# Base Belief Change for finitary monotonic logics

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**Abstract.** We slightly improve on characterization results already in the literature for base revision. We show that in order to axiomatically characterize revision operators in a logic the only conditions this logic is required to satisfy are: finitarity and monotonicity. A characterization of limiting cases of revision operators, full meet and maxichoice, is also offered. In the second part of the paper, as a particular case, we focus on the class of graded fuzzy logics and distinguish two types of bases, naturally arising in that context, exhibiting different behavior.

## Introduction

This paper is about (multiple) base belief change, in particular our results are mainly about base revision, which is characterized for a broad class of logics. The original framework of Alchourrón, Gärdenfors and Makinson (AGM) [1] deals with belief change operators on deductively closed theories. This framework was generalized by Hansson [11, 12] to deal with *bases*, i.e. arbitrary set of formulas, the original requirement of logical closure being dropped. Hansson characterized revision and contraction operators in, essentially, monotonic compact logics with the deduction theorem property. These results were improved in [13] by Hansson and Wassermann: while for contraction ([13, Theorem 3.8]) it is shown that finitarity and monotony of the underlying logic suffice, for revision (Theorem [13, Theorem 3.17]) their proof depends on a further condition, *Non-contravention*: for all sentences $\varphi$, if $\neg\varphi \in \mathrm{Cn}_{\mathcal{S}}(T \cup \{\varphi\})$, then $\neg\varphi \in \mathrm{Cn}_{\mathcal{S}}(T)$.

In this paper we provide a further improvement of Hansson and Wassermann's results by proving a characterization theorem for base revision in any finitary monotonic logic. Namely, in the context of partial meet base revision, we show that *Non-contravention* can be dropped in the characterization of revision if we replace the notion of unprovability (remainders) by consistency in the definition of partial meet, taking inspiration from [4]. This is the main contribution of the paper, together with its extension to the characterization of the revision operators corresponding to limiting cases of selection functions, i.e. full meet and maxichoice revision operators.

In the second part of the paper, as a particular class of finitary monotonic logics, we focus on graded fuzzy logics. We introduce there a distinction in basehood and observe some differences in the behavior of the corresponding base revision operators.

This paper is structured as follows. First we introduce in Section 1 the necessary background material on logic and partial meet base belief change. Then in Section 2 we set out the main characterization results for base revision, including full meet and maxichoice revision operators. Finally in Section 3 we briefly introduce fuzzy graded logics, present a natural distinction between bases in these logics (whether or not they are taken to be closed under truth-degrees) and compare both kinds of bases.

## 1 Preliminaries on theory and base belief change

We introduce in this section the concepts and results needed later. Following [8], we define a logic $\mathcal{S}$ as a finitary and structural consequence relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathbf{Fm}) \times \mathbf{Fm}$, for some algebra of formulas $\mathbf{Fm}$[3].

Belief change is the study of how some theory $T$ (non-necessarily closed, as we use the term) in a given language $L$ can adapt to new incoming information $\varphi \in L$ (inconsistent with $T$, in the interesting case). The main operations are: *revision*, where the new input must follow from the revised theory, which is to be consistent, and *contraction* where the input must not follow from the contracted theory. In the classical paper [1], by Alchourrón, Gärdenfors and Makinson, partial meet revision and contraction operations were characterized for closed theories in, essentially, monotonic compact logics with the deduction property[4]. Their work put in solid grounds this newly established area of research, opening the way for other formal studies involving new objects of change, operations (see [16] for a comprehensive list) or logics. We follow [1] and define change operators by using partial meet: *Partial meet* consists in (i) generating all logically maximal ways to adapt $T$ to the new sentence (those subtheories of $T$ making further information loss logically unnecessary), (ii) selecting some of these possibilities, (iii) forming their meet, and, optionally, (iv) performing additional steps (if required by the operation). Then a set of axioms is provided to capture these partial meet operators, by showing equivalence between satisfaction of these axioms and being a partial meet operator[5]. In addition, new axioms may be introduced to characterize the limiting cases of selection in step (ii), full meet

[3] That is, $\mathcal{S}$ satisfies (1) If $\varphi \in \Gamma$ then $\Gamma \vdash_{\mathcal{S}} \varphi$, (2) If $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_{\mathcal{S}} \varphi$, (3) If $\Gamma \vdash_{\mathcal{S}} \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash_{\mathcal{S}} \psi$ then $\Delta \vdash_{\mathcal{S}} \varphi$ (*consequence relation*); (4) If $\Gamma \mathcal{S} \varphi$ then for some finite $\Gamma_0 \subseteq \Gamma$ we have $\Gamma_0 \vdash_{\mathcal{S}} \varphi$ (*finitarity*); (5) If $\Gamma \vdash_{\mathcal{S}} \varphi$ then $e[\Gamma] \vdash_{\mathcal{S}} e(\varphi)$ for all substitutions $e \in Hom(\mathbf{Fm}, \mathbf{Fm})$ (*structurality*). We will use throughout the paper relational $\vdash_{\mathcal{S}}$ and functional $\mathrm{Cn}_{\mathcal{S}}$ notation indistinctively, where $\mathrm{Cn}_{\mathcal{S}}$ is the consequence operator induced by $\mathcal{S}$. We will further assume the language of $\mathcal{S}$ contains symbols for conditional $\rightarrow$ and *falsum* $\overline{0}$, and logic $\mathcal{S}$ to contain the *modus ponens* rule: $T \vdash_{\mathcal{S}} \varphi \rightarrow \psi$ then $T \cup \{\varphi\} \vdash_{\mathcal{S}} \psi$.

