Chapter 1

SMOOTH FINITE T-NORMS AND THEIR EQUATIONAL AXIOMATIZATION

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Abstract In this paper, as homage to Professor Gaspar Mayor in his 70 anniversary, we present a summary of results on BL-algebras and related structures that, using the one-to-one correspondence between divisible finite t-norms and finite BL-chains, allows us to provide an equational characterization of any divisible finite t-norm.

1. Introduction

In the early 90's, Mayor and Torrens introduced in [15] the notion of *divisible finite t-norms* and proved they can be represented as finite ordinal sums of copies of finite Łukasiewicz and finite Gödel t-norms. Some years later, Hájek introduced in his influential monograph [14] his Basic Fuzzy logic (BL), that has become the reference system in Mathematical fuzzy logic, and showed it was complete with respect to the class of linearly ordered BL-algebras, or BL-chains. BL-chains were characterized by Hájek [13] and Cignoli et al. [7] as ordinal sums of Łukasiewicz, Gödel and Product linearly ordered algebras, but also as ordinal sums of Wajsberg hoops by Aglianò and Montagna [1]. Moreover the variety generated by a finite BL-chains has been proved to be finitely axiomatizable e.g. by Busaniche and Montagna [5].

In this paper, as homage to Professor Gaspar Mayor in his 70 anniversary, we present a summary of these results that, using the one-to-one correspondence between divisible finite t-norms and finite BL-chains, allows us how to provide an equational characterization of any divisible finite t-norm. In more detail, after this short introduction, we first overview in Section 2 the main results by Mayor and Torrens on finite t-norms, while in Section 3 we focus on the relationship between finite t-norms and their residua. Then in the first part of Section 4 we recall the decomposition of finite t-norms as ordinal sums of Wajsberg hoops, which is used in the second part to show how to derive a set of equations that characterize a given finite divisible t-norm. We end up with some conclusions.

2. Mayor and Torrens' results on t-norms over finite chains

In the paper [15] Mayor and Torrens study *directed algebras* over totally ordered finite sets, inspired by the structures on the real unit interval $[0,1]$ called De Morgan triplets and defined by a t-norm, a strong negation and its dual t-conorm.

Definition 2.1 *A* directed algebra *is a structure* $\langle L, \leq, 0, 1, T, S, N\rangle$, *where:*

(1) $(L, \leq, 0, 1)$ *is a bounded linearly ordered finite set,*

(2) T, S *are associative and commutative binary operations on* L *such that* $T(1, x) = x$ *and* $S(0, x) = x$*,*

(3) N *is an order-reversing involution,*

(4) for all $x, y \in L$*,* $N(T(x, y)) = S(N(x), N(y))$*,*

(5) T *and* S *are divisible, that is, for all* $x, y \in L$*,*
$x \leq y$ *if and only if there exists* $z \in L$ *such that* $x = T(y, z)$*, and*
$x \leq y$ *if and only if there exists* $z \in L$ *such that* $y = S(x, z)$*.*

Since L is finite and linearly ordered, N is obviously univocally defined on L, and S is also determined from T and N by duality (item (3) of the definition). Therefore, a directed algebra over a finite chain L is univocally defined by a binary operation T on L satisfying conditions (2) and (5). Moreover, as the authors observe, T satisfies all the conditions of a *continuous* t-norm but over a finite set instead of $[0,1]$, and, dually, S satisfies all the conditions of a continuous t-conorm in a finite setting. Notice also that the *divisibility condition* in item (5) stipulates that any element $x \in L$ in the interval $[0, y]$ belongs to the image of the unary operation $T(y, \cdot) : L \to L$. In fact, in $[0,1]$ this condition is equivalent to the continuity for a t-norm (see e.g. [2] for a proof).

Consider the following definition of a *finite* t-norm operation.

Definition 2.2 *Let C be the chain $a_0 < a_1 < \ldots < a_n$. A* finite t-norm *over C is a binary operation $* : C \times C \to C$ such that:*

- *the operation $*$ is associative, commutative and non-decreasing in each variable,*
- *a_0 is an absorbent element, i.e., for all $x \in C$, $x * a_0 = a_0$,*
- *a_n is a neutral element, i.e., for all $x \in C$, $x * a_n = x$.*

Therefore, the operation T in a directed algebra $\langle L, \leq, 0, 1, T, S, N\rangle$ is nothing but a divisible finite t-norm in L. Main examples of divisible finite t-norms on a chain $C = \{a_0 < a_1 < \ldots < a_n\}$ are the $(n+1)$-valued Łukasiewicz t-norm

$$a_i *_{\text{Ł}} a_j = a_{\max(0,i+j-n)},$$

and the $(n+1)$-valued minimum t-norm

$$a_i *_{\min} a_j = a_{\min(i,j)}.$$

The notion of ordinal sum of t-norms naturally extends to the finite setting.

