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Revisiting fuzzy multisets: clarifying ambiguities and the underlying algebraic structure

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Abstract
We revisit the theory of fuzzy multisets to address formal inconsistencies in existing definitions. Building on Miyamoto's intuitive membership sequences, we propose two revised frameworks on a common foundation. The first one (Option 1) retains explicit multiplicities even at null memberships, thereby distinguishing clearly between "no assignment" and "zero assignment", while the second one (Option 2) restricts attention to the subfamily of fuzzy multisets that uniformly enforce zero multiplicity to null memberships. Under both frameworks, the new operations of union, intersection, and inclusion satisfy the usual algebraic laws, most notably the absorption law, which fails in the original treatment, and endow the family of fuzzy multisets with a lattice structure. Moreover, within Option 2 we define a complement operation by invoking a top multiset; this complement is order-reversing and involutive, thereby equipping bounded fuzzy multisets with a De Morgan algebra structure. Finally, we prove that n-dimensional fuzzy sets are isomorphic to n-bounded fuzzy multisets, and we study the formal connections between the lattices of general fuzzy multisets and multidimensional fuzzy sets.

Keywords Multiset · Fuzzy multiset · De Morgan algebra · n-dimensional fuzzy sets

Mathematics Subject Classification 06D72 · 06D30 · 03B52

1 Introduction

A *multiset* or *bag* is an unordered collection of elements in which repetitions of identical elements may appear (Dershowitz and Manna 1979; Hickman 1980). Any multiset over a universe U is identified with a function that assigns to each element of the universe the number of times it occurs (multiplicity). One of the initial uses was handling duplicate items

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efficiently in sorting and searching algorithms (Knuth 1973). Regarding the operations and algebraic properties of the set of multisets, Knuth (1991) considers the inclusion relation and the operations of union, intersection, and addition, but does not mention the complement or the difference of multisets. Later (Hickman 1980) mentions that intersection and union are idempotent, associative, commutative, and distributive over finite collections.

Later, Yager (1986) proposed in 1986 the first generalisation of multisets to the fuzzy context. According to his definition, a *fuzzy multiset* or fuzzy bag associates, to each element of the universe U, a crisp multiset of [0, 1]. For each element $u \in U$, each degree of membership $\alpha \in (0, 1]$ can be repeated a finite number of times, and it is assumed that the membership $\alpha = 0$ will always have multiplicity 0. The author introduces some binary operations between fuzzy bags, such as addition, union or intersection, based respectively on the sum, the maximum and the minimum of the multiplicities associated to each membership value $\alpha \in (0, 1]$, without mentioning any definition of complement. In this context, (crisp) multisets can be seen as particular cases of fuzzy multisets, for which the multiplicity associated to any membership value $\alpha \neq 1$ is zero.

Miyamoto (2001, 2005) proposed definitions different from those of Yager for the usual binary operations and for the inclusion order. His definitions also generalize the operations originally considered by Knuth and Hickman, but in an alternative way. To define them, he relies on an alternative representation of fuzzy multisets, rather than representing them in terms of their multiplicities. Specifically, he represents the crisp multiset of [0, 1] associated with each element $u \in U$ as a sequence of membership values in [0, 1], ordered from largest to smallest. Before operating with the pairs of membership sequences associated with the respective fuzzy multisets, he considers the maximum of both lengths and extends the shorter sequence with the membership value 0, repeating it as many times as necessary to match the lengths. Once this is done, to determine the union, intersection, or inclusion relation, Miyamoto takes the maximum, minimum, or comparison of the membership values, respectively, component-wise. The operations defined in this way are compatible with the corresponding operations between the so-called α-cuts of the fuzzy multisets. The α-cut of a fuzzy multiset A is the crisp multiset of U obtained by considering the elements of U associated with membership values not lower than α, repeated as many times as they appear with these memberships.

Miyamoto's context faces some formal issues. Initially it considers the possibility of assigning an arbitrary multiplicity value to the membership 0. However, when applying binary union and intersection operations, this distinction is lost, by "completing" with repetitions of null memberships the shortest sequence, until the size of the longest sequence is reached. In this treatment, an ambiguity arises between an assignment of membership 0 repeatedly and no assignment at all. Because of this ambiguity, the operations thus defined do not satisfy basic properties such as the absorption law, since the intersection of a fuzzy multiset A with another fuzzy multiset B that includes A is not necessarily included in A. Moreover, the operations thus defined do not endow the family of fuzzy multisets with lattice structure.

In this paper, we will modify Miyamoto's definitions of union, intersection and inclusion to avoid these formal problems, and furthermore to admit the possibility of differentiating, without ambiguity, between no membership assignments and (possibly repeated) null membership assignments. We will see that in the new context, the family of fuzzy multisets has a lattice structure, and that union and intersection univocally determine the order of inclusion. Even in the cases where it is not necessary to refer to the multiplicity of the null membership (because it has no relevant semantic meaning in a particular application), Miyamoto's construction can be simplified, without resorting to sequence completions by means of artificial repetitions of the 0-membership.

We will further analyze the notion of complement. First, we will review the considerations regarding the complement of crisp and fuzzy multisets in Miyamoto (2000, 2004a, b). Miyamoto (2000) addresses the concept of the difference between two crisp and fuzzy multisets. In this work, he starts from Yager's union and intersection operations, so that the proposed definition of the difference of fuzzy multisets is compatible with these operations, but it is not compatible with the formulation he proposes later. He furthermore argues that it would not make sense, in this context, to define the "complement" of a (crisp) multiset because the "world" of multisets is "open" (in the sense that there is no "top" multiset against which we could define the complement of a multiset as the difference with respect to that "top" element). Miyamoto (2004a) revisits the issue of defining the complement of a crisp multiset and suggests that, in order to introduce this definition, it is necessary to extend the set of possible values for the multiplicity function to include the value ∞. According to this definition, the complement of a (crisp) multiset assigns multiplicities of 0 or ∞ to the elements of the universe, so it does not strictly reverse the inclusion order nor is it self-inverse. In a paper from the same year, Miyamoto (2004b) considers a new definition for the difference between two fuzzy multisets, but this one is not compatible with his definitions of union and intersection, nor with those previously introduced by Yager. In this case, he also refrains from defining the complement of a fuzzy multiset due to the absence of a "top" element in the family of fuzzy multisets with respect to the inclusion order.

In this paper, we will consider a difference operation between fuzzy multisets that is compatible with the (corrected) Miyamoto definitions of union and intersection. In order to subsequently define the concept of complement of a fuzzy multiset, we must assume the existence of a "top" multiset (the crisp multiset that repeats each element of the universe at least as many times as it is repeated by any of the fuzzy multisets involved in the study). In this context, the complement does reverse the inclusion order and is self-inverse. However, it is not possible to distinguish between the two situations mentioned in the previous paragraphs (repetitions of 0-membership assignment vs. no assignment), and furthermore it is necessary to consider a top element. We will refer to this family of fuzzy multiset as "bounded fuzzy multisets."

We will show that the family of bounded fuzzy multisets is formally equivalent to the family of n-dimensional fuzzy sets introduced by Shang et al. (2010). We will show that both are De Morgan algebras, and we will show that there is an isomorphism between them. Moreover, we will see that there is a natural bijection between the general family of fuzzy multisets and the family of multidimensional fuzzy sets of Lima et al. (2021). We will show that the family of partial orders defined by Lima et al. for multidimensional fuzzy sets does not include the partial order between fuzzy multisets as a special case.

The paper is structured as follows. Section 2 recalls some basic notions regarding the algebraic structures mentioned throughout the work. We also review the main proposals concerning fuzzy multisets found in the literature, particularly those by Yager and Miyamoto, along with the concepts of n-dimensional and multidimensional fuzzy sets. In Sect. 3, we address the formal issues previously identified in Miyamoto's definitions and propose two alternative formulations that aim to preserve their intuitive appeal while resolving those problems. Section 4 compares our proposal with the original definitions by Yager and Miyamoto. In Sect. 5, we introduce a notion of complement that is compatible with the second of these two alternative formulations, within the framework of bounded fuzzy multisets. Section 6 explores the formal relationships between our two alternatives and the notions of n-dimensional and multidimensional fuzzy sets, including an analysis of the partial orders involved in each context. Finally, Sect. 7 concludes the paper with some final remarks and directions for future research.

2 Preliminaries

In this work, $\mathbb{N}$ will denote the set of natural numbers including 0, that is, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. On the other hand, $\mathbb{N}^*$ will refer to the set of natural numbers excluding 0, that is, $\mathbb{N}^* = \{1, 2, 3, \ldots\}$.

2.1 Basic algebraic notions

Throughout the article, we will refer to some algebraic structures associated with families of multisets and fuzzy multisets. In this subsection, we recall some basic definitions.

A *partially ordered set* (or *poset*) $(L, \leq)$ is a set L equipped with a partial order, that is, a binary relation $\leq$ that is reflexive, antisymmetric, and transitive. An *upper bound* of a subset $S \subseteq L$ is an element $x \in L$ satisfying $s \leq x, \ \forall s \in S$. The *infimum (resp. supremum)* of a subset S of L is the greatest (lowest) element in L that is less than or equal (resp. greater than or equal) to all elements of S if such an element exists. A *lattice* is a poset $(L, \leq)$ in which every pair of elements $a, b \in L$ has a supremum, called the *join* (denoted $a \vee b$), and an infimum, called the *meet* (denoted $a \wedge b$). A lattice is said to be *bounded* if it has a least element 0 and a greatest element 1.

Alternatively, a *lattice* may be defined as a set L equipped with two binary operations $\vee, \wedge : L \times L \to L$ satisfying, for all $x, y, z \in L$, the identities

- *Commutativity:* $x \vee y = y \vee x, x \wedge y = y \wedge x$.
- *Associativity:* $x \vee (y \vee z) = (x \vee y) \vee z, x \wedge (y \wedge z) = (x \wedge y) \wedge z$.
- *Idempotence:* $x \vee x = x, x \wedge x = x$.
- *Absorption:* $x \vee (x \wedge y) = x, x \wedge (x \vee y) = x$.

One then defines a relation $a \leq b \iff a \vee b = b$ (equivalently $a \wedge b = a$), and shows that this relation $\leq$ is a partial order on L for which $\vee$ and $\wedge$ coincide with the supremum and infimum of each pair. This algebraic viewpoint is equivalent to the poset-theoretic one and justifies that the order is univocally determined by $\vee$ (resp. by $\wedge$).

An *involution* on a lattice L is a self-inverse unary operation $\neg : L \to L$ (i.e such that for all $a \in L$, we have $\neg(\neg a) = a$) A *De Morgan algebra* is a bounded lattice $(L, \vee, \wedge, 0, 1)$ equipped with an involution $\neg$ that satisfies the De Morgan laws:

$$\neg(a \vee b) = \neg a \wedge \neg b, \quad \neg(a \wedge b) = \neg a \vee \neg b$$

for all $a, b \in L$. If $(L, \vee, \wedge, 0, 1, \neg)$ is a De Morgan algebra, then the involution reverses the associated order, i.e. $a \leq b \Leftrightarrow \neg b \leq \neg a$.

