Non-falsity and general threshold-preserving companions of MTL logics

Joan Gispert[2], Lluís Godo[1], and Francesc Esteva[1]

[1] Artificial Intelligence Research Institute (IIIA - CSIC)
Campus de la Univ. Autònoma de Barcelona, 08193 Bellaterra, Spain.
Email: {esteva,godo}@iiia.csic.es
[2] Dept. Maths and Computer Science, Univ. de Barcelona, 08007 Barcelona, Spain
Email: jgispert@ub.edu

Abstract. In this paper we study the definition and axiomatisation of different (finitary) threshold preserving companions of several extensions of the Monoidal t-norm based fuzzy logic MTL. More in detail, we first focus on the *non-falsity preserving* logics, where a conclusion follows from a set of premises if, whenever the premises are non-false (i.e., have a truth degree greater than 0), the conclusion is also non-false. We then introduce a new type of companions, namely logics that preserve the notion of *acceptability*, where a formula is called acceptable whenever it is deemed to be more true than false, or in other words it is more true than its negation. Finally, we also consider a more general stance and consider logics that preserve some intermediate truth-value $0 < a < 1$, so that a can be understood as a (strict or non-strict) threshold above which a formula is considered as valid. All these types of *threshold-preserving* companions of a given MTL logic can be seen as particular cases of matrix logics defined by lattice filters.

Keywords: Mathematical fuzzy logic, monoidal t-norm based logic, non-falsity preserving logics, acceptance logics, threshold preserving logics

1 Introduction

Systems of mathematical fuzzy logic are formal systems designed to handle *graded truth*, making them particularly suitable for reasoning with imprecise or vague information. Their defining feature is the interpretation of formulas over a linearly ordered scale of truth values, typically within the real unit interval $[0, 1]$. This framework is especially well-suited for capturing the gradual nature of vagueness. A significant body of work on fuzzy logics has been developed within the framework of mathematical fuzzy logic (MFL) [26, 10].

In MFL, deductive systems have been traditionally defined based on the principle of *full truth-preservation*, where a formula is considered a consequence of a set of premises if, under every algebraic evaluation that assigns the truth value 1 (absolute truth) to all premises, the conclusion also receives the value 1. This generalizes the classical notion of logical consequence.

In recent years, there have been at least two proposals of variants of systems of MFL that involve a shifting from the full truth-preserving to alternative paradigms. Namely, in the logical systems within the *degree-preserving paradigm* [5], a conclusion follows from a set of premises if, under all evaluations, its truth degree is at least as great as those of all the premises. This paradigm leads to define a notion of consequence relation generally weaker than the one based on the truth-preservation. In particular, although the set of theorems or valid formulas coincide with the one of the truth-preserving logic, the inference rule of Modus Ponens is not generally sound, unless its use is restricted to apply when the implication is a theorem.

A different kind of companion logic, introduced in [1] for the case of Łukasiewicz logic, is based on *non-falsity preservation*. Under this paradigm, a conclusion follows from a set of premises if, whenever the premises are non-false (i.e., have a truth degree greater than 0), the conclusion is also non-false. This yields a notion of consequence relation that is not weaker than the one based on truth-preservation but it is generally stronger than the one based on the degree-preservation paradigm. Indeed, while all tautologies of a truth-preserving logic remain valid under non-falsity preservation, inference rules such as Modus Ponens may fail to be sound, similarly to the case of degree-preserving variants. On the other hand, for example, the excluded middle axiom $\varphi \vee \neg\varphi$ becomes valid in the non-falsity preserving setting, as it always evaluates to a positive truth value, in contrast to the truth- or degree-preserving variants. One of the main objetives of this paper is to investigate and axiomatically characterise the non-falsity preserving companions for a wide family of MTL logics, notably extending the results for the particular case of the Nilpotent Minimum logic in [25].

In this paper we also introduce yet a new type of companion logics, namely the logics that preserve the notion of acceptability. We call a formula φ *acceptable*, for a given evaluation e, when φ is more true than false, that is, when $e(\varphi) > e(\neg\varphi)$. We also consider the dual notion, a formula φ is called *tolerable* when $\neg\varphi$ is not acceptable, i.e. when $e(\neg\neg\varphi) \geq e(\neg\varphi)$. We study the properties of these *acceptability* and *tolerance-preserving* companions and their relationship to their truth and non-falsity preserving companions, as well as their axiomatisation in some particular cases.

Actually, the (full) truth-preserving logics and both the non-falsity and acceptability/tolerance preserving logics can be seen as particular cases of matrix logics defined by lattice filters. Indeed, suppose L is a (truth-preserving) system of MFL which is complete with respect to a standard algebra $[\mathbf{0}, \mathbf{1}]_*$, this means that L is the logic of the matrix $M_L = \langle [0,1]_*, F_1 \rangle$ where the lattice filter is $F_1 = \{1\}$. Its non-falsity preserving variant is nothing but the logic of the logical matrix $M_{(0} = \langle [0,1]_*, F_{(0} \rangle$, where the lattice filter $F_{(0}$ is now the semi-open interval $(0, 1]$. These are somehow two extreme cases. On the other hand, if the residual negation $\neg(x) = x \to 0$ has a fixpoint, call it r, then the acceptance-preserving and tolerance-preserving logics correspond to the logics of the matrices $\langle [0,1]_*, (r,1] \rangle$ and $\langle [0,1]_*, [r,1] \rangle$, respectively.

One then can also take a more general stance and consider logics that preserve some intermediate truth-value $0 < a < 1$, so that a can be understood as a threshold above which a formula is considered as valid. In other words, it makes sense to study logics related to matrices $M_a = \langle [0,1]_*, F_a \rangle$ and $M_{(a} = \langle [0,1]_*, F_{(a} \rangle$ where the lattice filters is are $F_a = [a, 1]$ and $F_{(a} = (a, 1]$. We will generally call these logics *threshold-preserving* logics, while we will keep referring to *non-falsity preserving* (resp. *acceptance/tolerance preserving*) logic in the particular case the lattice filter of the matrix logic is the semi-open interval $(0, 1]$ (resp. the semi-open $(r, 1]$ / the closed interval $[r, 1]$). In the final part of the paper, we explore the definition and syntactic characterization of logics that preserve an arbitrary but fixed truth-value threshold $a \in (0, 1]$, focusing on the several prominent extensions of MTL, including Łukasiewicz, Product and Gödel logics, but not only them.[1]

The structure of the paper is as follows. Section 2 provides preliminaries on t-norm based fuzzy logics and the basic definitions of the matrix logics based on MTL-chains and lattice filters as designated values. Section 3 focuses on the paraconsistent non-falsity preserving companions of MTL logics, mainly for involutive MTL logics and MTL logics validating a suitable inference rule. In Section 4 we deal with variants to reason about acceptable or tolerable propositions. Finally, Section 5 addresses general threshold-preserving logics for different classes of MTL extensions. We conclude with some final remarks and prospects for future work in Section 6.

Throughout this paper we assume the MTL logics L we deal with are standard complete, that is, they are complete with respect to a family of standard L-algebras.

This paper is a revised and notably extended version of the two conference papers [18] and [19].

2 Preliminaries

2.1 T-norm based fuzzy logics

The most well-known and extensively studied systems within mathematical fuzzy logic are the so-called *t-norm based fuzzy logics*. These correspond to formal many-valued calculi with truth values in the real unit interval $[0, 1]$, where conjunction and implication are interpreted, respectively, by a (left-)continuous t-norm and its residuum. This class includes, for instance, the well-known infinitely-valued Łukasiewicz, Produc and Gödel logics, which are based on the Łukasiewicz, product and min t-norms, respectively.

The foundational system in this family is MTL (Monoidal t-norm based Logic), introduced in [21]. The theorems of MTL correspond to the common

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[1] The case of the expansion of Gödel logic with an involution was already considered to some detail in [15], leaving as an open problem whether the logics were finitary, that was positively solved later by Frittella and Kozhemiachenko in [24].

tautologies of all many-valued calculi defined by a left-continuous t-norm and its residuum [28].

