi \wedge \psi] = 0\}$. B is clearly coherent. However, this assignment is *meaningless* (rather than incoherent) in the context of monotonic and complete **pekf** because some of the propositions appearing in it are w-inaccessible, and hence are not w-events.

As our main result shows, a coherent w-book B can be extended, either by w-facts or by w-inaccessible propositions to a coherent propositional book B' only if the newly added propositions are given their actual truth value.

Theorem 1. *Let $B = \{[\varphi_i] = \beta_i : i = 1, \ldots, n\}$ be a coherent w-book, let $[\psi_1], \ldots, [\psi_r]$ be propositions that are not w-events, and let $B' = B \cup \{[\psi_j] = \gamma_j : j = 1, \ldots, r\}$ be a propositional book extending B. Then the following hold:*

(1) If all propositions $[\psi_j]$ are w-facts, then B' is coherent if and only if for every $j = 1, \ldots, r, \gamma_j = \nu_w(\psi_j)$;
(2) If all propositions $[\psi_j]$ are w-inaccessible, then B' is coherent if and only if for every $j = 1 \ldots, r$, $\gamma_j = 0$.

Proof. We only prove the direction from left-to-right, the converse being is immediate in both cases.

(1) Suppose, to the contrary, that exists j such that, $\gamma_j \neq \nu_w(\psi_j)$, and in particular suppose that $\nu_w(\psi_j) = 1$, so that $\gamma_j < 1$. Then, the gambler can secure sure win by betting a positive α on ψ_j. In this case in fact, since the **pekf** is monotonic by the definition of w-book, $\nu_{w'}(\varphi_i) = 1$ holds in every world w' which is accessible from w. Thus the gambler pays $\alpha \cdot \gamma_j$ in order to surely receive α in any w' accessible from w. Conversely, if $\nu_w(\psi_j) = 0$, then $\gamma_j > 0$ and in that case it is easy to see that a sure-winning choice for the gambler consists in swapping payoffs with the bookmaker, i.e. to bet a negative amount of money on $[\psi_j]$.

(2) As above suppose to the contrary that $\gamma_j > 0$ for some j, and that the gambler bets $-\alpha$ on $[\psi_j]$. By contract, this means that the bookmaker must pay

$\alpha \cdot \gamma_j$ to the gambler, thus incurring sure loss, since $[\psi_j]$ will not be decided in any world w' such that $R(w, w')$.

Remark 2. This result provides a formalisation of the key property identified by de Finetti in his informal characterisation of events, namely that *no monetary betting is rational* unless the conditions under which the relevant events will be decided are known to the bookmaker and the gambler. This naturally raises the question of whether a rational betting behaviour can be identified if we drop the assumption that betting is monetary. Put otherwise, is it possible to define rational degrees of belief for an event which includes w-inaccessible propositions? We shall come back to this extremely interesting question in the concluding section of this paper.

4.1 Fully accessible worlds

A gambler and a bookmaker interpreted on a complete and monotonic **pekf** are guaranteed that: (1) as soon as a proposition is realized in w, this information preserved across the frame to all the accessible worlds from w (memory), and (2) for every sentence φ, there exists a world w that realizes $[\varphi]$ (completeness). In this Section we focus on the accessibility of $[\varphi]$ for every $\varphi \in SL$. In accordance with the above informal discussion of the betting problem, not only gamblers and bookmaker must agree that a world w in which the events of interest are realized exists. They also must agree on the conditions under which this will happen, as captured by Theorem 1.

Definition 6. *A* **pekf** (W, e, R) *is* fully accessible *if* R *satisfies:*

($\mathbf{A}$) *for all* $w, w' \in W$*, if* $\nu_w \subseteq \nu_{w'}$*, then* $R(w, w')$*.*

The following shows that Example 2 cannot follow in a fully accessible **pekf**.