2.2 Crisp and fuzzy multisets

A *multiset* (also called a *bag*) is a generalization of a set where elements can appear multiple times. Unlike classical sets, which only consider the presence or absence of an element, multisets allow repeated occurrences. Formally, a multiset A over a universe U is represented by a multiplicity (count) function: $C_A : U \to \mathbb{N}$, where $C_A(x)$ denotes the number of times $x \in U$ appears in A. If $C_A(x) = 0$, then x is said not to be in the multiset.[1] The *cardinality*

[1] Hickman (1980) uses an equivalent definition but with a slightly different formulation. He initially refers to the “domain of the multiset” (which is a subset of the universe) and states that the multiset is determined by a function defined on this domain with values in $\mathbb{N}^*$.

of a multiset M is defined as: $|A| = \sum_{x \in U} C_A(x)$. The *union* $A_1 \cup A_2$ of two multisets A_1 and A_2 is defined by the count function

$$C_{A_1 \cup A_2}(x) = \max(C_{A_1}(x), C_{A_2}(x)).$$

Similarly, the *intersection* $A_1 \cap A_2$ of A_1 and A_2 is defined by the count function

$$C_{A_1 \cap A_2}(x) = \min(C_{A_1}(x), C_{A_2}(x)).$$

Let us denote by $MS(U)$ the set of multisets on U. It then readily follows that $(MS(U), \cap, \cup)$ forms a distributive lattice, cf. Hickman (1980). The corresponding partial order of inclusion, denoted $\subseteq$, is defined by: $A_1 \subseteq A_2$ if and only if $C_{A_1}(x) \leq C_{A_2}(x)$ for all $x \in U$.

Fuzzy multisets (or *fuzzy bags*) were introduced by Yager (1986) as an extension of both multisets and fuzzy sets. In a fuzzy multiset, elements can appear multiple times, and each occurrence has a degree of membership. Formally, a fuzzy multiset A over a universe U is represented by a multiplicity function

$$C_A : U \times [0, 1] \to \mathbb{N},$$

where $C_A(x, \alpha)$ denotes the number of times element $x \in U$ appears with membership value $\alpha \in [0, 1]$. Equivalently, one can see a fuzzy multiset A on U as assigning to each element x of U a (crisp) multiset $A(x)$ of membership values whose count function is $C_{A(x)} : [0, 1] \to \mathbb{N}$ defined as $C_{A(x)}(\alpha) = C_A(x, \alpha)$ for every $\alpha \in [0, 1]$. Yager assumes that $C_A(x, 0) = 0,\ \forall x \in U$.

From this definition it is clear that fuzzy multisets generalise both crisp multisets (when $C_A(x, \alpha) = 0$ for every $\alpha < 1$) and fuzzy sets (when $C_A(x, \alpha) \in \{0, 1\}$ for every α).

According to Yager (1986), the *union* of two fuzzy multisets A_1 and A_2 is defined as:

$$C_{A_1 \cup_Y A_2}(x, \alpha) = \max(C_{A_1}(x, \alpha), C_{A_2}(x, \alpha))$$

and the *intersection* is defined as:

$$C_{A_1 \cap_Y A_2}(x, \alpha) = \min(C_{A_1}(x, \alpha), C_{A_2}(x, \alpha)).$$

A fuzzy multiset A_1 is included in another fuzzy multiset A_2 in the sense of Yager, denoted $A_1 \subseteq_Y A_2$, if and only if, for all $x \in U$:

$$C_{A_1}(x, \alpha) \leq C_{A_2}(x, \alpha),\ \forall \alpha \in (0, 1].$$

Miyamoto (2001) reconsiders the notion of fuzzy multisets and proposes different definitions for the inclusion relation and the binary operations of union and intersection. To introduce these definitions, he needs a representation of a fuzzy multiset A alternative (but equivalent) to its multiplicity function C_A. Namely, for each $x \in U$, he represents its (crisp) multiset of membership values $A(x)$ as a decreasing sequence including repetitions (he calls it *membership sequence*):

$$\mu_A(x) = (\mu_A^1(x), \ldots, \mu_A^p(x))$$

with $\mu_A^1(x) \geq \cdots \geq \mu_A^p(x)$, for some $p \in \mathbb{N}$, that we will call *size* of the sequence.[2] He also defines the *length* of the sequence as follows:

$$L(x; A) = \max\{j : \mu_A^j(x) \neq 0\},\ \forall x \in U. \tag{1}$$

Observe that the length of a sequence is always smaller than or equal to its size; thus $\mu_A^j(x) > 0$ for $1 \leq j \leq L(x; A)$ and zeros appear only for $L(x; A) < j \leq p$. In this context, to define

[2] Actually p depends on x, but for the sake of shorter notation we will simply write p and not write $p(x)$.

the union or intersection of two fuzzy multisets A_1 and A_2, Miyamoto first ensures that, for each $x \in U$, the two sequences be of the same size by extending the shorter sequence with as many repetitions of membership 0 as necessary until both have the same size.

Then, he defines the combined length of the sequences for A_1 and A_2 as follows. $L(x; A_1, A_2) = \max\{L(x; A_1), L(x; A_2)\}$ and considers the corresponding two sequences of size $L(x; A_1, A_2)$ by adding the necessary zeros in the shorter sequence. Once the two sizes are made equal, he defines the union and intersection operators in terms of membership sequences as follows:

$$\mu_{(A_1 \cup_M A_2)}(x) = (\max(\mu^1_{A_1}(x), \mu^1_{A_2}(x)), \ldots, \max(\mu^{\bar{l}(x)}_{A_1}(x), \mu^{\bar{l}(x)}_{A_2}(x)))$$
$$\mu_{(A_1 \cap_M A_2)}(x) = (\min(\mu^1_{A_1}(x), \mu^1_{A_2}(x)), \ldots, \min(\mu^{\bar{l}(x)}_{A_1}(x), \mu^{\bar{l}(x)}_{A_2}(x))) \quad (2)$$

where we have used $\bar{l}(x)$ as a shorthand for $L(x; A_1, A_2)$. These operations clearly preserve the decreasing property of the membership sequences.

The concept of α-cut (Miyamoto 2001) provides a way to obtain a nested family of crisp multisets from a fuzzy multiset by fixing threshold values $\alpha \in (0, 1]$.

In Miyamoto's framework, the weak α-cut $[A]_\alpha$ is the crisp multiset of all occurrences whose membership values satisfy $\mu^j_A(x) \geq \alpha$. As mentioned above, the count function of this crisp multiset, $C_{[A]_\alpha}(x)$, returns the number of times x appears in $[A]_\alpha$.

For example,[3] let $U = \{x, y\}$ and take A with $\mu_A(x) = (0.8, 0.5, 0.5)$ and $\mu_A(y) = (1, 0.4)$. For $\alpha = 0.5$, $[A]_{0.5} = [x, x, x, y]$ so $C_{[A]_{0.5}}(x) = 3$ and $C_{[A]_{0.5}}(y) = 1$.

More formally, adapting the definition from Miyamoto (1997, 2001), the count function associated to the weak α-cut $[A]_\alpha$ is defined as follows:

$$C_{[A]_\alpha}(x) = \sum_{\beta \geq \alpha} C_A(x, \beta), \forall x \in U, \ \alpha \in (0, 1]$$

Analogously, the count function associated to the strong α-cut $(A)_\alpha$ is defined as:

$$C_{(A)_\alpha}(x) = \sum_{\beta > \alpha} C_A(x, \beta), \forall x \in U, \ \alpha \in [0, 1).$$

Here $C_A(x, \beta)$ refers to the multiplicity function $C_A : U \times [0, 1] \to \mathbb{N}$ earlier introduced by Yager (1986).[4]

Miyamoto's definitions of the union and intersection operations are compatible with the notion of α-cut in the sense that we have

$$[A_1 \cup_M A_2]_\alpha = [A_1]_\alpha \cup [A_2]_\alpha, \text{ and } [A_1 \cap_M A_2]_\alpha = [A_1]_\alpha \cap [A_2]_\alpha, \ \forall \alpha \in (0, 1]$$

for weak α-cuts, and

$$(A_1 \cup_M A_2)_\alpha = (A_1)_\alpha \cup (A_2)_\alpha, \text{ and } (A_1 \cap_M A_2)_\alpha = (A_1)_\alpha \cap (A_2)_\alpha, \ \forall \alpha \in [0, 1).$$

for strong α-cuts, see e.g. (Miyamoto, 2001, Prop. 3).

Finally, according to Miyamoto, A_1 is said to be included in A_2, denoted $A_1 \subseteq_M A_2$, if $[A_1 \cup_M A_2]_\alpha = [A_2]_\alpha$ (or equivalently if $[A_1 \cap_M A_2]_\alpha = [A_1]_\alpha$), $\forall \alpha \in (0, 1]$.

[3] Here the notation $[x, x, x, y]$ is not meant as an ordered tuple but as a compact way of writing the crisp multiset where x appears three times and y once. The order of appearance in the list is only conventional, and any permutation such as $[y, x, x, x]$ would represent the same multiset; we follow the chosen enumeration of U.

[4] In the following, if A and B are crisp multisets with count functions C_A and C_B resp., we will write $A = B$ in the sense that their count functions coincide, i.e. when $C_A = C_B$.

It is worth mentioning that the family of fuzzy multisets as defined by Yager, together with the operations $\cap_Y$ and $\cup_Y$ and the distinguished element $\overline{0}$ (the empty fuzzy multiset which $C_{\overline{0}}(x, \alpha) = 0$ for any $x \in U$ and $\alpha \in [0, 1]$) forms a (non-upper bounded) distributive lattice, inherited from the lattice structure over the natural numbers $(\mathbb{N}, \min, \max, 0)$, and the associated lattice order is the one described previously.

On the other hand, as we will show in Example 2, the inclusion relation proposed by Miyamoto does not determine a lattice on the family of fuzzy multisets, since it is not even a partial order, but only a pre-order, due to the failure of the antisymmetry property.

We will address this formal issue in Sect. 3 by introducing a modified inclusion relation, along with corresponding definitions of intersection and union, which yield a (non-upper bounded) lattice structure. This construction preserves the semantic intuition of Miyamoto's approach, particularly in allowing the consideration of α-cuts, while ensuring the formal properties required for a lattice, as is also the case in Yager's framework.

2.3 *n*-dimensional and multidimensional fuzzy sets

In Sect. 6 we will analyze the formal relationships between the concepts of fuzzy multiset and the most recent notions of n-dimensional fuzzy set (Shang et al. 2010) and multidimensional fuzzy set (Lima et al. 2021), which we are going to recall in this subsection.

According to Shang et al. (2010), an n-dimensional set A over a universe U is identified with an n-dimensional membership function, $\mu_A = (\mu_{A_1}, \dots, \mu_{A_n})$, where the components are ordered in increasing order, that is: $\mu_{A_i}(x) \leq \mu_{A_{i+1}}(x)$, $\forall i = 1, \dots, n-1$, for all $x \in U$. In other words, the membership function is a mapping $\mu_A : U \to L_n$, where $L_n = \{(a_1, \dots, a_n) \in [0, 1]^n : a_1 \leq \dots \leq a_n\}$. The set L_n together with the usual partial order (the component-wise order or product order) has a lattice structure, and the (component-wise) minimum and maximum are the corresponding meet and join. Furthermore, we can define the following involution on L_n:

$$\neg(a_1, \dots, a_n) = (1 - a_n, \dots, 1 - a_1),\ \forall (a_1, \dots, a_n) \in L_n.$$

The set L_n together with the above operations of meet, join and negation is a De Morgan algebra. The operations of union, intersection and complement of n-dimensional fuzzy sets (Shang et al. 2010) are based on the above operations over L_n. The inclusion relation is based on the above partial order on L_n. The family of n-dimensional fuzzy sets together with those operations inherits the structure of De Morgan algebra from L_n.