Definition 2.3 *Let C be the chain $a_0 < a_1 < \ldots < a_m < a_{m+1} < \ldots < a_n$ and let $*_1$ be a finite t-norm on the sub-chain $C_1 = \{a_0 < a_1 < \ldots < a_m\}$, and let $*_2$ be a finite t-norm on sub-chain $C_2 = \{a_m < a_{m+1} < \ldots < a_m\}$. Then the ordinal sum of $*_1$ and $*_2$ is the finite t-norm on C defined as follows:*

$$x *_{1,2} y = \begin{cases} x *_i y, & \text{if } x, y \in C_i \\ \min(x,y), & \text{otherwise} \end{cases}$$

The main result of Mayor and Torrens's paper [15] is the characterization of *divisible finite t-norms.*

Theorem 2.4 ([15]) *The only divisible finite t-norms over a chain of n elements are the Łukasiewicz n-valued t-norm ($*_{L_n}$), the minimum n-valued t-norm ($\min_n$) and ordinal sums of copies of finite Łukasiewicz and minimum t-norms.*

This is a result that extends to divisible finite t-norms the well-known Mostert and Shields ordinal sum representation theorem of continuous t-norms.[1]

[1]Take into account that there are no finite product chains different from the Boolean chain of two elements.

On the other hand, in a previous paper [10], with the goal of avoiding arbitrary numerical representations of linguistically expressed uncertainty, Godo and Sierra considered operators over a linearly ordered, finite set of linguistic terms or labels. In fact, in [10] the authors introduced what they called *r-smooth t-norms* over finite chains $C = \{a_0 < a_1 < ... < a_n\}$ to model conjunction operators. These are finite t-norms $* : C \times C \to C$ such that, for any $a_i, a_j, a_k, a_s \in C$,

$$\text{If } a_i * a_j = a_k \text{ and } a_i * a_{j+1} = a_s, \text{ then } s - k \leq r.$$

Here we will be interested in 1-*smooth* t-norms that, for simplicity, will be simply called *smooth* in what follows.

In [16], Mayor and Torrens prove a very interesting fact for our purposes.

Theorem 2.5 ([16]) *A finite t-norm is smooth if and only if it is divisible.*

The basic idea of the proof is that the two properties are equivalent to the fact that, given a finite t-norm $* : C \times C \to C$, for any $x \in C$, the x-row of the table of $*$ has to contain all the elements of the interval $[a_0, x]$. In some sense, these properties correspond to the continuity of a t-norm operation with respect to the order topology in any infinite complete chain, like $[0, 1]$, where the divisibility is equivalent to the continuity (see [2, 11] for a complete study of this problem). As a consequence we have the following result.

Theorem 2.6 *A finite t-norm is smooth if and only if it is a finite ordinal sum of copies of finite Łukasiewicz and minimum t-norms.*

As a direct onsequence of this result, Mayor and Torrens further prove the following results.

Proposition 2.7

(i) A smooth (divisible) finite t-norm $*$ is univocally determined by the set I_* of its idempotent elements.

(ii) There are as many smooth t-norms over a chain $C = \{a_0 < a_1 < ... < a_n\}$ as subsets of the set $C \setminus \{a_0, a_n\}$, i.e. 2^{n-1}.

The first result (that it is not true for divisible t-norms in general) follows from the fact that the set of idempotent elements univocally determines the structure the t-norm, i.e. the sequence of Łukasiewicz and Gödel components. In particular, maximal intervals in I_* correspond to Gödel components, and the rest of intervals correspond to Łukasiewicz components. The second result is an easy consequence of the first one.

3. About smooth (divisible) finite t-norms and their residua

As usual in logic, in order to define a logical calculus over a finite set of truth-values or linguistic terms, it is necessary to have some form of implication operation defined. In fuzzy logic two main types of implications are usually considered: S-*implications* and R-*implications*.

