Relating fuzzy autoepistemic logic and Łukasiewicz KD45 modal logic

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In this paper, we propose an axiomatization for a Kripke-style possible world semantics related to the fuzzy autoepistemic logic defined in [1], and we also provide an equivalent algebraic semantics.

Autoepistemic logic is a formalism of nonmonotonic reasoning that was originally intended to model an ideally rational agent reflecting upon his own beliefs [7]. These beliefs are sets of sentences in a propositional language augmented by a modal operator $\Box$. If φ is a formula, then $\Box\varphi$, which has to be interpreted as "φ is believed", is a formula as well. Hence, in this language nested modal operators are allowed; it is possible to have beliefs about beliefs. Let A be any set of formulas in this language. In [7], a *stable expansion* of A is defined as a set of formulas T such that the following fix-point condition holds:

$$T = \{\varphi \mid A \cup \{\Box\psi \mid \psi \in T\} \cup \{\neg\Box\psi \mid \psi \notin T\} \vdash \varphi\}, \tag{1}$$

where $\vdash$ denotes the notion of proof for classical propositional logic and each formula $\Box\varphi$ is considered as a new variable. Informally, a stable expansion of A is the closed set of beliefs of an ideal rational agent based on the premises A.

In [1] a fuzzy generalization (from a semantical point of view) of autoepistemic logic is defined and it was shown that the important relation between autoepistemic logic and answer set programming established in [6] is preserved: answer sets of a fuzzy answer set program can be equivalently determined by fuzzy stable expansions of a corresponding set of fuzzy autoepistemic formulas. Given a particular fuzzy propositional calculus L with truth constants, let $\mathcal{L}$ denote its language and let $\mathcal{L}_\Box$ denote its expansion with $\Box$. Then a *fuzzy stable expansion* of a set of formulas A of the latter is now a fuzzy set of formulas $T : \mathcal{L}_\Box \to [0,1]$ satisfying the following fix-point condition:

$$T(\varphi) = \inf\left\{v(\varphi) \mid v \in \Omega, v \text{ is a model}^3 \text{ of } A \cup \left\{\Box\psi \leftrightarrow \overline{T(\psi)} \mid \psi \in \mathcal{L}_\Box\right\}\right\}, \tag{2}$$

where Ω is the set of L-evaluations over $\mathcal{L}_\Box$ (treating each formula $\Box\varphi$ as a new variable), and $\overline{T(\psi)}$ denotes the truth-constant of value $T(\psi)$. Notice that (2) generalizes the semantical version of (1).

[3] v is a model of a set of formulas B if $v(\psi) = 1$ for all $\psi \in B$.

Generalizing [8], a class of fuzzy Kripke-style models is introduced in [1] in order to evaluate formulas of $\mathcal{L}_\Box$. For each $v \in \Omega$ and $\emptyset \neq S \subseteq \Omega$,[4] the truth-value of φ in the structure (v, S), denoted $\|\varphi\|_{v,S}$, is defined as follows:

- If φ is a formula of $\mathcal{L}$, then $\|\varphi\|_{v,S} = v(\varphi)$.
- If $\varphi = \Box\psi$, then $\|\varphi\|_{v,S} = \inf_{w \in S} \|\psi\|_{w,S}$.
- The truth-value of compound formulas is defined using truth-functions of L.

We will denote the class of all structures (v, S) by $\mathcal{M}^{ae}$.

A fuzzy autoepistemic model of a set of formulas A of $\mathcal{L}_\Box$ is a set $E \subseteq \Omega$, such that

$$E = \{v \in \Omega \mid \|\varphi\|_{v,E} = 1 \text{ for each } \varphi \in A\}.$$

In [1] it is shown that for each fuzzy stable expansion T of a set of formulas A there exists a fuzzy autoepistemic model E of A, such that, for each φ, $T(\varphi) = \inf_{v \in E} \|\varphi\|_{v,E}$.

In this contribution, we provide an axiomatization for the above introduced class of models $\mathcal{M}^{ae}$ for fuzzy autoepistemic logic in the particular case where the propositional calculus L is the $(k+1)$-valued Łukasiewicz logic with truth-constants $Ł_k^c$. In what follows we will denote by Ω_k the set of propositional $Ł_k^c$-evaluations and by $\mathcal{M}_k^{ae}$ the class of models resulting from $\mathcal{M}^{ae}$ by taking L $= Ł_k^c$.

