On completeness results for the expansions with truth-constants of some predicate fuzzy logics

Francesc Esteva
AI Research Institute (IIIA)
Spanish Research Council (CSIC)
08193 Bellaterra, Catalonia, Spain
esteva@iiia.csic.es

Lluís Godo
AI Research Institute (IIIA)
Spanish Research Council (CSIC)
08193 Bellaterra, Catalonia, Spain
godo@iiia.csic.es

Carles Noguera
Dept. Comp. Science
University of Lleida
25001 Lleida, Catalonia, Spain
cnoguera@diei.udl.cat

Abstract

In this paper we study generic expansions of predicate logics of some left-continuous t-norms (mainly Gödel and Nilpotent Minimum predicate logics) with a countable set of truth-constants. Using known results on t-norm based predicate fuzzy logics we obtain results on the conservativeness and completeness for the expansions of some predicate fuzzy logics. We describe the problem for the cases of Łukasiewicz and Product predicate logics and prove that the expansions of Gödel and Nilpotent Minimum predicate logics are canonical complete for tautologies, and strong standard complete for deduction upon any set of premisses.

Keywords: Monoidal T-norm based Logic MTL, Gödel, Łukasiewicz, Product and Nilpotent Minimum propositional and predicate logics, t-norm-based logic, Rational Pavelka Logic, Gödel, Product and Nilpotent Minimum logics with truth-constants, completeness results.

1 Introduction

In the context of expansions of propositional t-norm based fuzzy logics with truth-constants, an algebraic analysis has been recently used to establish different completeness results (with respect to a finitary notion of deduction) for a number of propositional logics, among them Gödel and Nilpotent Minimum logics [5], Product logic [17], logics a continuous t-norm [4] and logics of Weak Nilpotent Minimum t-norms [6].[1]

This approach contrasts with the one related to Łukasiewicz logic, initiated by Pavelka [16] for the propositional case, and extended by Nóvak [14, 15] for the first-order case. This latter approach, based on a infinitary notion of provability, strongly relies on the continuity of the truth functions of Łukasiewicz logic (and hence not applicable to other t-norm based logics), and it was simplified by Hájek in [7], both for the propositional and first-order cases. In particular, Hájek defines what he calls *Rational Pavelka Predicate Logic*, RPL$\forall$ for short, as the expansion of Łukasiewicz predicate logic Ł$\forall$ by introducing in the language a truth-constant $\overline{r}$ for each rational r of $[0,1]$ and by adding the well-known book-keeping axioms

$$\overline{r}\&\overline{s} \leftrightarrow \overline{\max(0, r+s-1)}$$
$$(\overline{r} \to \overline{s}) \leftrightarrow \overline{\min(1, 1-r+s)}$$

Hájek shows that RPL$\forall$ enjoys the so-called *Pavelka style completeness*, which means that for any theory T and formula φ, one has

$$\|\varphi\|_T \ = \ |\varphi|_T,$$

where $\|\varphi\|_T = \inf\{\|\varphi\|_{\mathbf{M}} \mid \mathbf{M} \text{ model of } T\}$ is the truth degree of φ in T and $|\varphi|_T = \sup\{r \mid T \vdash_{\mathrm{RPL}\forall} \overline{r} \to \varphi\}$ is the provability degree of φ from T.

In this paper, following the algebraic approach mentioned above, we consider the expansions with truth-constants of the corresponding predicate logics, with special attention to the cases of Gödel and Nilpotent Minimum. In fact, to the best of our knowledge, until now only the expansion of Łukasiewicz predicate logic with truth-constants had been considered in the literature. A nice and deep result contained in [10] proves that Rational Pavelka Predicate logic RPL$\forall$ is a conservative expansion of Łukasiewicz predicate logic Ł$\forall$ and, since Ł$\forall$ is not recursively axiomatizable with respect to the standard semantics, so neither is RPL$\forall$. This is a negative result. In this paper we show other negative results, but also two positive new results. Namely, after some preliminary definitions and results in next section, we first prove that the expansions of predicate Product, Gödel, Nilpotent Minimum logics

[1] For a number these logics their complexity issues have been recently studied in [8].