Lima et al. (2021) extend the concept of n-dimensional fuzzy set to the case in which a fixed dimension n for the images of the membership function is not determined. According to the authors, a multidimensional fuzzy set A over U is identified with a membership function μ_A from U to $\mathcal{L}_\infty = \cup_{n \in \mathbb{N}^*} L_n$. Given a function $F : \mathcal{L}_\infty \to [0, 1]$, a sequence of partial orderings $(\leq_n)_{n \in \mathbb{N}^*}$ (each defined on the corresponding L_n and $\alpha \in [0, 1]$, the authors propose the following partial order over $\mathcal{L}_\infty$:

$$x \leq^{\infty}_{\alpha, F} y \iff \begin{cases} F(x) < F(y), \\ F(x) = F(y) \leq \alpha \text{ and } |y| < |x|, \\ F(x) = F(y) > \alpha \text{ and } |x| < |y|, \\ F(x) = F(y),\ |x| = |y| = n \text{ and } x \leq_n y \end{cases} \tag{3}$$

The authors define two operations, join and meet, on $\mathcal{L}_\infty$ that are compatible with the above order. They prove that, under certain restrictions about F and $(\leq_n)_{n \in \mathbb{N}^*}$, the set $\mathcal{L}_\infty$ together with the above partial order has a lattice structure. They define the inclusion order and the union and intersection of multidimensional fuzzy sets based on the above partial order on $\mathcal{L}_\infty$ and the corresponding meet and join.

There is a close relationship between the concepts of fuzzy multiset and multidimensional fuzzy set. As we mentioned in Sect. 2.2, Miyamoto identifies a fuzzy multiset with a mapping that, to each element x of the universe U, associates a sequence of values from [0, 1] ordered from highest to lowest. If we reverse the order of the components in the case of multidimensional fuzzy sets, we obtain sequences of that type. In Sect. 6, we will study the connection between the inclusion of multidimensional fuzzy sets defined by Lima et al. (2021) and the inclusion of fuzzy multisets.

3 Formal challenges in Miyamoto's definitions

When Miyamoto refers to the concepts of "fuzzy multiset" and of weak and strong α-cut of a fuzzy multiset, it seems that they are implying that these α-cuts uniquely determine a fuzzy subset of the family of (crisp) multisets. However, it is easy to observe that given $x \in U$, the decreasing family of multiplicities $\{C_{[A]_\alpha}(x)\}_{\alpha\in[0,1]}$ represents a gradual number (Fortin et al. 2008) and not a fuzzy number. Similarly, the families of strong and weak α-cuts of a fuzzy multiset constitute gradual multisets (to each α, a crisp multiset is assigned), and they are not truly the α-cuts of a fuzzy subset of multisets.

In general, it is well known that a fuzzy set is uniquely determined by the family of weak α-cuts, for $\alpha \in (0, 1]$ (or equivalently, by the family of strong α-cuts, for $\alpha \in [0, 1)$). However, a fuzzy multiset is not uniquely determined by the family of weak α-cuts $\{[A]_\alpha\}_{\alpha\in(0,1]}$, but also depends on the weak 0-cut. Indeed, the weak 0-cut does not coincide with any "top" multiset (in fact, it is not even required that there be a "top" element exist in the family of multisets). In general, it does not coincide with the strong 0-cut either, unless it is required that the multiplicity of membership 0 is zero. And even if this last restriction is assumed, it becomes necessary to fix a problem in the formulation of Miyamoto's union and intersection, in order to eliminate the need to complete sequences with repetitions of membership 0, which implicitly contradicts this restriction. In this section, we will clarify in more detail the formal issues of Miyamoto's original formulation and carry out this correction. We now show a simple numerical example to clarify that neither the family of strong α-cuts nor the family of weak α-cuts for $\alpha \in (0, 1]$ uniquely determine the fuzzy multiset.

Example 1 Consider the universe $U = \{u\}$ (a singleton). Consider the multisets A and B respectively represented by the multiplicity functions $C_A, C_B : U \times [0, 1] \to \mathbb{N}$ defined as follows:

$$C_A(u, 0.2) = 1, C_A(u, 0) = 1, C_A(u, \alpha) = 0, \ \forall \alpha \notin \{0, 0.2\},$$
$$C_B(u, 0.2) = 1, C_B(u, 0) = 4, C_B(u, \alpha) = 0, \ \forall \alpha \notin \{0, 0.2\}.$$

We can easily see that the families of strong α-cuts of both fuzzy multisets coincide. The respective weak α-cuts also coincide with each other, for all $\alpha \in (0, 1]$. To differentiate A and B based on the α-cuts, we need to explicitly consider the 0-cuts. The 0-cut of A assigns multiplicity 2 to the element u, while the 0-cut of B assigns multiplicity 5. Thus, although A and B coincide on all α-cuts for $\alpha > 0$, they differ only in the number of trailing zeros, hence the need to include the zero-cut.

In the following result we will clarify that every fuzzy multiset is univocally determined by the families of strong and weak α-cuts, provided that among the latter the 0-cut is taken into account. In fact, although (Miyamoto 2001) restricts attention to weak α-cuts for $\alpha > 0$, we also require the weak 0-cut, defined analogously by

$$C_{[A]_0}(x) \;=\; \sum_{\beta \geq 0} C_A(x, \beta), \quad \forall x \in U.$$

Theorem 1 *The families of weak and strong α-cuts (including the weak 0-cut) univocally determine a fuzzy multiset. In fact, the multiplicity function of a fuzzy multiset A, $C_A : U \times [0,1] \to \mathbb{N}$, can be recovered as follows from those two families:*

$$C_A(x, \alpha) = C_{[A]_\alpha}(x) - C_{(A)_\alpha}(x),\ \forall \alpha \in [0, 1),\ C_A(x, 1) = C_{[A]_1}(x),\ \forall x \in U.$$

The proof is immediate. Next, we prove that every strong α-cut is determined by the family of weak cuts of the fuzzy multiset:

Theorem 2 *The family of strong α-cuts is univocally determined by the family of weak α-cuts. In fact,*

$$C_{(A)_\alpha}(x) = \sup_{\beta > \alpha} C_{[A]_\beta}(x) = \max_{\beta > \alpha} C_{[A]_\beta}(x),\ \forall \alpha \in [0, 1).$$

Proof Let us arbitrarily select $\alpha \in [0, 1)$ and $x \in U$. By hypothesis, there is a finite number of values $\beta > \alpha$ such that $C_A(x, \beta) > 0$. Let $\beta^*(\alpha)$ be the minimum of that set of values, $\beta^*(\alpha) := \min\{\beta > \alpha : C_A(x, \beta) > 0\}$. It is easy to check that $C_{(A)_\alpha}(x) = C_{[A]_{\beta^*(\alpha)}}(x)$, and this is true for any $x \in U$. □

Corollary 3 *Any fuzzy multiset is univocally determined by the family of weak α-cuts (including the weak 0-cut).*

Miyamoto defines the union and intersection operations of fuzzy multisets according to the procedure outlined in Eq. (2), once the lengths $L(x; A_1)$ and $L(x; A_2)$ have been made equal. This procedure introduces a formal issue concerning the multiplicity of membership 0. Below we illustrate this problem with a numerical example.

Example 2 Consider again the pair of fuzzy multisets A and B from Example 1. According to the definition of union recalled in Eq. (2), $A \cup B = B$ which means that the union $A \cup B$ assigns multiplicity 4 to membership 0. According to the definition of intersection recalled in the same equation, we can also see that $A \cap B = B$ (that is, the intersection also assigns multiplicity 4 to membership 0, instead of assigning it multiplicity 1, as assigned by the fuzzy multiset A). In fact, according to Miyamoto's definition of inclusion, $A \subseteq_M B$ but also $B \subseteq_M A$. Thus, the inclusion relation defined by Miyamoto (1997) is not an order (it does not satisfy the antisymmetric property) unless we assume that A and B are the same fuzzy multiset. However, their multiplicity functions do not coincide, since $C_A(x, 0) = 1 < C_A(x, 0) = 4$.

The above example highlights a formal problem with Miyamoto's definitions. We cannot simultaneously identify a fuzzy multiset A with the function $C_A : U \times [0, 1] \to \mathbb{N}$ and the family of weak α-cuts, excluding the weak 0-cut. Nevertheless, it is possible to keep the essence of Miyamoto's definitions, while correcting the formal problem. There are two possibilities to do so, depending on the semantic interpretation of the 0-membership:

- **Option 1:** To identify a fuzzy multiset with the multiplicity mapping $C : U \times [0, 1] \to \mathbb{N}$, which explicitly records the multiplicities $C(u, 0)$ for each element $u \in U$ corresponding to membership 0. This approach distinguishes between having a specific number of occurrences of an element u with membership 0 and the element being absent in the fuzzy multiset, thereby allowing for a different semantic treatment.

- **Option 2:** To identify a fuzzy multiset with a mapping $C^* : U \times (0, 1] \to \mathbb{N}$, thus denying any reference to the multiplicity of membership 0. This option is formally equivalent to defining the multiplicity function $C : U \times [0, 1] \to \mathbb{N}$, with additional constraint $C(u, 0) = 0,\ \forall u \in U$, and considering the operations introduced in the next Sect. 3.1 unchanged.

In order to accommodate the different treatments of the multiplicity for membership 0, Miyamoto's original definitions of inclusion, union, and intersection must be revised. Although both representations yield consistent frameworks, Option 1 is more general as it allows an arbitrary multiplicity for membership 0, while Option 2 restricts this multiplicity to be zero. Consequently, the next section presents a detailed formalization of union, intersection, and inclusion according to Option 1, with a brief discussion of Option 2 provided at the end of the section.

3.1 Option 1: incorporating zero membership multiplicity in fuzzy multiset models

We will show three alternative equivalent representations, presented respectively in terms of the weak α-cuts (Definition 1), the sequence of decreasing memberships (Definition 2), and the multiplicity function (Remark 3.1). Note that, unlike (Miyamoto 2001), who defines weak α-cuts only for $\alpha > 0$, we extend the definition to all $\alpha \in [0, 1]$, in particular including the zero-cut.

Definition 1 Consider two fuzzy multisets A_1 and A_2 over U and denote by $[A_1]_\alpha$ and $[A_2]_\alpha$ respectively their weak α-cuts, for $\alpha \in [0, 1]$.

- We define the union and intersection of A_1 and A_2 as follows:
  - $A_1 \sqcup A_2$ is the fuzzy multiset determined by the weak α-cuts defined as:

    $$[A_1 \sqcup A_2]_\alpha := [A_1]_\alpha \cup [A_2]_\alpha,\ \forall \alpha \in [0, 1]$$

    that is, $C_{[A_1 \sqcup A_2]_\alpha}(x) = \max(C_{[A_1]_\alpha}(x), C_{[A_2]_\alpha}(x)),\ \forall \alpha \in [0, 1],\ x \in U.$
  - $A_1 \sqcap A_2$ is the fuzzy multiset determined by the weak α-cuts defined as:

    $$[A_1 \sqcap A_2]_\alpha := [A_1]_\alpha \cap [A_2]_\alpha,\ \forall \alpha \in [0, 1]$$

    that is, $C_{[A_1 \sqcap A_2]_\alpha}(x) = \min(C_{[A_1]_\alpha}(x), C_{[A_2]_\alpha}(x)),\ \forall \alpha \in [0, 1]\, x \in U.$
- The inclusion relation is defined as follows:

  $$A_1 \sqsubseteq A_2 \text{ if } [A_1]_\alpha \subseteq [A_2]_\alpha,\ \forall \alpha \in [0, 1]$$

  that is, if $C_{[A_1]_\alpha} \le C_{[A_2]_\alpha},\ \forall \alpha \in [0, 1],\ x \in U.$

We now present an equivalent constructive procedure for computing the union and intersection of pairs of fuzzy multisets, which allows for a direct comparison between the corrected definitions and Miyamoto's original formulation. This revised procedure avoids the need to identify a fuzzy multiset with other multisets that assign different multiplicities to membership 0—an identification that, in the original formulation, depended on the specific fuzzy multiset being compared or combined.