The language of MTL consists of countably many propositional variables $p_1, p_2, \ldots$, binary connectives $\wedge$, $\&$, and $\rightarrow$, and the truth constant $\overline{0}$. Formulas, denoted by lowercase Greek letters $\varphi, \psi, \chi, \ldots$, are defined recursively in the usual way. Additional connectives and constants can be defined: for instance, $\neg\varphi$ abbreviates $\varphi \rightarrow \overline{0}$, and $\overline{1}$ is defined as $\neg\overline{0}$.

A Hilbert-style calculus for MTL was introduced in [21], with the following axioms:

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $\varphi \,\&\, \psi \rightarrow \varphi$
(A3) $\varphi \,\&\, \psi \rightarrow \psi \,\&\, \varphi$
(A4) $\varphi \wedge \psi \rightarrow \varphi$
(A5) $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$
(A6) $\varphi \,\&\, (\varphi \rightarrow \psi) \rightarrow \varphi \wedge \psi$
(A7a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \,\&\, \psi \rightarrow \chi)$
(A7b) $(\varphi \,\&\, \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A8) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A9) $\overline{0} \rightarrow \varphi$

The only inference rule is *modus ponens*: from φ and $\varphi \rightarrow \psi$, infer ψ.

MTL is an algebraizable logic in the sense of Blok and Pigozzi [3], with its equivalent algebraic semantics given by the variety of MTL-algebras. These algebras can be characterised as commutative, bounded, integral residuated lattices $\langle A, \wedge, \vee, *, \rightarrow, \overline{0}, \overline{1}\rangle$ satisfying the prelinearity condition: $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$. An associated negation operator $\neg$ (also called residual negation) is usually defined as $\neg a = a \rightarrow \overline{0}$. Of particular interest is the class of MTL-algebras over the real unit interval $[0, 1]$, called *standard algebras*. In fact any standard MTL-algebra is of the form $[\mathbf{0}, \mathbf{1}]_* = ([0, 1], \min, \max, *, \rightarrow, 0, 1)$, where $*$ is a left-continuous t-norm,[2] $\rightarrow$ is its residuum,[3] and $\neg x$ is defined as $x \rightarrow 0$.

Table 1 summarizes several important axiomatic extensions of MTL and their corresponding additional axioms. Of particular interest in this paper is the logic IMTL (Involutive MTL), which extends MTL with the axiom (Inv), enforcing involutiveness of negation [21]. IMTL-algebras are MTL-algebras satisfying $x = \neg\neg x$. The well-known Łukasiewicz logic is obtained by further adding the divisibility axiom (Div), while Nilpotent Minimum logic NM is the extension of IMTL with the (WNM) axiom and SMTL is the extension with the pseudo-complementation axiom (PC). On the other hand, Product logic *Π* is the extension of MTL with (Div) and the Cancellation axiom (C), and Gödel logic extends MTL with the contraction axiom (Con) [26]. Algebras for Łukasiewicz logic,

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[2] A left-continuous t-norm is a binary mapping $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that is commutative, associative, non-decreasing in each variable, $1 * x = x$ and $0 * x = 0$ for all $x \in [0, 1]$, and it is left-continuous.

[3] $x \rightarrow y = \max\{z \in [0, 1] \mid x * z \leq y\}$.

known as MV-algebras, are IMTL-algebras satisfying $x * (x \to y) = x \wedge y$, NM-algebras are IMTL-algebras satisfying the equation $\neg(x*y) \vee (x \wedge y \to x*y) = 1$, SMTL-algebras are pseudocomplented MTL-algebras, Product algebras are divisible SMTL-algebras that further satisfy cancellation, and finally Gödel algebras are MTL-algebras satisfying $x * y = x \wedge y$.

Axiom Schema	Name
$\neg\neg\varphi \to \varphi$	(Inv)
$\neg\varphi \vee ((\varphi \to \varphi \,\&\, \psi) \to \psi)$	(C)
$\varphi \to \varphi \,\&\, \varphi$	(Con)
$\varphi \wedge \psi \to \varphi \,\&\, (\varphi \to \psi)$	(Div)
$\neg(\varphi \wedge \neg\varphi)$	(PC)
$\neg(\varphi \,\&\, \psi) \vee (\varphi \wedge \psi \to \varphi \,\&\, \psi)$	(WNM)
$\varphi \vee \neg\varphi$	(EM)

Logic	Additional Axioms
Strict MTL (SMTL)	(PC)
Involutive MTL (IMTL)	(Inv)
Nilpotent Minimum (NM)	(Inv), (WNM)
Basic Logic (BL)	(Div)
Strict Basic Logic (SBL)	(Div), (PC)
Łukasiewicz Logic (Ł)	(Div), (Inv)
Product Logic (Π)	(Div), (C)
Gödel Logic (G)	(Con)
Classical Logic (CL)	(EM)

Table 1. Axiomatic extensions of MTL and their corresponding additional axioms.

As a consequence of their algebraizability, all the logics listed in Table 1 enjoy, not only strong completeness with respect to their corresponding classes of algebras, but also strong completeness with respect to their subclasses of linearly-ordered algebras, due to the fact that the pre-linearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$ is a theorem of MTL. In fact this hods for any axiomatic extension of MTL. Moreover, all the logics in Table 1 also enjoy *standard completeness* for deductions from a finite set of premises with respect to their corresponding subclasses of standard algebras (over the real-unit interval $[0, 1]$). Standard completeness for MTL was established in [28], and for IMTL in [20]. Moreover, Łukasiewicz, Product, Gödel and Nilpotent Minimum logics are complete with respect to a single standard algebra: the standard MV-algebra, Product-algebra, Gödel algebra and Nilpotent Minimum-algebra, respectively [26, 21].

A rather useful expansion of any MTL logic L is with the so-called Baaz-Monteiro Δ operator. This unary operator, over MTL-chains, behaves in the following form: $\Delta(\overline{1}) = \overline{1}$ and $\Delta(x) = \overline{0}$ if $x \neq \overline{1}$. Thus, Δ is a Boolean operator that can isolate the full truth, a property that cannot be captured by any combination of the primitive connectives in any MTL axiomatic extension. Such an expansion (in the expanded language of MTL with a new unary connective Δ), denoted L_Δ, can be axiomatically defined as follows: axioms of L_Δ are those of L plus the following ones:

($\Delta1$) $\Delta\varphi \vee \neg\Delta\varphi$
($\Delta2$) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$
($\Delta3$) $\Delta\varphi \to \varphi$
($\Delta4$) $\Delta\varphi \to \Delta\Delta\varphi$
($\Delta5$) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

and the rules of L_Δ are Modus Ponens and necessitation for Δ: from φ derive $\Delta\varphi$. Then, if L was complete wrt a class of standard algebras, L_Δ keeps being complete wrt the same class of algberas expanded with the Δ operator in its signature and defined as above, see e.g. [26, 27] for further details in the case.

Finally, let us mention the general notions of core and Δ-core fuzzy logics, introduced in [27], as very general expansions of MTL and MTL_Δ respectively that keep satisfying most of the desirable properties of MTL. We take the following definitions from [2]:[4]

- A finitary logic L is a *core fuzzy logic* if: i) it expands MTL; ii) for any formulas φ, ψ, χ, it holds that $\varphi \leftrightarrow \psi \vdash_L \chi \leftrightarrow \chi'$, where χ' is a formula resulting from χ by replacing some occurrences of its subformula φ by a formula ψ; and iii) L has the local deduction theorem, i.e. for any set of formulas $T \cup \{\varphi, \psi\}$, it holds that $T, \varphi \vdash_L \psi$ iff there is n such that $T \vdash_L \varphi^n \to \psi$.
- A finitary logic L is a *Δ-core fuzzy logic* if: i) it expands MTL_Δ; ii) for any formulas φ, ψ, χ, it holds that $\varphi \leftrightarrow \psi \vdash_L \chi \leftrightarrow \chi'$, where χ' is a formula resulting from χ by replacing some occurrences of its subformula φ by a formula ψ; and iii) L has the Δ-deduction theorem, i.e. for any set of formulas $T \cup \{\varphi, \psi\}$, it holds that $T, \varphi \vdash_L \psi$ iff $T \vdash_L \Delta\varphi \to \psi$.

