Phase Transition in Realistic Random SAT Models[1]

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Abstract. Phase-transition in random SAT formulas is one of the properties best studied by theoretical SAT researchers. There exists a constant r_k depending on k such that, if we choose randomly a k-SAT formula over n variables and m clauses, it will be satisfiable with high probability, if $m/n < r$, and unsatisfiable, otherwise. However, this criterion is useless in practice, because real-world or industrial instances have some properties not shown in random formulas. In the last years, several models of realistic random formulas have been proposed.

Here we discuss about the phase transition in these models, and about the size of unsatisfiability proofs. We observe that in these models, like in real-world formulas, there is not a sharp phase transition, the transition occurs for smaller values of r, and the proofs on unsatisfiable formulas are smaller than in the classical random model. We also discuss about the strategies used by modern SAT solvers to exploit these properties.

Keywords. Satisfiability, SAT, Complex Networks, Phase Transition

1. Introduction

SAT is a crucial problem in computer science, artificial intelligence and even in complex systems theory. From a theoretical perspective, the (in)possibility of deciding SAT in polynomial time is probably the most famous open question in computer science. From an AI perspective, the translation of many constraint satisfaction problems like scheduling, planning, hardware verification into SAT, and the use of SAT solvers has shown to be a successful approach, that defines the state-of-the-art in many applications. In recent years, SAT has also been studied from a complex systems and complex networks perspective. This has made SAT the object of study of several scientific communities, often disjoint and with very different techniques and sensibilities.

One of the best studied phenomena in SAT is the satisfiability phase transition. If we chose a random k-SAT formula over n variables and m clauses with

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uniform probability, we fix the ratio $\alpha = m/n$ and make n tend to infinite, the probability that the formula is satisfiable will tend to 0, when $\alpha < r_k$, and to 1 when $\alpha > r_k$, where r_k is a constant that only depends on k. The existence of this satisfiability threshold has been shown experimentally [17]. It has also been shown that most difficult to solve random SAT instances are located around this satisfiability threshold. This motivates the generation of instances close to the satisfiability threshold when using a random model to generate SAT instances for solvers evaluation. For $k = 2$, the phase transition has been rigorously proved for $r_2 = 1$. For $k = 3$, experimentally it has been estimated $r_3 \approx 4,27$, and several upper and lower bounds for r_3 have been proved. Using the *cavity method*, a technique applied in statistical mechanics to analyze the Ising models and other physical systems, Mézard and Zecchina [20], Mézard et al. [21] found an exact value for r_3. However, this cannot be considered a rigorous prove since it is based in some heuristics. The same technique is behind the Braunstein et al. [10] survey propagation algorithm, that is the most efficient SAT solving algorithm when dealing with random SAT instances near the phase transition threshold.

The main difference difference between real-world SAT instances and random instances is the presence of a *structure*. This structure may be revealed representing them as graphs. Real-world instances use to be *scale-free* [2], which means that the number of occurrences of variables k follows a power-law distribution $P(k) \sim k^{-\delta}$. They are also very *modular* [4], and have a small *fractal dimension* [5]. The space of tree-like resolution proofs, also called *hardness*, of these instances use to be small, when compared with random instances [1, 9].

The scale-free structure of real-world networks is usually explained as the result of *preferential attachment* [8]. If we have a growing network where new nodes get connected to old nodes with probability proportional to the number of edges they already have, we get a network where $P(k) \sim k^{-2}$. Most real-world SAT instances are not the result of a growing process. In this case, the scale-free structure may be the result of a communication-optimization process. Modularity may be the result of a *local* structure or low fractal dimension. For instance, a network with n nodes, where every node is only connected to two neighbors, forming a ring of dimension 1, has modularity close to 1, when partitioning it into modules of $\sqrt{n}$ consecutive nodes. The low fractal dimension has a more intriguingly explanation. As we will see, the satisfiability phase transition in SAT instances is related with the *percolation threshold* in graphs. It is known that the biggest connected component of a graph, in the percolation threshold, has a low fractal dimension [12]. If we assume that real-world SAT instances are also in the satisfiability threshold, the same reason may explain their low fractal dimension.