We start from the logic $\Lambda(\mathbf{CFr}, Ł_k^c)$ defined in [2], that is the minimal modal logic over $Ł_k^c$ axiomatizing the set of valid formulas over the class of Kripke frames with crisp accessibility relations. The language of $\Lambda(\mathbf{CFr}, Ł_k^c)$ is obtained by enlarging the one of $Ł_k^c$ by a unary modality $\Box$, and defining well formed formulas in the usual inductive manner: (1) every formula of $Ł_k^c$ is a formula; (2) if φ and ψ are formulas, then $\Box\varphi$, $\varphi \odot \psi$, and $\varphi \to \psi$, are formulas as well. Truth is defined relative to Kripke structures $M = (W, e, R)$ where W is a non-empty set of possible worlds, $R : W \times W \to \{0, 1\}$ is a crisp accessibility relation and $e : \mathcal{L} \times W \to S_k$ is such that for every $w \in W$, $e(\cdot, w) \in \Omega_k$. Given a formula φ, the truth value of φ in a world $w \in W$ is defined as follows:

- If φ is a formula of $Ł_k^c$, then $\|\varphi\|_{M,w} = e(\varphi, w)$.
- If $\varphi = \Box\psi$, then $\|\varphi\|_{M,w} = \inf\{\|\psi\|_{M,v} \mid v \in W, R(w, v) = 1\}$.
- The truth values of compound formulas is defined according to $Ł_k^c$ truth functions.

The notion of logical consequence is defined as usual: given a set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \models_\Box \varphi$, iff for every Kripke model $M = (W, e, R)$ and for every $w \in W$ we have that if $\|\psi\|_{M,w} = 1$ for every $\psi \in \Gamma$, then $\|\varphi\|_{M,w} = 1$ holds as well.

The following are the axioms for the logic $\Lambda(\mathbf{CFr}, Ł_k^c)$ (see [2]):

- All the axioms from $Ł_k^c$.

[4] Actually, in the original proposal [8] and in its fuzzy generalization of [1] the set is not required to be non-empty.

- (K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $(\Box 2)$ $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$
- $(\Box 3)$ $\Box(\overline{r} \to \varphi) \leftrightarrow (\overline{r} \to \Box\varphi)$, for each $r \in S_k$
- $(\Box 4)$ $\Box(\overline{r} \vee \varphi) \leftrightarrow (\overline{r} \vee \Box\varphi)$
- $(\Box 5)$ $(\Box\varphi \oplus \Box\varphi) \leftrightarrow \Box(\varphi \oplus \varphi)$

and its inference rules are modus ponens (from φ and $\varphi \to \psi$, infer ψ) and necessitation for $\Box$ (from φ infer $\Box\varphi$).

One can easily show that in this proof system, the monotonicity rule for $\Box$ (from $\varphi \to \psi$ infer $\Box\varphi \to \Box\psi$) and the rule of substitution of equivalent formulas are derivable rules.

Now, we consider the logic $KD45(\mathbf{CFr}, \text{Ł}_k^{\mathbf{c}})$ obtained from $\Lambda(\mathbf{CFr}, \text{Ł}_k^{\mathbf{c}})$ by adding the axioms:

- (D) $\Diamond\overline{1}$
- (4) $\Box\varphi \to \Box\Box\varphi$
- (5) $\Diamond\Box\varphi \to \Box\varphi$

where $\Diamond\varphi = \neg\Box\neg\varphi$. Then we have the following theorem:

Theorem 1. *$KD45(\mathbf{CFr}, Ł_k^{\mathbf{c}})$ is sound and complete with respect to the class $\mathcal{KD}45$ of Kripke models (W, e, R) where $R : W \times W \to \{0, 1\}$ is serial, transitive and euclidean.*

To prove this theorem, in particular for the completeness part, we make a connection with $\text{Ł}_k^{\mathbf{c}}$, since it is strong complete. This connection is as follows. As was done for (fuzzy) autoepistemic logic, each formula φ in $KD45(\mathbf{CFr}, \text{Ł}_k^{\mathbf{c}})$ can be seen as a formula φ^* in $\text{Ł}_k^{\mathbf{c}}$ by treating every subformula $\Box\alpha$ as a new propositional variable. Explicitly, for a variable p, truth-constant $\overline{r}$, formulas φ and ψ and $\cdot \in \{\odot, \to\}$, we define:

- $p^* = p$, $\overline{r}^* = \overline{r}$,
- $(\varphi \cdot \psi)^* = \varphi^* \cdot \psi^*$,
- $(\Box\varphi)^* = p_\varphi$ with p_φ a new variable.

We can now prove the following lemma:

Lemma 1. *Let $T \cup \{\alpha\}$ be a set of formulas in $KD45(\mathbf{CFr}, Ł_k^{\mathbf{c}})$. Let $T^* = \{\varphi^* \mid \varphi \in T\}$ and $\Lambda = \{\varphi^* \mid \varphi$ axiom in $KD45(\mathbf{CFr}, Ł_k^{\mathbf{c}})\} \cup \{(\Box\varphi)^* \mid \vdash_\Box \varphi\}$. Then it holds that*

$$T \vdash_\Box \alpha \ \text{ iff } \ T^* \cup \Lambda \vdash \alpha^*$$

where $\vdash_\Box$ denotes the notion of proof in $KD45(\mathbf{CFr}, Ł_k^{\mathbf{c}})$ and $\vdash$ the notion of proof in $Ł_k^{\mathbf{c}}$.