All logics listed in Table 1 are core fuzzy logics, while the expansion of SMTL logics –particularly Gödel logic (G) and Product logic (Π)– with an additional involutive negation, that will be considered later in this paper, are Δ-core fuzzy logics.

2.2 Degree-preserving logics $\mathbf{L}^{\leq}$

According to [5], given a ($\triangle$-)core fuzzy logic L, the consequence relation $\models_L^{\leq}$ of its degree-preserving companion $\mathrm{L}^{\leq}$ has the following semantics: for every finite set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \models_L^{\leq} \varphi \quad \text{if,} \quad \begin{array}{l} \text{every L-chain } \mathbf{A}, \text{ every } a \in A, \text{ and every } \mathbf{A}\text{-evaluation } e, \\ \text{if } a \leq e(\psi) \text{ for every } \psi \in \Gamma, \text{ then } a \leq e(\varphi). \end{array}$$

For this reason $\mathrm{L}^{\leq}$ is known as the *degree-preserving companion* of L. As regards to axiomatization, if L is a core fuzzy logic, i.e. with Modus Ponens as the unique inference rule, then the logic $\mathrm{L}^{\leq}$ admits a Hilbert-style axiomatization having the same axioms as L and the following deduction rules [5]:

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[4] In the context of the two definitions below, the authors of [2] they use the term *finitary logic* to refer to a logic defined as Hilbert-style system by a set of axioms and inference rules with finitely many premises.

- Rule of Adjunction: (Adj) $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- Restricted Modus Ponens: (r-MP) $\dfrac{\varphi, \quad \vdash_L \varphi \to \psi}{\psi}$

Note that in the rule (MP-r), $\varphi \to \psi$ is required to be theorem of L. Thus, if the set of theorems of L is decidable, then the above is in fact a recursive Hilbert-style axiomatization of $\mathrm{L}^{\leq}$.

In general, if L is a semilinear expansion of MTL (i.e. a logic which is complete with respect to some class of linearly-ordered expansions of MTL-algebras, see e.g. [10]) with a set of new inference rules,

- (R_i) $\dfrac{\Gamma_i}{\varphi_i}$,

for $i \in I$, then $\mathrm{L}^{\leq}$ is axiomatised by adding to the axioms of L the above two inference rules plus the following restricted rules

- (r-R_i) $\dfrac{\vdash_{\mathrm{L}} \Gamma_i}{\varphi_i}$,

In particular, if L is a $\triangle$-core fuzzy logic, then the only rule one should add to $\mathrm{L}^{\leq}$ is the following restricted necessitation rule for $\triangle$:

- (r-$\triangle$): $\dfrac{\vdash_{\mathrm{L}} \varphi}{\triangle\varphi}$

The key relationship between L and $\mathrm{L}^{\leq}$ is given by the following equivalence: for any formulas $\varphi_1, \ldots, \varphi_n, \psi$, it holds

$$\varphi_1, \ldots, \varphi_n \vdash_{\mathrm{L}^{\leq}} \psi \text{ iff } \vdash_{\mathrm{L}} (\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi.$$

This relation points out that, indeed, deductions from a finite set of premises in $\mathrm{L}^{\leq}$ exactly correspond to theorems in L. In particular, both logics share the same theorems: $\vdash_{\mathrm{L}} \varphi$ iff $\vdash_{\mathrm{L}^{\leq}} \varphi$. Moreover, this also implies that if L' is a conservative expansion of L, then $\mathrm{L}'^{\leq}$ is also a conservative expansion of $\mathrm{L}^{\leq}$.

2.3 Logics Defined by Matrices with Lattice Filters

In the systems of mathematical fuzzy logic discussed above, the standard notion of logical consequence is defined in terms of the preservation of *truth*, represented by the top element of the corresponding algebras. Let L be any axiomatic extension of MTL, assumed to be complete with respect to the class $\mathcal{C}_L = \{[\mathbf{0},\mathbf{1}]_* \mid [\mathbf{0},\mathbf{1}]_* \text{ is an } L\text{-algebra}\}$ of standard L-algebras. Then, the usual semantic consequence relation is defined as follows for any set of formulas $\Gamma \cup \{\varphi\}$:

$$\begin{array}{lll} \Gamma \models_L \varphi & \text{if,} & \text{for any } [0,1]_* \in \mathcal{C}_L \text{ and any } [0,1]_*\text{-evaluation } e, \\ & & \text{if } e(\psi) = 1 \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) = 1 \text{ as well.} \end{array}$$

This notion can be generalised by considering logics defined by logical matrices $M = \langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is a standard L-chain and F is a non-trivial lattice filter of $\mathbf{A}$. Specifically, F may be a closed interval $F_a = [a, 1]$ for some $a \in (0, 1]$, or a semi-open interval $F_{(a} = (a, 1]$ for some $a \in [0, 1)$. Treating these filters as sets of designated values, we define the following companion logics: for any set of formulas $\Gamma \cup \{\varphi\}$,

$$\begin{array}{lll} \Gamma \models^a_L \varphi & \text{if,} & \text{for any } [0,1]_* \in \mathcal{C}_L \text{ and any } [0,1]_*\text{-evaluation } e, \\ & & \text{if } e(\psi) \geq a \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) \geq a \text{ as well.} \\ \Gamma \models^{(a}_L \varphi & \text{if,} & \text{for any } [0,1]_* \in \mathcal{C}_L \text{ and any } [0,1]_*\text{-evaluation } e, \\ & & \text{if } e(\psi) > a \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) > a \text{ as well.} \end{array}$$

The extreme cases of this framework are particularly noteworthy: the logic $\models^1_L$ coincides with the standard truth-preserving logic $\models_L$, which is explosive, $\models^{(0}_L$ corresponds to the non-falsity preserving logic, which is paraconsistent with respect to $\neg$, while the degree-preserving logic is nothing but the intersection of all lattice filter logics, i.e. $\models^{\leq}_L = \bigcap_{a \in (0,1]} \models^a_L$, which is paraconsistent as well.[5]

The question of whether logics $\models^a_L$ and $\models^{(a}_L$, for $0 < a < 1$ are explosive or paraconsistent is addressed next. Notice that in any standard MTL algebra $[\mathbf{0}, \mathbf{1}]_* = ([0, 1], \min, \max, *, \to, 0, 1)$, the negation function defined as $\neg x = x \to 0$ is left-continuous, hence either $\neg x$ has a fixpoint, or the set $\{x \in [0, 1] \mid \neg x > x\}$ has a minimum. Therefore, let $s_* = \min\{x \in [0, 1] \mid \neg x \geq x\}$. This point s_* is such that:

(i) $x > s_*$ iff $\neg x < x$ (hence, $x \leq s_*$ iff $\neg x \geq x$)
(ii) if $x < s_*$ then $\neg x > x$
(iii) if $x = s_*$ then either $\neg x = x$ if the s_* is the negation fixpoint, or $\neg x > x$ otherwise.

Lemma 1. *Let L be the logic of a matrix $[\mathbf{0}, \mathbf{1}]_*$. Then, the companion logics $\models^a_L$ and $\models^{(a}_L$ are such that:*

- *If $a > s_*$ then $\models^a_L$ and $\models^{(a}_L$ are explosive and $\neg$-paracomplete.*
- *If $a < s_*$ then $\models^a_L$ and $\models^{(a}_L$ are $\neg$-paraconsistent and validate the excluded-middle axiom (EM).*
- *If $a = s_*$ then $\models^a_L$ is $\neg$-paraconsistent and satisfies (EM), while $\models^{(a}_L$ is explosive and $\neg$-paracomplete.*

Proof. 1. Let $a > s_*$. Then $\neg a < a$. Hence if $e(\varphi) \geq a$ then $e(\neg\varphi) \leq \neg a < a$, hence $\varphi, \neg\varphi \models^a_L \bot$, so $\models^a_L$ is explosive. Similarly with $\models^{(a}_L$. On the other hand, let p a propositional variable and e an evaluation such that $s_* < e(p) < a$. Then, by (i) above, $e(\neg p) = \neg(e(p)) < e(p)$, hence $e(p \vee \neg p) = e(p) < a$, hence $\not\models^a_L p \vee \neg p$ as well as $\not\models^{(a}_L p \vee \neg p$, and thus, both $\models^a_L$ and $\models^{(a}_L$ are $\neg$-paracomplete.