In this article, we will focus on two new models of random SAT instances that capture the nature of real-world SAT instances better than the traditional uniformly-random model: the scale-free SAT model [3, 6] and the popularity-similarity SAT model [19]. We will discuss on the satisfiability threshold on these models, and its relation with the proof-size of unsatisfiable instances. We will relate the satisfiability threshold with the percolation threshold on graphs and we will describe a method that may help to characterize the satisfiability threshold based on techniques already developed for networks.

2. Phase Transition in 1-SAT

There is a close relationship between percolation threshold in graphs and satisfiability threshold in SAT instances. *Percolation theory* describes the behavior of connected components in a graph when we remove edges randomly (or we construct it adding random edges). Erdös and Rényi [14], in a seminal paper, proposed a random graph model $G(n, m)$ where all graphs with n nodes and m edges are selected with the same probability. Gilbert [18] proposed a similar model $G(n, p)$ where n is also the number of nodes, and every $\binom{n}{2}$ possible edge is selected with probability p. For not very sparse graphs (when $p\,n^2 \to \infty$), both models have basically the same properties taking $m = \binom{n}{2}p$. Erdös and Rényi [15] also studied the connectivity on these graphs and proved that

- when $m < n/2$, a random graph almost surely has no connected component larger than $\mathcal{O}(\log n)$,
- when $m = n/2$ a largest component of size $n^{2/3}$ almost surely emerges. Later, [12] proved that this connected component has a small fractal dimension. And,
- when $m > n/2$, the graph almost surely contains a unique giant component with a fraction of the nodes and no other component contains more than $\mathcal{O}(\log n)$ nodes.

In next section, we will see that the phase transition observed in graphs is responsible for the abrupt satisfiability transition observed in random SAT. There exists a relation between the existence of connected components (in the formula represented as a graph) and the existence of a unsatisfiability core of clauses (i.e. of a unsatisfiability proof) that makes the formula unsatisfiable. The existence of a big connected component in a SAT formula implies the existence of a contradiction, i.e. the existence of a *big* unsatisfiability proof. But, what happens with small connected components? Could not them make the formula unsatisfiable? In classical models of random formulas, existence of *small* minimal unsatisfiability cores of clauses (*cores* for short) is much less probable than existence of *large cores*. Therefore, in these models, we can add clauses and we do not get a contradiction until we get a giant connected component. These models show then an abrupt phase transition. However, there are other models where the unsatisfiability of the formula is due to the existence of small cores. In this second case, there is not a proper phase transition threshold. In order to analyze this second phenomena, we will study the phase transition in 1-SAT formulas.

1-SAT formulas are conjunctions of one-literal clauses. The only possible minimally unsatisfiability core is $\{a, \neg a\}$, for some variable a. Therefore, in this model all cores are small. There are satisfiable 1-SAT formulas of any size (since we can have repeated clauses). However, a random formula with just one repeated variable has probability $1/2$ to be unsatisfiable. In order to find the phase transition threshold, if it really exists, we may compute the probability of a formula with m clauses over n variables to not contain repeated variables:

$$P(n, m) = \frac{n-1}{n}\,\frac{n-2}{n}\cdots\frac{n-m+1}{n} = \frac{(n-1)!}{(n-m)!\,n^{m-1}}$$