Before presenting the procedure, we will redefine the *length* of the decreasing sequence $\mu_A^1(x) \ge \cdots \ge \mu_A^p(x)$ of memberships associated to an element $x \in U$ as "p", which formally corresponds to the following value:

$$L^*(x; A) = \sum_{\alpha \in [0,1]} C_A(x, \alpha).$$

This definition is valid, since, for every $x \in U$, there is only a finite number of levels $\alpha \in [0, 1]$ with $C_A(x, \alpha) > 0$ by hypothesis. We will start from this definition instead of the definition of $L(A; x)$ proposed in Eq. (1), which counts the number of non-zero elements of the sequence $\mu_A^1(x)$, instead of counting the total number of elements. Next, we will detail the procedure for constructing the union and intersection of two fuzzy multisets A_1 and A_2:

Definition 2 Consider two fuzzy multisets A_1 and A_2 over U. For every $x \in U$, let $k(x), \underline{l}^*(x), \overline{l}^*(x) \in \mathbb{N}$ respectively denote the following quantities: $k(x) = \arg\max_{i=1,2} L^*(x; A_i)$, $\overline{l}^*(x) = \max\{L^*(x; A_1), L^*(x; A_2)\}$ and $\underline{l}^*(x) = \min\{L^*(x; A_1), L^*(x; A_2)\}$.

- The union $A_1 \sqcup A_2$ and the intersection $A_1 \sqcap A_2$ are defined respectively as follows:
  - For all $x \in U$, $L^*(x; A_1 \sqcup A_2) := \overline{l}^*(x)$ and

$$\mu^i_{A_1 \sqcup A_2}(x) := \begin{cases} \max\{\mu^i_{A_1}(x), \mu^i_{A_2}(x)\}, & i = 1, \ldots, \underline{l}^*(x), \\ \mu^i_{A_{k(x)}}(x), & i = \underline{l}^*(x) + 1, \ldots, \overline{l}^*(x). \end{cases}$$

  - For all $x \in U$, $L^*(x; A_1 \sqcap A_2)) := \underline{l}^*(x)$ and

$$\mu^i_{A_1 \sqcap A_2}(x) := \min\{\mu^i_{A_1}(x), \mu^i_{A_2}(x)\},\ i = 1, \ldots, \underline{l}^*(x).$$

- The inclusion relation is defined as follows: $A_1 \sqsubseteq A_2$ if for all $x \in U$,

$$L^*(x; A_1) \le L^*(x; A_2) \text{ and } \mu^i_{A_1}(x) \le \mu^i_{A_2}(x),\ \forall i = 1, \ldots, \underline{l}^*(x).$$

Next we show that indeed both definitions are equivalent.

Theorem 4 *Definitions 1 and 2 are equivalent.*

Proof Consider an arbitrary pair of fuzzy multisets A_1 and A_2. Let us see that the two procedures for constructing the fuzzy multiset $A_1 \sqcup A_2$ described in Definitions 1 and 2 are equivalent to each other. According to the procedure proposed in Definition 2, given $x \in U$, the decreasing sequence of memberships associated with $A_1 \cup A_2$ is given by

$$\mu^i_{A_1 \sqcup A_2}(x) := \begin{cases} \max\{\mu^i_{A_1}(x), \mu^i_{A_2}(x)\}, & i = 1, \ldots, \underline{l}^*(x), \\ \mu^i_{A_k}(x), & i = \underline{l}^*(x) + 1, \ldots, \overline{l}^*(x). \end{cases}$$

The multiplicity function associated with the weak α-cut of the fuzzy multiset $A_1 \cup A_2$ thus constructed is $C_{[A_1 \cup A_2]_\alpha} : U \to \mathbb{N}$ computed as follows:

$$\begin{aligned} & C_{[A_1 \sqcup A_2]_\alpha}(x) \\ & \quad = \#\{i \in \{1, \ldots, \underline{l}^*(x)\} : \max\{\mu^i_{A_1}(x), \mu^i_{A_2}(x)\} \ge \alpha\} \\ & \qquad + \#\{i \in \{\underline{l}^*(x) + 1, \ldots, \overline{l}^*(x)\} : \mu^i_{A_k}(x) \ge \alpha\} \\ & \quad = \#\{i \in \{1, \ldots, \overline{l}^*(x)\} : \mu^i_{A_1}(x) \ge \alpha \text{ or } \mu^i_{A_2}(x) \ge \alpha\} \\ & \quad = \max\left\{\#\{i \in \{1, \ldots, \overline{l}^*(x)\} : \mu^i_{A_1}(x) \ge \alpha\}, \#\{i \in \{1, \ldots, \overline{l}^*(x)\} : \mu^i_{A_2}(x) \ge \alpha\}\right\} \end{aligned}$$

$$= \max\left\{C_{[A_1]_\alpha}(x), C_{[A_2]_\alpha}(x)\right\}.$$

The last term corresponds to the multiplicity of the (crisp) multiset $[A]_\alpha \cup [A_2]_\alpha$. The above equalities are valid for all $\alpha \in [0, 1]$. In this way we have proved that both procedures give rise to the same fuzzy multiset.

Similar proofs as above would serve to show that the procedures proposed in Definitions 1 and 2 for determining the intersection $\sqcap$ or inclusion relation $\sqsubseteq$ are also equivalent. □

There is a third way to introduce union and intersection operations, namely in terms of multiplicities. The result follows straightforwardly from Theorems 1 and 2:

Remark 3.1 Consider two fuzzy multisets A_1 and A_2 over the same universe U and let C_{A_1}, C_{A_2} denote their multiplicity functions. Now let $C_{A_1 \sqcup A_2} : U \times [0, 1] \to \mathbb{N}$ and $C_{A_1 \sqcap A_2} : U \times [0, 1] \to \mathbb{N}$ denote the multiplicity functions associated respectively to their union and intersection, according to Definition 1. Then:

- For $\alpha < 1$,

$$C_{A_1 \sqcup A_2}(x, \alpha) = \max\left\{\sum_{\beta \geq \alpha} C_{A_1}(x, \beta), \sum_{\beta \geq \alpha} C_{A_2}(x, \beta)\right\} - \max\left\{\sum_{\beta > \alpha} C_{A_1}(x, \beta), \sum_{\beta > \alpha} C_{A_2}(x, \beta)\right\}.$$

$$C_{A_1 \sqcap A_2}(x, \alpha) = \min\left\{\sum_{\beta \geq \alpha} C_{A_1}(x, \beta), \sum_{\beta \geq \alpha} C_{A_2}(x, \beta)\right\} - \min\left\{\sum_{\beta > \alpha} C_{A_1}(x, \beta), \sum_{\beta > \alpha} C_{A_2}(x, \beta)\right\}.$$

- For $\alpha = 1$,

$$C_{A_1 \sqcup A_2}(x, 1) = \max\{C_{A_1}(x, 1), C_{A_2}(x, 1)\},$$
$$C_{A_1 \sqcap A_2}(x, 1) = \min\{C_{A_1}(x, 1), C_{A_2}(x, 1)\}.$$

Moreover, $A_1 \sqsubseteq A_2$ whenever $C_{A_1}(x, \alpha) \leq C_{A_2}(x, \alpha)$ for all $x \in U$ and $\alpha \in [0, 1]$.

Example 3 In this example, we will show the difference between the binary operations defined by Miyamoto ($\cup_M$ and $\cap_M$) and the corresponding corrected binary operations ($\sqcup$ and $\sqcap$), as they are considered in Definitions 1 and 2 or in the formulation presented in Remark 3.1 (all three formulations are equivalent). We consider the universe $U = \{x, y\}$ and the fuzzy multisets A and B determined by the multiplicity functions $C_A, C_B : U \times [0, 1] \to \mathbb{N}$ defined as follows:

$$\begin{array}{llll}
C_A(x, 0.2) = 2, & C_A(x, 0.3) = 1, & C_A(x, 0) = 1, & C_A(x, \alpha) = 0 \quad \forall \alpha \notin \{0, 0.2, 0.3\} \\
C_A(y, 0.1) = 1, & C_A(y, 0.2) = 2; & C_A(y, \alpha) = 0 \quad \forall \alpha \notin \{0.1, 0.2\} & \\
C_B(x, 0.3) = 2, & C_B(x, 0.4) = 1, & C_B(x, \alpha) = 0 \quad \forall \alpha \notin \{0, 0.3, 0.4\} & \\
C_B(y, 0.2) = 2, & C_B(y, 0.3) = 1, & C_B(y, 0) = 1, & C_B(y, \alpha) = 0 \quad \forall \alpha \notin \{0, 0.2, 0.3\}
\end{array}$$

The alternative representation of these two fuzzy multisets in terms of decreasing sequences of membership values is as follows:

$$A(x) = (0.3, 0.2, 0.2, 0), \quad A(y) = (0.2, 0.2, 0.1)$$
$$B(x) = (0.4, 0.3, 0.3), \quad B(y) = (0.3, 0.2, 0.2, 0)$$

The union and intersection of these two fuzzy multisets, according to Miyamoto's definitions, are identified by the following sequences:

$$(A \cup_M B)(x) = (0.4, 0.3, 0.3, 0), \quad (A \cup_M B)(y) = (0.3, 0.2, 0.2, 0)$$
$$(A \cap_M B)(x) = (0.3, 0.2, 0.2, 0), \quad (A \cap_M B)(y) = (0.2, 0.2, 0.1, 0)$$

Alternatively, if we use either of the Definitions 1 or 2, or the equivalent formulation presented in Remark 3.1, the fuzzy multiset $A \sqcup B$ coincides with Miyamoto's union fuzzy multiset, but, the fuzzy multiset intersection, $A \sqcap B$, results in shorter sequences of membership values. Specifically:

$$(A \sqcup B)(x) = (0.4, 0.3, 0.3, 0), \quad (A \sqcup B)(y) = (0.3, 0.2, 0.2, 0)$$
$$(A \sqcap B)(x) = (0.3, 0.2, 0.2), \quad (A \sqcap B)(y) = (0.2, 0.2, 0.1)$$

We can easily observe that the union and intersection defined by Miyamoto do not satisfy the absorption laws. Thus, for example we see that $(A \cup_M B) \cap_M A \neq A$. Indeed:

$$[(A \cup_M B) \cap_M A](x) = (0.3, 0.2, 0.2, 0) = A(x), \quad \text{but}$$
$$[(A \cup_M B) \cap_M A](y) = (0.2, 0.2, 0.1, 0) \neq A(y).$$

Although the definitions of union and intersection of crisp multisets (Hickman 1980) do satisfy the absorption laws, and Miyamoto's operations are compatible with the corresponding operations between the weak α-cuts, for all $\alpha \in (0, 1]$, they are not compatible, in general, with the corresponding operations between the weak 0-cuts. Thus, the lack of equality between $(A \cup_M B) \cap_M A$ and A in this example lies in the lack of equality between the weak 0-cut $[(A \cup_M B) \cap_M A]_0$ and the intersection of the weak 0-cuts $[A \cup_M B]_0$ and $[A]_0$. In fact, $C_{[(A\cup_M B)\cap_M A]_0}(y) = 4$ while $\min\{C_{[A\cup_M B]_0}(y), C_{[A]_0}(y)\} = 3$.

It can be easily checked that the operations considered in Definitions 1 and 2 and in the formulation of Remark 3.1 do satisfy the absorption laws, since they are compatible with the corresponding operations between weak α-cuts, for all $\alpha \in [0, 1]$.

With Miyamoto's original formulation, the inclusion relation is not an order relation because it does not satisfy the antisymmetric property, as we have shown in Example 2. With the corrected formulation, it is, as shown in the following result.