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[5] A logic with a negation $\neg$, given by a consequence relation $\models$, is called $\neg$-*paraconsistent* when there exist formulas φ such that $\varphi, \neg\varphi \not\models \bot$, and it is called $\neg$-*paracomplete* when there exist formulas φ such that $\not\models \varphi \vee \neg\varphi$. If it is not $\neg$-paraconsistent, it is called *explosive*.

2. Let $a < s_*$. Then $\neg a > a$. Let p and e such that $s_* > e(p) > a$. Then, by (ii), $e(\neg p) = \neg e(p) > e(p) > a$, hence $p, \neg p \not\models_L^a \bot$ and $p, \neg p \not\models_L^{(a} \bot$. Thus both $\models_L^a$ and $\models_L^{(a}$ are $\neg$-paraconsistent. On the other hand, let $b = e(\varphi)$, then we have two cases: if $b > s^* > a$, it is clear that $e(\varphi \vee \neg\varphi) > a$; if $b \leq s^*$ then $\neg b \geq \neg s_* \geq s_* > a$, hence $e(\varphi \vee \neg\varphi) > a$ as well. Therefore both $\models_L^a$ and $\models_L^{(a}$ satisfy the (EM).

3. The final case $a = s_*$ easily follows from the two previous cases by selecting the suitable inequalities for $\models_L^a$ and $\models_L^{(a}$. □

From a semantical point of view, we recall that a logic L defined by a logical consequence relation $\models_L$ is called *finitary* when, for each set of formulas $\Gamma \cup \{\varphi\}$, it holds that $\Gamma \models_L \varphi$ iff there is a finite subset $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \models_L \varphi$.

Note that in the case of core or Δ-core fuzzy logics L with semantics based on a class of L-algebras on the real unit interval $[0,1]$, it is not uncommon that the truth-preserving logic $\models_L$ is not finitary. For instance, this is the case e.g. of Łukasiewicz, Product, BL or SMTL logics. On the other hand, $\models_L$ is finitary e.g. in the case of L being MTL, Gödel or Nilpotent Minimum logics.

Given a (possibly infinitary) consequence relation $\models_L$, we will refer to its *finitary companion* to the consequence relation $\models_L^f$ defined as follows: for any set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \models_L^f \varphi \text{ if there is a finite subset } \Gamma_f \subseteq \Gamma \text{ such that } \Gamma_f \models_L \varphi.$$

For the sake of simplicity, in the rest of the paper we will refer to $\models_L^f$ as the "finitary $\models_L$".

3 Non-falsity preserving companions of MTL logics

Interestingly, although very different from a semantic point of view, the finitary versions $\models_L^1$ of a truth-preserving logic L and of its non-falsity preserving companion $\models_L^{(0}$ are closely related. Indeed, the following proposition shows that $\models_L^{(0}$ can always be interpreted in $\models_L^1$, while in the other way round, $\models_L^1$ is interpretable in $\models_L^{(0}$ whenever $\models_L^1$ satisfies an additional condition.

Proposition 1. *Let* L *be a core fuzzy logic. The following conditions hold:*

(i) $\models_{\mathrm{L}}^{(0}$ *can always be interpreted in* $\models_{\mathrm{L}}^1$*, that is, for any formulas* φ *and* ψ*, the following condition holds:*

$$\text{(Int1)} \qquad \varphi \models_{\mathrm{L}}^{(0} \psi \quad \textit{iff} \quad \neg\psi \models_{\mathrm{L}}^1 \neg\varphi.$$

(ii) On the other way round, $\models_{\mathrm{L}}^1$ *is interpretable in* $\models_{\mathrm{L}}^{(0}$*, i.e. for any formulas* φ *and* ψ *it holds that*

$$\text{(Int2)} \qquad \varphi \models_{\mathrm{L}}^1 \psi \quad \textit{iff} \quad \neg\psi \models_{\mathrm{L}}^{(0} \neg\varphi,$$

if, and only if, $\models_{\mathrm{L}}^1$ *is such that* $\neg\neg\varphi \models_{\mathrm{L}}^1 \varphi$ *holds for every* φ*.*

Proof. As for (i), (Int1) follows directly from the fact that in any MTL-chain, for any element x, we have $x > 0$ if and only if $\neg x < 1$, or equivalently, $\neg x = 1$ if and only if $x = 0$.

As for (ii), assume first the condition $\neg\neg\varphi \models^1_L \varphi$ holds for $\models^1_L$, which amounts to assume that L-chains satisfy the condition "$\neg\neg x = 1$ implies $x = 1$". Note that the condition "$x = 1$ implies $y = 1$" is equivalent to "$y < 1$ implies $x < 1$, and then, using the assumption we have "$x < 1$ implies $\neg\neg x < 1$", and thus "$y < 1$ implies $\neg\neg x < 1$", which is in turn equivalent to "$\neg y > 0$ implies $\neg x > 0$", since the equality $\neg\neg\neg x = \neg x$ holds in any MTL-algebra. Hence (Int2) holds.

Conversely, assume (Int2) holds. Since, $\neg\varphi$ is logically equivalent to $\neg\neg\neg\varphi$, $\neg\varphi \models^{(0}_L \neg\neg\neg\varphi$ obviously holds, and then by (Int2) we also have $\neg\neg\varphi \models \varphi$. □

Since the general problem for any axiomatic extension of MTL looks quite involved, in this section we study the axiomatisation of the non-falsity preserving companions of different families of MTL logics. We start with the general case of MTL logics enjoying a global deduction theorem, including the Δ-core fuzzy logics, which is rather straighforward. In the second subsection we deal with the class of IMTL logics, that is, extensions of MTL for which the negation is involutive. In the third subsection, we consider non-SMTL logics extended with a new inference rule that ensures a good behavior of the negation and for which the results for IMTL can be easily extended. Finally, in the fourth subsection we study the case of the expansion of SMTL with an involutive negation.

3.1 The case of MTL logics with a global deduction theorem

To begin with, let us assume L is a core fuzzy logic, hence an axiomatic expansion of MTL, that is complete with respect to a class $\mathcal{C}_L \subseteq \{[\mathbf{0},\mathbf{1}]_* \mid [\mathbf{0},\mathbf{1}]_* \text{ is an } L\text{-algebra}\}$ of standard L-algebras, with associated consequence relation $\models_L$. And let us further assume L satisfies some form of global Deduction Theorem in the sense that there exists a term t such that the following form global of deduction theorem holds: for any set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \cup \{\varphi\} \vdash_L \psi \quad \text{iff} \quad \Gamma \vdash_L t(\varphi) \to \psi.$$

By Proposition 1, we have then the following chain of equivalences:

$$\varphi \models^{(0}_L \psi \quad \text{iff} \quad \neg\psi \models_L \neg\varphi \quad \text{iff} \quad \models_L t(\neg\psi) \to \neg\varphi \quad \text{iff} \quad \vdash_L t(\neg\psi) \to \neg\varphi.$$

Moreover, note that MTL proves the equivalence $(\varphi \to \neg\psi) \leftrightarrow (\psi \to \neg\varphi)$, hence the condition $\vdash_L t(\neg\psi) \to \neg\varphi$ is equivalent to $\vdash_L \varphi \to \neg t(\neg\psi)$. This, allows us to devise the following syntactic Hilbert-style system for the non-falsity preserving companion of L.