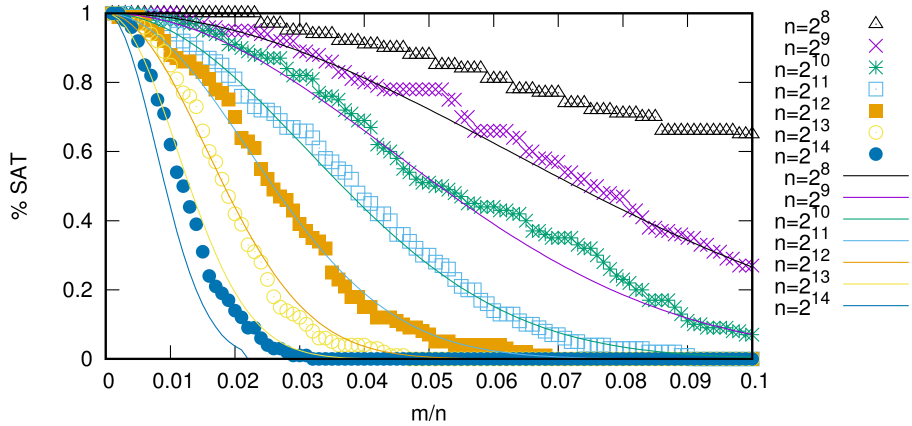

Figure 1. Fraction of satisfiable random 1-SAT formulas as a function of clause/variable for n between 2^8 and 2^{14}. Dots represent experimental data, and continuous lines the prediction $P(n,m)$ for the probability of not repeated variables. The experimental fraction is approximated repeating the experiment for 50 formulas.

Using Stirling's approximation and $\lim_{n\to\infty}(1-1/n)^n = e^{-1}$, for big values of $n-m$, this probability can be approximated as:

$$
\begin{aligned}
P(n,m) &\approx \frac{\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}}{\sqrt{2\pi(n-m)}\left(\frac{n-m}{e}\right)^{n-m} n^{m-1}} \\
&= e^{1-m}\,(1-1/n)^{n-1/2}\,(1-m/n)^{m-n-1/2} \\
&\approx e^{-m}\,(1-m/n)^{m-n-1/2} = \frac{1}{\sqrt{1-m/n}}\left(\frac{1}{e^{m/n}\,(1-m/n)^{1-m/n}}\right)^n
\end{aligned}
$$

If we replace $m = r\,n$, with $r < 1$, we get $P(n, r\,n) \approx \frac{1}{\sqrt{1-r}}\left(\frac{1}{e^r\,(1-r)^{1-r}}\right)^n$. When $n \to \infty$, the function $P(n, r\,n)$ has a phase transition, but it is located at the critical value of r solving: $e^r\,(1-r)^{1-r} = 1$. However, the unique solution of this equation is $r = 0$.

Therefore, 1-SAT has a phase transition point, when $m = r\,n$ and $n \to \infty$, but it is located at $r = 0$. In Figure 1 we represent the fraction of satisfiable formulas with respect to m/n found experimentally and the theoretical prediction for the occurrence of the fist repeated variable. Since the repetition of variables does not imply the unsatisfiability of the formula, the experimental data are moved to the right of theoretical data.

In Section 4, we will see another example of formulas with a phase transition at $r = 0$, and where cores are small. We conjecture that, in a random SAT model, when small cores are more probable than large cores, the phase transition threshold is $r = 0$. And, when large cores are more probable than small cores, then the percolation threshold, obtained with our criterion, and the phase transition threshold are equal.

3. A Criterion for Phase Transition in 2-SAT

In this section we consider clauses of up to 2 literals. It may be argued that, since 2-SAT is a polynomial problem, these formulas are not interesting. However, this only means that we cannot observe the easy-hard-easy pattern on the difficulty of solving the formulas. But we can observe the sharp and smooth satisfiability transitions, and analyze the size of unsatisfiable cores.

Unsatisfiability proofs of 2-SAT formulas are characterized by *bicycles* [7, 11]. We define a *cycle* in a 2-SAT formula F as a sequence of literals $x_1, \ldots, x_n$ such that, $\neg x_i \vee x_{i+1}$, for any $i = 1, \ldots, n-1$, and $\neg x_n \vee x_1$ are clauses of F. We define a *bicycle* in a 2-SAT formula as a cycle $x_1, \ldots, x_n$ such that there exists a variable a satisfying $\{a, \neg a\} \subseteq \{x_1, \ldots, x_n\}$. A 2-SAT formula is unsatisfiable if, and only if, it contains a bicycle [7, 11].