(and more generally any pseudo-complemented t-norm based logic) are conservative expansions of their corresponding predicate logics. Moreover, we prove (canonical) standard completeness results for the expansions with truth-constants of Gödel and Nilpotent Minimum predicate logics. The paper ends with some conclusions and remarks.

2 Preliminaries

2.1 Propositional expansions with truth-constants

The basic logic we will use is the Monoidal t-norm based logic MTL introduced in [3] and proved to be the logic of left-continuous t-norms and their residua in [11]. In this setting, given a left-continuous t-norm $*$ we will denote by $[0,1]_*$ the standard MTL-chain given by the left-continuous t-norm $*$ and its residuum $\Rightarrow$, i. e. $[0,1]_* = \langle [0,1], *, \Rightarrow, \min, \max, 0, 1\rangle$, and by L_* the axiomatic extension of MTL whose equivalent algebraic semantics is $\mathbf{V}([0,1]_*)$, i. e. the variety generated by $[0,1]_*$. Well-known examples of these logics are the cases when $*$ is the minimum t-norm ($\mathrm{L}_*=$ G), the Łukasiewicz t-norm ($\mathrm{L}_* =$ Ł), the product t-norm ($\mathrm{L}_* = \Pi$) or the nilpotent minimum t-norm (L_* = NM).

Now, given a left-continuous t-norm $*$ and its corresponding logic L_*, let $\mathcal{C} = \langle C, *, \Rightarrow, \min, \max, 0, 1\rangle$ be a *countable* subalgebra of the standard L_*-algebra $[0,1]_*$. Then, the logic $\mathrm{L}_*(\mathcal{C})$ is defined as follows:

(i) the language of $\mathrm{L}_*(\mathcal{C})$ is the one of L_* expanded with a new propositional variable $\overline{r}$ for each $r \in C \setminus \{0,1\}$,

(ii) the axioms of $\mathrm{L}_*(\mathcal{C})$ are those of L_* plus the book-keeping axioms:

$$\overline{r}\&\overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow_* s}$$

for each $r, s \in C$. The algebraic counterpart of the $\mathrm{L}_*(\mathcal{C})$ logic consists of the class of $\mathrm{L}_*(\mathcal{C})$-algebras, defined as structures

$$\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$$

such that:

1. $\langle A, \&, \to, \wedge, \vee, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ is an L_*-algebra, and

2. for every $r, s \in C$ the following identities hold:

$$\overline{r}^{\mathcal{A}}\&\overline{s}^{\mathcal{A}} = \overline{r * s}^{\mathcal{A}}$$
$$\overline{r}^{\mathcal{A}} \to \overline{s}^{\mathcal{A}} = \overline{r \Rightarrow s}^{\mathcal{A}}.$$

The $\mathrm{L}_*(\mathcal{C})$-chains defined over the real unit interval $[0,1]$ are called *standard*. Among them there is one which reflects the intended semantics, the so-called *canonical standard* $\mathrm{L}_*(\mathcal{C})$-chain

$$[0,1]_{\mathrm{L}_*(\mathcal{C})} = \langle [0,1], *, \Rightarrow, \min, \max, \{r : r \in C\}\rangle,$$

i. e. the standard chain where the truth-constants are interpreted by themselves. Note that, for a logic $\mathrm{L}_*(\mathcal{C})$ there may exist multiple standard chains as soon as there exist different ways of interpreting the truth-constants on $[0,1]$ respecting the book-keeping axioms. For instance, let $C = [0,1] \cap \mathbb{Q}$ and let $*$ be a *pseudo-complemented t-norm*, i.e. a left-continuous t-norm $*$ whose definable negation $\neg x = x \Rightarrow 0$ is the so-called Gödel negation, i.e. $\neg x = 0$ for all $x \neq 0$ and $\neg 0 = 1$. In such a case, if $*$ is closed on C, it is easy to check that the algebra $\mathcal{A} = \langle [0,1], *, \Rightarrow, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$ where

