When Belief Functions and Lower Probabilities are Indistinguishable

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Abstract

This paper reports on a geometrical investigation of de Finetti's Dutch Book method as an operational foundation for a wide range of generalisations of probability measures, including lower probabilities, necessity measures and belief functions. Our main result identifies a number of non-limiting circumstances under which de Finetti's coherence fails to lift from less to more general models. In particular our result shows that rich enough sets of events exist such that the coherence criteria for belief functions and lower probability collapse.

Keywords: Belief functions, coherence, geometric interpretation, imprecise probabilities, lower probabilities, uncertain reasoning.

1. Introduction

The research reported in the present paper is rooted in de Finetti's foundation of probability [4]. We assume the reader to be familiar with the approach, and limit ourselves to recall only the details of the Dutch Book method which are directly relevant to our contribution.

Suppose that $\psi_1, \ldots, \psi_n$ are elements of *SL*, the set of sentences built recursively from a finite set of propositional variables as usual, which are interpreted as the events of interest to a bookmaker *B*. Suppose further that this interest materialises with the publication of a *book* $\beta\colon \psi_1 \mapsto \beta_1, \ldots, \psi_n \mapsto \beta_n$ where for $i = 1, \ldots, n$, $\beta_i \in [0, 1]$. A gambler *G* then chooses real-valued stakes $\sigma_1, \ldots, \sigma_n$ and for $i = 1, \ldots, n$, pays $\sigma_i \beta_i$ to *B*. *G* will then receive back $\sigma_i v(\psi_i)$ where $v(\psi_i) = 1$, if ψ_i is true and $v(\psi_i) = 0$ otherwise. Thus, *B*'s payoff is $\sum_{i=1}^n \sigma_i(\beta_i - v(\psi_i))$.

This setup is sufficient to put forward an operational definition of "rational degrees of belief": the *book* published by B is *coherent* if there is no choice of (possibly negative) stakes which *G* can make, exposing *B* to a sure loss. More precisely, for every $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$ there is a valuation v such that,

$$\sum_{i=1}^{n} \sigma_i(\beta_i - v(\psi_i)) \geq 0. \tag{1}$$

As shown by de Finetti, coherence is necessary and sufficient for the existence of a finitely additive measure P that extends, over the boolean algebra of the events, the assessment β. That is, there is a probability function P such that for $i = 1, \ldots, n$ $P(\psi_i) = \beta_i$.

We consider books defined over generic sets of events. As shown in [14, 13], the definition of books over boolean algebras can be generalized to other algebraic structures that are more suitable to represent many-valued events. Furthermore, more refined notions of coherence, such as that of *strict coherence*, have been recently treated for both classical and many-valued events in [10].

Over the past decades considerable attention has been devoted to showing that the criterion captured by (1) can be used to provide a foundation to a much broader class of uncertainty measures than probability. The seminal contribution of Walley [16] is a particularly telling example, as it is largely motivated by concerns entirely analogous to those of de Finetti: tying uncertainty quantification to decision-making. Particularly relevant to our contribution is [17], in which Walley treats sets of desirable gambles, a model for representing imprecise probabilities.

The connection of these sets with the definition of books over a set of events is clear but not direct. Desirable gambles are defined over a finite set of outcomes $\Omega = \{\omega_1, \ldots, \omega_n\}$ such that there is an unknown outcome value belonging to Ω. A *gamble* over Ω is a bounded mapping from Ω to $\mathbb{R}$, i.e. $X : \Omega \to \mathbb{R}$. If an agent A accepts a gamble X, then $X(\omega_i)$ is the reward A obtains if the outcome of the experiment is ω_i. If $\mathscr{L}$ denotes the set of all gambles defined over Ω and $X, Y \in \mathscr{L}$, then $X > Y$ means that $X(\omega_i) \geq Y(\omega_i)$ for all $\omega_i \in \Omega$ and for at least one $\omega_i \in \Omega$, $X(\omega_i) > Y(\omega_i)$. A subset $\mathscr{D}$ of $\mathscr{L}$ is a *coherent set of desirable gamble* if it satisfies the following axioms:

D1. $0 \notin \mathscr{D}$.
D2. If $X \in \mathscr{L}$ and $X > 0$, then $X \in \mathscr{D}$.
D3. If $X \in \mathscr{D}$ and $c \in \mathbb{R}_{>0}$, then $cX \in \mathscr{D}$.
D4. If $X \in \mathscr{D}$ and $Y \in \mathscr{D}$, then $X + Y \in \mathscr{D}$.