We will also consider random graphs with n nodes and m edges and connected components, defined as subsets of nodes such that any pair of them is connected by a path inside the component. A random graph of size n is said to contain a *giant connected component* if almost surely[2] it contains a connected component with a positive fraction of the nodes. Given a model of random graphs, we say that r is the *percolation threshold* if any random graph with n nodes and more than $r\,n$ edges almost surely contains a giant component. In a random graph, the degree of a node x, noted k_x, is a random variable. The random variable k represents the degree of a random node chosen with uniform probability.[3]

As we comment in the introduction, there is a parallelism between graphs and SAT formulas, and between the percolation threshold and the satisfiability threshold. We can consider the graph of all literals of a formula, where every clause clause $a \vee b$ is represented as an edge $a \leftrightarrow b$ between nodes a and b. However, notice that a connected component in the graph is not a bicycle in the formula (in a bicycle, the clause $a \vee b$ is connected to $\neg b \vee c$, whereas in a connected component the edge $a \leftrightarrow b$ is connected to $b \leftrightarrow c$).

Theorem 1 establishes a criterion for the existence of a giant bicycle in a formula. The proof of the theorem resembles Cohen et al. [13], Molloy and Reed [22]'s criterion for the existence of a giant component in a graph when $\frac{E[k^2]}{E[k]} > 2$, where k is the degree of a random node, and E denotes expectation. However, notice that, in Theorem 1, the 2 of Molloy-Reeds criterion is replaced by a 3.

Theorem 1 *Let F be a random 2-SAT formula generated with a model where (1) the number of occurrences of literals x and $\neg x$, noted k_x and $k_{\neg x}$, follow the same and independent probability distribution*[4] *and (2) the probability of clause $x \vee y$ only depends on the probability distributions for k_x and k_y. Then, F contains a giant bicycle if, and only if, $\frac{E[k^2]}{E[k]} + E[k] > 3$. Or, equivalently: $\frac{E[K^2]}{E[K]} > 3$, where $K_x = k_x + k_{\neg x}$, for every variable x.*

[2] *Almost surely* or *with high probability* means that, in the model of random graph, as $n \to \infty$, the probability tends to 1.

[3] In some of the models of random graphs that we will consider, not all degrees of nodes follow the same probability distribution. Therefore, we will distinguish between k and k_x.

[4] The distribution is the same for k_x and $k_{\neg x}$, but it can be different for the number of occurrences of distinct variables k_x and k_y.

PROOF: A similar theorem is proved in detail in [6], following similar arguments as in [22]. Here we only sketch the proof following arguments of [13], that are more in the style of physics.

In percolation theory we get a giant connected component when a node i, connected to a node j, is also connected in average to at least one other node. Formally, when the expected degree of i, conditioned to the fact that i and j are connected, is $E[k_i \mid i \leftrightarrow j] = 2$.

In our case, in order to emerge a giant cycle, when there is a clause $x \vee y$, we have to find another clause containing $\neg x$. It is difficult to find a criterion expressing such condition dealing with literals. Instead, we will reason about variables. When we have a clause containing variable x, i.e. $x \vee y$ or $\neg x \vee y$, for some literal y, we have to find another clause that contains $\neg x$, or x, respectively, allowing us to continue the construction of the cycle. In this situation, the expected number of other clauses containing x is 2, that added to the original clause gives the 3. Let $\pm x \vee y$ express the fact "$x \vee y \in F$ or $\neg x \vee y \in F$, for some literal y, and let $K_x = k_x + k_{\neg x}$ be the number of occurrences of variable x. Formally, our criterion can be written as $E[K_x \mid \pm x \vee y] > 3$. This criterion is the necessary and sufficient condition to continue the construction of a set of clauses, ensuring that the probability that this set contains a fraction of the literals tends to one.