Definition 1. *The system* `nf`*-L is defined by the same axioms of L plus the rules:*

- *Rule of Adjunction: (Adj)* $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$

- *Restricted Modus Ponens: (r-MP)* $\dfrac{\varphi, \quad \vdash_L \varphi \to \psi}{\psi}$,
- *Restricted GDT-rule: (GDT-MP)* $\dfrac{\varphi, \quad \vdash_L \varphi \to \neg t(\neg\psi)}{\psi}$

Theorem 1. *Let L be core fuzzy logic as above. The system* nf-*L is a sound and complete axiomatization of the finitary logic* $\models_L^{(0}$ *defined by the class of matrices* $\mathcal{C}_L^0 = \{\langle [0,1]_*, F_{(0}\rangle \mid \langle [0,1]_*, F_1\rangle \in \mathcal{C}_L\}$. *That is, for any formulas* $\varphi_1, ..., \varphi_n, \psi$, *it holds:*

$$\varphi_1, ..., \varphi_n \models_L^{(0} \psi \quad \textit{iff} \quad \varphi_1, ..., \varphi_n \vdash_{\mathsf{nf}-L} \psi.$$

Proof. Let $\Gamma = \{\varphi_1, ..., \varphi_n\}$ and assume $\Gamma \models_L^{(0} \psi$. This is obviously equivalent to $\varphi \models_L^{(0} \psi$, where $\varphi = \varphi_1 \wedge ... \wedge \varphi_n$. As observed above, by completeness of L and the deduction theorem, this holds iff $\vdash_L \varphi \to \neg t(\neg\psi)$. Hence, there is a proof in L of $\varphi \to \neg t(\neg\psi)$, namely a sequence $\langle \Pi_1, \Pi_2, \ldots, \Pi_r = \varphi \to \neg t(\neg\psi)\rangle$, where each Π_i, for $i = 1, \ldots, r$, is either an axiom of L or has been obtained from two previous Π_j and Π_k (with $j, k < i$) by application of the modus ponens rule. Then, in order to get a proof of ψ from Γ in nf-L, we only need add a previous step Π_0 with $n-1$ applications of the Adjunction rule in order to get φ from Γ, and a final step Π_{r+1} to get ψ by application of the GDT-rule to φ and $\varphi \to \neg t(\neg\psi)$, which has been proved to be a theorem of L. Therefore, $\langle \Pi_0 = \varphi, \Pi_1, \Pi_2, \ldots, \Pi_r, \Pi_{r+1} = \psi\rangle$ is a correct proof in nf-L since the applications of the modus ponens are in fact applications of the (r-MP) since they all apply to theorems of L. ☐

A notable family of MTL logics that enjoy a global deduction theorem are the so-called *n-contractive* fuzzy logics [8]. An n-contractive core fuzzy logic is an extension of a MTL logic proving the n-contraction axiom

$$(C_n) \quad \varphi^{n-1} \to \varphi^n,$$

where φ^n stands for $\varphi \& \overset{n}{\ldots} \& \varphi$, already considered in [7]. A distinguished feature of n-contractive MTL logics is that they enjoy the following form of global deduction theorem:

$$\varphi, \psi \vdash_L \chi \quad \text{iff} \quad \varphi \vdash_L \psi^n \to \chi.$$

In fact, by general algebraization results, it follows that n-contractive core fuzzy logics are exactly those core fuzzy logics that enjoy some form of global deduction theorem. Among n-contractive core fuzzy logics we may mention the following ones. Gödel logic is the extension of MTL logic with the axiom (C_2) while the extension of IMTL with (C_2) is classical propositional logic. Nilpotent minimum logic (NML) is the extension of IMTL proving (C_3). On the other hand, finite-valued Łukasiewicz logics Ł$_n$ also prove (C_n). Also, logics of ordinal sums of n-contractive left-continuous t-norms (see [29] for a complete description) are again n-contractive. Therefore, the above Theorem 1 directly applies to all these logics where the rule (GDT-MP) takes this particular form:

$$\frac{\varphi, \quad \vdash_L \varphi \to n\psi}{\psi}$$

where $n\psi := \neg(\neg\psi)^n$.

In the second part of this section, let us now consider the case that L is a Δ-core fuzzy logic, where for simplicity we assume that L has as inference rules only Modus Ponens and necessitation for Δ. As mentioned in the Preliminaries, Δ-core fuzzy logic enjoy the global Δ-deduction theorem:

$$\varphi, \psi \vdash_L \chi \quad \text{iff} \quad \varphi \vdash_L \Delta\psi \to \chi$$

In such a logic we can introduce the following definable connective:

$$\nabla\varphi := \neg\Delta\neg\varphi.$$

Note that, for any interpretation in any L-chain, $e(\nabla\varphi) = 1$ iff $e(\varphi) > 0$, and hence ∇ is a Boolean connective that exactly captures whether a formula is non-false. This connective allows us to axiomatically define the non-falsity preserving companion of L, denoted nf-L again, as follows. Axioms and rules of nf-L are:

- Axioms of L
- Restricted forms of the inference rules of L, including : (r-MP) $\dfrac{\varphi, \quad \vdash_L \varphi \to \psi}{\psi}$
- Adjunction rule: (Adj) $\dfrac{\varphi, \psi}{\varphi \wedge \psi}$
- Restricted Δ-necessitation: (r-Δ) $\dfrac{\vdash_L \varphi}{\Delta\varphi}$
- Positive MP for Δ: (∇-MP) $\dfrac{\varphi, \quad \nabla\varphi \to \nabla\psi}{\psi}$

It is clear that $\models_L^{(0}$ validates all the rules above. In particular, note that, for every L-evaluation e, $e(\nabla\varphi \to \nabla\psi) > 0$ iff $e(\nabla\varphi \to \nabla\psi) = 1$, since the ∇ connective only takes values 0 or 1. Then, in such a case, if $e(\varphi) > 0$ then $e(\psi) > 0$ as well.

Theorem 2. *Let L be a Δ-core fuzzy logic. Then the system nf-L is a sound and complete axiomatization of the finitary logic $\models_L^{(0}$ defined by the class of matrices $\mathcal{C}_L^0 = \{\langle [0,1]_*, F_{(0}\rangle \mid \langle [0,1]_*, F_1\rangle \in \mathcal{C}_L\}$. That is, for any formulas $\varphi_1, ..., \varphi_n, \psi$, it holds:*

$$\varphi_1, ..., \varphi_n \models_L^{(0} \psi \quad \textit{iff} \quad \varphi_1, ..., \varphi_n \vdash_{\mathsf{nf}-L} \psi.$$

Proof. Soundness is straightforward. As for completeness, assume $\varphi_1, ..., \varphi_n \models_L^{(0} \psi$. This is equivalent to assume $\nabla\chi \models_L \nabla\psi$, where $\chi = \varphi_1 \wedge ... \wedge \varphi_n$. By completness of L, this is equivalent to $\nabla\chi \vdash_L \nabla\psi$ and, by the Δ deduction theorem, to $\vdash_L \Delta(\nabla\chi) \to \nabla\psi$. Moreover, since $\nabla\chi$ is a negation of a Δ formula, $\vdash_L \Delta(\nabla\chi) \equiv \nabla\chi$, hence we can safely assume $\vdash_L \nabla\chi \to \nabla\psi$.

Now, this means there is a proof $\langle \Pi_1, \ldots, \Pi_r = \nabla\chi \to \nabla\psi\rangle$, where all the Π_i's are theorems of L (either axioms of L or deduced from them using the inference rules of L).

Therefore, to obtain a proof in nf-L of ψ from $\varphi_1, ..., \varphi_n$, we only need to add: (i) a previous step to derive $\Pi_0 : \psi = \varphi_1 \wedge ... \wedge \varphi_n$ using the Adjunction rule, and (ii) a final step $\Pi_{r+1} = \psi$ where ψ is derived by the rule (∇-IR) from χ and $\nabla\chi \to \nabla\psi$, the latter being proved to be a theorem of L. in step Π_r. Note that the inference rules of L used in the above proof steps Π_1, ..., Π_r in L, in fact correspond to the restricted rules (r-MP) and (r-Δ) in the proof in nf-L. □.