As consequence of the above axioms we have that if $X < 0$, then $X \notin \mathscr{D}$ (*avoiding partial loss*). Let $\mathscr{D}$ be a set of gambles on Ω. A *lower prevision* on $\mathscr{D}$ is a functional

$\underline{P} : \mathscr{D} \to \mathbb{R}$. For any gamble X in $\mathscr{D}$, $\underline{P}(X)$ represents A's supremum acceptable buying price for X, i.e. A is willing to pay $\underline{P}(X) - \varepsilon$ with $\varepsilon \in \mathbb{R}_{>}$ for the uncertain reward determined by X and the outcome of the experiment. A buying price c for X is acceptable if $X - c$ is desirable. (See Chapter 3 of [15] and [12] for more details.)

Lower previsions are subject to coherence constraints inspired by (1). In particular, a positive linear combination of acceptable gambles should never result in A losing money independently of the outcome of the experiment. This generalisation of de Finetti's criterion is known in the area as *avoiding sure loss* and can be formalised as follows. For every $i = 1, \dots, n$ and $X_i \in \mathscr{D}$, we should have

$$\sup_{\omega \in \Omega} \sum_{i=1}^{n} [X_i(\omega) - \underline{P}(X_i)] \geq 0. \tag{2}$$

If we understand the set of possible outcomes as the events of a book, we have that every outcome of the experiment over Ω corresponds to a valuation v such that $v(\omega_i) = 1$, if the result of the experiment is ω_i and $v(\omega_i) = 0$ otherwise. This last property does not hold for a generic book, i.e. on assignments on an arbitrary subset of the whole algebra of events.

Coherent books and desirable gambles share much conceptual ground. Mathematically though, criteria (1) and (2) are clearly distinct: the former is defined over all the events of the book, while the latter asserts that there is at least one $\omega \in \Omega$ such that A has positive gain if ω is the result of the experiment. Moreover, the requirement of avoiding partial loss which follows from the axioms of coherent desirable gambles is more general than, and cannot be reduced to, de Finetti's criterion. For in this case, A's gain is negative regardless of the outcome of the experiment, i.e. for $i = 1, \dots, n$ whenever the experiment's outcome is ω_i, A must pay $X(\omega_i)$ to the bookmaker.

The methodological framework we adopt in this paper is rooted in the geometric perspective put forward by Paris in [14], whose key results show how geometrical tools can be used to generalise de Finetti's method to non-boolean events and, more importantly for the present paper, to non (finitely) additive measures of uncertainty. We can also find connections with the geometric perspective presented in [2, 3]. By using a geometric approach similar to Paris's we will recall in Section 3 results on the characterization of books that can be extended to probabilities, normalized necessity measures, and belief functions. Then, in Section 4, we will tackle the same extendability problem but in the more general setting of lower probabilities. There we prove our main result to the effect that partial assignments on events exists for which it is impossible to tell whether they are coherent in the sense of lower probability theory but fail coherence according to belief functions. Thus, in logical terms, our result suggests that there are non negligible limits to the expressive power of coherence.

2. Background on Uncertainty Measures

We shall assume the reader acquainted with basic notions and results of (finitely additive) probability theory. In particular, since we will only consider measures on finite boolean algebras, we shall often identify a probability measure P on an algebra $\mathbf{A}$ with the distribution p obtained by restricting P on the atoms of $\mathbf{A}$.

As for the other uncertainty measures we will deal with in the next sections, it is convenient to recall some basic definitions and results that we will take from [1, 6, 11, 12].

As we have already declared, we will only consider finite, and hence atomic, boolean algebras as the domain for uncertainty measures. Boolean algebras will be understood as described in the signature $\{\wedge, \vee, \neg, \bot, \top\}$ and their elements will be denoted by lower-case greek letters with possible subscripts. In particular, atoms of an algebra will be indicated as $\alpha_1, \alpha_2, \dots$.

Definition 1 *A* normalized necessity measure N *on an algebra* $\mathbf{A}$ *is a* $[0,1]$*-valued map satisfying the following equations:*

(N1) $N(\top) = 1, N(\bot) = 0$;

(N2) $N(\psi_1 \wedge \psi_2) = \min\{N(\psi_1), N(\psi_2)\}$

If π is a normalized possibility distribution on the atoms $\alpha_1, \dots, \alpha_t$ of $\mathbf{A}$ (i.e., $\pi(\alpha_i) \in [0,1]$ and $\max\{\pi(\alpha_i) \mid i = 1, \dots, t\} = 1$), then the map defined as follows is a normalized necessity measure on $\mathbf{A}$

$$N(\psi) = \bigwedge_{j=1}^{t} (1 - \pi(\alpha_t)) \wedge \psi(\alpha_j). \tag{3}$$

Furthermore, every normalized necessity measure on $\mathbf{A}$ can be obtained by a normalized possibility distribution as in (3) above.