Following [2], the proof of Theorem 1 makes use of a canonical model construction and the following truth-lemma: for each formula φ of $KD45(\mathbf{CFr}, \text{Ł}_k^{\mathbf{c}})$ and every $v \in W_{can}$ we have $e_{can}(\varphi, v) = \|\varphi\|_{M_{can}, v}$, where $M_{can} = (W_{can}, R_{can}, e_{can})$ is the canonical model of $KD45(\mathbf{CFr}, \text{Ł}_k^{\mathbf{c}})$ defined as follows:

- $W_{can} = \{v \mid v$ is a $Ł_k^c$-evaluation that is a model of $\Lambda\}$,
- $R_{can}(v_1, v_2) = 1$ if, for every formula φ, $\|(\Box\varphi)^*\|_{v_1} = 1$ implies $\|\varphi^*\|_{v_2} = 1$, and $R_{can}(v_1, v_2) = 0$ otherwise,
- $e_{can}(\varphi, v) = v(\varphi)$, for each (non-modal) $Ł_k^c$-formula φ.

Using classical techniques (see e.g. [9]), one can prove that $KD45(\mathbf{CFr}, Ł_k^c)$ is also complete with respect to the class of *semi-universal* models of the form (W, e, R) where $R = W \times E$ with $\emptyset \neq E \subseteq W$. From this, next corollary easily follows.

Corollary 1. *$KD45(\mathbf{CFr}, Ł_k^c)$ is sound and complete with respect to the class of structures $\mathcal{M}_k^{ae}$.*

Notice that if one allows $\mathcal{M}_k^{ae}$ to contain structures (v, S) with $S = \emptyset$, then one is led to drop the axiom (D) from the logic in order to keep the above completeness result, i.e. in that case the logic to be considered would be $K45(\mathbf{CFr}, Ł_k^c)$ rather than $KD45(\mathbf{CFr}, Ł_k^c)$.

Inspired by [5], we can also introduce an algebraic semantics for $KD45(\mathbf{CFr}, Ł_k^c)$ by considering MV-algebras endowed with an *internal necessity operator*. We will henceforth consider finite MV-algebras A as algebras of functions $(S_k)^X$ where X is a finite set, and where S_k denotes the finite MV-chain over $\{0, 1/k, \ldots, (k-1)/k, 1\}$. In this frame, for every $r \in S_k$, $\overline{r}$ will always denote the constant function $\overline{r} : x \in X \mapsto r \in S_k$.

A (finite) *MV-algebra with internal necessity operator* (NMV-algebra, for short), is then a pair (A, N) where A is a finite MV-algebra of the kind $(S_k)^X$, and $N : A \to A$ satisfies the following equations:

$$\begin{array}{ll}
(A1)\ N(\overline{r}) = \overline{r} \text{ for every } r \in S_k & (A4)\ N(f) \oplus N(f) = N(f \oplus f) \\
(A2)\ N(f \to g) \leq N(f) \to N(g) & (A5)\ N(\overline{r} \to f) = \overline{r} \to N(f) \\
(A3)\ N(f \wedge g) = N(f) \wedge N(g) & (A6)\ N(N(f)) = N(f)
\end{array}$$

Let $\mathfrak{p}$ be any prime MV-filter A, and let $\eta_\mathfrak{p}$ be the canonical homomorphism of A in $A/\mathfrak{p}$. Notice that in this case $A/\mathfrak{p}$ is linear and isomorphic to S_k. Then the map $\hat{N} : A \to S_k$ such that, for every $f \in A$,

$$\hat{N}(f) = \eta_\mathfrak{p}(N(f))$$

is a (external) necessity measure in the sense of [4]. Moreover one can show that there exists a finite (crisp) subset E of homomorphisms of A into S_k such that, for every a in A, $\hat{N}(a) = \inf_{h \in E} h(a)$.

Conversely, given any finite MV-algebra A, and an (external) necessity measure $\hat{N} : A \to S_k$, the map

$$N : f \in A \mapsto \overline{\hat{N}(f)} \in A$$

makes (A, N) a NMV-algebra.

From this construction, the following further algebraic completeness theorem can be proved.

Theorem 2. *The class $Taut(NMV)$ of tautologies for the class of finite NMV-algebras, equals the class $Taut(KD45)$ of tautologies for the models in $KD45$.*

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