$$\overline{r}^{\mathcal{A}} = \begin{cases} 1, & \text{if } r > 0 \\ 0, & \text{otherwise} \end{cases}$$

is always a standard $\mathrm{L}_*(\mathcal{C})$ algebra. This is the case e.g. of minimum and product t-norms. Furthermore, in the particular case of $* = \min$, for any $\alpha > 0$, the algebra $\mathcal{A} = \langle [0,1], *, \to, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$ where

$$\overline{r}^{\mathcal{A}} = \begin{cases} 1, & \text{if } r \geq \alpha \\ 0, & \text{otherwise} \end{cases}$$

is also standard $\mathrm{G}_*(\mathcal{C})$.

Since the additional symbols added to the language are 0-ary, $\mathrm{L}_*(\mathcal{C})$ is also an algebraizable logic and its equivalent algebraic semantics is the variety of $\mathrm{L}_*(\mathcal{C})$-algebras. This, together with the fact that $\mathrm{L}_*(\mathcal{C})$-algebras are representable as a subdirect product of $\mathrm{L}_*(\mathcal{C})$-chains, leads to the following general completeness result of $\mathrm{L}_*(\mathcal{C})$ with respect to the class of $\mathrm{L}_*(\mathcal{C})$-chains. In the following, for any set $\Gamma \cup \{\varphi\}$ of $\mathrm{L}_*(\mathcal{C})$-formulae and any class $\mathbb{K}$ of $\mathrm{L}_*(\mathcal{C})$-chains, we write $\Gamma \models_{\mathbb{K}} \varphi$ to denote that, for each $\mathcal{A} \in \mathbb{K}$, $e(\varphi) = \overline{1}^{\mathcal{A}}$ for all $\mathcal{A}$-evaluation e model of Γ

Theorem 1. *For any set* $\Gamma \cup \{\varphi\}$ *of* $\mathrm{L}_*(\mathcal{C})$*-formulae, it holds that* $\Gamma \vdash_{\mathrm{L}_*(\mathcal{C})} \varphi$ *if, and only if,* $\Gamma \models_{\{\mathrm{L}_*(\mathcal{C})-chains\}} \varphi$.

The issue of studying when a logic $\mathrm{L}_*(\mathcal{C})$ is also complete with respect to the class of standard $\mathrm{L}_*(\mathcal{C})$-chains (called *standard completeness* property) or with respect to the canonical standard $\mathrm{L}_*(\mathcal{C})$-chain (called *canonical standard completeness* property) has been addressed in the literature for some logics L_*. Hájek already proved in [7] the canonical completeness of the expansion of Łukasiewicz logic with rational truth-constants for finite theories. More recently, the expansions of Gödel (and of some t-norm based logic related to the nilpotent minimum t-norm) and of Product logic with countable sets of truth-constants have been proved to enjoy the canonical standard completeness for theorems in [5] and in [17] respectively. A rather exhaustive description of completeness results for the logics $L_*(\mathcal{C})$ can be found in [4] and in [6].

2.2 Core predicate fuzzy logics

Predicate versions of the propositional t-norm based logics described above have also been defined and studied in the literature. Following [9] we give below a general definition of the predicate logic L∀ for any (propositional) *core fuzzy logic* L. A finitary logic L in a countable language is a *core fuzzy logic* [1] if:

(i) L expands MTL;

(ii) L satisfies the congruence condition: $\varphi \leftrightarrow \psi \vdash_L \chi(\varphi) \leftrightarrow \chi(\psi)$, for every φ, ψ, χ;

(iii) L satisfies the following local deduction theorem: $\Gamma, \varphi \vdash_L \psi$ iff there a is natural number n such that $\Gamma \vdash_L \varphi \& \overset{n}{\ldots} \& \varphi \to \psi$.

Note that the logics $L_*(\mathcal{C})$ introduced above are core fuzzy logics, so what follows also applies to them.

Given a core fuzzy logic L, the language $\mathcal{PL}$ of L∀ is built from the propositional language $\mathcal{L}$ of L by enlarging it with a set of predicates $Pred$, a set of object variables Var and a set of object constants $Const$, together with the two classical quantifiers ∀ and ∃. The notion of formula trivially generalizes taking into account that now, if φ is a formula and x is an object variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulae as well.