DEFINITION 3.1 *Let $\langle C, \leq \rangle$ be a complete (bounded) chain.*

- *A S-*implication *on C is a binary operation defined as $x \rightarrow_S y = \neg_C x \oplus y$, where $\neg_C$ is an involutive negation on C and $\oplus$ is a t-conorm on C.*
- *A R-*implication *on C is a binary operation defined as $x \rightarrow_R y =$* $\sup\{z \mid x * z \leq y\}$*, where $*$ is a t-norm on C.*

Some fuzzy logicians (see e.g. [14]) argue that S-implications are not adequate since, in general, they are not compatible with the (linear) order of the chain of truth values, and hence they advocate the use of R-implications (i.e. residuated implications) as they have a better behaviour in this respect.

DEFINITION 3.2 *Let $\langle C, \leq \rangle$ be a complete (bounded) chain and let $*$ be a t-norm over C. Then, the* residuum *of $*$ is a binary operation $\rightarrow_*$ on C such that the following property is satisfied for all $x, y, z \in C$:*

$$x * y \leq z \textit{ if and only if } x \leq y \rightarrow_* z \quad \text{(Residuation condition)}.$$

The residuum of a t-norm does not always exist. Indeed, if $C = [0, 1]$, a t-norm $*$ on C has residuum if and only if the t-norm is left-continuous. This condition makes clear that the residuum of $*$ and the R-implication associated to $*$ are not exactly the same notion, as the R-implication always exists since $[0, 1]$ is complete, but if the residuum exists (i.e. if $*$ is left-continuous) then they do coincide. Indeed an easy computation shows that a t-norm and its associated R-implication satisfy the residuation condition if and only if the supremum in the definition of the R-implication (see Def. 3.1) is, in fact, a maximum.

It is easy to check that if $*$ is left-continuous then:

- its residuum $\rightarrow_*$ is univocally defined as

$$x \rightarrow_* y = \max\{z \mid x * z \leq y\};$$

- $x \rightarrow_* y = 1$ if and only if $x \leq y$.

Therefore if a t-norm $*$ has a residuum, we will denote it as $\to_*$. Nevertheless we will write only $\to$ if there is no possibility of confusion.

Finally, in [14] it is proved that if a t-norm $*$ has residuum, then the divisibility condition is equivalent to both the continuity of $*$ and to the satisfaction of the following equation:

$$x * (x \to_* y) = \min(x, y) \qquad (\textit{Divisibility equation}).$$

This equivalence is well known but, for the reader's convenience, we will reproduce the proof for the case of divisible finite t-norms. Suppose $*$ is a finite and divisible t-norm. Then, for each pair $x, y \in C$ such that $x \geq y$, there exists z such that $x * z = y$. Then, if $x \geq y$, by definition of the residuum (that clearly exists for any finite t-norm), it must hold that $x * (x \to y) = y = \min(x, y)$. On the other hand, it is clear that if $x \leq y$, then $x * (x \to y) = x * 1 = x = \min(x, y)$. Notice the interest of this equivalence for t-norms on $[0, 1]$, since a topological property like continuity can be equivalently expressed by an equation, the divisibility equation.

In the case of C being a finite chain, the residuum of a (finite) t-norm always exists (the supremum is always a maximum) but, as we have already observed (see Theorem 2.4), not all finite t-norms are divisible, as the following example shows:

EXAMPLE 3.3 *Let $*$ be the t-norm on the finite set $C = \{0, a, b, 1\}$ with $0 < a < b < 1$, defined by $a * b = a * a = 0$ and $b * b = b$, i.e. the nilpotent minimum over a four elements chain. Obviously $*$ is not divisible since $a < b$ and there is no $x \in C$ such that $b * x = a$.*

4. Axiomatizing finite divisible t-norms

In this section we describe how to obtain a finite equational characterization of any finite divisible t-norm, with equations in the language $\langle *, \to, \wedge, \vee, 0, 1 \rangle$, that is, using symbols not only for the t-norm operation but also for its residuum. Actually, the reader can wonder whether one could do it with equations in the restricted language $\langle *, \wedge, \vee, 0, 1 \rangle$ without the residuum $\to$. And it turns out that, as shown by Bou in [3], equations in this language cannot distinguish for instance on a chain of four elements the finite t-norm $Ł_2 \oplus Ł_3$ from the t-norm $Ł_3 \oplus Ł_2$. Indeed, Bou shows [3, Lemma 4] that an equation in the restricted language is valid on an ordinal sum of hoops $\mathbf{A} \oplus \mathbf{B}$ if, and only if, it is valid both in $\mathbf{A}$ and in $\mathbf{B}$. Indeed, this proves that the variety generated by an ordinal sum is indistinguishable from the one generated by any permutation of the components in the ordinal sum. Therefore there is no hope to obtain an equational characterization of any (divisible) t-norm different from the minimum t-norm with equations in the restricted language $\langle *, \wedge, \vee, 0, 1\rangle$.[2]