Theorem 5 *The inclusion relation* $\sqsubseteq$ *introduced in Definition 1 is a partial order.*

Proof Reflexivity, transitivity and antisymmetry are easily derived from the corresponding properties of the inclusion of (crisp) multisets (Blizard 1989; Hickman 1980) and from the fact that the family of weak α-cuts of a fuzzy multiset (including the weak 0-cut) uniquely determines the fuzzy multiset (see Corollary 3). □

Furthermore, this inclusion order gives the family of fuzzy multisets a lattice structure, as shown below.

Theorem 6 *The family of all fuzzy multisets of* U*, with the inclusion relation introduced in Definition 1, has a lattice structure. The union and intersection operations are the meet and join operators of such a lattice.*

Proof We have already proved that the inclusion relation considered in Definition 1 is a partial order over the family of all fuzzy multiset. We must just prove that the union and intersection of two arbitrary fuzzy multisets A_1 and A_2 are respectively their supremum and infimum with respect to this partial order. Considering how the union is defined in Definition 1, to prove that every pair of elements in the poset has supremum, and it coincides with their union, it suffices to prove that, for every $\alpha \in [0, 1]$, (a) the union of the weak α-cuts $[A_1]_\alpha \cup [A_2]_\alpha$ includes both $[A_1]_\alpha$ and $[A_2]_\alpha$ and that (b) every (crisp) multiset B that simultaneously includes $[A_1]_\alpha$ and $[A_2]_\alpha$ also includes $[A_1]_\alpha \cup [A_2]_\alpha$. Both follow immediately from the fact that the multiplicity of the union of two crisp multisets coincides with the maximum of the multiplicities. Analogously, it can be proved that every pair of elements of the poset of fuzzy multisets has infimum, and that this coincides with its intersection. □

We derive the following corollary from the previous two results.

Corollary 7 *Let A and B be two arbitrary fuzzy multisets. The following three conditions are equivalent:*

1. $A \sqsubseteq B$.
2. $A \sqcap B = A$.
3. $A \sqcup B = B$.

To sum up, the definitions of union and intersection of fuzzy multisets introduced by Miyamoto are partially compatible with those of the corresponding α-cuts. In particular, for any $\alpha \neq 0$, the union and intersection of the weak α-cuts coincide with the α-cut of the union and intersection, respectively. However, Miyamoto's approach does not take the weak 0-cut into account, and effectively identifies all fuzzy multisets whose non-zero cuts coincide, leaving the multiplicity of membership 0 in a kind of state of limbo. As a result, the union or intersection of any pair of fuzzy multisets assigns a 0-cut multiplicity greater than or equal to that of the original multisets, which ultimately causes Miyamoto's operations to violate the absorption laws. The formulation introduced in this paper differs from Miyamoto's definitions precisely in its treatment of the weak 0-cut; with this adjustment, the absorption laws are satisfied, the inclusion relation is a true partial order, and the family of fuzzy multisets is endowed with a lattice structure.

3.2 Option 2: revised formulation of Miyamoto fuzzy multiset models

At the beginning of this section, we mentioned that there are two options for modifying Miyamoto's definitions in order to be formally correct and to yield a lattice structure over the family of fuzzy multisets. Option 1 has been discussed in the previous section. Option 2 consists of excluding occurrences (whether repeated or not) of the membership value 0. In other words, it amounts to identify a fuzzy multiset C^* with a mapping $C^* : U \times (0, 1] \to \mathbb{N}$, and consider the same operations introduced in Definition 1, except for the case $\alpha = 0$ (i.e. without referring to the weak 0-cut). This approach does not allow for a semantic distinction between assigning a membership of value 0 and the absence of an assignment.

Option 2 is semantically and formally equivalent to considering Option 1 multiplicity functions $C : U \times [0, 1] \to \mathbb{N}$ with the additional constraint $C(u, 0) = 0,\ \forall u \in U$, while keeping the operations introduced in Sect. 3.1 unchanged. However, even under this restricted scenario, Miyamoto's original operations require formal corrections following the approach described in Sect. 3.1. Specifically, in the case of the intersection, instead of extending the shorter sequence by adding zeros until reaching the size of the longer sequence, one must retain the minimum of the two lengths. It is straightforward to verify that, with this correction, the resulting family of fuzzy multisets—equipped with the inclusion relation and the corrected intersection and union operations—has also the structure of a distributive lattice.

From an algebraic point of view, the following is a formal, but elementary, summary of the relations between Option 1 and Option 2 notions and structures of fuzzy multisets.

Let us identify the set of fuzzy multisets according to Option 1 with the set of functions $\mathrm{FMS}(U) = \mathbb{N}^{C\times[0,1]}$, and the set of fuzzy multisets according to Option 2 with the set of functions $\mathrm{FMS}^*(U) = \mathbb{N}^{C\times(0,1]}$. On the other hand, consider the subset of $\mathrm{FMS}(U)$ defined as $\mathrm{FMS}_0(U) = \{C \in \mathrm{FMS}(U) : C(x, 0) = 0, \forall x \in U\}$.

As we have seen in the previous section $\mathcal{FMS}(U) = (\mathrm{FMS}(U), \sqcap, \sqcup)$ is a (distributive) lattice. Since $\sqcap$ and $\sqcap$ are closed on $\mathrm{FMS}_0(U)$, then $\mathcal{FMS}_0(U) = (\mathrm{FMS}_0(U), \sqcap, \sqcup)$ is a sublattice of $\mathcal{FMS}(U)$.

On $\mathrm{FMS}^*(U)$ let us slightly adapt Definition 1 of $\sqcup$ and $\sqcap$ for $\mathrm{FMS}(U)$ in terms of weak α-cuts:

- $A_1 \sqcup^* A_2$ is determined by the α-cuts as: $[A_1 \sqcup^* A_2]_\alpha := [A_1]_\alpha \cup [A_2]_\alpha,\ \forall \alpha \in (0, 1]$
- $A_1 \sqcap^* A_2$ is determined by the α-cuts as: $[A_1 \sqcap^* A_2]_\alpha := [A_1]_\alpha \cap [A_2]_\alpha,\ \forall \alpha \in (0, 1]$
- The inclusion relation is defined as: $A_1 \sqsubseteq^* A_2$ if $[A_1]_\alpha \subseteq [A_2]_\alpha,\ \forall \alpha \in (0, 1]$

It is easy to check that $\mathcal{FMS}^*(U) = (FMS^*(U), ...)$ is a lattice as well. Now, let us define a mapping $h : \mathrm{FMS}^*(U) \to \mathrm{FMS}_0(U)$ as follows. Let $C \in \mathrm{FMS}^*(U)$, and let $h(C) : U \times [0, 1] \to \mathbb{N}$ defined as follows: $h(C)(x, \alpha) = C(x, \alpha)$ for all $\alpha > 0$, and $h(C)(x, 0) = 0$. It is clear that h is bijective and indeed it is a lattice morphism, that is, $h(A_1 \sqcup^* A_2) = h(A_1) \sqcup h(A_2)$ and $h(A_1 \sqcap^* A_2) = h(A_1) \sqcap h(A_2)$. The obvious reason is that, if $C \in \mathrm{FMS}_0(U)$, then $[C]_0 = (C)_0$. Therefore, $\mathcal{FMS}^*(U)$ is isomorphic, as a lattice, to $\mathcal{FMS}_0(U)$.

3.3 Fuzzy multisets represented by multidimensional membership functions: revisiting the two options

In this section, we reformulate fuzzy multisets and the basic binary operations and the inclusion relation considered in Sect. 3.1 in terms of multidimensional memberships. As indicated in Sect. 2.2, Miyamoto used an alternative representation for fuzzy multisets in terms of multidimensional membership functions. Thus, given a fuzzy multiset A, we can uniquely represent it via a function $\mu_A : U \to \mathcal{M}_\infty$, where $\mathcal{M}_\infty = \bigcup_{p=1}^{\infty} M_p$ and

$$M_p = \Big\{(a_1, \ldots, a_p) \in [0, 1]^p : a_1 \geq a_2 \geq \cdots \geq a_p\Big\}.$$

Alternatively we can consider the set $\mathcal{M}^*_\infty = \bigcup_{p=1}^{\infty} M^*_p$ with

$$M^*_p = \Big\{(a_1, \ldots, a_p) \in (0, 1]^p : a_1 \geq a_2 \geq \cdots \geq a_p\Big\}.$$

Let

$$\vec{a} = (a_1, \ldots, a_p) \in M_p \quad \text{and} \quad \vec{b} = (b_1, \ldots, b_q) \in M_q$$

be two vectors in $\mathcal{M}_\infty$.

Next we introduce a meet and a join operations and an order relation on $\mathcal{M}_\infty$

Meet operator: Define the meet $\vec{a} \wedge \vec{b}$ as the vector of length

$$r = \min\{p, q\},$$

with components given by

$$\vec{a} \wedge \vec{b} = \Big(\min(a_1, b_1),\ \min(a_2, b_2),\ \ldots,\ \min(a_r, b_r) \Big).$$

Join operator: Let

$$r = \min\{p, q\} \quad \text{and} \quad s = \max\{p, q\}.$$

Then, the join $\vec{a} \vee \vec{b}$ is defined as the vector of length s by

$$(\vec{a} \vee \vec{b})_i = \begin{cases} \max(a_i, b_i), & \text{for } 1 \le i \le r, \\ \text{if } p \le q: \quad (\vec{a} \vee \vec{b})_i = b_i, & \text{for } r < i \le s, \\ \text{if } p > q: \quad (\vec{a} \vee \vec{b})_i = a_i, & \text{for } r < i \le s. \end{cases}$$

Order relation associated with the meet and join operators: For any two vectors

$$\vec{a} = (a_1, \ldots, a_p) \in M_p \quad \text{and} \quad \vec{b} = (b_1, \ldots, b_q) \in M_q,$$

we define the binary relation $\le$ by

$$\vec{a} \le \vec{b} \quad \Longleftrightarrow \quad p \le q \text{ and } a_i \le b_i \text{ for all } i = 1, \ldots, p$$

which turns to be a partial order. Equivalently, this order can be characterized in terms of the meet and join operators as follows:

$$\vec{a} \le \vec{b} \quad \Longleftrightarrow \quad \vec{a} \wedge \vec{b} = \vec{a} \quad \Longleftrightarrow \quad \vec{a} \vee \vec{b} = \vec{b}.$$

With these operations, $(\mathcal{M}, \le)$ has a lattice structure whose meet and join operations are the ones defined above. Moreover, since the meet and join operations are closed on $\mathcal{M}^*$, $(\mathcal{M}^*, \le)$ is a sublattice of $(\mathcal{M}, \le)$.

Both the union and intersection operations, as well as the inclusion relation introduced in Sect. 3.1, can also be expressed in terms of the meet, join, and order relation as follows. Given two fuzzy multisets A and B represented respectively by the multidimensional membership functions $\mu_A : U \to \mathcal{M}$ and $\mu_B : U \to \mathcal{M}$, the membership functions associated with the fuzzy multisets $A \cap B$ and $A \cup B$ are given, respectively, by the following formulas:

$$\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x), \quad \forall x \in U.$$
$$\mu_{A \cup B}(x) = \mu_A(x) \vee \mu_B(x), \quad \forall x \in U.$$

Moreover, $A \subseteq B$ if and only if

$$\mu_A(x) \le \mu_B(x), \quad \forall x \in U.$$

Formally, the difference between Option 1 and Option 2, as discussed in Sects. 3.1 and 3.2, lies in the fact that Option 1 considers the family of all multisets (i.e., membership functions with images in $\mathcal{M}$), while Option 2 considers the subfamily of multisets corresponding to membership functions with images in $\mathcal{M}^*$. The operations and order relations remain the same. Both families of fuzzy multisets—the global family associated with $\mathcal{M}$ and the subfamily associated with $\mathcal{M}^*$—inherit a lattice structure from $\mathcal{M}$ and $\mathcal{M}^*$ respectively. Therefore, one can look at fuzzy multisets over a universe U as a particular class of lattice-valued fuzzy sets (L-fuzzy sets), i.e. fuzzy sets over U taking values in the lattice $\mathcal{M}$ (Option 1) or $\mathcal{M}^*$ (Option 2).