[4] That is, logics satisfying the Deduction Theorem: $\varphi \vdash_{\mathcal{S}} \psi$ iff $\vdash_{\mathcal{S}} \varphi \rightarrow \psi$.

[5] Other known formal mechanisms defining change operators can be classified into two broad classes: *selection*-based mechanisms include selection functions on remainder sets and incision functions on kernels; *ranking*-based mechanisms include entrench-

and maxichoice selection types. Finally, results showing the different operation types can be defined each other are usually provided too.

A *base* is an arbitrary set of formulas, the original requirement of logical closure being dropped. Base belief change, for the same logical framework than AGM, was characterized by Hansson (see [11], [12]). The results for contraction and revision were improved in [13] (by Hansson and Wassermann): for contraction ([13, Theorem 3.8]) it is shown that finitarity and monotony suffice, while for revision ([13, Theorem 3.17]) their proof depends on a further condition, *Non-contravention*: for all sentences $\varphi$, if $\neg\varphi \in \mathrm{Cn}_{\mathcal{S}}(T \cup \{\varphi\})$, then $\neg\varphi \in \mathrm{Cn}_{\mathcal{S}}(T)$. Observe this condition holds in logics having (i) the deduction property and (ii) the structural axiom of Contraction[6]. We show *Non-contravention* can be dropped in the characterization of revision if we replace unprovability (remainders) by consistency in the definition of partial meet (see next section; see also [5] for a comparison in theory change).

The main difference between base and theory revision is syntax-sensitivity (see [14] and [3] for a discussion): two equivalent bases may output different solutions under a fixed revision operator and input (compare e.g. $T = \{p, q\}$ and $T' = \{p \wedge q\}$ under revision by $\neg p$, which give $\{\neg p, q\}$ and $\{\neg p\}$ respectively). Another difference lies in maxichoice operations: for theory revision it was proved in [2] that: non-trivial revision maxichoice operations $T \circledast \varphi$ output complete theories, even if $T$ is far from being complete. This was seen as an argument against maxichoice. For base belief change, in contrast, the previous fact is not the case, so maxichoice operators may be simply seen as modeling optimal knowledge situations for a given belief change problem.

## 2 Multiple base revision for finitary monotonic logics.

Partial meet was originally defined in terms of unprovability of the contraction input sentences: *remainders* are maximal subsets of $T$ failing to imply $\varphi$. This works fine for logics with the deduction theorem, where remainders and their consistency-based counterparts (defined below) coincide. But, for the general case, remainder-based revision does not grant consistency and it is necessary to adopt the consistency-based approach. Observe we also generalize revision operators to the *multiple* case, where the input of revision is allowed to be a base, rather than just a single sentence.

**Definition 1.** *([17], [4]) Given some monotonic logic $\vdash_{\mathcal{S}}$ let $T_0, T_1$ be theories. We say $T_0$ is* consistent *if $T_0 \nvdash_{\mathcal{S}} \overline{0}$, and define the set* $\mathrm{Con}(T_0, T_1)$ *of* subsets *of* $T_0$ maximally consistent with $T_1$ *as follows: $X \in Con(T_0, T_1)$ iff:*

ments and systems of spheres. For the logical framework assumed in the original developments (compact -and monotonic- closure operators satisfying the deduction property), all these methods are equivalent (see [16] for a comparison). These equivalences between methods need not be preserved in more general class of logics.

[6] If $T \cup \{\varphi\} \vdash_{\mathcal{S}} \varphi \to \overline{0}$, then by the deduction property $T \vdash_{\mathcal{S}} \varphi \to (\varphi \to \overline{0})$; i.e. $T \vdash_{\mathcal{S}} (\varphi \& \varphi) \to \overline{0}$. Finally, by transitivity and the axiom of contraction, $\vdash_{\mathcal{S}} \varphi \to \varphi \& \varphi$, we obtain $T \vdash_{\mathcal{S}} \varphi \to \overline{0}$.

*(i)* $X \subseteq T_0$,
*(ii)* $X \cup T_1$ *is consistent, and*
*(iii) For any* $X'$ *such that* $X \subsetneq X' \subseteq T_0$, *we have* $X' \cup T_1$ *is inconsistent*

Now we prove some properties[7] of $\mathrm{Con}(\cdot,\cdot)$ which will be helpful for the characterization theorems of base belief change operators for arbitrary finitary monotonic logics.