3.2 The case of extensions of IMTL

Inspired by the results in [25] for the Nilpotent Minimum logic NM expanded with a consistency operator $\circ$, in this section we focus on the characterisation of logics defined by (sets of) matrices of the form $\langle [0,1]_*, F_{(0} \rangle$, for $*$ being an IMTL t-norm. We remind that this means that $*$ is left-continuous and that the residual negation $\neg$, defined as $\neg x = x \to 0 = \max\{y \in [0,1] \mid x * y = 0\}$, is such that $\neg(\neg x) = x$. Notable examples of IMTL t-norms are Łukasiewicz t-norm (which is continuous) and Nilpotent Minimum t-norm.

Assume L is an axiomatic extension of IMTL, complete with respect to a class of standard algebras $\mathcal{C}_L$, and whose corresponding notion of proof is denoted $\vdash_L$.

Then our aim is to axiomatise the finitary logic defined by the class of matrices $\mathcal{C}_L^0 = \{\langle [0,1]_*, F_{(0} \rangle \mid \langle [0,1]_*, F_1 \rangle \in \mathcal{C}_L\}$. We syntactically define the system nf-L, the non-falsity preserving companion of L, as follows.

Definition 2. *The logic* nf-*L is defined by taking as axioms those of L together with the following inference rules:*

- *Rule of Adjunction: (Adj)* $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- *Restricted Modus Ponens: (r-MP)* $\dfrac{\varphi, \quad \vdash_L \varphi \to \psi}{\psi}$
- *Reverse Modus Ponens: (MP^r)* $\dfrac{\neg\psi \vee \chi}{\neg\varphi \ \vee \ \neg(\varphi \to \psi) \ \vee \ \chi}$

The corresponding notion of proof will be denoted by $\vdash_{\mathsf{nf}\text{-}L}$.

The distinctive rule in the system nf-L is the rule (MP^r) rule captures the following form of reverse of modus ponens: if $\neg\psi$ is non-false then either $\neg\varphi$ is non-false or $\neg(\varphi \to \psi)$ is non-false. It is the involutiveness of the negation that guarantees the soundness of that rule. Indeed, for all $x, y \in [0,1]$, the MP condition "if $x = 1$ and $x \to y = 1$ then $y = 1$", by contraposition, is equivalent to the condition "if $y < 1$ then either $x < 1$ or $x \to y < 1$". Since the condition $z < 1$ iff $\neg z > 0$ is valid due to the fact that $\neg$ is involutive, the previous condition can be equivalently rephrased as "if $\neg y > 0$ then either $\neg x > 0$ or $\neg(x \to y) > 0$", which in turn it is equivalent to "if $\neg y > 0$ then $(\neg x) \vee \neg(x \to y) > 0$". The addition of the disjunct χ both in the premise and in the conclusion of the rule is needed to properly keep track of successive applications of (MP^r), as it will be made clear in Example 1 below.

It is straightforward to check that the logic nf-L is sound w.r.t. the class of matrices $\mathcal{C}_L^0$. Only notice that, on the one hand, if a rule φ/ψ is sound for a matrix $\mathcal{M} = \langle [\mathbf{0},\mathbf{1}]_*, \{1\}\rangle \in \mathcal{C}_L$ then the rule $\neg\psi \vee \chi / \neg\varphi \vee \chi$ is automatically sound for the matrix $\mathcal{M}' = \langle [\mathbf{0},\mathbf{1}]_*, (0,1]\rangle \in \mathcal{C}_L^0$.

In order to show the logic nf-L is complete, we first prove in the following proposition a syntactic counterpart of Lemma 1, relating proofs in L and proofs in nf-L.

Proposition 2. *If* $\{\psi_1, \ldots, \psi_n\} \vdash_L \varphi$ *then* $\neg\varphi \vdash_{\mathsf{nf}\text{-}L} \neg\psi_1 \vee \ldots \vee \neg\psi_n$.

Proof. Let $\Gamma = \{\psi_1, \ldots, \psi_n\}$ and suppose $\Gamma \vdash_L \varphi$. Then in L there is a proof $\langle \Pi_1, \ldots \Pi_r\rangle$, where $\Pi_1 = \psi_1$, $\Pi_r = \varphi$, and where each Π_i (with $1 < i \leq r$) either:

- is a formula of Γ, or
- is an axiom of L, or
- has been obtained from previous Π_k, Π_j $(k, j < r)$ by the application of the Modus ponens rule (MP).

We show next, in a non-totally formal way, how we can build a proof for $\neg\psi_1 \vee \ldots \vee \neg\psi_n$ from $\neg\varphi$ in nf-L. We define:

(1) $\Sigma_1 = \neg\Pi_r = \neg\varphi$.

(2) By induction, let $j > 1$ and assume Σ_j is defined and let us define Σ_{j+1} as follows.

We can further assume that Σ_j will be of the form $\Sigma_j = \neg\Pi_{j_1} \vee \ldots \neg\Pi_{j_m}$ for some set of indices $\{j_1, \ldots, j_m\} \subseteq \{1, \ldots, r\}$. Then we define $\Sigma^+_{j+1} = (\neg\Pi_{j_1})^* \vee \ldots \vee (\neg\Pi_{j_m})^*$, where, for each $i = 1, m$:

- If $\Pi_{j_i} \in \Gamma$ or is an axiom of L, then we set $(\neg\Pi_{j_i})^* = \neg\Pi_{j_i}$
- If Π_{j_i} has been obtained by Modus Ponens from previous Π_k and Π_l, then we set $(\neg\Pi_{j_i})^* = \neg\Pi_k \vee \neg\Pi_l$. This means that $\neg\Pi_k \vee \neg\Pi_l$ is the result of applying the Reverse Modus Ponens to $\neg\Pi_{j_i}$.

(3) As a result of the previous step, Σ^+_{j+1} is the form $\Sigma^+_{j+1} = \neg\Pi_{(j+1)_1} \vee \ldots \vee \neg\Pi_{(j+1)_p}$, where some of the $\neg\Pi_{(j+1)_i}$'s are such that $\Pi_{(j+1)_i}$ are axioms of L. Taking into account that if ϕ is an axiom, then $\Psi \vee \neg\phi \to \Psi$ is a theorem of L, we can apply the (r-MP) as many times as needed to remove all these clauses $\neg\Pi_{(j+1)_i}$ from Σ^+_{j+1}.

Finally, we define Σ_{j+1} to be such disjunction which is free of negated axiom clauses.

Note that Σ_{j+1} may result in not of a single proof step but of several, as many as applications of the (MP^r) and (r-MP) rules have been needed to deal with Σ^+_{j+1}.

(4) If $\Sigma_{j+1} = \Sigma_j$ it means that in the step (2) there is no i for which Π_{j_i} has been obtained by Modus Ponens. Then we stop since it means that Σ_{j+1} is already the disjunction of the negated formulas of Γ and hence the sequence

$$\langle \Sigma_1, \ldots, \Sigma_j\rangle$$

is a proof although, as noted above, each Σ_i can encapsulate more than one single proof step.

□

We provide next a small example of translating a proof in a IMTL logic L into a proof in the logic nf-L, showing the use of the reverse Modus Ponens rule (MP^r).

Example 1. Let L be a axiomatic extension of IMTL, and consider the following derivation

$$\varphi, \psi, \varphi \wedge \psi \to \chi \vdash_{\mathrm{L}} \chi$$

which clearly holds, so the sequence

- $\Pi_1 = \varphi$, premise
- $\Pi_2 = \psi$, premise
- $\Pi_3 = \varphi \to (\psi \to \varphi \wedge \psi)$, axiom of MTL
- $\Pi_4 = \psi \to \varphi \wedge \psi$, by application of (MP) to Π_1 and Π_3
- $\Pi_5 = \varphi \wedge \psi$, by application of (MP) to Π_2 and Π_4
- $\Pi_6 = (\varphi \wedge \psi) \to \chi$, premise
- $\Pi_7 = \chi$, by application of (MP) to Π_5 and Π_6

is a proof of χ from $\{\varphi, \psi, \varphi \wedge \psi \to \chi\}$ in L.