4 Connections between Yager's and Miyamoto's definitions

The definitions of union, intersection, and inclusion originally introduced by Yager differ from both Miyamoto's later definitions and our corrected versions (Option 1, see Sect. 3.1). In what follows, we demonstrate the formal relationships between these operations and prove that our union and intersection yield more precise bounds for the combined fuzzy multisets than Yager's. In consonance with this, our inclusion order is a refinement of Yager's; specifically, if a fuzzy multiset A is included in B in Yager's sense, then it is also included according to our definition.

Example 4 Consider the universe $U = \{u\}$ (a singleton). Consider the multisets A and B respectively determined by the multiplicity functions $C_A, C_B : U \times [0,1] \to \mathbb{N}$ defined as follows:

$$C_A(u, 0.2) = 1, C_A(u, 0.3) = 1, C_A(u, \alpha) = 0, \ \forall \alpha \notin \{0.2, 0.3\},$$
$$C_B(u, 0.3) = 1, C_B(u, 0.5) = 2, C_B(u, \alpha) = 0, \ \forall \alpha \notin \{0.4, 0.5\}.$$

Equivalently, in terms of Miyamoto's decreasing sequences, $\mu_A(u) = (0.3, 0.2)$ and $\mu_B(u) = (0.5, 0.5, 0.3)$. We can observe that $A \subseteq B$ and yet $A \nsubseteq_Y B$. Indeed, the weak α-cuts of these two fuzzy multisets are given by the following expressions:

$$C_{[A]_\alpha}(u) = \begin{cases} 2, & \text{if } \alpha \leq 0.2, \\ 1, & \text{if } 0.2 < \alpha \leq 0.3, \\ 0, & \text{else.} \end{cases}$$

$$C_{[B]_\alpha}(u) = \begin{cases} 3, & \text{if } \alpha \leq 0.3, \\ 2, & \text{if } 0.3 < \alpha \leq 0.5, \\ 0, & \text{else.} \end{cases}$$

We observe that $[A]_\alpha \subseteq [B]_\alpha, \ \forall \alpha \in [0,1]$, and therefore $A \sqsubseteq B$ in accordance with Definition 1. However, $A \nsubseteq_Y B$. In fact $C_A(u, 0.2) = 1$ and $C_B(u, 0.2) = 0$, and therefore $C_A(u, \alpha) \nleq C_B(u, \alpha), \ \forall \alpha \in (0, 1]$. In fact, $A \cup_Y B$ does not coincide with B, but gives rise to the decreasing sequence of memberships $\mu_{A \cup_Y B}(u) = (0.5, 0.5, 0.3, 0.2)$, which is strictly longer than the sequence associated with $\mu_B(u) = (0.5, 0.5, 0.2)$. To determine the union of two fuzzy multisets, according to Yager's definition, we consider, for each α the highest number of repeats appearing in the two combined sequences, which allows the sequences associated with the union to be strictly longer than either of the two combined sequences. This does not occur in the case of Definition 1.

The order introduced in Definition 2 is a refinement of Yager's order. In other words, any two sequences that are comparable under Yager's order are also comparable under Miyamoto's (corrected) order; however, the converse does not hold. Moreover, Miyamoto's (corrected) intersection and union operations yield bounds that are more precise than those obtained through Yager's operations.

Theorem 8 *Let $\sqcup$ and $\sqcap$ respectively denote the union and the intersection considered in Definitions 1 and 2. The following relations hold for every $\alpha \in [0,1]$, and every $x \in U$:*

(a) $C_{A \sqcup B}(x, \alpha) \leq C_{A \cup_Y B}(x, \alpha)$
(b) $C_{A \sqcap B}(x, \alpha) \geq C_{A \cap_Y B}(x, \alpha)$

Proof We prove both inequalities.

(a) According to Theorem 1, the multiplicity of the of Yager's union can be expressed alternatively:

$$C_{A\cup_Y B}(x,\alpha)=\max\left\{\sum_{\beta\geq\alpha}C_A(x,\beta)-\sum_{\beta>\alpha}C_A(x,\beta),\sum_{\beta\geq\alpha}C_B(x,\beta)-\sum_{\beta>\alpha}C_B(x,\beta)\right\}.$$

According to Remark 3.1, the α-cut based union is characterized by the following multiplicity function:

$$C_{A\sqcup B}(x,\alpha)=\max\left\{\sum_{\beta\geq\alpha}C_A(x,\beta),\sum_{\beta\geq\alpha}C_B(x,\beta)\right\}$$
$$-\max\left\{\sum_{\beta>\alpha}C_A(x,\beta),\sum_{\beta>\alpha}C_B(x,\beta)\right\}.$$

For simplicity, let us use the notation $a=\sum_{\beta\geq\alpha}C_A(x,\beta)$, $b=\sum_{\beta\geq\alpha}C_B(x,\beta)$, $a^-=\sum_{\beta>\alpha}C_A(x,\beta)$ and $b^-=\sum_{\beta>\alpha}C_B(x,\beta)$. To prove this results, it suffices to notice that:

$$a-a^-\geq a-\max\{a^-,b^-\}\text{ and }b-a^-\geq b-\max\{a^-,b^-\}.$$

Thus we can deduce that

$$\max\{a-a^-,b-b^-\}\geq\max\{a-\max\{a^-,b^-\},b-\max\{a^-,b^-\}\}$$
$$=\max\{a,b\}-\max\{a^-,b^-\}.$$

(b) The proof is analogous to the proof in item (a). We just have to swap max and min and reverse the order of the inequalities.

□

Theorem 9 $A\subseteq_Y B\Rightarrow A\sqsubseteq B$.

Proof Considering how Yager defines the union operation and the inclusion order, it immediately follows that $A\subseteq_Y B$ if and only if $A\cup_Y B=B$. Thus, if $A\subseteq_Y B$, then $A\cup_Y B=B$, and therefore, according to part (a) of Theorem 8, we have $C_{A\sqcup B}\leq C_{A\cup_Y B}=C_B$. Hence, for all $\alpha\in[0,1]$, $C_{[A\sqcup B]_\alpha}\leq C_{[B]_\alpha}$, and thus $[A\sqcup B]_\alpha=[B]_\alpha$, which means $A\sqcup B=B$, and by Corollary 7 this is equivalent to $A\sqsubseteq B$. □

Remark 4.1 As we have shown in Example 4, the reciprocal of Theorem 9 is not satisfied. In that example, $A\sqsubseteq B$ but $A\not\subseteq_Y B$.

Corollary 10 *Let* $\cup$ *and* $\cap$ *respectively denote the union and the intersection considered in Definitions 1 and 2. The following conditions hold:*

(a) $A\sqcup B\subseteq_Y A\cup_Y B$
(b) $A\sqcup B\sqsubseteq A\cup_Y B$
(c) $A\sqcap B\supseteq_Y A\cap_Y B$
(d) $A\sqcap B\sqsupseteq A\cap_Y B$.

Proof The results in (a) and (c) are a direct consequence of Theorem 8. The results in (b) and (d) follow, respectively, from (a) and (c), after applying the result of Theorem 9. □

5 Bounded fuzzy multisets

In this section, we consider the notions of difference and complementation for fuzzy multisets. However, before proceeding, we briefly recall existing results in the literature.

Yager (1986) does not address any of these operations in his original paper. Miyamoto, on the other hand, explicitly refers to both operations in several publications. Chronologically, Miyamoto (2000) proposes a fuzzy multiset difference compatible with Yager's operations. In this context, Miyamoto argues that defining complementation is not meaningful because the family of fuzzy multisets is *open* in the sense that it lacks a *top* element with respect to Yager's inclusion order. Subsequently, Miyamoto (2004a) points out that even the definition of complement for crisp multisets presents challenges. As a last-resort solution, he suggests extending the set of possible values for multiplicity functions to include the value ∞. According to this definition, the complement of a crisp multiset assigns multiplicities of either 0 or ∞ to elements of the universe as follows:

$$C_{A^c}(x) = \begin{cases} 0 & \text{if } C_A(x) > 0, \\ \infty & \text{if } C_A(x) = 0. \end{cases} \tag{4}$$

Clearly, as defined, this operation neither strictly reverses the inclusion order nor is self-inverse. In that paper, Miyamoto does not explicitly discuss the complementation of fuzzy multisets.

Later, Miyamoto (2004b) revisits the concept and proposes a new definition for the difference between two fuzzy multisets. However, this definition is incompatible with his earlier definitions of union and intersection as well as with those previously introduced by Yager. Specifically, Miyamoto defines a nonstandard difference between two fuzzy multisets in terms of decreasing membership sequences as follows:

$$\mu^j_{A-B}(x) = \begin{cases} \mu^j_A(x), & \text{if } \mu^1_B(x) = 0, \\ 0, & \text{if } \mu^1_B(x) > 0, \end{cases} \quad j = 1, \ldots, L(x; A, B). \tag{5}$$

Again, Miyamoto refrains from defining the complement of a fuzzy multiset, attributing this decision to the absence of a *top* element in the family of fuzzy multisets concerning the inclusion order.

Subsequently, Riesgo et al. (2018) consider the following definition of complementation for a fuzzy multiset. Given a fuzzy multiset A, with multiplicity function $C_A : U \times [0, 1] \to \mathbb{N}$, the complement A^c, is the fuzzy multiset associated with the multiplicity function $C_{A^c} : U \times [0, 1] \to \mathbb{N}$ defined by

$$C_{A^c}(x, \alpha) = C_A(x, 1 - \alpha), \quad \forall x \in U, \alpha \in [0, 1]. \tag{6}$$

Below we show an example in which the complement thus defined does not reverse the order of inclusion.

Example 5 Consider the universe $U = \{x, y\}$, and two fuzzy multisets A and B defined by the following memberships:

$$A(x) = (0.7, 0.2), \; A(y) = (0.8, 0.2), \; B(x) = (0.7, 0.2, 0), \; B(y) = (0.8, 0.2, 0.2).$$

It is easy to observe that A is strictly included in B according to Miyamoto's original definition (and also according to the inclusion definition introduced in Definition 1). According to the definition of complement recalled above, we have:

$$A^c(x) = (0.8, 0.3),\ A^c(y) = (0.8, 0.2),\ B^c(x) = (1, 0.8, 0.3),\ B^c(y) = (0.8, 0.8, 0.2).$$

Therefore, A^c is also strictly included in B^c. We thus have an example where $A \subsetneq B$ and $A^c \subsetneq B^c$.

The problem with this definition is twofold. On the one hand, we must restrict ourselves to the case where the multiplicity of membership 0 is zero, i.e. $C(x, 0),\ \forall x \in U$. On the other side, we need to assume the existence of a "top" element in the poset of fuzzy multisets. This top element will be a crisp multiset that assigns to each $x \in U$ element a multiplicity greater than the multiplicity assigned by any of the (fuzzy) multisets considered in a particular situation.