**Lemma 1.** *Let* $\mathcal{S}$ *be some finitary logic and* $T_0$ *a theory. For any* $X \subseteq T_0$, *if* $X \cup T_1$ *is consistent, then* $X$ *can be extended to some* $Y$ *with* $Y \in \mathrm{Con}(T_0, T_1)$.

*Proof.* Let $X \subseteq T_0$ with $X \cup T_1 \nvdash_{\mathcal{S}} \overline{0}$. Consider the poset $(T^*, \subseteq)$, where $T^* = \{Y \subseteq T_0 : X \subseteq Y \text{ and } Y \cup T_1 \nvdash_{\mathcal{S}} \overline{0}\}$. Let $\{Y_i\}_{i\in I}$ be a chain in $(T^*, \subseteq)$; that is, each $Y_i$ is a subset of $T_0$ and consistent with $T_1$. Hence, $\bigcup_{i\in I} Y_i \subseteq T_0$; since $\mathcal{S}$ is finitary, $\bigcup_{i\in I} Y_i$ is also consistent with $T_1$ and hence is an upper bound for the chain. Applying Zorn's Lemma, we obtain an element $Z$ in the poset with the next properties: $X \subseteq Z \subseteq T$ and $Z$ maximal w.r.t. $Z \cup \{\varphi\} \nvdash_{\mathcal{S}} \overline{0}$. Thus $X \subseteq Z \in \mathrm{Con}(T, \varphi)$.

*Remark 1.* Considering $X = \emptyset$ in the preceding lemma, we infer: if $T_1$ is consistent, then $\mathrm{Con}(T_0, T_1) \neq \emptyset$.

For simplicity, we assume that the input base $T_1$ (to revise $T_0$ by) is consistent. Now, the original definition of selection functions is modified according to the consistency-based approach.

**Definition 2.** *Let* $T_0$ *be a theory. A* selection function *for* $T_0$ *is a function*

$$\gamma : \mathcal{P}(\mathcal{P}(\mathbf{Fm})) \setminus \{\emptyset\} \longrightarrow \mathcal{P}(\mathcal{P}(\mathbf{Fm})) \setminus \{\emptyset\}$$

*such that for all* $T_1 \subseteq \mathbf{Fm}$, $\gamma(\mathrm{Con}(T_0, T_1)) \subseteq \mathrm{Con}(T_0, T_1)$ *and* $\gamma(\mathrm{Con}(T_0, T_1))$ *is non-empty.*

Thus, selection functions and revision operators are defined relative to some fixed base $T_0$. Although, instead of writing $\circledast^{T_0} T_1$, we use the traditional infix notation $T_0 \circledast T_1$ for the operation of revising base $T_0$ by $T_1$.

### 2.1 Base belief revision.

The axioms we propose (inspired by [4]) to characterize (multiple) base revision operators for finitary monotonic logics $\mathcal{S}$ are the following, for arbitrary sets $T_0, T_1$:

[7] Note that $\mathrm{Con}(T_0, T_1)$ cannot be empty, since if input $T_1$ is consistency, then in the worst case, we will have $\emptyset \subseteq T_0$ to be consistent with $T_1$.

(F1) $T_1 \subseteq T_0 \circledast T_1$ (Success)

(F2) If $T_1$ is consistent, then $T_0 \circledast T_1$ is also consistent. (Consistency)

(F3) $T_0 \circledast T_1 \subseteq T_0 \cup T_1$ (Inclusion)

(F4) For all $\psi \in \mathbf{Fm}$, if $\psi \in T_0 - T_0 \circledast T_1$ then, there exists $T'$ with $T_0 \circledast T_1 \subseteq T' \subseteq T_0 \cup T_1$ and such that $T' \nvdash_{\mathcal{S}} \overline{0}$ but $T' \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0})$ (Relevance)

(F5) If for all $T' \subseteq T_0$ $(T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0} \Leftrightarrow T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0})$ then $T_0 \cap (T_0 \circledast T_1) = T_0 \cap (T_0 \circledast T_2)$ (Uniformity)

Given some theory $T_0 \subseteq \mathbf{Fm}$ and selection function $\gamma$ for $T_0$, we define the partial meet revision operator $\circledast_\gamma$ for $T_0$ by $T_1 \subseteq \mathbf{Fm}$ as follows:

$$T_0 \circledast_\gamma T_1 \;=\; \bigcap \gamma(\mathrm{Con}(T_0, T_1)) \cup T_1$$

**Definition 3.** *Let $\mathcal{S}$ be some finitary logic, and $T_0$ a theory. Then $\circledast : \mathcal{P}(\mathbf{Fm}) \to \mathcal{P}(\mathbf{Fm})$ is a* revision operator *for $T_0$ iff for any $T_1 \subseteq \mathbf{Fm}$, $T_0 \circledast T_1 = T_0 \circledast_\gamma T_1$ for some selection function $\gamma$ for $T_0$.*