In first-order fuzzy logics it is usual to restrict the semantics to L-chains only. For each L-chain $\mathcal{A}$ an L-interpretation for a predicate language $\mathcal{PL} = (Pred, Const)$ of L∀ is a structure

$$\mathbf{M} = (M, (r_P)_{P \in Pred}, (m_c)_{c \in Const})$$

where $M \neq \emptyset$, $r_P : M^{ar(P)} \to A$ and $m_c \in M$ for each $P \in Pred$ and $c \in Const$. For each evaluation of variables $v : Var \to M$, the truth-value $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v}$ of a formula (where $v(x) \in M$ for each variable x) is defined inductively from

$$\|P(x, \cdots, c, \cdots)\|^{\mathcal{A}}_{\mathbf{M},v} = r_P(v(x), \cdots, m_c \cdots),$$

taking into account that the value commutes with connectives, and defining

$$\begin{aligned}\|(\forall x)\varphi\|^{\mathcal{A}}_{\mathbf{M},v} &= \inf\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables } y, \text{ except } x\}\\ \|(\exists x)\varphi\|^{\mathcal{A}}_{\mathbf{M},v} &= \sup\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables } y, \text{ except } x\}\end{aligned}$$

if the infimum and supremum exist in $\mathcal{A}$, otherwise the truth-value(s) remain undefined. An structure $\mathbf{M}$ is called $\mathcal{A}$-safe if all infs and sups needed for definition of the truth-value of any formula exist in $\mathcal{A}$. Then, the truth-value of a formula φ in a safe $\mathcal{A}$-structure $\mathbf{M}$ is just

$$\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = \inf\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} \mid v : Var \to M\}.$$

When $\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = 1$ for a $\mathcal{A}$-safe structure $\mathbf{M}$, the pair $(\mathbf{M}, \mathcal{A})$ is said to be a model for φ, written $(\mathbf{M}, \mathcal{A}) \models \varphi$. Sometimes we will call the pair $(\mathbf{M}, \mathcal{A})$ an L∀-structure.

The axioms for L∀ are the axioms resulting from those of L by substitution of propositional variables with formulae of $\mathcal{PL}$ plus the following axioms on quantifiers (the same used in [7] when defining BL∀):

(∀1)	$(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$)
(∃1)	$\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$)
(∀2)	$(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ (x not free in ν)
(∃2)	$(\forall x)(\varphi \to \nu) \to ((\exists x)\varphi \to \nu)$ (x not free in ν)
(∀3)	$(\forall x)(\varphi \vee \nu) \to ((\forall x)\varphi \vee \nu)$ (x not free in ν)

The rules of inference of L∀ are modus ponens and generalization: from φ infer $(\forall x)\varphi$.

A completeness theorem for first-order BL was proven in [7] and the completeness theorems of other predicate fuzzy logics defined in the literature have been proven in the corresponding papers where the propositional logics are introduced. The following general formulation of completeness for predicate core fuzzy logics is from the paper [9].

Theorem 2. *For any core fuzzy logic* L *over a predicate language* $\mathcal{PL}$*, it holds that*

$$T \vdash_{L\forall} \varphi \text{ iff } (\mathbf{M}, \mathcal{A}) \models \varphi \text{ for each model } (\mathbf{M}, \mathcal{A}) \text{ of } T,$$

for any set of sentences T *and formula* φ *of the predicate language* $\mathcal{PL}$*.*

3 Types of completeness properties and their relationships

We will use the following terminology and notation to refer to the usual three notions of completeness for core fuzzy logics.