Hence we are led to consider equations over a language including an operation for the residuum of the t-norm as well. In doing so, we are actually prompted in fact to consider enriched algebraic structures of the kind $\langle A, \wedge, \vee, *, \rightarrow_*, 0, 1\rangle$, where the lattice reduct $\langle A, \wedge, \vee, 0, 1\rangle$ is indeed a finite linearly ordered set, $*$ is a finite divisible t-norm on A and $\rightarrow_*$ is its residuum. These structures are examples of linearly ordered BL-algebras, or BL-chains. BL-algebras are bounded, integral, commutative, pre-linear and divisible residuated lattices, and they are the algebraic counterpart of Hájek's BL logic [14], a logic capturing the common 1-tautologies of all the many-valued calculi on $[0, 1]$ defined by a continuous t-norm and its residuum.

Before describing how to get an equational characterization of (the BL-chain defined by) a finite divisible t-norm, mainly based on results from [5], we first recall an alternative ordinal sum decomposition of a finite BL-chain that has advantages for our purposes.

4.1 An alternative decomposition of a finite divisible t-norm as ordinal sum of hoops

First of all we consider an example in order to stress a problem concerning the ordinal sum of (finite) t-norms when the residuated implication is involved. Let $*$ be a divisible finite t-norm over a chain $\mathbf{A}$ that is an ordinal sum of two non-trivial components $*_1$ and $*_2$, i.e. $* = *_1 \oplus *_2$. Suppose now that $x \leq y$ are elements of the first component. Then, clearly, $x \rightarrow_* y = 1$, but 1 is not an element of the first component. This means that, as an ordinal sum of BL-chains $\mathbf{A} = \langle A_1, \wedge, \vee, *_1, \rightarrow_{*_1}, a_0, a_{n_1}\rangle \oplus \langle A_2, \wedge, \vee, *_2, \rightarrow_{*_2}, a_{n+1}, a_m\rangle$, the first component $\mathbf{A_1}$ is not a subalgebra of the algebra $\mathbf{A}$ defined over the full chain.

As a particular case of a more general result of Aglianò and Montagna in [1], we recall a slightly different notion of ordinal sum for finite linearly-ordered *Wajsberg hoops*. Actually, a *hoop* is an algebra $\mathbf{A} = \langle A, *, \rightarrow, 1\rangle$ such that $\langle A, *, 1\rangle$ is a commutative monoid and for all $x, y, z \in A$ the following equations hold: $x \rightarrow x = 1$, $x * (x \rightarrow y) = y * (y \rightarrow x)$, $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$. A *Wajsberg hoop* is a hoop satisfying the equation: $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$. A bounded hoop

[2]Note however, that Bou has shown [4] that there is at least one equation in the language $(*, \wedge, \vee, 0, 1)$ that is valid for all finite divisible t-norms but fails in some finite non-divisible t-norm. In particular the exhibited equation in [4] has 9 variables and it fails on a t-norm over a chain of 33 elements.

is an algebra $\mathbf{A} = (A, *, \rightarrow, 1, 0)$ such that $\langle A, *, \rightarrow, 1\rangle$ is a hoop and $0 \leq x$ for all $x \in A$, where by definition $x \leq y$ if and only if $x \rightarrow y = 1$. Then it turns out that bounded Wajsberg hoops are termwise equivalent to MV-algebras, or in other words, BL-algebras satisfying the equation $\neg\neg x = x$, where $\neg x = x \rightarrow 0$. Particularly relevant examples of finite Wajsberg hoops are the following.

Lemma 4.1 *Any linearly ordered finite (bounded) Wajsberg hoop of n elements is isomorphic to the hoop $\mathbf{Ł}_n = \langle Ł_n, *, \rightarrow, 1\rangle$, where*

- *the support of L_n is the set $\{0, \frac{1}{n-1}, ..\frac{n-2}{n-1}, 1\}$,*
- *$*$ is the n-valued Łukasiewicz t-norm, i.e., $x*y = \max(0, x+y-1)$,*
- *$\rightarrow$ is the corresponding residuum, i.e. , $x \rightarrow y = \min(1, 1-x+y)$.*

Therefore, from now on, when speaking about finite linearly ordered Wajsberg hoops, we will directly refer to the hoops $\mathbf{Ł}_n$. Notice that $\mathbf{Ł}_2$ coincide with the two-element Boolean algebra.