**Lemma 2.** *The condition* $\mathrm{Con}(T_0, T_1) = \mathrm{Con}(T_0, T_2)$ *is equivalent to the antecedent of Axiom* (F5)

$$\forall T' \subseteq T_0\ (T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0} \Leftrightarrow T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0})$$

*Proof.* (<u>If-then</u>) Assume $\mathrm{Con}(T_0, T_1) = \mathrm{Con}(T_0, T_2)$ and let $T' \subseteq T_0$ with $T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0}$. By Lemma 1, $T'$ can be extended to $X \in \mathrm{Con}(T_0, T_1)$. Hence, by assumption we get $T' \subseteq X \in \mathrm{Con}(T_0, T_2)$ so that $T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0}$ follows. The other direction is similar. (<u>Only if</u>) This direction follows from the definition of $\mathrm{Con}(T_0, \cdot)$.

Finally, we are in conditions to prove the main characterization result for partial meet revision.

**Theorem 1.** *Let $\mathcal{S}$ be a finitary monotonic logic. For any $T_0 \subseteq \mathbf{Fm}$ and function $\circledast : \mathcal{P}(\mathbf{Fm}) \to \mathcal{P}(\mathbf{Fm})$:*

$$\circledast \textit{ satisfies } (\mathrm{F1}) - (\mathrm{F5}) \text{ iff } \circledast \textit{ is a revision operator for } T_0$$

*Proof.* (<u>Soundness</u>) Given some partial meet revision operator $\circledast_\gamma$ for $T_0$, we prove $\circledast_\gamma$ satisfies (F1) – (F5).

(<u>F1</u>) – (<u>F3</u>) hold by definition of $\circledast_\gamma$. (<u>F4</u>) Let $\psi \in T_0 - T_0 \circledast_\gamma T_1$. Hence, $\psi \notin T_1$ and for some $X \in \gamma(\mathrm{Con}(T_0, T_1))$, $\psi \notin X$. Simply put $T' = X \cup T_1$: by definitions of $\circledast_\gamma$ and Con we have (i) $T_0 \circledast_\gamma T_1 \subseteq T' \subseteq T_0 \cup T_1$ and (ii) $T'$ is consistent (since $T_1$ is). We also have (iii) $T' \cup \{\psi\}$ is inconsistent (otherwise $\psi \in X$ would follow from maximality of $X$ and $\psi \in T_0$, hence contradicting our previous step $\psi \notin X$). (<u>F5</u>) We have to show, assuming the antecedent of(F5), that $T_0 \cap (T_0 \circledast_\gamma T_1) = T_0 \cap (T_0 \circledast_\gamma T_2)$. We prove the $\subseteq$ direction only since the other is similar. Assume, then, for all $T' \subseteq T_0$,

$$T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0} \;\Leftrightarrow\; T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0}$$

and let $\psi \in T_0 \cap (T_0 \circledast_\gamma T_1)$. This set is just $T_0 \cap (\bigcap \gamma(Con(T_0, T_1)) \cup T_1)$ which can be transformed into $(T_0 \cap \bigcap \gamma(Con(T_0, T_1)) \cup (T_0 \cup T_1)$, i.e. $\bigcap \gamma(Con(T_0, T_1)) \cup (T_0 \cup T_1)$ (since $\bigcap \gamma(Con(T_0, T_1)) \subseteq T_0$). Case $\psi \in \bigcap \gamma(Con(T_0, T_1))$. Then we use Lemma 2 upon the assumption to obtain $\bigcap \gamma(Con(T_0, T_1)) = \bigcap \gamma(Con(T_0, T_2))$, since $\gamma$ is a function. Case $\psi \in T_0 \cap T_1$. Then $\psi \in X$ for all $X \in \gamma(Con(T_0, T_1))$, by maximality of $X$. Hence, $\psi \in \bigcap \gamma(Con(T_0, T_1))$. Using the same argument than in the former case, $\psi \in \bigcap \gamma(Con(T_0, T_2))$. Since we also assumed $\psi \in T_0$, we obtain $\psi \in T_0 \cap (T_0 \circledast_\gamma T_2)$.
(Completeness) Let $\circledast$ satisfy (F1) – (F5). We have to show that for some selection function $\gamma$ and any $T_1$, $T_0 \circledast T_1 = T \circledast_\gamma T_1$. We define first

$$\gamma(\mathrm{Con}(T_0, T_1)) = \{X \in \mathrm{Con}(T_0, T_1) : X \supseteq T_0 \cap (T_0 \circledast T_1)\}$$

We prove that (1) $\gamma$ is well-defined, (2) $\gamma$ is a selection function and (3) $T_0 \circledast T_1 = T \circledast_\gamma T_1$.