Using Bayes, we have

$$\begin{aligned} E[K_x \mid \pm x \vee y] &= \sum_{k=0}^{\infty} k\, P(K_x = k \mid \pm x \vee y) = \sum_{k=0}^{\infty} k\, \frac{P(K_x = k \wedge \pm x \vee y)}{P(\pm x \vee y)} \\ &= \sum_{k=0}^{\infty} k\, \frac{P(\pm x \vee y \mid K_x = k) P(K_x = k)}{P(\pm x \vee y)} \end{aligned}$$

Under the conditions of the theorem we have that the probability of a clause conditioned to the fact that the number of occurrences of one of its variables is k is $P(\pm x \vee y \mid K_x = k) = \frac{k}{n-1}$ and, the probability of any clause is $P(\pm x \vee y) = \frac{E[K_x]}{n-1}$. Therefore

$$E[K_x \mid \pm x \vee y] = \sum_{k=0}^{\infty} k\, \frac{\frac{k}{n-1} P(K_x = k)}{\frac{E[K_x]}{n-1}} = \frac{\sum_{k=0}^{\infty} k^2\, P(K_x = k)}{E[K_x]} = \frac{E[K_x^2]}{E[K_x]}$$

Now, since $K_x = k_x + k_{\neg x}$ and k_x and k_x follow the same distribution, we have $E[K_x] = 2\, E[k_x]$ and $E[K_x^2] = 2\, E[k_x^2] + 2\, E^2[k_x]$. Therefore, $\frac{E[K^2]}{E[K]} = \frac{E[k^2]}{E[k]} + E[k]$.

Finally, we prove that any giant cycle in a formula is, with high probability, a bicycle. I.e. any set of literals containing a fraction of all literals will almost surely contain also a literal a and its negation $\neg a$. We are in the presence of $2n$ literals and a giant cycle of size $r\, n$ (where $0 < r < 1$). The probability of having two given literals in the giant cycle is roughly r^2. But then, the giant cycle can have any of the n pairs of the form $(x, \neg x)$ with probability $1 - (1 - r^2)^n$, which tends exponentially fast to 1. ■

The previous theorem ensures that, when $\frac{E[K^2]}{E[K]} > 3$, there is a giant bicycle containing a fraction of the literals, and the formula is unsatisfiable. However, if the formula is unsatisfiable, it can be due to a *small* bicycle, and we can not conclude $\frac{E[K^2]}{E[K]} > 3$. In other words, Theorem 1 establish a sufficient (but not necessary) condition for unsatisfiability of random 2-SAT formulas, which result into an upper bound for the phase transition point. However, we conjecture that, either giant bicycles are more probable than small bicycles and the percolation threshold (obtained with the criterion) is equal to the phase transition point, or, if small bicycles are more probable, the phase transition point is at $r = 0$, like for 1-SAT.

4. Scale-free 2-SAT Formulas

Random scale-free formulas were introduced by Ansótegui et al. [3]. They are parametric on an exponent $\beta \in [0, 1]$. Clauses are chosen independently, with possible repetitions, like in the classical random model. However, the probability to be chosen is not uniform, and depends on the probability of their literals $P(x_1 \vee \ldots \vee x_n) \approx \prod_{i=1}^{n} P(x_i)$, being zero when the clause contains repeated variables. The probability of a literal and its negation is the same $P(x) = P(\neg x)$, and the probability of variable x_i is $P(x_i) \sim i^{-\beta}$.

Ansótegui et al. [3] proved that in this model, the number of occurrences K of a random variable follows a power-law distribution $P(K) \sim K^{-\delta}$, where $\delta = 1 + 1/\beta$.