Definition 4.2 (Ordinal sums of Wajsberg hoops)
Let $Ł_{k_i} = \langle Ł_{k_i}, *_i, \rightarrow_i, 1\rangle$ for $1 \leq i \leq m$ be a finite family of finite linearly ordered Wajsberg hoops such that $Ł_{k_i} \cap Ł_{k_j} = \{1\}$ for all $i \neq j$. The ordinal sum (as hoops) of that family is the hoop

$$Ł_{k_1} \oplus Ł_{k_2} \oplus \cdots \oplus Ł_{k_n} = \langle \bigcup_{i=1}^{n} Ł_{k_i}, *, \rightarrow, 1\rangle,$$

where:

- *the order is defined by: $x \leq y$ if either both x and y belong to the same component and $x \leq y$, or $y = 1$, or $x \in Ł_{k_i}$ and $y \in Ł_{k_j}$ and $i < j$.*
- *$x * y = x *_i y$ if $x, y \in Ł_{k_i}$, and $x * y = \min(x, y)$ otherwise.*
- *$x \rightarrow y$ is either $x \rightarrow_i y$ if $x, y \in Ł_{k_i}$, or 1 if $x \leq y$, or y if x, y belong to different components and $x > y$.*

A main advantage of this kind of decomposition is that the components $\langle Ł_{k_i}, *_i, \rightarrow_{*_i}, 1\rangle$ are substructures (i.e., subhoops) of the whole hoop structure $Ł_{k_1} \oplus Ł_{k_2} \oplus \cdots \oplus Ł_{k_n}$.

From this definition it is easy to prove the following hoop decomposition theorem for finite divisible t-norms.

Theorem 4.3 *For any given finite divisible t-norm, its corresponding finite BL-chain is (isomorphic to) an ordinal sum of a finite family of finite linearly ordered Wajsberg hoops.*

Proof: Given a finite divisible t-norm we know, by the Mayor and Torrens result, that it is an ordinal sum (as t-norms) of copies of finite minimum t-norm and finite Łukasiewicz t-norms. Take for each minimum component as many Ł$_2$ as elements has the component minus 1, and for each finite Łukasiewicz component take the corresponding Wajsberg hoop of the same cardinal. An easy computation shows that the structure $\langle C, *, \to_*, 1\rangle$ is in fact an ordinal sum (as hoops) of components of the type Ł$_k$ defined before. □

EXAMPLE 4.4 *Take the t-norm $*$ defined by* $\mathrm{G}_3\oplus$ *Ł$_5$ as ordinal sum of t-norms over a finite chain of 7 elements C. Then the (hoop) structure $\langle C, *, \to_*, 1\rangle$ is the ordinal sum of hoops: Ł$_2\oplus$ Ł$_2\oplus$ Ł$_5$ (see Figure 1.1). As noticed, the components $\langle$Ł$_k, *, \to_*, 1\rangle$ are subhoops of $\langle C, *, \to_*, 1\rangle$.*

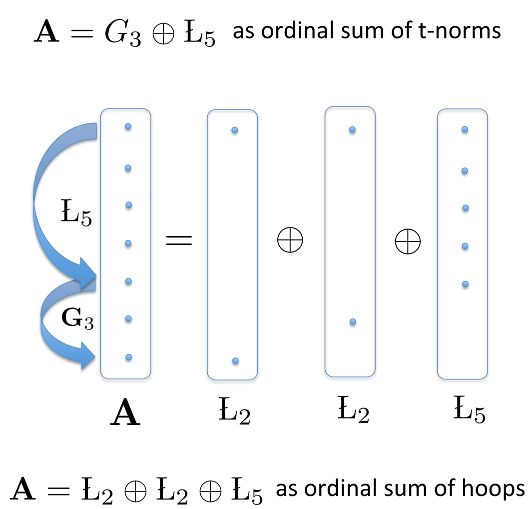

Figure 1.1. t-norm ordinal sum versus hoop ordinal sum.