Definition 2 *A* belief function B *on an algebra* $\mathbf{A}$ *is a* $[0,1]$*-valued map satisfying*

(B1) $B(\top) = 1, B(\bot) = 0$;

$$(B2)\quad B\left(\bigvee_{i=1}^{n} \psi_i\right) \geq \sum_{i=1}^{n} \sum_{\{J \subseteq \{1,\dots,n\} : |J| = i\}} (-1)^{i+1} B\left(\bigwedge_{j \in J} \psi_j\right)$$

for $n = 1, 2, 3, \dots$.

Belief functions on boolean algebras can be characterized in terms of *mass functions* as follows. Let $\mathbf{A}$ be any finite boolean algebra with atoms $\alpha_1, \dots, \alpha_t$. A mass function is a map m that assigns to each subset X of atoms, a real number such that $m(\emptyset) = 0$ and $\sum_X m(X) = 1$. Given a mass function m, the map

$$B(\psi) = \sum_{X \subseteq \{\alpha_i \mid \alpha_i \leq \psi\}} m(X)$$

is a belief function and every belief function on $\mathbf{A}$ can be defined in this way.

Definition 3 *A* lower probability $\underline{P}$ *on an algebra* $\mathbf{A}$ *is a monotone* $[0,1]$*-valued map satisfying*

(L1) $\underline{P}(\top)=1$, $\underline{P}(\bot)=0$*;*

(L2) For all natural numbers n,m,k *and all* $\psi_1,\ldots,\psi_n$*, if* $\{\{\psi_1,\ldots,\psi_n\}\}$ *is an* (m,k)*-cover of* $(\varphi,\top)$[1]*, then* $k+m\underline{P}(\varphi)\geq\sum_{i=1}^n\underline{P}(\psi_i)$*.*

Although the definition above does not make clear why those measures are called *lower* probabilities, [1, Theorem 1] characterizes them as follows: Let $\underline{P}:\mathbf{A}\to[0,1]$ be a lower probability and call $\mathscr{M}(\underline{P})=\{P:\mathbf{A}\to[0,1]\mid P$ is a probability function and $\forall\psi\in A,\underline{P}(\psi)\leq P(\psi)\}$. Then, for all $\psi\in A$,

$$\underline{P}(\psi)=\min\{P(\psi)\mid P\in\mathscr{M}(\underline{P})\}.$$

Lower probabilities are more general than belief functions. The following result characterizes those lower probability that are belief functions.

Remark 4 *A lower probability* $\underline{P}$ *on an algebra* $\mathbf{A}$ *is a belief function iff* $\underline{P}$ *satisfies (B2), namely*

$$\underline{P}\left(\bigvee_{i=1}^n\psi_i\right)\geq\sum_{i=1}^n\sum_{\{J\subseteq\{1,\ldots,n\}:|J|=i\}}(-1)^{i+1}\underline{P}\left(\bigwedge_{j\in J}\psi_j\right)\quad(4)$$

for all $n=1,2,\ldots$*.*

The following, which is an immediate consequence of the above characterization, gives a minimal algebraic requirement to distinguish belief functions and lower probabilities. It will be useful to justify our main result and its consequences that we will show in Section 4.

Corollary 5 *Let* $\mathbf{A}$ *be a boolean algebra. Then, every lower probability on* $\mathbf{A}$ *is a belief function iff* $\mathbf{A}$ *has two atoms.*

Proof Assume α_1,α_2 be the unique atoms of $\mathbf{A}$ and let $\underline{P}$ be a lower probability on $\mathbf{A}$. Discarding trivial cases, let us focus on the non-trivial events of $\mathbf{A}$: α_1 and α_2. Then, $\underline{P}(\alpha_1\vee\alpha_2)=\underline{P}(\top)=1$ and $\underline{P}(\alpha_1)+\underline{P}(\alpha_2)-\underline{P}(\alpha_1\wedge\alpha_2)=\underline{P}(\alpha_1)+\underline{P}(\alpha_2)-\underline{P}(\bot)=\underline{P}(\alpha_1)+\underline{P}(\alpha_2)$. Moreover α_1 and α_2 are disjoint, so $\underline{P}(\alpha_1)+\underline{P}(\alpha_2)\leq1$, whence (4) is satisfied.

Conversely, assume that $\mathbf{A}$ has more than two atoms. Then the claim just follows from Example 2 below. ■

1. An element φ of a boolean algebra $\mathbf{A}$ is said to be covered m times by a multiset $\{\{\psi_1,\ldots,\psi_n\}\}$ of elements of A if every homomorphisms of $\mathbf{A}$ to $\{0,1\}$ that maps φ to 1, also maps to 1 at least m propositions from $\psi_1,\ldots,\psi_n$ as well. An (m,k)-cover of $(\varphi,\top)$ is a multiset $\{\{\psi_1,\ldots,\psi_n\}\}$ that covers $\top$ k times and covers φ $n+k$ times.