Recently, Friedrich et al. [16] have proved that scale-free random 2-SAT formulas with exponent $\delta > 3$ and clause/variable ratio $m/n < \frac{(\delta-1)(\delta-3)}{(\delta-2)^2}$ are satisfiable with probability $1 - o(1)$.This gives a lower bound for a possible phase transition point, in terms of δ. They conjecture that this bound is tight and that this phase transition exists. Replacing $\delta = 1+1/\beta$ (according to [3, 6]) in [16], we get: Scale-free random 2-SAT formulas with exponent $\beta < 1/2$ and clause/variable ratio $m/n < \frac{1-2\beta}{(1-\beta)^2}$ are satisfiable with probability $1 - o(1)$. The first statement of the following theorem states that, when the clause/variable ratio exceeds this value, formulas are almost surely unsatisfiable.

Theorem 2 ([6]) *Scale-free random 2-SAT formulas with exponent* $\beta < 1/2$ *and clause/variable ratio* $m/n > \frac{1-2\beta}{(1-\beta)^2}$ *are unsatisfiable with probability* $1 - o(1)$.
Scale-free random 2-SAT formulas over n *variables, exponent* $1/2 < \beta < 1$, *and* $m > \frac{1}{(1-\beta)^2\zeta(2\beta)} n^{2(1-\beta)} + \mathcal{O}(n^{1-\beta})$ *distinct clauses, are unsatisfiable with probability* $1 - o(1)$.

5. When Formulas Have Small Refutations

In the previous section, we have seen that, in random scale-free 2-SAT formulas, when $1/2 < \beta < 1$ the number of clauses needed to make the formula unsatisfiable is sub-linear: $m = \mathcal{O}(n^{2(1-\beta)})$. Therefore, the satisfiability threshold –understood as a constant r such that, on the limit $n \to \infty$, formulas with less that $r\,n$ clauses are satisfiable and those with more than $r\,n$ clauses are unsatisfiable– is $r = 0$. In this section, we go further and prove that, when β exceeds a certain bound, scale-free formulas become unsatisfiable due to a small subset of clauses containing variables with small indexes. Moreover, this result holds for clauses of any size.

Theorem 3 ([6]) *A random scale-free formula over n variables, exponent β and $\mathcal{O}(n^{(1-\beta)k})$ clauses of size k is unsatisfiable with probability $1-o(1)$.*

To prove the previous result, it suffices to compute the probability of a clause only containing the smallest k variables:

$$P(x_1 \vee \ldots \vee x_k) \geq P(x_1) \cdots P(x_k)\,(1/2)^k = \frac{1^{-\beta} \cdots k^{-\beta}}{\left(\sum_{i=1}^n i^{-\beta}\right)^k}\,(1/2)^k$$
$$\approx \frac{(k!)^{-\beta}}{\left(2\int_1^n i^{-\beta}\,\mathrm{d}i\right)^k} = (k!)^{-\beta}\left(\frac{1-\beta}{2(n^{1-\beta}-1)}\right)^k$$

In the limit $n \to \infty$, when the number of clauses is $\mathcal{O}(n^{(1-\beta)k})$, the formula will contain all 2^k clauses of the form $\pm x_1 \vee \ldots \vee \pm x_k$ with probability $1-o(1)$.

As in classical random formulas, the expected number of truth assignments that satisfy a scale-free random formula is $2^n(1-2^{-k})^m$. This imposes a linear upper bound on the number of clauses of satisfiable scale-free formulas, i.e. a random scale-free formula with $m = r\,n$ clauses of size k over n variables such that $r > 2^k \log 2$ is unsatisfiable with probability $1-o(1)$. Therefore, the bound in Theorem 3 only *improves* this other linear bound when $(1-\beta)k < 1$, hence when $\beta > 1 - 1/k$.