(1) Assume (i) $\mathrm{Con}(T_0, T_1) = \mathrm{Con}(T_0, T_2)$; we prove that $\gamma(\mathrm{Con}(T_0, T_1)) = \gamma(\mathrm{Con}(T_0, T_2))$. Applying Lemma 2 to (i) we obtain the antecedent of (F5). Since $\circledast$ satisfies this axiom, we have (ii) $T_0 \cap (T_0 \circledast T_1) = T_0 \cap (T_0 \circledast T_2)$. By the above definition of $\gamma$, $\gamma(\mathrm{Con}(T_0, T_1)) = \gamma(\mathrm{Con}(T_0, T_2))$ follows from (i) and (ii).

(2) Since $T_1$ is consistent, by Remark 1 we obtain $\mathrm{Con}(T_0, T_1)$ is not empty; we have to show that $\gamma(\mathrm{Con}(T_0, T_1))$ is not empty either (since the other condition $\gamma(\mathrm{Con}(T_0, T_1)) \subseteq \mathrm{Con}(T_0, T_1)$ is met by the above definition of $\gamma$). We have $T_0 \cap T_0 \circledast T_1 \subseteq T_0 \circledast T_1$; the latter is consistent and contains $T_1$, by (F2) and (F1), respectively; thus, $(T_0 \cap T_0 \circledast T_1) \cup T_1$ is consistent; from this and $T_0 \cap T_0 \circledast T_1 \subseteq T_0$, we deduce by Lemma 1 that $T_0 \cap T_0 \circledast T_1$ is extensible to some $X \in \mathrm{Con}(T_0, T_1)$. Thus, exists some $X \in \mathrm{Con}(T_0, T_1)$ such that $X \supseteq T_0 \cap T_0 \circledast T_1$. In consequence, $X \in \gamma(\mathrm{Con}(T_0, T_1)) \neq \emptyset$.

For (3), we prove first $T_0 \circledast T_1 \subseteq T_0 \circledast_\gamma T_1$. Let $\psi \in T_0 \circledast T_1$. By (F3), $\psi \in T_0 \cup T_1$. Case $\psi \in T_1$: then trivially $\psi \in T_0 \circledast_\gamma T_1$ Case $\psi \in T_0$. Then $\psi \in T_0 \cap T_0 \circledast T_1$. In consequence, for any $X \in \mathrm{Con}(T_0, T_1)$, if $X \supseteq T_0 \cap T_0 \circledast T_1$ then $\psi \in X$. This implies, by definition of $\gamma$ above, that for all $X \in \gamma(\mathrm{Con}(T_0, T_1))$ we have $\psi \in X$, so that $\psi \in \bigcap \gamma(\mathrm{Con}(T_0, T_1)) \subseteq T_0 \circledast_\gamma T_1$. In both cases, we obtain $\psi \in T_0 \circledast_\gamma T_1$.

Now, for the other direction: $T_0 \circledast_\gamma T_1 \subseteq T_0 \circledast T_1$. Let $\psi \in \bigcap \gamma(\mathrm{Con}(T_0, T_1)) \cup T_1$. By (F1), we have $T_1 \in T_0 \circledast T_1$; then, in case $\psi \in T_1$ we are done. So we may assume $\psi \in \bigcap \gamma(\mathrm{Con}(T_0, T_1))$. Now, in order to apply (F4), let $X$ be arbitrary with $T \circledast T_1 \subseteq X \subseteq T_0 \cup T_1$ and $X$ consistent. Consider $X \cap T_0$: since $T_1 \subseteq T_0 \circledast T_1 \subseteq X$ implies $X = X \cup T_1$ is consistent, so is $(X \cap T_0) \cup T_1$. Together with $X \cap T_0 \subseteq T_0$, by Lemma 1 there is $Y \in \mathrm{Con}(T_0, T_1)$ with $X \cap T_0 \subseteq Y$. In addition, since $T_0 \circledast T_1 \subseteq X$ implies $T_0 \circledast T_1 \cap T_0 \subseteq X \cap T_0 \subseteq Y$ we obtain $Y \in \gamma(\mathrm{Con}(T_0, T_1))$, by the definition of $\gamma$ above. Condition $X \cap T_0 \subseteq Y$ also implies $(X \cap T_0) \cup T_1 \subseteq Y \cup T_1$. Observe that from $X \subseteq X \cup T_1$ and $X \subseteq T_0 \cup T_1$ we infer that $X \subseteq (X \cup T_1) \cap (T_0 \cup T_1)$. From the latter being identical to $(X \cap T_0) \cup T_1$ and the fact that $(X \cap T_0) \cup T_1 \subseteq Y \cup T_1$, we obtain that $X \subseteq Y \cup T_1$. Since $\psi \in Y \in \mathrm{Con}(T_0, T_1)$, we have $Y \cup T_1$ is consistent with $\psi$, so its subset $X$ is also consistent with $\psi$. Finally, we may apply *modus tollens* on Axiom (F4) to obtain that $\psi \notin T_0 - T_0 \circledast T_1$, i.e. $\psi \notin T_0$ or $\psi \in T_0 \circledast T_1$. But since the former is false, the latter must be the case.