Let us close this section observing that, although the algebra with two atoms does not distinguish lower probabilities and belief functions, it does distinguish probability functions from normalized necessities and both probability and necessities from belief functions. Indeed, it is easy to see that the necessity N whose possibility distribution π maps $\pi(\alpha_1)=\pi(\alpha_2)=1$ cannot be a probability function and the probability function P given by the distribution $p(\alpha_1)=p(\alpha_2)=1/2$ does not define a normalized necessity measure. Moreover, the mass assignment m that maps $m(\{\alpha_1,\alpha_2\})=1$ and the rest of subsets to 0 gives a belief function that is neither a probability, nor a normalized necessity.

3. A Geometric View on Coherence and Extendibility

Let $\Psi=\{\psi_1,\ldots,\psi_n\}$ be a finite set of events (i.e., elements of a finite boolean algebra $\mathbf{A}$). Let us denote by $\mathbb{V}=\{v_1,\ldots,v_t\}$ the finite set of all possible homomorphisms of $\mathbf{A}$ to the boolean chain on the two-element set $\{0,1\}$. For every $j=1,\ldots,t$, call $\mathbf{e}_j$ the binary vector

$$\mathbf{e}_j=(v_j(\psi_1),\ldots,v_j(\psi_n))\in\{0,1\}^n.\qquad(5)$$

Given this basic construction, we can characterize in geometric terms the extendability problem for books on Ψ to finitely additive probability measures, normalized necessity measures and belief functions. The unique further ingredient is the notion of Euclidean convex hull $\overline{\text{co}}(X)$ of a subset $X\subseteq\mathbb{R}^t$, that reduces to $\text{co}(X)$ in case X is finite, and the less common *tropical* convex hull of X (see [5]) that we recall in the next.

Definition 6 *Let* $\mathbf{x}_1,\ldots,\mathbf{x}_t\in[0,1]^n$*. The* tropical hull *of the* $\mathbf{x}_j$*'s is the subset* $\text{co}_{\wedge,+}(\mathbf{x}_1,\ldots,\mathbf{x}_t)$ *of all points* $\mathbf{y}$ *of* $[0,1]^n$ *for which there exist parameters* $\lambda_1,\ldots,\lambda_t\in[0,1]$ *such that* $\bigwedge_{j=1}^t\lambda_j=1$ *and*

$$\mathbf{y}=\bigwedge_{j=1}^t\lambda_j+\mathbf{x}_j.$$

The symbol $\wedge$ *stands for the minimum and* $+$ *for the ordinary addition in the tropical semiring* $(\mathbb{R},\wedge,+)$*. Given* $\lambda\in[0,1]$ *and* $\mathbf{x}\in[0,1]^n$*,* $\lambda+\mathbf{x}=(\lambda+x_1,\ldots,\lambda+x_n)$ *and the* $\bigwedge$ *operator is defined component-wise.*

Now, for $\mathbf{e}_1,\ldots,\mathbf{e}_t$ being defined as above from the formulas ψ_i's in Ψ, let us consider the following sets:

1. $\mathscr{P}_\Psi=\text{co}(\mathbf{e}_1,\ldots,\mathbf{e}_t)$, where co denotes the usual Euclidean convex hull;

2. $\mathscr{N}_\Psi=\text{co}_{\wedge,+}(\mathbf{e}_1,\ldots,\mathbf{e}_t)$, where $\text{co}_{\wedge,+}$ is as in Definition 6;

3. $\mathscr{B}_\Psi = \overline{\text{co}}(\mathscr{N}_\Psi)$, where, in this case, being $\mathscr{N}_\Psi$ usually uncountable, $\overline{\text{co}}$ denotes the topological closure of the Euclidean convex hull co.

The following theorem recalls known results that have been proved in [4, 7, 8]

Theorem 7 *Let $\Psi = \{\psi_1, \ldots, \psi_n\}$ be a finite set of events and let $\beta : \Psi \to [0,1]$ be a book. Then,*

1. *β extends to a probability measure iff $(\beta(\psi_1), \ldots, \beta(\psi_n)) \in \mathscr{P}_\Psi$;*
2. *β extends to a normalized necessity measure iff $(\beta(\psi_1), \ldots, \beta(\psi_n)) \in \mathscr{N}_\Psi$;*
3. *β extends to a belief function iff $(\beta(\psi_1), \ldots, \beta(\psi_n)) \in \mathscr{B}_\Psi$.*

It is worth noticing that the previous characterization also allows to easily distinguish the uncertainty measures appearing in the theorem above. Indeed, assume the set of events Ψ we start with is not *trivial*, i.e., it neither is $\{\top, \bot\}$ on which all uncertainty measures coincide, nor it is itself a boolean algebra on which all uncertainty measures can be easily distinguished. In general, $\mathscr{P}_\Psi$ and $\mathscr{N}_\Psi$ are both strictly included in $\mathscr{B}_\Psi$ (i.e., $\mathscr{P}_\Psi \subset \mathscr{B}_\Psi$ and $\mathscr{N}_\Psi \subset \mathscr{B}_\Psi$) and this is expected because belief functions are strictly more general than both probabilities and normalized necessity measures. For the same reason, it is easy to see that $\mathscr{P}_\Psi$ and $\mathscr{N}_\Psi$ are usually incomparable. The next example clarifies this situation.