Now let us see how to get a corresponding proof in nf-L for the derivability

$$\neg\chi \vdash_{\mathsf{nf\text{-}L}} \neg\varphi \vee \neg\psi \vee \neg(\varphi \wedge \psi \to \chi).$$

Following the procedure described in the proof of the above proposition, we get that the following sequence

- $\Sigma_1 = \neg\Pi_7 = \neg\chi$
- $\Sigma_2 = \neg\Pi_5 \vee \neg\Pi_6 = \neg(\varphi \wedge \psi) \vee \neg(\varphi \wedge \psi \to \chi)$, by application of (MP^r) to $\neg\chi$, since $\Pi_7 = MP(\Pi_5, \Pi_6)$ and Π_6 is a premise
- $\Sigma_3 = \neg\Pi_2 \vee \neg\Pi_4 \vee \neg\Pi_6 = \neg\psi \vee \neg(\psi \to \varphi \wedge \psi) \vee \neg((\varphi \wedge \psi) \to \chi)$. Indeed, note that, since $\Pi_5 = MP(\Pi_2, \Pi_4)$ and Π_6 is a premise, $\Sigma_3^+ = (\neg\Pi_5)^* \vee (\neg\Pi_6)^* = \neg\Pi_2 \vee \neg\Pi_4 \vee \neg\Pi_6 = \neg\psi \vee \neg(\psi \to \varphi \wedge \psi) \vee \neg((\varphi \wedge \psi) \to \chi)$. Since there is no negated axiom clause in Σ_3^+, then $\Sigma_3 = \Sigma_3^+$.
- $\Sigma_4 = \neg\Pi_2 \vee \neg\Pi_1 \vee \neg\Pi_6 = \neg\psi \vee \neg\varphi \vee \neg(\varphi \wedge \psi \to \chi)$. Indeed, since $\Pi_4 = MP(\Pi_1, \Pi_3)$, Π_2 and Π_6 are premises and Π_3 is an axiom, $\Sigma_4^+ = (\neg\Pi_2)^* \vee (\neg\Pi_4)^* \vee \neg(\Pi_6)^* = \neg\Pi_2 \vee \neg\Pi_1 \vee \neg\Pi_3 \vee \neg\Pi_6 = \neg\psi \vee \neg\varphi \vee \neg(\varphi \to (\psi \to \varphi \wedge \psi)) \vee \neg((\varphi \wedge \psi)$. Now, since Π_3 is an axiom, we can remove the clause $\neg\Pi_3$ from Σ_4^+ by applying (MP-r). Hence $\Sigma_4 = \neg\Pi_2 \vee \neg\Pi_1 \vee \neg\Pi_6$.
- $\Sigma_5 = \Sigma_4$, since Π_1, Π_2 and Π_6 are all premises.

Therefore, $\langle \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4 \rangle$ is a proof of $\neg\psi \vee \neg\varphi \vee \neg(\varphi \to (\psi \to \chi))$ from $\neg\chi$ in nf-L. □

Theorem 3. *Let L be an axiomatic extension of IMTL. The above calculus* nf-*L is sound and complete w.r.t. to the finitary logic defined by the class of matrices* $\mathcal{C}_L^0$.

Proof. Soundness is easy and has already been mentioned above. As for completeness, suppose $\{\psi_1, ..., \psi_n\} \models_{\mathcal{M}} \varphi$ for every $\mathcal{M} \in \mathcal{C}_L^0$. This is equivalent to $\neg\varphi \models_{\mathcal{M}'} \neg(\psi_1 \wedge ... \wedge \psi_n)$ for every $\mathcal{M}' \in \mathcal{C}_L$. By completeness of L, there is a proof $\langle \Pi_1, \ldots \Pi_r \rangle$, where $\Pi_1 = \neg\varphi$, $\Pi_r = \neg\psi_1 \vee ... \vee \neg\psi_n$. Now, by the above Proposition 2, there is also a proof of $\neg\neg\varphi$ from $\neg\neg(\psi_1 \wedge ... \wedge \psi_n)$ in nf-L. Then, if $\Pi_1, ..., \Pi_r$, with $\Pi_1 = \neg\neg(\psi_1 \wedge ... \wedge \psi_n)$ and $\Pi_r = \neg\neg\varphi$, is a proof of $\neg\neg\varphi$ from $\neg\neg(\psi_1 \wedge ... \wedge \psi_n)$, to get a proof of φ from $\Gamma = \{\psi_1, ..., \psi_n\}$ it is enough to add:

- A previous step $\Pi_0 = \psi_1 \wedge ... \wedge \psi_n$, obtained by $n-1$ applications of the Adjunction rule (Adj) to the premises Γ. Then Π_1 is obtained by applying the (r-MP) rule to Π_0 and the theorem $\psi_1 \wedge ... \wedge \psi_n \to \neg\neg(\psi_1 \wedge ... \wedge \psi_n)$.
- And a final step $\Pi_{r+1} = \varphi$, obtained by applying the (r-MP) rule to Π_r and the theorem $\neg\neg\varphi \to \varphi$. □

Since both Łukasiewicz logic and of Nilpotent Minimum logic are axiomatic extensions of IMTL, as a direct corollary we obtain complete axiomatisations of the non-falsity preserving companions of Łukasiewicz logic and of Nilpotent Minimum logic, nf-Ł and nf-NML respectively.

Actually, regarding Łukasiewicz logic, we have mentioned in the introduction that Avron introduces in [1] a paraconsistent extension of the logic T of Anderson and Belnap called FT. This logic is presented as "a paraconsistent counterpart of Łukasiewicz Logic $Ł_\infty$ that preserves *non-falsity*" [1, pp. 75] in the sense that it takes the semi-open interval $(0,1]$ as set of designated values, i.e., all values from $[0,1]$ except the value 0 (falsity). The logic is firstly defined axiomatically over a propositional language with connectives $\wedge$, $\vee$, $\neg$ and $\to_{\mathrm{FT}}$, and then it is proved that FT is semantically characterised by the logic matrix $\langle \mathbf{M_{[0,1]}}, F \rangle$, where $\mathbf{M_{[0,1]}} = ([0,1], \wedge, \vee, \neg, \to_{\mathrm{FT}}, 0, 1)$ and the filter of designated values is $F = (0,1]$. In $\mathbf{M_{[0,1]}}$, the operations $\wedge$, $\vee$ and $\neg$ are as in Łukasiewicz logic (i.e. interpreted by min, max and $n(x) = 1-x$, respectively), but $\to_{\mathrm{FT}}$ is not Łukasiewicz implication (whose truth-function is $x \to y = \min(1, 1-x+y)$) but interpreted by the following truth-function:

$$x \to_{\mathrm{FT}} y = \begin{cases} \max(1-x, y), & \text{if } x \leq y \\ 0, & \text{otherwise.} \end{cases}$$

In fact, $\to_{\mathrm{FT}}$ captures the order since it satisfies the relation $x \to_{\mathrm{FT}} y > 0$ iff $x \leq y$. Avron shows nice properties for this logic (semi-relevance, variable-sharing, modus ponens, etc.), but FT is something else than the non-falsity companion of Łukasiewicz logic nf-Ł that we have presented above.

As for the case of Nilpotent Minimum logic NML, its non-falsity preserving companion nf-NML has been recently studied in [25].