Theorem 2. *Let* $K = (W, e, R)$ *be a complete* **pekf**. *If* K *is fully accessible, then for every* $w \in W$*,* $\mathcal{E}(w)$ *is closed under the classical connectives.*

Proof. Let w be any world such that $\mathcal{E}(w) \neq \emptyset$ and let $[\varphi_1]$ and $[\varphi_2]$ be w-events. We want to prove that $[\varphi_1] \star [\varphi_2] = [\varphi_1 \star \varphi_2]$ is a w-event for every $\star \in \{\wedge, \vee\}$ (the case for $\neg$ is clearly analogous, and omitted). Clearly $\nu_w \not\Vdash [\varphi_1 \star \varphi_2]$, moreover for every partial valuation $\nu' \in \Omega^P$ such that $\nu' \supseteq \nu_w$, and $\nu' \Vdash [\varphi_1 \star \varphi_2]$, ($\mathbf{C}$) guarantees the existence of a $w_{[\varphi_1 \star \varphi_2]}$ such that

$$\nu' = \nu_{w_{[\varphi_1 \star \varphi_2]}}, \nu_{w_{[\varphi_1 \star \varphi_2]}} \Vdash [\varphi_1 \star \varphi_2], \text{ and } \nu_w \subseteq \nu_{w_{[\varphi_1 \star \varphi_2]}} = \nu'.$$

Therefore, since since K is fully accessible, ($\mathbf{A}$) implies $R(w, w_{[\varphi_1 \star \varphi_2]})$.

The coherence of a propositional book B is usually characterized by the possibility of extending the assessment to a (finitely additive) probability measure on the Boolean algebra spanned by the events in B. Theorem 2 can be interpreted as identifying the necessary conditions (i.e. completeness and full accessibility) in order for the algebra generated by the w-events in a w-book B to contain *only* w-events.

5 Conclusion and future work

A central modelling assumption which appears only implicitly in de Finetti's Dutch book argument is that when betting on a particular event of interest, the (proposition representing the) event must be *unknown* to the bookmaker and the gambler. Yet, it is part of de Finetti's characterisation of events that at the time of betting, both parties agree on which facts will be necessary and sufficient to decide whether the event occurred or not. Building on the definitions of *events*, *facts* and *inaccessible propositions*, this note introduced a relational framework in which this modelling assumption can be given a logical formalisation.

The main result which arises in our framework, Theorem 1, captures the intuitive idea that under the assumption – central to de Finetti's betting problem– that bookmaker and gamblers are betting real money, no coherent book can be published if inaccessible propositions are involved. This means that under those circumstances, no rational degrees of belief can be defined. Note however, that this kind of incompleteness does not depend on the logical undecidability of the relevant facts – it only depends on the inaccessibility of the worlds at which the relevant facts are decided. This naturally suggests that by relaxing the assumption that betting involves archimedean quantities, a fuller formal model of rational belief may be put forward. With this respect, Theorem 1 suggests defining a generalised betting problem in which bookmakers and gamblers can make transactions with possibly "immaterial" assets –like for instance one's word– to which an infinitesimal value may be attached. We conjecture that in this extended betting problem agents might rationally be entitled to assign a positive value also to inaccessible propositions, thereby remeding to the descriptive incompleteness of the classical model. Providing a suitable formalisation of this extended betting problem is the first line of development that we envisage for the investigation initiated in this paper. This justifies the implicit redundancy of our notion of **pekf** (we could have simply let $W = \Omega^P(L)$, and defined R accordingly). However the *extended* betting problem is intrinsically multi-agent. Hence the generality of **pekf** (W, e, R) is no longer redundant: To each agent a_i, we attach a map e_i associating, to each $w \in W$, the partial valuation $e_i(w)$ whose intended meaning is to express the agent's a_i information about w (i.e. which propositions $[\varphi_j]$ are known by a_i in w).

A second, natural development, consists in extending the present framework to the case of *conditional* events. It is well known that de Finetti insisted on all probability being conditional (See, e.g. [4]), a conception which has many counterparts in Bayesian theory. This extension requires us to construct partially evaluated Kripke frames on top of a three-valued logic which is needed to account for the fact that conditional bets might be called-off if the conditioning event fails to become a fact. Indeed, the notion of an *event conditioned by a fact* arises naturally in the framework we introduced above. Further research is needed to flesh out the properties of this notion which must certainly be distinguished from the usual notion of conditional event, and which may lead to a fuller understanding of de Finetti's intuitions on conditional probability.

Finally, the framework introduced by this paper appears to have interesting connections with the *Ents model* of [9, 10]. On central idea of the Ents model is that agents construct their degrees of belief by taking into account increasingly more detailed “scenarios”, which is crucial towards the computational feasibility of the resulting model of belief. The framework of **pekf** certainly includes the idea that facts are built up by successively filling-in the gaps in a partially evaluated events, which suggests that our construction might be interpreted as a model of belief in the sense of Paris-Vencovská.

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