Let us fix a (crisp) multiset T with multiplicity function $C_T : U \to \mathbb{N}$ that we will call *top* multiset. Then, a fuzzy multiset A satisfying the following restrictions:

$$C_A(x, 0) = 0, \quad \forall x \in U, \quad (\text{i.e., } C_{[A]_0} = C_{(A)_0}) \tag{7}$$

$$C_{(A)_0}(x) = \sum_{\alpha \in (0,1]} C_A(x, \alpha) \le C_T(x), \quad \forall x \in U \tag{8}$$

will be called a T-*bounded* fuzzy multiset.?

If we do not assume the restriction in Eq. (7), situations like the one in Example 5, involving the multivalued memberships $A(x)$ and $B(x)$, may arise, where they differ in the number of times the membership value 0 appears. This leads to the complements $A^c(x)$ and $B^c(x)$ being different from each other.

On the other hand, if we do not assume the restriction in Eq. (8), and do not define the complement with respect to a top multiset, we may encounter two multivalued memberships $A(y)$ and $B(y)$ of different lengths, where all membership values lie strictly within the open interval (0, 1). In this case, under the previous definition of complement, it may happen that $L(y; A) < L(y; B)$, and also $L(y; A^c) < L(y; B^c)$, so the inclusion order is not reversed either.

In words, when we restrict ourselves to T-bounded fuzzy multisets, we actually restrict ourselves to the family of fuzzy multisets included in T according to the inclusion defined in Sect. 3.1, and we consider Option 2 mentioned in Sect. 3.2, i.e., we assume that the multiplicity of membership 0 is zero for every element in the universe. In this context, the complement of a T-bounded fuzzy multiset with respect to T can be defined as follows.

Definition 3 Consider an arbitrary crisp multiset T over a universe U with multiplicity function $C_T : U \to \mathbb{N}$. Given a T-bounded fuzzy multiset A, with multiplicity function $C_A : U \times [0, 1] \to \mathbb{N}$, we define the T-complement of A, A^c as the following T-bounded fuzzy multiset:

$$C_{A^c}(x, \alpha) = \begin{cases} C_A(x, 1 - \alpha), & \text{if } \alpha \in (0, 1) \\ C_T(x) - C_{(A)_0}(x), & \text{if } \alpha = 1. \end{cases} \tag{9}$$

In the following theorem, we will show that the above definition of T-complement is equivalent to the following formulation written in terms of α-cuts: for each $x \in U$,

$$C_{[A^c]_\alpha}(x) = C_T(x) - C_{(A)_{1-\alpha}}(x),\ \forall \alpha \in (0, 1]. \tag{10}$$

Theorem 11 *Consider a (crisp) multiset T on U with multiplicity function $C_T : U \to \mathbb{N}$. Consider the family of T-bounded fuzzy multisets (i.e., the family of fuzzy multisets satisfying conditions* (7) *and* (8)*). Then Eqs.* (10) *and* (9) *lead to the same definition of T-complement.*

Proof Consider an arbitrary T-bounded fuzzy multiset A. Consider the fuzzy multiset A^c defined from A according to Eq. (9). Select an arbitrary $\alpha \in (0, 1]$. We will prove that the multiplicity of the weak α-cut of A^c is determined by Eq. (10).

In fact, according to the definition of weak α-cut of a fuzzy multiset, we have:

$$C_{[A^c]_\alpha}(x) = \sum_{\beta \geq \alpha} C_{A^c}(x, \alpha), \ \forall x \in U.$$

Now according to Eq. (9), we have

$$\sum_{\beta \geq \alpha} C_{A^c}(x, \beta) = \sum_{\alpha \leq \beta < 1} C_A(x, 1-\beta) + C_T(x) - C_{(A)_0}(x), \ \forall x \in U.$$

Now, by changing the variable $\gamma = 1 - \alpha$, we can alternatively write the last equality as follows:

$$\begin{aligned}
\sum_{\beta \geq \alpha} C_{A^c}(x, \alpha) &= \sum_{0 < \gamma \leq 1-\alpha} C_A(x, \gamma) + C_T(x) - C_{(A)_0}(x) \\
&= C_T(x) - \left[C_{(A)_0}(x) - \sum_{0 < \gamma \leq 1-\alpha} C_A(x, \gamma) \right] \\
&= C_T(x) - \sum_{\gamma > 1-\alpha} C_A(x, \gamma) \\
&= C_T(x) - C_{(A)_\alpha}(x), \ \forall x \in U.
\end{aligned}$$

We conclude that $C_{[A^c]_\alpha}(x) = C_T(x) - C_{(A)_\alpha}(x)$ for every $x \in U$. □

Next, we will consider an alternative (equivalent) representation of the complement of a fuzzy multiset, in terms of multi-valued membership functions.

To do this, we will first refer to an involution defined over the set $\mathcal{M}^*_m = \bigcup_{p \leq m} M^*_p$, where

$$M^*_p = \{(a_1, \ldots, a_p) \in (0, 1]^p : a_1 \geq a_2 \geq \cdots \geq a_p\}.$$

In Sect. 3.3, we derived the intersection and union operations from the corresponding meet and join operations defined on the set $\mathcal{M} = \bigcup_{p=1}^{\infty} M_p$. We now proceed analogously for the unary operation of complement. Given an arbitrary but fixed natural number $m \in \mathbb{N}^*$, we define an m-involution on the set $\mathcal{M}^*_m$.

We define on this set a unary operation $\neg_m : \mathcal{M}^*_m \to \mathcal{M}^*_m$ as follows. Given $\vec{a} \in \mathcal{M}^*_m$, there exists $p \leq m$ such that $\vec{a} \in M^*_p$. That is, $\vec{a} = (a_1, \ldots, a_p)$ with $a_1 \geq \cdots \geq a_p$. Let us denote

$$p'(\vec{a}) = \max\{i \in \{1, \ldots, p\} \mid a_i = 1\}.$$

Clearly, $0 \leq p'(\vec{a}) \leq p \leq m$, hence $m - p'(\vec{a}) \leq m$. We define $\neg_m(\vec{a})$ as the vector of length $m - p'(\vec{a})$ constructed as follows:

$$\neg_m(\vec{a}) = (\underbrace{1, \ldots, 1}_{m-p}, 1 - a_p, \ldots, 1 - a_{p'(\vec{a})}).$$

It can be verified that $\neg_m : \mathcal{M}^*_m \to \mathcal{M}^*_m$ is self-inverse and order-reversing with respect to the relation $\leq$ introduced in Sect. 3.3 as we prove in Theorem 12.

Remark 5.1 When the function $\neg_m$ is extended from $\mathcal{M}^*_m$ (finite decreasing sequences in $(0, 1]$) to $\mathcal{M}_m$ (finite decreasing sequences in $[0, 1]$), it fails to be self-inverse because it is no longer one-to-one. In other words, there exist distinct sequences in $\mathcal{M}_m$ whose images under $\neg_m$ coincide.

For example, consider the case $m = 3$ and the sequences

$$\vec{a} = (1, 0) \quad \text{and} \quad \vec{b} = (1).$$

Both $\vec{a}$ and $\vec{b}$ belong to $\mathcal{M}_3$ since their entries lie in [0, 1] and they are decreasing.

For $\vec{a} = (1, 0)$, we have:

$$p(\vec{a}) = 2, \quad p'(\vec{a}) = \max\{\, i \in \{1, 2\} \mid a_i = 1\,\} = 1.$$

Thus,

$$\neg_3(\vec{a}) = \Big(\underbrace{1}_{3-2},\ 1 - a_2,\ 1 - a_1\Big) = \big(1,\ 1 - 0,\ 1 - 1\big) = (1, 1, 0)$$

For $\vec{b} = (1)$, we have:

$$p(\vec{b}) = 1, \quad p'(\vec{b}) = 1.$$

Thus,

$$\neg_3(\vec{b}) = \Big(\underbrace{1, 1}_{3-1},\ 1 - a_1\Big) = \big(1, 1,\ 1 - 1\big) = (1, 1, 0).$$

Even though $\vec{a} \neq \vec{b}$, we see that

$$\neg_3(\vec{a}) = \neg_3(\vec{b}) = (1, 1, 0).$$

This example illustrates that the extended function $\neg_m : \mathcal{M}_m \to \mathcal{M}_m$ is not injective, and hence it cannot be self-inverse.

Theorem 12 *The operation $\neg_m : \mathcal{M}_m^* \to \mathcal{M}_m^*$ is an involution, i.e. (i.e. self-inverse). Furthermore it reverses the order $\leq$ considered in Sect. 3.3.*

Proof We prove that $\neg_m$ is self-inverse and revers the order.

- **Self-inversion:** Let $\vec{a} = (a_1, \ldots, a_p) \in M_p^*$, where $p \leq m$, and denote by

  $$p'(\vec{a}) = \max\{i \in \{1, \ldots, p\} \mid a_i = 1\}$$

  the number of positions in $\vec{a}$ with value 1. By definition, $\neg_m(\vec{a})$ is a vector of length

  $$m - p'(\vec{a}) = (m - p) + (p - p'(\vec{a})),$$

  constructed by concatenating:

  1. A block of $m - p$ copies of 1, corresponding to the $m - p$ unoccupied positions, and
  2. A block of $p - p'(\vec{a})$ components given by the duals of the non-unity entries of $\vec{a}$, namely

     $$(1 - a_p, 1 - a_{p-1}, \ldots, 1 - a_{p'+1}),$$

     which are arranged in reverse order to restore the nonincreasing condition.

  When applying $\neg_m$ a second time, the block of 1's will now represent the unoccupied positions of the new vector, while the duals of the entries $1 - a_i$ (with $a_i < 1$) will yield $1 - (1 - a_i) = a_i$, and the reversal of the order re-establishes the original order of the non-unity components. Hence,

  $$\neg_m(\neg_m \vec{a}) = \vec{a}.$$

- **Order-reversal:** Assume $\vec{a} \leq \vec{b}$, where $\vec{a} \in M_p^*$ and $\vec{b} \in M_q^*$ with $p \leq q$, and such that $a_i \leq b_i$ for all $i = 1, \ldots, p$. Notice that if $a_i = 1$ then, since $a_i \leq b_i$, we must have $b_i = 1$, so the number of ones in $\vec{a}$, $p'(\vec{a})$, is less than or equal to that in $\vec{b}$, $p'(\vec{b})$. Consequently, the length of $\neg_m(\vec{a})$ is
$$m - p'(\vec{a}) \geq m - p'(\vec{b}),$$
which is the length of $\neg_m(\vec{b})$. Moreover, for the positions corresponding to non-unity entries, since $a_i \leq b_i$ it follows that $1 - a_i \geq 1 - b_i$. Together, these facts imply that
$$\neg_m(\vec{b}) \leq \neg_m(\vec{a}),$$
demonstrating that $\neg_m$ reverses the order.

□

It also follows that this new unary operator $\neg_m$, together with the meet and join operators defined in Sect. 3.3, induces a De Morgan algebra structure on the set $\mathcal{M}_m^*$.

Having established this, we return to the family of T-bounded fuzzy multisets and show that the T-complement defined in this section can alternatively be expressed in terms of the involution just introduced. Consider the top crisp multiset T, determined by the multiplicity function $C_T : U \to \mathbb{N}$. Consider the family of T-bounded multisets. Then the complement A^c of any T-bounded fuzzy multiset A is determined by the membership function μ_{A^c} constructed from μ_A as follows:
$$\mu_{A^c}(x) = \neg_{C_T(x)}\mu_A(x), \quad \forall x \in U.$$

From the previous results, it follows that the family of T-bounded fuzzy multisets, together with the operations of intersection, union, and complement, has the structure of a De Morgan algebra. We can also easily observe that the definition of T-complement simultaneously generalizes the notion of complement for fuzzy sets (considered as fuzzy multisets where $C(x) = 1,\ \forall x \in U$) and the notion of complement for (crisp) multisets given by Hickman (1980) as $C_{A^c}(x) = C_T(x) - C_A(x),\ \forall x \in U$.