**Full meet and maxichoice base revision operators.** The previous result can be extended to limiting cases of selection functions formally defined next.

**Definition 4.** *A revision operator for* $T_0$ *is* full meet *if it is generated by the identity selection function* $\gamma_{\mathrm{fm}} = \mathsf{Id}$*:* $\gamma_{\mathrm{fm}}(\mathrm{Con}(T_0, T_1)) = \mathrm{Con}(T_0, T_1)$*; that is,*

$$T_0 \circledast_{\mathrm{fm}} T_1 = (\bigcap \mathrm{Con}(T_0, T_1)) \cup T_1$$

*A revision operator for* $T_0$ *is* maxichoice *if it is generated by a selection function of type* $\gamma_{\mathrm{mc}}(\mathrm{Con}(T_0, T_1)) = \{X\}$*, for some* $X \in \mathrm{Con}(T_0, T_1)$*, and in that case* $T_0 \circledast_{\gamma_{\mathrm{mc}}} T_1 = X \cup T_1$*.*

To characterize *full meet* and *maxichoice* revision operators for some theory $T_0$ in any finitary logic, we define the next additional axioms:

(FM) For any $X \subseteq \mathbf{Fm}$ with $T_1 \subseteq X \subseteq T_0 \cup T_1$
$X \nvdash_{\mathcal{S}} \overline{0}$ implies $X \cup (T_0 \circledast T_1) \nvdash_{\mathcal{S}} \overline{0}$
(MC) For all $\psi \in \mathbf{Fm}$ with $\psi \in T_0 - T_0 \circledast T_1$ we have
$T_0 \circledast T_1 \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$

**Theorem 2.** *Let* $T_0 \subseteq \mathbf{Fm}$ *and* $\circledast$ *be a function* $\circledast : \mathcal{P}(\mathbf{Fm})^2 \to \mathcal{P}(\mathbf{Fm})$*. Then the following hold:*

(fm) $\circledast$ *satisfies* (F1) – (F5) *and* (FM) iff $\circledast = \circledast_{\gamma_{\mathrm{fm}}}$
(mc) $\circledast$ *satisfies* (F1) – (F5) *and* (MC) iff $\circledast = \circledast_{\gamma_{\mathrm{mc}}}$

*Proof.* We prove (fm) first. (Soundness): We know $\circledast_{\gamma_{\mathrm{fm}}}$ satisfies (F1) – (F5) so it remains to be proved that (FM) holds. Let $X$ be such that $T_1 \subseteq X \subseteq T_0 \cup T_1$ and $X \nvdash_{\mathcal{S}} \overline{0}$. From the latter and $X - T_1 \subseteq (T_0 \cup T_1) - T_1 \subseteq T_0$ we infer by Lemma 1 that $X - T_1 \subseteq Y \in \mathrm{Con}(T_0, T_1)$, for some $Y$. Notice $X = X' \cup T_1$ and that for any $X'' \in \mathrm{Con}(T_0, T_1) X'' \cup T_1$ is consistent and

$$T_0 \circledast_{\gamma_{\mathrm{fm}}} T_1 = (\bigcap \mathrm{Con}(T_0, T_1)) \cup T_1 \subseteq X' \subseteq X''$$

Hence $X \subseteq X''$, so that $T_0 \circledast_{\gamma_{\mathrm{fm}}} T_1 \cup X \subseteq X''$. Since the latter is consistent, $T_0 \circledast_{\mathrm{fm}} T_1 \cup X \nvdash_{\mathcal{S}} \overline{0}$. (Completeness) Let $\circledast$ satisfy (F1) – (F5) and (FM). It suffices to prove that $X \in \gamma(\mathrm{Con}(T_0, T_1)) \Leftrightarrow X \in \mathrm{Con}(T_0, T_1)$; but we already know that $\circledast = \circledast_\gamma$, for selection function $\gamma$ (for $T_0$) defined by: $X \in \gamma(\mathrm{Con}(T_0, T_1)) \Leftrightarrow T_0 \cap T_0 \circledast T_1 \subseteq X$. It is enough to prove, then, that $X \in \mathrm{Con}(T_0, T_1)$ implies $X \supseteq T_0 \cap T_0 \circledast T_1$. Let $X \in \mathrm{Con}(T_0, T_1)$ and let $\psi \in T_0 \cap T_0 \circledast T_1$. Since $\psi \in T_0$ and $X \in \mathrm{Con}(T_0, T_1)$, we have by maximality of $X$ that either $X \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$ or $\psi \in X$. We prove the former case to be impossible: assuming it we would have $T_1 \subseteq X \cup T_1 \subseteq T_0 \cup T_1$. By (FM), $X \cup T_1 \cup (T_0 \circledast T_1) \nvdash_{\mathcal{S}} \overline{0}$. Since $\psi \in T_0 \circledast T_1$, we would obtain $X \cup \{\psi\} \nvdash_{\mathcal{S}} \overline{0}$, hence contradicting the case assumption; since the former case is not possible, we have $\psi \in X$. Since $X$ was arbitrary, $X \in \mathrm{Con}(T_0, T_1)$ implies $X \subseteq T_0 \cap T_0 \circledast T_1$ and we are done.
For (mc): (Soundness) We prove (MC), since (F1) – (F5) follow from $\circledast_{\gamma_{\mathrm{mc}}}$ being a partial meet revision operator. Let $X \in \mathrm{Con}(T_0, T_1)$ be such that $T_0 \circledast_{\gamma_{\mathrm{mc}}} \varphi =$