Definition 3. *Let* L *be a core fuzzy and let* $\mathbb{K}$ *be a class of* L*-algebras. We define:*

- L *has the property of* strong $\mathbb{K}$-completeness, S$\mathbb{K}$C *for short, when for every set of* L*-formulae* Γ *and every* L*-formula* φ*,* $\Gamma \vdash_L \varphi$ *iff* $\Gamma \models_{\mathbb{K}} \varphi$*.*

- L *has the property of* finite strong $\mathbb{K}$-completeness, FS$\mathbb{K}$C *for short, when for every* finite *set of* L*-formulae* Γ *and every* L*-formula* φ*,* $\Gamma \vdash_L \varphi$ *iff* $\Gamma \models_{\mathbb{K}} \varphi$*.*

- L *has the property of* $\mathbb{K}$-completeness, $\mathbb{K}$C *for short, when for every* L*-formula* φ, $\vdash_{\mathrm{L}} \varphi$ *iff* $\models_{\mathbb{K}} \varphi$.

They are analogously defined for the predicate logics.

Definition 4. *Let* L *be a core fuzzy logic. We say that* L$\forall$ *has the* S$\mathbb{K}$C *if for each language* $\mathbf{\Gamma}$*, theory* T*, and formula* φ *the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $(\mathbf{M}, \mathcal{A}) \models \varphi$ *for each* $\mathcal{A} \in \mathbb{K}$ *and each model* $(\mathbf{M}, \mathcal{A})$ *of the theory* T.

We say that L$\forall$ *has the* FS$\mathbb{K}$C *if the above condition holds for* finite *theories. Finally, we say that* L$\forall$ *has the* $\mathbb{K}$C *if the above condition holds for the empty theory.*

When $\mathbb{K}$ is the class of standard algebras in the variety of L-algebras, then instead of $\mathbb{K}$-*completeness* properties we talk about *standard completeness* properties and we use the notation $\mathcal{R}$C instead of $\mathbb{K}$C (to stress that it is a completeness with respect to algebras defined of the real unit interval). Moreover, as mentioned above, when the considered core fuzzy logic is of the form $\mathrm{L}_*(\mathcal{C})$ we can think of further restricting the semantics to the canonical standard algebra. Thus, we also consider the three *canonical standard completeness* properties for these logics both in the propositional and in the first order case.

The completeness properties, their relations and algebraic equivalent (or sufficient) conditions have been studied in [2]. In particular, the following results for the S$\mathbb{K}$C have been proved.

Theorem 5 ([2])**.** *Let* L *be a core fuzzy logic and let* $\mathbb{K}$ *be a class of* L*-algebras. Then:*

1. L *has the* S$\mathbb{K}$C *if, and only if, every countable* L*-chain can be embedded into some chain from* $\mathbb{K}$.
2. L$\forall$ *has the* S$\mathbb{K}$C *if every countable* L*-chain can be* σ*-embedded (i.e. with an embedding which preserves existing suprema and infima) into some chain from* $\mathbb{K}$.

Now we recall a relation between completeness of a propositional core fuzzy logic L and completeness of its corresponding predicate logic L$\forall$.

Proposition 6 (cf. [2])**.** *If for some family* $\mathbb{K}$ *of* L*-chains* L$\forall$ *enjoys the* $\mathbb{K}$C *(*FS$\mathbb{K}$C, S$\mathbb{K}$C *resp.), then* L *enjoys the* $\mathbb{K}$C *(*FS$\mathbb{K}$C, S$\mathbb{K}$C *resp.) as well.*

This proposition yields a necessary condition for the completeness properties of predicate fuzzy logics that will be useful to refute some completeness results in the next section. In a similar way we will also use the following proposition relating completeness of two predicate logics when one is a conservative expansion of the other one.

Proposition 7. *Let* L *and* L$'$ *be two core fuzzy logics such that* L$'\forall$ *is a conservative expansion of* L$\forall$*. Let* $\mathbb{K}'$ *be a class of* L$'$*-chains and let* $\mathbb{K}$ *be the class of their* L*-reducts. If* L$'\forall$ *enjoys the* $\mathbb{K}'$C *(*FS$\mathbb{K}'$C, S$\mathbb{K}'$C *resp.), then* L$\forall$ *enjoys* $\mathbb{K}$C *(*FS$\mathbb{K}$C, S$\mathbb{K}$C *resp.) as well.*