6 Connections with the notions of *n*-dimensional and multidimensional fuzzy set

6.1 Connection with *n*-dimensional fuzzy sets

As we have mentioned in Sect. 2.2, Miyamoto identifies fuzzy multisets with memberships with multidimensional values. Formally, as discussed in Sect. 3.3, any fuzzy multiset can be represented by means of a mapping (membership function) $\mu_A : U \to \mathcal{M}$.

Let us now set $n \in \mathbb{N}$ and consider the top multiset T characterized by the constant multiplicity function $C_T : U \to \mathbb{N}$ defined as $C_T(x) = n,\ \forall x \in U$, and consider the family of T-bounded (or *n-bounded)* fuzzy multisets. Any element of this family is characterized by a membership function μ_A with values in $\mathcal{M}_n^* = \bigcup_{p \leq n} M_p^*$. Let us now consider the bijection $B_n : \mathcal{M}_n^* \to L_n$, where $L_n = \{(a_1, \ldots, a_n) \in [0, 1]^n : a_1 \leq \cdots \leq a_n\}$, defined as follows:
$$B_n(a_1, \ldots, a_p) = (0, \overset{n-p}{\ldots}, 0, a_p, \ldots, a_1),\ \forall (a_1, \ldots, a_p) \in M_p^*, \forall p = 1, \ldots, n.$$

Clearly, it is an injective and surjective function. Indeed, any vector $(a_1, \ldots, a_n) \in L_n$ is uniquely determined by the number of non-zero components $p^* = \#\{i \in \{1, \ldots, n\} :$

$a_i > 0\}$ and the p^*-dimensional vector composed by them arranged in ascending order $(a_n, \ldots, a_{n-p^*+1})$. Thus, the anti-image under B_n of any vector $(a_1, \ldots, a_n) \in L_n$ exists and is unique.

On the other hand, it is straightforward to verify that the meet and join operations over $\cup_{p\in\mathbb{N}} M^*_p$, considered in Sect. 3.3 when restricted to $\mathcal{M}^*_n$ are respectively mapped into meet and join operations over L_n according to the formulation recalled in Sect. 2.3. Finally, in the case we are concerned with—the n-bounded multisets—the involution on $\mathcal{M}^*_n$ defined in Sect. 5 is also mapped into the involution on L_n.

Therefore, B_n is an isomorphism between the two De Morgan algebras over $\mathcal{M}^*_n$ and L_n, and thus it also induces an isomorphism between the De Morgan algebras respectively associated with the classes of n-bounded fuzzy multisets and n-dimensional fuzzy sets over a given universe. In other words, both classes are formally equivalent constructions. Bedregal et al. (2012) had already pointed out that n-dimensional fuzzy sets could be seen as particular cases of fuzzy multisets.

6.2 Connection with multidimensional fuzzy sets

We consider again the whole family of all fuzzy multisets over a universe U, rather than restricting ourselves to the set of n-bounded fuzzy multisets. As we have mentioned, every fuzzy multiset A over U can be represented by a mapping $\mu_A : U \to \mathcal{M}_\infty$, where $\mathcal{M}_\infty = \bigcup_{p\in\mathbb{N}^*} M_p$ represents the set of tuples of an arbitrary length, where some components may be null. In this broader context compared to Sect. 6.1, we can refer to another mapping B from $\mathcal{M}$ to $\mathcal{L}_\infty = \bigcup_{n\in\mathbb{N}^*} L_n$ defined as follows:

$$B(a_1, \ldots, a_p) = (a_p, \ldots, a_1), \quad \forall (a_1, \ldots, a_p) \in \mathcal{M}_\infty.$$

Using reasoning similar to Sect. 6.1, we can observe that this is indeed a bijection.

In Sect. 3.3, we proved that $(\mathcal{M}_\infty, \wedge, \vee, \leq)$ has the structure of a lattice. In this way, the bijection B can be used to induce a lattice structure on the set $\mathcal{L}_\infty$. The resulting (partial) order on $\mathcal{L}_\infty$ is defined as follows:

$$x \leq y \text{ if } \begin{cases} |x| = |y| = n & \text{and } x \leq_n y, \\ k = |x| < |y| = n & \text{and } x \leq_k (y_{n-k+1}, \ldots, y_n). \end{cases} \tag{11}$$

In a more compact form:

$$x \leq y \text{ if } m = |x| \leq |y| = n \text{ and } (0, \overset{n-m}{\ldots}, 0, x_1, \ldots, x_m) \leq_n (y_1, \ldots, y_n), \tag{12}$$

i.e., where the vector $(0, \ldots, 0, x_1, \ldots, x_m)$ has $n - m$ leading zeros.

On the other hand, and independently of our approach, as recalled in the Preliminaries section, in the context of the study of so-called “multidimensional fuzzy sets”, Lima et al. (2021) defined the following family of partial orders on the set $\mathcal{L}$:

$$x \leq^\infty_{\alpha,F} y \iff \begin{cases} F(x) < F(y), \\ F(x) = F(y) \leq \alpha \text{ and } |y| < |x|, \\ F(x) = F(y) > \alpha \text{ and } |x| < |y|, \\ F(x) = F(y),\ |x| = |y| = n \text{ and } x \leq_n y \end{cases} \tag{13}$$

Each order in this family depends on a specific choice of a threshold α and a function $F : \mathcal{L}_\infty([0, 1]) \to [0, 1]$.

We observe that the intuition behind the order defined in Eq. (12) is fundamentally different from that underlying the family of orders proposed by Lima et al. Notably, the orders defined by Lima et al. allow situations where $x \leq y$ even when $|x| > |y|$, whereas this is not permitted by the order in Eq. (12). In the following, we show that the order in Eq. (12) cannot, in fact, be recovered as a particular case of the family of partial orders introduced by Lima et al. (2021).

Theorem 13 *The partial order defined in Eq.* (12) *is not included in the family of partial orders* $\leq_{\infty}^{\alpha,F}$ *defined in Eq.* (3).

Proof Let us set a function $F : \mathcal{L}_{\infty} \to [0, 1]$ and an arbitrary threshold $\alpha \in [0, 1]$. By definition of the partial order $\leq_{\infty}^{\alpha,F}$ defined in Eq. (3), any two sequences of different lengths $x, y \in \mathcal{L}_{\infty}$ are (strictly) comparable with respect to that order. However, for the partial order defined in Eq. (12), if $k = |x| < |y| = n$, then $x \not\geq y$. Furthermore, if $x \not\leq_k (y_{n-k+1}, \ldots, y_n)$, then $x \not\leq y$. In summary, all pairs of sequences such that $|x| < |y|$ and also $x \not\leq_k (y_{n-k+1}, \ldots, y_n)$ are incomparable with respect to Eq. (12), but they are comparable for the order $\leq_{\infty}^{\alpha,F}$. Thus, the partial order defined in Eq. (12) cannot coincide with (nor be a refinement of) any of the orders in the family defined in Eq. (3). □

Furthermore, for an arbitrary α and a function F comonotonic with respect to $\leq_n$, the partial order $\leq_{\alpha,F}^{\infty}$ is not necessarily a refinement of the order considered in Eq. (12). Indeed, if for such a function F there exist natural numbers $k < n$ and vectors $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_n)$ such that $F(x_1, \ldots, x_k) = F(y_1, \ldots, y_n) = \beta < 1$, then it suffices to consider a value $\alpha > \beta$ to conclude that $(x_1, \ldots, x_k) \not\leq_{\alpha,F}^{\infty} (y_1, \ldots, y_n)$, while $(x_1, \ldots, x_k) \leq (y_1, \ldots, y_n)$.

Lima et al. (2021) proved that, for any $\alpha \in [0, 1]$, if F is increasing with respect to each order $\leq_n$, and if $\leq_n$ endows the set L_n with a lattice structure for every $n \in \mathbb{N}^*$, then the set $\mathcal{L}_{\infty}$, equipped with the order $\leq_{\infty}^{\alpha,F}$, also forms a lattice. In the previous section, we have showed that the order defined over $\mathcal{M}_{\infty}$ also induces a lattice structure and so it does the order considered in Eq. (12) over the set $\mathcal{L}_{\infty}$. Such a lattice structure induces corresponding meet and join operators over $\mathcal{L}_{\infty}$. In other words, the order given in Eq. (12) also supports the definition suitable operations over the family of multidimensional fuzzy sets, alternative to those defined by Lima et al. (2021). This observation may lead to alternative approaches within the framework of multidimensional fuzzy sets.

7 Concluding remarks

Yager (1986) initially introduced fuzzy multisets along with associated union and intersection operations. However, these operations were replaced by alternative definitions proposed by Miyamoto (2001, 2005), which have since then been generally accepted by researchers. Miyamoto's approach, nevertheless, ambiguously handled the multiplicity of membership 0, creating formal difficulties in defining basic operations such as union and intersection.

In this paper, we have proposed two alternative formulations that rectify Miyamoto's original definitions, referred to as Option 1 and Option 2. Option 1 allows unrestricted specification of the multiplicity associated with membership 0, enabling clear differentiation between fuzzy multisets that differ solely in this aspect, while preserving formal correctness. In contrast, the more restrictive Option 2 assumes the multiplicity of membership 0 is always zero.

Miyamoto also explored various definitions for the complement of fuzzy multisets. Here, we have demonstrated that, within the context of Option 2 and assuming the existence of a top multiset T, it is possible to define a complement operation that satisfies the requirements necessary to equip the family of T-bounded fuzzy multisets with the structure of a De Morgan algebra.

Independently from fuzzy multisets, Shang et al. (2010) introduced the notion of n-dimensional fuzzy sets in 2010. Bedregal et al. (2012) later suggested connections between this concept and a particular case of fuzzy multisets, those in which each element is assigned exactly n membership values arranged in increasing order. Subsequently, Lima et al. (2021) proposed the notion of multidimensional fuzzy sets to represent variable-dimension membership functions, introducing a family of partial orderings along with union and intersection operations. In this paper, we have rigorously examined the formal relationships between all these concepts. Specifically, in the final section, we established that the family of n-dimensional fuzzy sets is formally equivalent (isomorphic) to the family of n-bounded fuzzy multisets, both equipped with the structure of a De Morgan algebra. Furthermore, we have shown that a new lattice structure can be defined on the family of multidimensional fuzzy sets, using an ordering distinct from the family considered by Lima et al. (2021). This observation may stimulate future synergies between research on fuzzy multisets and multidimensional fuzzy sets.

Our study has focused exclusively on the formal aspects of fuzzy multisets, aiming to address known limitations without delving into semantic interpretations or practical applications. Historically, one of the earliest uses of multisets was in natural language processing (NLP) (Johnson 2003), where documents are represented as bags of words, which ignore word order but preserve word frequency. In this context, fuzzy multisets can model semantic similarity through membership degrees. Allowing elements with zero membership to have different multiplicities reflects how often certain interpretations are ruled out, which helps manage ambiguity and disagreement in NLP. The distinction between the absence of a membership assignment and the repeated assignment of zero, may have meaningful semantic implications. If so, the practical relevance of Option 1, as introduced in this paper, would be clear.

As an open problem, it is worth investigating in more detail the differences between the families of orderings, meet and join operators, and strong negations defined over $L_\infty([0, 1])$, as considered by Lima et al. (2021) and Santiago and Bedregal (2022, 2024), and the corresponding definitions over $M_\infty([0, 1])$. While the former give rise to intersection, union, and negation operations on the family of multidimensional fuzzy sets, the latter serve as the basis for defining such operations and the inclusion relation within the family of fuzzy multisets.

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