$X \cup T_1$ and let $\psi \in T_0 - T_0 \circledast_{\gamma_{\mathrm{mc}}} T_1$. We have $\psi \notin X \cup T_1 = T_0 \circledast T_1$. Since $\psi \in T_0$ and $X \in \mathrm{Con}(T_0, T_1)$, $X \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$. Finally $T_0 \circledast T_1 \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$. (Completeness) Let $\circledast$ satisfy (F1) – (F5) and (MC). We must prove $\circledast = \circledast_{\gamma_{\mathrm{mc}}}$, for some maxichoice selection function $\gamma_{\mathrm{mc}}$. Let $X, Y \in \mathrm{Con}(T_0, T_1)$; we have to prove $X = Y$. In search of a contradiction, assume the contrary, i.e. $\psi \in X - Y$. We have $\psi \notin \bigcap \gamma(\mathrm{Con}(T_0, T_1))$ and $\psi \in X \subseteq T_0$. By MC, $T_0 \circledast T_1 \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$. Since $T_0 \circledast T_1 \subseteq X$, we obtain $X \cup \{\psi\}$ is also inconsistent, contradicting previous $\psi \in X \nvdash_{\mathcal{S}} \overline{0}$. Thus $X = Y$ which makes $\circledast = \circledast_{\gamma_{\mathrm{mc}}}$, for some maxichoice selection function $\gamma_{\mathrm{mc}}$.

## 3 The case of graded fuzzy logics.

The characterization results for base revision operators from the previous section required weak assumptions (monotony and finitarity) upon the consequence relation $\vdash_{\mathcal{S}}$. In particular these results hold for a wide family of systems of (mathematical) fuzzy logic. The distinctive feature of these logics is that they cope with graded truth in a compositional manner (see [10]). Graded truth may be dealt implicitly, by means of comparative statements, or explicitly, by introducing truth-degrees in the language. Here we will focus on a particular kind of fuzzy logical languages allowing for explicit representation of truth-degrees, that will be referred as *graded fuzzy logics*, and which are expansions of t-norm logics with countable sets of truth-constants, see e.g. [7]. These logics allow for occurrences of truth-degrees, represented as new propositional atoms $\overline{r}$ (one for each $r \in \mathcal{C}$) in any part of a formula. These truth-constants and propositional variables can be combined arbitrarily using connectives to obtain new formulas. The graded language obtained in this way will be denoted as $\mathbf{Fm}(\mathcal{C})$. A prominent example of a logic over a graded language is Hájek's Rational Pavelka Logic **RPL** [10], an extension of Łukasiewicz logic with rational truth-constants in $[0, 1]$; for other graded extensions of t-norm based fuzzy logics see e.g. [7]. In t-norm based fuzzy logics, due to the fact that the implication is residuated, a formula $\overline{r} \to \varphi$ gets value 1 under a given interpretation $e$ iff $r \leq e(\varphi)$. In what follows, we will also use the signed language notation $(\varphi, r)$ to denote the formula $\overline{r} \to \varphi$.

If $\mathcal{S}$ denotes a given t-norm logic, let us denote by $\mathcal{S}(\mathcal{C})$ the corresponding expansion with truth-constants from a suitable countable set $\mathcal{C}$ such that $\{0, 1\} \subset \mathcal{C} \subseteq [0, 1]$. For instance if $\mathcal{S}$ is Łukasiewicz logic and $\mathcal{C} = \mathbb{Q} \cap [0, 1]$, then $\mathcal{S}(\mathcal{C})$ would refer to **RPL**. For these graded fuzzy logics, besides the original definition of a base as simply a set of formulas, it makes sense to consider another natural notion of basehood, where bases are closed by lower bounds of truth-degrees. We call them $\mathcal{C}$-closed bases.

**Definition 5.** *(Adapted from [11]) Given some (monotonic) t-norm fuzzy logic* $\mathcal{S}$ *with language* $\mathbf{Fm}$ *and a countable set* $\mathcal{C} \subset [0, 1]$ *of truth-constants, let* $T \subseteq \mathbf{Fm}(\mathcal{C})$ *be a base in* $\mathcal{S}(\mathcal{C})$. *We define* $\mathrm{Cn}_{\mathcal{C}}(T) = \{(\varphi, r') : (\varphi, r) \in T,$ *for* $r, r' \in \mathcal{C}$ *with* $r \geq r'\}$. *A base* $T \subseteq \mathbf{Fm}(\mathcal{C})$ *is called* $\mathcal{C}$*-closed when* $T = \mathrm{Cn}_{\mathcal{C}}(T)$.