Proof: Assume that L$'\forall$ enjoys the $\mathbb{K}$C and we prove that L$\forall$ also enjoys it. Suppose that $\not\vdash_{\mathrm{L}\forall} \varphi$ for some formula φ in language of L$\forall$. Then, since L$'\forall$ is a conservative expansion of L$\forall$ we also have $\not\vdash_{\mathrm{L}'\forall} \varphi$, hence there is some structure $(\mathbf{M}, \mathcal{A}')$ with $\mathcal{A}' \in \mathbb{K}'$ such that $(\mathbf{M}, \mathcal{A}') \not\models \varphi$. Let $\mathcal{A}$ be the L-reduct of $\mathcal{A}'$. Since φ is an L$\forall$-formula, it is clear that $(\mathbf{M}, \mathcal{A}) \not\models \varphi$. □

4 Completeness results for some $\mathrm{L}_*\forall(\mathcal{C})$ logics

In the following, given a left-continuous t-norm $*$ and its corresponding logic L_*, and a *countable* subalgebra $\mathcal{C}$ of the standard L_*-algebra $[0,1]_*$, we will denote by $\mathrm{L}_*\forall(\mathcal{C})$ the predicate version of the (core fuzzy) logic $\mathrm{L}_*(\mathcal{C})$ according to the definitions in Section 2.2.

Remark about the notation used. In the way we have just defined $\mathrm{L}_*\forall(\mathcal{C})$, we should have rather used the notation $\mathrm{L}_*(\mathcal{C})\forall$, since we have started by the expanded logic $\mathrm{L}_*(\mathcal{C})$ and then we have defined the predicate variant over it. But in fact, starting with the $\mathrm{L}_*\forall$ logic and then expanding it with the truth constants from $\mathcal{C}$ leads exactly to the same predicate logic and thus we will keep using $\mathrm{L}_*\forall(\mathcal{C})$. Moreover, we will make also use of the notations G$\forall$, Ł$\forall$, $\Pi\forall$ and NM$\forall$ when referring to $\mathrm{L}_*\forall$ logic when the t-norm $*$ is the minimum, Łukasiewicz, product and nilpotent minimum t-norm respectively.

In the case of expansions of $\mathrm{L}_*\forall$ logics with truth-constants, it was already proved by Hájek *et al.* in [10] that RPL$\forall$ (Rational Pavelka predicate logic[2]) is a conservative expansion of Ł$\forall$. Next theorem proves the analogous result for other logics.

Proposition 8. *If* $*$ *is a pseudo-complemented t-norm or the nilpotent minimum t-norm, then* $\mathrm{L}_*\forall(\mathcal{C})$ *is a conservative expansion of* $\mathrm{L}_*\forall$.

Proof: Let φ be an $\mathrm{L}_*\forall$-formula such that $\not\vdash_{\mathrm{L}_*\forall} \varphi$. We must show that $\not\vdash_{\mathrm{L}_*\forall(\mathcal{C})} \varphi$. By hypothesis, there is some $\mathrm{L}_*\forall$-structure $(\mathbf{M}, \mathcal{A})$ such that $(\mathbf{M}, \mathcal{A}) \not\models \varphi$. It is enough to show that $\mathcal{A}$ can be expanded to an

[2]In our notation RPL$\forall$ corresponds to Ł$\forall(\mathcal{C})$ when $C = [0,1] \cap \mathbb{Q}$.

$L_*(\mathcal{C})$-chain. If $*$ is a pseudo-complemented t-norm we can define the interpretation of every truth-constant $\overline{r}$ such that $r \neq 0$ as $\overline{1}^{\mathcal{A}}$ (see Section 2.1). Assume now that $*$ is the nilpotent minimum t-norm. If $\mathcal{C}$ has no negation fixpoint, we define the interpretation of a truth-constant $\overline{r}$ as $\overline{1}^{\mathcal{A}}$ when $\neg r < r$, and we define it as $\overline{0}^{\mathcal{A}}$ when $\neg r > r$. If $\mathcal{C}$ has the negation fixpoint $\frac{1}{2}$, we can suppose that $\mathcal{A}$ also has a negation fixpoint a (otherwise it could be added as shown in [13]), and then we interpret $\overline{\frac{1}{2}}$ and the rest of the constants as in the previous case. □

This result, together with the one above mentioned by Hájek *et al.*, shows that when $*$ is one of the three basic continuous t-norms (Łukasiewicz, product and minimum), $L_*\forall(\mathcal{C})$ is a conservative expansion of $L_*\forall$ for every subalgebra $\mathcal{C}$ of truth-constants, except for the case of Łukasiewicz t-norm where the result has only been proved for $C = [0,1] \cap \mathbb{Q}$.