4.2 Equational characterization of a divisible finite t-norm

As a necessary first step, let us focus on the equational characterization of the finite linearly ordered Wajsberg hoops **Ł**$_n$. In what follows we will denote by x^n the result of the operations $x * \overset{n}{\cdots} * x$, and by $n.x$ the result of the operation $x \oplus \overset{n}{\cdots} \oplus x$, where $\oplus$ is the bounded sum operation (the dual of the Łukasiewicz t-norm), that is definable in each Wajsberg hoop as $x \oplus y := \neg(\neg x * \neg y)$.

Notice that the Wajsberg hoops of the family $Ł_n$, besides satisfying the typical equations of t-norms:

$$x * y = y * x \tag{1.1}$$
$$x * (y * z) = (x * y) * z \tag{1.2}$$
$$1 * x = x \tag{1.3}$$
$$(x \wedge y) * z = (x * z) \wedge (y * z), \tag{1.4}$$

they also satisfy the divisibility equation:

$$x * (x \rightarrow_* y) = \min(x, y), \tag{1.5}$$

the involution equation for the negation:

$$\neg\neg x = x, \tag{1.6}$$

and the $\vee$-definability equation:

$$(x \rightarrow_* y) \rightarrow_* y = \max(x, y). \tag{1.7}$$

Actually, to fully characterize the basic Wajsberg hoops $Ł_n$ we have at hand the axiomatization provided by Grigolia [12] of the $Ł_n$'s as finite MV-algebras (see also [8]). Indeed, $Ł_n$ is equationally characterized as MV-algebra by the (finite) set of equations of axiomatizing the variety of MV-algebras (see e.g. [6]), together with the following equations in one variable:

$$x^n = x^{n-1}, \tag{τ_n}$$

and, if $n \geq 4$:

$$(p \cdot x^{p-1})^n = n \cdot x^p, \tag{$\tau\nu_{np}$}$$

for every $p \in \{2, \ldots, n-2\}$ that does not divide $n-1$.

Since an equation of the kind $t(x) = s(x)$ can be rewritten, using the double implication, as $t(x) \leftrightarrow s(x) = 1$, the above finite set of equations $\{(\tau_n)\} \cup \{(\tau\nu_{np}) : p \in \{2, \ldots, n-2\}$ not dividing $n-1\}$ can be equivalently expressed, using the conjunction, as a single equation on one variable:

$$t_n(x) = 1. \tag{t_n}$$

Therefore, the equations characterizing Ł$_n$ will be those of MV-algebras plus (t_n). For example, for Ł$_3$, the equation (t_3) is:

$$x^3 \leftrightarrow x^2 = 1, \qquad (t_3)$$

while the equation (t_4) is:

$$((2x)^4 \leftrightarrow 4x^2) \wedge (x^4 \leftrightarrow x^3) = 1. \qquad (t_4)$$

Notice that a set of equations defines a *variety* of algebras, and thus the equations given above actually define the variety of Wajsberg hoops generated by Ł$_n$. This implies that the equations characterizing Ł$_n$ are also satisfied by the subalgebras of Ł$_n$, i.e., by Ł$_k$, where k divides n (but the proper subalgebras satisfy equations that Ł$_n$ does not). Nevertheless, they do provide a univocal characterization in the following sense: given a chain $\langle C, \leq \rangle$ of n elements, then Ł$_n$ is the unique Wajsberg hoop defined by a t-norm on C that satisfies the above equations.

In order to axiomatize any finite ordinal sum of finite Wajsberg hoops we need some preliminary results. In the following, for any natural k we will denote by $t^*_k(x)$ the term obtained from $t_k(x)$ by replacing the constant 0 by x^k.

LEMMA 4.5 (CF. [5, LEMMA 5.4.1]) *Let $A_1 \oplus \ldots \oplus A_n$ be an ordinal sum of finite l.o. Wajsberg hoops and assume A_i is a component with k elements. Then A_i is isomorphic to Ł$_k$ if and only if the equation*

$$t^*_k(x) = 1. \qquad (t^*_k)$$

is valid in A_i.