For random scale-free 2-SAT formulas, Theorem 3 predicts that the number of clauses in a satisfiable cannot grow faster that $\mathcal{O}(n^{2(1-\beta)})$, due to the emergence of constant size cores. When $1/2 < \beta < 1$, the second statement of Theorem 2 predicts exactly the same exponent $2(1-\beta)$ for the emergence of a giant bicycle. This suggest that, in this range of $\beta \in [1/2,\ 1]$, the probability of existence of a small and a giant unsatisfiable core of clauses is similar (at least in $\mathcal{O}$-approximation). However, experimental results suggest that the SAT-UNSAT transition is quite smooth, like in classical 1-SAT. This suggest that small cores are, in fact, more prominent.

6. The Effect of Locality

According to the conclusions of the previous section, it seems that the best we can do for solving a random scale-free SAT instance is to instantiate first those variables that occur more frequently. When $\beta > 1$, variables with smaller indexes are a constant fraction of all variable occurrences. This implies that by instantiating them first, we would get a polynomial algorithm. For $\beta < 1$, the complexity of this algorithm is an open question (nobody has been able to prove that it is polynomial, and nobody has been able to prove either that they require linear tree-like refutation space, which would be an indicator of its hardness). In any case, this is not the heuristics used by most industrial-specialized SAT solvers. The reason is that the scale-free model does not capture another property of real-world SAT instances. Clauses in those instances tend to relate *similar* variables (according to some hidden metrics), i.e. variables that are *close* or local. This leaded Giráldez-Cru and Levy [19] to define a more sophisticated random model called *popularity-similarity* model. In this model, the probability that a clause relates a pair of variables depends on the popularity of those variables (popularity is the responsible of the scale-free structure) and their similarity (proximity in this hidden metrics). There is a parameter, the temperature, that adds some randomness or entropy to the resulting instance. Modern SAT solvers, using the VSID heuristics, are able to use (a combination of) both properties, i.e. they tend to instantiate most popular variables (the ones that occur more frequently) but also closest variables (the ones that are more similar to the ones that have been instantiated recently).

At low temperatures, clauses only relate closest variables. This means that we can have small cores only containing most popular variables (like in the scale-free model) or only containing a small subset of local variables, or a combination of both. The use of the VSID heuristics would be an efficient way to search for those small unsatisfiable cores (and proofs). The analysis of the phase transitions of this model is proposed as a future research. However, the experiments support the hypothesis of the existence of these small cores at low temperature, and the existence of a smooth phase transition.

Finally, we conjecture that the existence of (a sharp) phase transition is only possible when the number of clauses that make the formula unsatisfiable is linear on the number of variables. In this situation, we conjecture that the tree-like refutational space (the hardness) is linear (i.e. that the proofs found by the solvers are linear). When the set of clauses is sub-linear we conjecture that, like in the scale-free model, the satisfiability transition is smooth (there is not properly a phase transition), and the unsatisfiable cores are also sub-linear.

7. Conclusions

In this paper, we have shown that percolation-based or, in general, mean field techniques are useful tools for the analysis of phase transition in SAT. We have applied these techniques to analyze the phase transition threshold in random scale-free 2-SAT formulas. These model of random formulas are more realistic describing real-world SAT instances. They have a parameter that regulates the homogeneity on the number of occurrences of variables. When all variables have a similar number of occurrences, the model shows a satisfiability phase transition, like in classical random formulas. However, when the number of occurrences has more variability the satisfiability transition becomes smoother and the size of unsatisfiable cores of clauses become smaller (just like it is observed in real-world SAT instances).

We also discuss the effect of the locality of variables on the phase transition, following the popularity-similarity random model. In this model, with low temperature, we obtain small unsatisfiable cores of clauses that relate very popular and close variables. The VSID heuristics used by most SAT solvers, exploit this topology in the space of variables to find these cores efficiently.

Finally, we conjecture that phase transition in SAT is only present when the size of unsatisfiable cores are linear on the number of variables, and the phenomena is related to the percolation phenomena in graphs. Otherwise, the models generate smooth satisfiability transitions.

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