Now we are prepared to deal with the standard completeness properties of predicate logics with truth-constants. It is well known that Product and Łukasiewicz predicate logics do not enjoy the standard completeness. Therefore, by Propositions 7 and 8, Ł$\forall(\mathcal{C})$ and $\Pi\forall(\mathcal{C})$ do not have the $\mathbb{K}$C when $\mathbb{K}$ is the class of all standard Ł$\forall(\mathcal{C})$-chains and the class of all standard $\Pi\forall(\mathcal{C})$-chains, respectively; hence these logics do not enjoy the canonical standard completeness neither. The same reasoning would also hold for every logic based on a pseudo-complemented t-norm $*$ for which we know that $L_*\forall$ fails to enjoy the standard completeness.[3] This is not the case for Gödel and Nilpotent Minimum predicate logics which, in fact, satisfy the strong standard completeness. Next we show that these completeness properties extend to their expansions with truth-constants.

Theorem 9. *The logics* $G\forall(\mathcal{C})$ *and* $NM\forall(\mathcal{C})$ *enjoy the* $S\mathcal{R}C$.

Proof: As stated in the preliminaries, the strong standard completeness is equivalent to the property of σ-embeddability. Since Gödel logic already satisfies the strong standard completeness, we know that any countable G-chain is σ-embeddable into $[0,1]_G$, thus every countable $G(\mathcal{C})$-chain is also σ-embeddable in a standard $G(\mathcal{C})$-chain. Indeed, given a countable $G(\mathcal{C})$-chain $\mathcal{A}$ let f be the σ-embedding of its G-reduct into $[0,1]_G$. Then $\mathcal{A}$, as $G(\mathcal{C})$-chain, is also σ-embeddable into the standard $G(\mathcal{C})$-chain defined over $[0,1]_G$ interpreting each truth-constant $\overline{r}$ as $f(\overline{r}^{\mathcal{A}})$. The proof for $NM\forall(\mathcal{C})$ is completely analogous. □

[3] This is the case of all pseudo-complemented continuous t-norms except for $[0,1]_G$, which enjoys the standard completeness (see [12]).

A natural question here is whether these completeness results can be improved by restricting the semantics to the canonical standard algebra. As a matter of fact, the canonical FS$\mathcal{R}$C fails for the logics $G(\mathcal{C})$ and $NM(\mathcal{C})$, as shown in [4, 6]. Therefore, by Proposition 6, this completeness property also fails for their predicate versions. Nevertheless, we can still prove the canonical standard completeness for these logics.

Theorem 10. *The logics* $G\forall(\mathcal{C})$ *and* $NM\forall(\mathcal{C})$ *enjoy the canonical* $\mathcal{R}$C, *i.e. the provable formulae coincide with the 1-tautologies of the canonical standard chain* $[0,1]_{G(\mathcal{C})}$ *and of* $[0,1]_{NM(\mathcal{C})}$ *respectively.*

Proof: We only give a sketch of the proof for $G\forall(\mathcal{C})$ (the proof for $NM\forall(\mathcal{C})$ is similar with the obvious changes). Soudness is obvious as usual. For the converse direction we will argue by contraposition, i.e. we will prove that if $\nvdash_{G\forall(\mathcal{C})} \varphi$ for some formula φ, then there is a $G\forall(\mathcal{C})$-structure $(\mathbf{M}, [0,1]_{G(\mathcal{C})})$ such that $(\mathbf{M}, [0,1]_{G(\mathcal{C})}) \not\models \varphi$.