Proof: The basic difference between A_i as component of the ordinal sum and Ł$_k$ is the minimum element. The minimum of A_i is not 0 (the minimum of the ordinal sum) but it can be recovered taking x^k for any element $x < 1$ of A_i. Then the result follows. □

LEMMA 4.6 *Let C be a finite chain with n elements, and let $*$ be a divisible t-norm defined on C. Then the equation*

$$\bigwedge_{i=1}^{n}((x_{i+1} \to x_i) \to x_i) \leq \bigvee_{i=1}^{n+1} x_i. \qquad (\lambda_n)$$

is valid on the hoop $\langle C, *, \to, 1\rangle$ if and only if its decomposition as ordinal sum of hoops **Ł**$_k$*'s has a number of components less or equal than n.*

Proof: Observe first that if $x_{i+1} \leq x_i$ and they belong to a different component then $((x_{i+1} \to x_i) \to x_i) = 1 \to x_i = x_i$ and thus the inequality holds. Moreover if x_i, x_{i+1} belong to the same component then $((x_{i+1} \to x_i) \to x_i) = x_i \vee x_{i+1}$, and thus the inequality holds as well. Thus, in order to check whether the inequality does not hold, we only need to take into account a sequence of $n+1$ elements x_i such that they are strictly increasing and each x_i belonging to a different component. If the number of components is less or equal than n then such a sequence does not exist, and thus the inequality holds. However, if the number of components is greater than n then an strictly increasing sequence x_i where each element belong to a different component and $x_{n+1} \neq 1$ exists. But for this sequence and for each $i \in \{1, \dots, n\}$, $((x_{i+1} \to x_i) \to x_i) = 1$ and $\bigvee_{i=1}^{n+1} x_i = x_{n+1} \neq 1$. Thus the inequality does not hold. $\square$

LEMMA 4.7 *Let C be a finite chain and let $*$ be a divisible t-norm defined on C such that $\langle C, *, \to, 1\rangle = Ł_{k_1} \oplus Ł_{k_2} \oplus \dots \oplus Ł_{k_n}$, i.e. the ordinal sum decomposition has n components. Then the equation*

$$\bigwedge_{i=1}^{n-1}((x_{i+1} \to x_i) \to x_i) \leq \bigvee_{i=1}^{n} x_i \vee (\bigwedge_{i=1}^{n} t^*_{k_i}(x_i)) \qquad (\epsilon_n)$$

is valid on the hoop $\langle C, *, \to, 1\rangle$,.

Proof: Like in the proof of the previous lemma, the inequality clearly holds in the case that either $x_{i+1} \leq x_i$, and then $(x_{i+1} \to x_i) \to x_i = x_i$, or both x_i, x_{i+1} belong to the same component and then $(x_{i+1} \to x_i) \to x_i = x_i \vee x_{i+1}$. Then, since the number of components is n, a strictly increasing sequence x_i where each element belong to a different component with $x_n \neq 1$ exists. Then, for each x_i the corresponding equation $t^*_{k_i}(x_i)$ defining $Ł_{k_i}$, has to hold. $\square$

Therefore, the problem is to fix that the number of components is exactly n, but this is not definable directly because a set of equations defines a variety and if a variety contain $Ł_k$ have to contain its subalgebras in particular $Ł_r$ for r divisor of k. In the paper [1], Aglianò and Montagna solve the problem in the following way.

LEMMA 4.8 *Let C be a finite chain and let $*$ be a divisible t-norm defined on C such that $\langle C, *, \to, 1\rangle = Ł_{k_1} \oplus Ł_{k_2} \oplus \dots \oplus Ł_{k_n}$ Then $(C, *, \to, 1, 0)$ is characterized by the equations:*

$$\bigwedge_{i=1}^{n}((x_{i+1} \to x_i) \to x_i) \leq \bigvee_{i=1}^{n+1} x_i \qquad (\lambda_n)$$

$$t^*_{k_1}(\neg\neg x) = 1 \qquad (t^*_{k_1})$$

together with the set of equations (ϵ_r) for $r = 2, \ldots, n$:

$$\bigwedge_{i=1}^{r-1}((x_{i+1} \to x_i) \to x_i) \to \bigvee_{i=1}^{r} x_i \vee \bigvee_{\sigma_r}(t^*_{k_{\sigma_r(1)}}(x_1) \wedge \ldots \wedge t^*_{k_{\sigma_r(r)}}(x_r)) = 1 \qquad (\epsilon_r)$$

where for every r, σ_r ranges over increasing sequences of r elements out of n

Proof: By the previous lemmas, we know that $*$ is an ordinal sum with less than $n+1$ components and that if it has n components they have to be the components of $*$. It only remains to prove that the satisfaction of equations (ϵ_r) implies that $*$ cannot have less than n components. Suppose that $*$ has $r < n$ components. Then we can define a strictly increasing sequence $x_1 < x_2 < \ldots < x_r$ and by equation (ϵ_r) we know that there is a sequence σ such that $(t_{k_{\sigma(1)}}(x_1) \wedge \ldots \wedge (t_{k_{\sigma(r)}})(x_r)) = 1$. This implies that the r components are $Ł_{k_{\sigma(1)}}, \ldots, Ł_{k_{\sigma(r)}}$, but the sum of the number of elements of these components is less than n and thus $Ł_{k_{\sigma(1)}} \oplus \ldots \oplus Ł_{k_{\sigma(r)}}$ does not define a t-norm over C. □

To finish the paper we give two examples, the latter being a simpler axiomatic system for the particular case that the decomposition of $*$ as hoops has only two components.