Example 1 *Let $\mathbf{A}$ be the boolean algebra of 8 elements and 3 atoms $\{\alpha_1, \alpha_2, \alpha_3\}$ and consider the non-trivial set of events $\Psi = \{\psi_1, \psi_2, \psi_3\} \subset A$ where $\psi_1 = \alpha_1 \vee \alpha_2$, $\psi_2 = \alpha_2 \vee \alpha_3$ and $\psi_3 = \alpha_1 \vee \alpha_3$. The algebra $\mathbf{A}$ has 3 homomorphisms to $\{0,1\}$. Computing the points $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as in (5), we hence obtain*

$$\mathbf{e}_1 = (1,0,1); \mathbf{e}_2 = (1,1,0); \mathbf{e}_3 = (0,1,1).$$

The subsets $\mathscr{P}_\Psi$, $\mathscr{N}_\Psi$ and $\mathscr{B}_\Psi$ are hence as in Figures 1, 2 and 3 respectively. Notice that, although Ψ does not coincide with the whole algebra $\mathbf{A}$, it allows to distinguish those books that are either extendible to a probability or a normalized necessity, from those extendible to belief functions. Indeed both $\mathscr{P}_\Psi$ and $\mathscr{N}_\Psi$ are strict subsets of $\mathscr{B}_\Psi$.

The question we raise is hence if similar results on the possibility of distinguishing uncertainty theories, via coherence, still hold when we consider more general uncertainty models and in particular lower probabilities. A partial yet surprising result will be presented in the next section in which we will study if coherence is sufficiently robust to distinguish lower probability from belief functions.

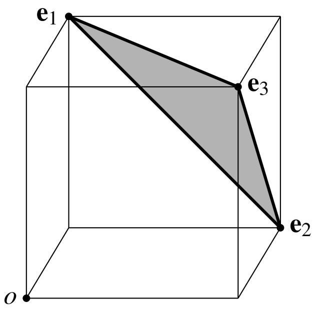

Figure 1: The polytope $\mathscr{P}_\Psi$.

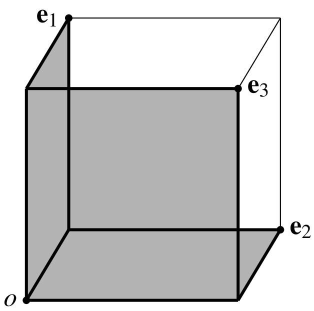

Figure 2: The tropical polytope $\mathscr{N}_\Psi$.

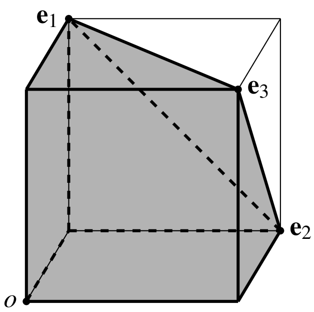

Figure 3: The polytope $\mathscr{B}_\Psi$.

4. On Coherent Books: From Belief Functions to Lower Probabilities

This section aims to show that, in general, the geometric characterization of coherence we presented in Theorem 7 seems not to be sufficiently robust to distinguish books that are extendible to lower probabilities from those that are extendible to belief functions.

As it will appear clear in a while, a major role in this sense is played by the chosen set of events $\Psi \subseteq A$ (the domain of the boolean algebra with start with). Due to what we proved in Corollary 5, we will henceforth assume that $\mathbf{A}$ has a number of atoms strictly greater than 2.

Proposition 8 *For every algebra* $\mathbf{A}$ *and for every subset* Ψ *of* A*, if* $\mathscr{P}_\Psi \cap \mathscr{N}_\Psi \neq \mathscr{P}_\Psi$ *and* $\mathscr{P}_\Psi \cap \mathscr{N}_\Psi \neq \mathscr{N}_\Psi$*, then* $\mathscr{P}_\Psi \subset \mathscr{B}_\Psi$ *and* $\mathscr{N}_\Psi \subset \mathscr{B}_\Psi$*.*

Proof For every algebra $\mathbf{A}$ and for every subset X of A, $\mathscr{P}_X \subseteq \mathscr{B}_X$ and $\mathscr{N}_X \subseteq \mathscr{B}_X$. Then, assume that $\mathscr{P}_\Psi \cap \mathscr{N}_\Psi \neq \mathscr{P}_\Psi$ and $\mathscr{P}_\Psi \cap \mathscr{N}_\Psi \neq \mathscr{N}_\Psi$. We have to prove that hence the above inclusions are proper.