Notice that, using Gerla's framework of abstract fuzzy logic [9], Booth and Ricther [4] defines revision operators for bases which are closed with respect to truth-values in some complete lattice $W$.

The following results prove $\circledast_\gamma$ operators preserve $\mathcal{C}$-closure, thus making $\mathcal{C}$-closed revision a particular case of base revision under Theorem 1.

**Proposition 1.** *If* $T_0, T_1$ *are* $\mathcal{C}$*-closed graded bases, for any partial meet revision operator* $\circledast_\gamma$, $T_0 \circledast_\gamma T_1$ *is also a* $\mathcal{C}$*-closed graded base.*

*Proof.* Since $T_0$ is $\mathcal{C}$-closed, by maximality of $X \in \gamma(\mathrm{Con}(T_0, T_1))$ we have $X$ is also $\mathcal{C}$-closed, for any such $X$. Let $(\psi, s) \in \bigcap \gamma(\mathrm{Con}(T_0, T_1))$ and $s' <_{\mathcal{C}} s$ for some $s' \in \mathcal{C}$. Then $(\psi, s) \in X$ for any $X \in \gamma(\mathrm{Con}(T_0, T_1))$ implies $(\psi, s') \in X$ for any such $X$. Hence $\bigcap \gamma(\mathrm{Con}(T_0, T_1))$ is $\mathcal{C}$-closed. Finally, since $T_1$ is $\mathcal{C}$-closed, we deduce $\bigcap \gamma(\mathrm{Con}(T_0, T_1)) \cup T_1$ is also $\mathcal{C}$-closed.

Let $\mathcal{P}_{\mathcal{C}}(\mathbf{Fm})$ be the set of $\mathcal{C}$-closed sets of $\mathbf{Fm}$ sentences. We introduce an additional axiom (F0) for revision of $\mathcal{C}$-closed bases by $\mathcal{C}$-closed inputs:

$$\text{(F0) } T_0 \circledast T_1 \text{ is } \mathcal{C}\text{-closed, if } T_0, T_1 \text{ are}$$

**Corollary 1.** *Assume* $\mathcal{S}$ *and* $\mathcal{C}$ *are as before and let* $\circledast : \mathcal{P}_{\mathcal{C}}(\mathbf{Fm}) \to \mathcal{P}(\mathbf{Fm})$. *Then,* $\circledast$ *satisfies* (F0) – (F5) *iff for some selection function* $\gamma, T_0 \circledast T_1 = T_0 \circledast_\gamma T_1$ *for every* $T_1 \in \mathcal{P}_{\mathcal{C}}(\mathbf{Fm})$.

For the case of **RPL**, where the negation operator $\neg$ is interpreted by the negation function on $[0, 1]$ defiend as $n(x) = 1 - x$, both approaches (non-$\mathcal{C}$-closed, $\mathcal{C}$-closed) differ in the revision output.

*Example 1.* (In **RPL**) Let $\mathcal{C} = \mathbb{Q} \cap [0, 1]$ and define $T_0 = \{(\varphi, 0.5), (\varphi, 0.7)\}$.In each case, there in only a possible selection function, call them $\gamma_0$ and $\gamma_1$; revision results in:

$$T_0 \circledast_{\gamma_0} (\neg\varphi, 0.4) = \{(\varphi, 0.5), (\neg\varphi, 0.4)\}, \text{ while}$$
$$\mathrm{Cn}_{\mathcal{C}}(T_0) \circledast_{\gamma_1} \mathrm{Cn}_{\mathcal{C}}(\{(\neg\varphi, 0.4)\}) = \mathrm{Cn}_{\mathcal{C}}(\{(\varphi, 0.6), (\neg\varphi, 0.4)\})$$

Observe that, while standard revision only preserves graded formulas that have been explicitly added to $T$, $\mathcal{C}$-closed revision will search for lower-than-actual degrees which are optimal relative to the revision input.

## 4 Conclusions and future work

We improved Hansson and Wassermann characterization of the revision operator by dropping one of their conditions, implicitly characterizing revision operators for the class logics with the deduction property. Apart from the general theorem, standard results for full meet and maxichoice revision operators are also provided. Then we moved to the field of graded fuzzy logics, in contradistinction to the approach by Booth and Richter in [4]; their work inspired us to prove similar results for a more general logical framework, including t-norm based fuzzy logics from Hájek. Finally, we observed the differences between bases if they are assumed to be closed under truth-degrees.

Several problems are left open for future research: mainly, whether the present (consistency-based) results can be used to characterize contraction as well. Presumably, the standpoint adopted in this paper would lead to a definition of contraction with slightly different properties than that proposed in [13].

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