EXAMPLE 4.9 *Suppose the decomposition of $\langle C, *, \to, 1\rangle$ as ordinal sum is $L_s \oplus L_t \oplus L_r$. Then the following equations determine $*$:*

$$((x_4 \to x_3) \to x_3) \wedge ((x_3 \to x_2) \to x_2) \wedge ((x_2 \to x_1) \to x_1) \leq x_1 \vee x_2 \vee x_3 \vee x_4 \quad (\lambda_3)$$

$$t^*_s(\neg\neg x) = 1 \qquad (t^*_s)$$

$$((x_2 \to x_1) \to x_1) \to [x_1 \vee x_2 \vee (t^*_s(x_1) \wedge t^*_t(x_2)) \vee (t^*_s(x_1) \wedge t^*_r(x_2)) \vee (t^*_t(x_1) \wedge t^*_r(x_2))] = 1 \qquad (\epsilon_2)$$

$$[((x_3 \to x_2) \to x_2) \wedge ((x_2 \to x_1) \to x_1)] \to [x_1 \vee x_2 \vee x_3 \vee (t^*_s(x_1) \wedge t^*_t(x_2) \wedge t^*_r(x_3))] = 1 \qquad (\epsilon_3)$$

EXAMPLE 4.10 *When the decomposition as hoops of a finite t-norm has only two components, then there is also the following simplified equational characterization with only two equations. Namely, let C be a finite chain of n elements and $*$ be a divisible t-norm over C such that the decomposition of $\langle C, *, \to, 1\rangle$ as ordinal sum of hoops is $L_s \oplus L_t$, i.e., it has only two components. Then, the following simplified pair of equations determine $*$:*

$$t^*_s(\neg\neg x) = 1. \qquad (1.8)$$

$$t_t^*(\neg\neg x \to x) = 1. \tag{1.9}$$

The proof is very easy since all the elements of the first component L_s are of the form $\neg\neg x$, with $x \in C$, while all the elements of the second component L_t are of the form $\neg\neg x \to x$, with $x \in C$. In other words, $C = \{\neg\neg x \mid x \in C\} \cup \{\neg\neg x \to x \mid x \in C\}$. To finish the proof, take into account that the chain defined by only the first component would satisfy these equations as well, but it would not be a t-norm over C, since it should coincide with L_s, and $s < n$.

5. Conclusions

The paper has overviewed an approach to characterize divisible t-norms on finite chains by a finite set of equations that use not only the t-norm itself but also its residuum. Thus, in fact, these equations characterize the class (variety) of algebraic structures over finite sets defined by them, namely finite BL-chains.

If we move from finite divisible t-norms to continuous (or divisible) t-norms over $[0,1]$, then each continuous t-norm defines a *standard* BL-chain, namely the structure $[0,1]_* = \langle [0,1], *, \to_*, 0, 1\rangle$. In [9] it is proved that there is a finite set of equations (using the t-norm itself but also its residuum) defining the variety $V([0,1]_*)$ generated by a standard BL-chain $[0,1]_*$. Nevertheless, only when the t-norm $*$ is a *finite* ordinal sum of copies of Łukasiewicz, product and minimum t-norms, the equations actually characterize the t-norm, since the only standard BL-chain contained in the variety $V([0,1]_*)$ is $[0,1]_*$ itself. However when $*$ is an infinite ordinal sum of of copies of Łukasiewicz, product and minimum t-norms, there exist an infinite number of continuous t-norms $\circ$ such that $[0,1]_\circ \in V([0,1]_*)$.

Dedication

This short note is dedicated to Gaspar Mayor in the occasion of his 70th birthday. We have taken as starting point the research line initiated in his early works about representation of finite divisible t-norms and we have ended with the equational characterization of them. Thanks a lot for your inspiring work and congratulations Gaspar!

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