The current hypothesis states that there are $\beta_1, \beta_2 : X \to [0,1]$ such that $\beta_1 \in \mathscr{P}_\Psi$, $\beta_2 \in \mathscr{N}_\Psi$, $\beta_1 \notin \mathscr{N}_\Psi$ and $\beta_2 \notin \mathscr{P}_\Psi$. Since $\mathscr{P}_\Psi, \mathscr{N}_\Psi \subseteq \mathscr{B}_\Psi$, one has that $\beta_1, \beta_2 \in \mathscr{B}_\Psi$ and therefore $\mathscr{P}_\Psi, \mathscr{N}_\Psi \subset \mathscr{B}_\Psi$. ∎

One could expect that, under the same hypothesis, a similar behaviour lifts to the realm of lower probabilities. However, as our main result shows, this is not the case. First, we need the following result where we will indicate by $\mathscr{L}_\Psi$ the set of all books β on Ψ that extend to a lower probability $\underline{P}$.

Lemma 9 *Let* $\mathbf{A}$ *be a finite boolean algebra and* $\Psi = \{\psi_1, \ldots, \psi_n\} \subseteq A$*. A book* β *on* Ψ *belongs to* $\mathscr{L}_\Psi$ *iff there are* $\beta_1, \ldots, \beta_n \in \mathscr{P}_\Psi$ *such that, for all* $\psi_i \in \Psi$*,* $\beta(\psi_i) = \min\{\beta_j(\psi_i) \mid j = 1, \ldots, n\}$*.*

Proof The right-to-left direction is trivial. Let us hence assume that β extends to a lower probability $\underline{P}$. Let $\mathscr{M}(\underline{P}) = \{P \mid P(a) \geq \underline{P}(a), \forall a \in A\}$ as in Section 2 and then, for all $\psi_i \in \Psi$,

$$\underline{P}(\psi_i) = \min\{P(\psi_i) \mid P \in \mathscr{M}(\underline{P})\}.$$

For all $P \in \mathscr{M}(\underline{P})$, call β_P the (necessarily coherent) book on Ψ obtained from P by restriction. Then, obviously,

$$\beta(\psi_i) = \min\{\beta_P(\psi_i) \mid P \in \mathscr{M}(\underline{P})\}.$$

Finally, since Ψ is finite, for every $\psi_i \in \Psi$ fix a book $\beta_{P(i)}$ among the β_P's such that

$$\beta_{P(i)}(\psi_i) = \beta(\psi_i) = \min\{\beta_P(\psi_i) \mid P \in \mathscr{M}(\underline{P})\}.$$

For every i, $\beta_{P(i)}$ exists. Then the claim follows since $\beta(\psi_i) = \min\{\beta_P(\psi_i) \mid P = P(i)\}$. In other words $\beta = \min\{\beta_{P(1)}, \ldots, \beta_{P(k)}\}$. ∎

In the light of Corollary 5 and previous observation, let us introduce the following notion of "adequate" set of events Ψ which allows us to discard those cases that we already know does not allow us to distinguish $\mathscr{B}_\Psi$ from $\mathscr{L}_\Psi$.

Definition 10 *Let* $\mathbf{A}$ *be a boolean algebra. A non-empty subset* Ψ *of* A *is said to be* adequate *if* Ψ *is a strict subset of* $A \setminus \{\bot, \top\}$ *and the subalgebra* $\mathbf{A}_\Psi$ *of* $\mathbf{A}$ *generated by* Ψ *has at least 3 atoms.*

Then, our main result reads as follows.

Theorem 11 *For every algebra* $\mathbf{A}$ *with at least three atoms there exists an adequate subset* Ψ *of* A *such that* $\mathscr{P}_\Psi \cap \mathscr{N}_\Psi \neq \mathscr{P}_\Psi$ *and* $\mathscr{P}_\Psi \cap \mathscr{N}_\Psi \neq \mathscr{N}_\Psi$*, but* $\mathscr{B}_\Psi = \mathscr{L}_\Psi$*.*

Proof Let us assume without loss of generality that $\alpha_1, \ldots, \alpha_n$ ($n \geq 3$) are the atoms of $\mathbf{A}$ and let us fix the subset Ψ of A made of the following elements: $\psi_1 = \alpha_1 \vee \alpha_2$, $\psi_2 = \alpha_1 \vee \alpha_3$ and $\psi_3 = \alpha_2 \vee \alpha_3$. Clearly Ψ is adequate in the sense of Definition 10.