If $\nvdash_{G\forall(\mathcal{C})} \varphi$, then there exists a $G\forall(\mathcal{C})$-structure $(\mathbf{M}, \mathcal{A})$ over a countable G-chain $\mathcal{A}$ and an evaluation v such that $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} < \overline{1}^{\mathcal{A}}$. Take $s = \min\{r \in C \mid \overline{r}^{A} = \overline{1}^{\mathcal{A}}, r \text{ appears in } \varphi\}$ and define $g : A \to [0,1]$ by taking $g(\overline{1}^{\mathcal{A}}) = 1$ and $g\mid_{A\setminus\{\overline{1}^{\mathcal{A}}\}}$ a bijection of $A \setminus \{\overline{1}^{\mathcal{A}}\}$ into $[0,s)$ preserving existing suprema and infima, and such that $g(\overline{r}^{\mathcal{A}}) = r$ for every truth-constant appearing in φ such that $\overline{r}^{\mathcal{A}} \neq \overline{1}^{\mathcal{A}}$. Using the mapping g we can produce an structure $(\mathbf{M}', [0,1]_{G(\mathcal{C})})$ in such a way that for every evaluation e on $(\mathbf{M}, \mathcal{A})$ and every evaluation e' on $(\mathbf{M}', [0,1]_{G(\mathcal{C})})$ it holds that

$$\|P(t_1, t_2, .., t_n)\|^{[0,1]_{G(\mathcal{C})}}_{M',e'} = g(\|P(t_1, t_2, .., t_n)\|^{A}_{M,e})$$

for each predicate symbol P.

Now by induction we can prove that given any $\mathbf{M}$ and e and their associated $\mathbf{M}'$ and e', the following statement is true for all ψ subformula of φ, then:
a) If $\|\varphi\|^{A}_{M,e} = \overline{1}^{A}$, then $\|\varphi\|^{[0,1]_{G(\mathcal{C})}}_{M',e'} \geq s$,
b) If $\|\varphi\|^{A}_{M,e} \neq \overline{1}^{A}$, then $\|\varphi\|^{[0,1]_{G(\mathcal{C})}}_{M',e'} = g(\|\varphi\|^{A}_{M,e})$.
From this result the theorem is easily proved. □

5 Conclusions

In this short paper we have considered the (canonical) standard completeness properties for several prominent predicate fuzzy logics enriched with constants for intermediate truth-values. Some of these properties have been denied by showing that the standard completeness already fails for the corresponding logic without additional truth-constants, while in some other cases the answer has turned out to be positive by some ad hoc proofs. The following tables collect these results.

Logic	$\mathcal{R}$C	FS$\mathcal{R}$C	S$\mathcal{R}$C
Ł$\forall(\mathcal{C})$	No	No	No
$\Pi\forall(\mathcal{C})$	No	No	No
G$\forall(\mathcal{C})$	Yes	Yes	Yes
NM$\forall(\mathcal{C})$	Yes	Yes	Yes

Logic	Can. $\mathcal{R}$C	Can. FS$\mathcal{R}$C	Can. S$\mathcal{R}$C
Ł$\forall(\mathcal{C})$	No	No	No
$\Pi\forall(\mathcal{C})$	No	No	No
G$\forall(\mathcal{C})$	Yes	No	No
NM$\forall(\mathcal{C})$	Yes	No	No

As open problems that we plan to address in a forthcoming paper we can mention the following:

1. For which left-continuous t-norms $*$ is $L_*\forall(\mathcal{C})$ a conservative extension of $L_*\forall$?
2. Is it possible to prove the result of Proposition 8 for Łukasiewicz logic when the language is expanded with truth-constants for irrational values?
3. Is the canonical FS$\mathcal{R}$C true for G$\forall(\mathcal{C})$ and NM$\forall(\mathcal{C})$ when the language is restricted to evaluated formulae (i.e. formulae of the kind $\overline{r} \to \varphi$, where there are no new truth-constants in φ)?
4. Investigate completeness results for the expansions of the logics $L_*\forall(\mathcal{C})$ with the projection connective Δ.

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