First, let us show that $\mathscr{P}_\Psi \cap \mathscr{N}_\Psi \neq \mathscr{P}_\Psi$ and $\mathscr{P}_\Psi \cap \mathscr{N}_\Psi \neq \mathscr{N}_\Psi$.

Since every belief function is, in particular, a lower probability, $\mathscr{B}_\Psi \subseteq \mathscr{L}_\Psi$. Let β be a book in $\mathscr{L}_\Psi$. We want to prove that $\beta \in \mathscr{B}_\Psi$. Let $\underline{P}$ be a lower probability on $\mathbf{A}$ such that, for all $i = 1, \ldots, 3$, $\underline{P}(\psi_i) = \beta(\psi_i)$. Let us also assume that $\underline{P}$ is not a probability, that is to say, that β does not belong to $\mathscr{P}_\Psi$, otherwise, the claim would be trivial.

Now we prove the following.

Fact 1 $\beta \in M = \mathrm{co}(\min\{\mathbf{e}_1, \mathbf{e}_2\}, \min\{\mathbf{e}_2, \mathbf{e}_3\}, \min\{\mathbf{e}_1, \mathbf{e}_3\}, \min\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$.

Proof (of Fact 1). Assume, by way of contradiction, that $\beta \notin M$. Thus, $\beta \in [0,1]^3 \setminus M$, that is to say, $\beta \in \mathrm{co}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \max\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$. In other words, there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (with $\lambda_4 > 0$) such that $\sum_i \lambda_i = 1$ and

$$\beta = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 + \lambda_4 \max\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

The expression above is equal to $\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 + \max\{\lambda_4 \mathbf{e}_1, \lambda_4 \mathbf{e}_2, \lambda_4 \mathbf{e}_3\}$ and since $a + \max\{b, c\} = \max\{a + b, a + c\}$, one has

$$\beta = \max\{\beta_1, \beta_2, \beta_3\}$$

where $\beta_1 = (\lambda_1 + \lambda_4)\mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3$, $\beta_2 = \lambda_1 \mathbf{e}_1 + (\lambda_2 + \lambda_4)\mathbf{e}_2 + \lambda_3 \mathbf{e}_3$, $\beta_3 = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + (\lambda_3 + \lambda_4)\mathbf{e}_3$. Thus, $\beta_1, \beta_2, \beta_3 \in \mathscr{P}_\Psi$. Letting P_i, for $i = 1, 2, 3$ such that P_i extends β_i, we conclude that β extends to an upper probability. Therefore, by assumption β extends to a lower probability. In addition, β extends to an upper probability, thus β extends to a probability that is absurd by a previous hypothesis. ∎

Now, we go back to the proof of the main claim and we prove that $\min\{\mathbf{e}_1, \mathbf{e}_2\}$, $\min\{\mathbf{e}_2, \mathbf{e}_3\}$, $\min\{\mathbf{e}_1, \mathbf{e}_3\}$,

$\min\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \in \mathscr{N}_\Psi$. The claim is indeed easy to show by direct computation. For instance, check that $\min\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = (0,0,0)$ is $(N(\psi_1), N(\psi_2), N(\psi_3))$ where N is the necessity measure computed as in (3) and given by the normalized possibility distribution $\pi : \alpha_t = 1$ for all $t = 1, \ldots, n$.

Therefore β is a convex combination of points belonging to $\mathscr{N}_\Psi$. Hence it extends to a belief function. ■

Notice that the above result does not say that if β extends to a lower probability $\underline{P}$, then $\underline{P}$ is necessarily a belief function, rather it shows that if β on events ψ_1, ψ_2, ψ_3 extends to lower probability $\underline{P}$, then there exists a belief function Bel that agrees with $\underline{P}$ on the ψ_i's but which is in general different from $\underline{P}$ on the remaining events in the algebra. The following example clarifies this claim.

Example 2 *Consider an algebra* $\mathbf{A}$ *with atoms* $\alpha_1, \ldots, \alpha_t$ *($t \geq 3$) and probability distributions* $p_1(\alpha_1) = q$, $p_1(\alpha_2) = 1-q$, $p_1(\alpha_i) = 0$ *for all* $i \neq 1,2$; $p_2(\alpha_2) = q$, $p_2(\alpha_3) = 1-q$, $p_2(\alpha_i) = 0$ *for all* $i \neq 2,3$; $p_3(\alpha_1) = 1-q$, $p_3(\alpha_3) = q$, $p_3(\alpha_i) = 0$ *for all* $i \neq 1,3$ *where* q *is any value* $1/3 < q \leq 1/2$. *For every* $j = 1, \ldots, 3$, *denote by* P_j *the probability given by the distribution* p_j *and let* $\underline{P}$ *the lower probability such that, for all* $\psi \in A$,

$$\underline{P}(\psi) = \min\{P_j(\psi) \mid j = 1,2,3\}. \quad (6)$$

Let us consider events $\psi_1 = \alpha_1 \vee \alpha_2, \psi_2 = \alpha_2 \vee \alpha_3, \psi_3 = \alpha_1 \vee \alpha_3$ *as in the proof of Theorem 11, and the book* $\beta : \psi_i \mapsto q$ *for every* $i = 1,2,3$.

Since $q \leq 1/2$, $q \leq 1-q$ *and hence* $\underline{P}(\psi_1) = \underline{P}(\psi_2) = \underline{P}(\psi_3) = q$. *Thus, the lower probability* $\underline{P}$ *defined as in (6) extends* β.

Furthermore, $\underline{P}$ *is not a belief function. Indeed,* $\underline{P}(\psi_1) + \underline{P}(\psi_2) + \underline{P}(\psi_3) - \underline{P}(\psi_1 \wedge \psi_2) - \underline{P}(\psi_2 \wedge \psi_3) - \underline{P}(\psi_1 \wedge \psi_3) + \underline{P}(\psi_1 \wedge \psi_2 \wedge \psi_3)$. *Now,* $\psi_1 \wedge \psi_2 \wedge \psi_3 = \perp$, *whence* $\underline{P}(\psi_1 \wedge \psi_2 \wedge \psi_3) = 0$ *and, by definition of the* P_i*'s,* $\underline{P}(\psi_1 \wedge \psi_2) = \underline{P}(\psi_2 \wedge \psi_3) = \underline{P}(\psi_1 \wedge \psi_3) = 0$. *Therefore, since* $q > 1/3$, *the above expression reduces to* $\underline{P}(\psi_1) + \underline{P}(\psi_2) + \underline{P}(\psi_3) = 3q > 1 = \underline{P}(\psi_1 \vee \psi_2 \vee \psi_3)$ *showing that* $\underline{P}$ *does not satisfy (4).*

However, the belief function B *whose mass assignments that gives* $m(\{\alpha_1\}) = m(\{\alpha_2\}) = m(\{\alpha_3\}) = q/2$, $m(\{\alpha_1, \ldots, \alpha_t\}) = 1 - \frac{3}{2}q$ *and* $m(X) = 0$ *otherwise, extends the same book* β *to* $\mathbf{A}$.

We conclude the present paper with the following observation.

Remark 12 *Let us point out a couple of questionable points that one could reasonably raise in the light of Theorem 11 and the above Example 2.*

The first one is the following: our main result shows that, over that particular subset of formulas Ψ *of* A, *it is impossible to distinguish books that are extendible to lower probabilities from those that are extendible to belief functions:* $\mathscr{B}_\Psi = \mathscr{L}_\Psi$. *It is worth remarking that* Ψ *is not the unique adequate subset of* A *on which we can observe such a behavior. For instance, the same result holds for* $\Psi' = \{\alpha_1, \alpha_2, \alpha_3\}$.

This leads to the second observation: one may be tempted to improve Theorem 11 showing that $\mathscr{B}_\Psi = \mathscr{L}_\Psi$ for every *non-trivial subset of* A, *i.e., every strict subset of* $A \setminus \{\perp, \top\}$. *However, this is false in most cases. Indeed, take* $\mathbf{A}$ *with more than 4 atoms, let* $\mathbf{A}'$ *be any subalgebra of* $\mathbf{A}$ *with more than 3 atoms and let* Ψ *be* $A' \setminus \{\perp, \top\}$. *Then Corollary 5 shows that* $\mathscr{B}_\Psi \subset \mathscr{L}_\Psi$ *(strict inclusion) and hence* Ψ *is a non-trivial subset of* A *that distinguishes those sets.*

5. Conclusion and Future Work

This paper illustrated how through a geometric characterization of coherence, books which are extendable to lower probabilities cannot be distinguished from those which are extendable to belief functions. To the best of our knowledge, this is a new result. As a consequence we are not at present able to tell whether the observed phenomenon can be appreciated also outside the geometric settings.

An interesting question for further investigation is to characterize the adequate subsets of events of a given algebra $\mathbf{A}$ that do not distinguish belief functions from lower probabilities and provide a geometric characterization of $\mathscr{L}_\Psi$. In addition, we intend to investigate these subsets of events considering coherence criteria defined in terms of (proper) scoring rules.

Finally, following the approach put forward in [9], we intend to explore the consequences for the betting games therein defined, of the fact that certain belief functions cannot be distinguished from lower probabilities.

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