3.3 The case of MTL logics validating the rule $(\mathbf{R}^{\neg\neg})$

In this section we show that to prove the results in the previous section the requirement of the negation $\neg$ to be involutive, as it happens in IMTL logics, can be significantly weakened. Indeed, let MTL$^{\neg\neg}$ be the (non-axiomatic) extension of MTL with the rule

$$(\mathrm{R}^{\neg\neg}) \quad \frac{\neg\neg\varphi}{\varphi},$$

introduced in [13]. The algebraic semantics of MTL$^{\neg\neg}$ consists of the quasivariety generated by the class of MTL-chains $\mathbf{A}$ such that its negation $\neg$ is such that, for any $a \in A$, $\neg a = 0$ iff $a = 1$, or equivalently $\neg a > 0$ iff $a < 1$.

If L is an axiomatic extension of MTL, let us denote by L$^{\neg\neg}$ the extension of L with the rule (R$^{\neg\neg}$). Note that if L is an IMTL logic, then L$^{\neg\neg}$ = L. If L is complete w.r.t. a class of standard matrices $\mathcal{C}_L$ over the real unit interval $[0, 1]$, then L$^{\neg\neg}$ is complete w.r.t. the class of matrices $\mathcal{C}_{L^{\neg\neg}} = \{\langle [0, 1]_*, \{1\}\rangle \mid \langle [0, 1]_*, \{1\}\rangle \in \mathcal{C}_L$ s.t. for all $x, \neg x = 0$ iff $x = 1\}$.

It is worth observing that, although it keeps being semilinear, L$^{\neg\neg}$ is not necessarily a core fuzzy logic, see e.g. [9, Th. 3.29]. Indeed, take for instance L to be the so-called Weak Nilpotent Minimum (WNM) logic and consider the standard WNM-algebra $[\mathbf{0}, \mathbf{1}]_*$ where $*$ is the WNM t-norm defined by the negation $\neg$ such that $\neg(x) = 1 - x$ for $x \in [0, 1/3] \cup [2/3, 1]$ and $\neg(x) = 1/3$ for $x \in (1/3, 2/3)$. This is a WNM$^{\neg\neg}$-algebra, where $\neg(\neg(x)) = x$ for $x \in [0, 1/3] \cup [2/3, 1]$ and $\neg(\neg(x)) = 2/3$ for $x \in (1/3, 2/3)$. Take $a = 1/2$, hence $\neg(\neg(x) = 2/3 > 1/2$, but there is no natural n such that $(2/3)^n = (2/3) * \overset{n}{\ldots} * (2/3) \leq 1/2$, since $(2/3)^n = 2/3$ for every n. This means that $\neg\neg\varphi \vdash_{WNM^{\neg\neg}} \varphi$ but $\nvdash_{WNM^{\neg\neg}} (\neg\neg\varphi)^n \to \varphi$ for any n. Thus, WNN$^{\neg\neg}$ does not satisfy the local deduction theorem, and hence it is not a core fuzzy logic.

By Proposition 1, in L$^{\neg\neg}$ we keep having at the semantical level the equivalence between the 1-preserving logic and the non-falsity preserving logic, in the following sense.

Lemma 2. *For any logic L extension of MTL, the following conditions hold:*

(i) $\varphi \models^{1}_{L^{\neg\neg}} \psi$ *iff* $\neg\psi \models^{(0}_{L^{\neg\neg}} \neg\varphi$, *(ii)* $\varphi \models^{(0}_{L^{\neg\neg}} \psi$ *iff* $\neg\psi \models^{1}_{L^{\neg\neg}} \neg\varphi$.

Then one can define the non-falsity preserving companion of a MTL$^{\neg\neg}$-logic and prove its completeness as follows. In fact, we can restrict ourselves to extensions of non-SMTL logics validating the rule (R$^{\neg\neg}$). Indeed, note that if L is a SMTL logic, then L$^{\neg\neg}$ collapses into classical logic. This is so because, using the rule (R$^{\neg\neg}$), from axiom (PC) $\neg(\varphi \wedge \neg\varphi)$, which can be equivalently expressed in MTL as $\neg\varphi \vee \neg\neg\varphi$, L$^{\neg\neg}$ then derives the Excluded-Middle axiom $\neg\varphi \vee \varphi$.

Theorem 4. *Let* L *be a non-SMTL logic. Then the calculus* nf-$L^{\neg\neg}$ *defined by the following axioms and rules:*

- *Axioms of L*
- *The rule (R$^{\neg\neg}$)*
- *The rule of adjunction (Adj)*
- *The rule of Restricted Modus Ponens (r-MP)*
- *The rule of Reverse Modus Ponens (MP^r)*

is a sound and complete axiomatisation w.r.t. to the finitary logic defined by the class of matrices $\mathcal{C}^0_{L^{\neg\neg}}$.

The proof is an easy adaptation of the proof of Theorem 3 for IMTL logics.

3.4 The case of SMTL logics and their expansions with an involutive negation

In this last subsection we consider the case of SMTL logics. A first straightforward observation is that if L is an extension of SMTL, the residual negation $\neg\varphi = \varphi \to 0$ is in fact Gödel negation, whose interpretation in any L-chain is the mapping defined by $\neg x = 1$ if $x = 0$ and $\neg x = 0$ otherwise. Hence $x > 0$ iff $\neg\neg x = 1$. Also, note that the monoidal operation $*$ (strong conjunction) in any SMTL-chain has no zero divisors, i.e. if $x * y = 0$ then either $x = 0$ or $y = 0$.

Let us recall as well that any axiomatic extension L is complete with respect to the class of L-chains, i.e. wrt the set of matrices $C_L = \{\langle \mathbf{A}, F_1\rangle \mid \mathbf{A} \text{ is a L-chain}\}$. As before we will let $C_L^0 = \{\langle \mathbf{A}, F_{(0}\rangle \mid \mathbf{A} \text{ is a L-chain}\}$. Then, the following lemma holds.

Lemma 3. *For any axiomatic extension L of SMTL, the following hold:*
(i) $\Gamma \models_{C_L^0} \psi$ *iff* $\Gamma^{\neg\neg} \vdash_L \neg\neg\psi$, *where* $\Gamma^{\neg\neg} = \{\neg\neg\varphi \mid \varphi \in \Gamma\}$
(ii) Modus ponens is a valid rule in $\models_{C_L^0}$.

Proof. Property (i) follows from the above observation that any SMTL-evaluation e in a L-chain $\mathbf{A}$ is such that $e(\varphi) > 0$ iff $e(\neg\neg\varphi) = 1$. As for (ii) note that, by definition, $e(\varphi \to \psi) = \sup\{a \in A \mid e(\varphi) * a \leq e(\psi)\}$. So if $e(\varphi \to \psi) > 0$, there exists $a > 0$ such that $e(\varphi) * a \leq e(\psi)$. Now, if $e(\varphi) > 0$ then, since $*$ has no zero-divisors, necessarily $0 < e(\varphi) * a \leq e(\psi)$. □

These are nice properties, however they imply that the falsity-preserving companion of a SMTL logic collapses into classical propositional logic CPL.

Lemma 4. *Let* $\mathbf{A}$ *be a SMTL-chain and let the matrix* $M = (\mathbf{A}, F_{(0})$. *Then* $\varphi \models_M \psi$ *iff* $\varphi \vdash_{CPL} \psi$.

Proof. Since the matrix of classical propositional logic $\langle \mathbf{2}, \{1\}\rangle$, where $\mathbf{2}$ is the 2-element Boolean algebra on $\{0, 1\}$, is a submatrix of $M = \langle \mathbf{A}, F_{(0)}\rangle$, then $\varphi \models_M \psi$ implies $\varphi \vdash_{CL} \psi$. Conversely, let the mapping $h : A \to \{0, 1\}$ be defined as $h(0) = 0$ and $h(x) = 1$ if $x > 0$. Then it is easy to check that h is a homomorphism of SMTL-algebras. Therefore, if $\varphi \not\models_M \psi$, there is a $\mathbf{A}$-evaluation e such that $e(\varphi) > 0$ and $e(\psi) = 0$. But then, the evaluation $e' = h \circ e$ is a $\mathbf{2}$-evaluation such that $e'(\varphi) = 1$ and $e(\psi) = 0$, hence $\varphi \not\vdash_{CL} \psi$. □

As an immediate consequence of (i) of Lemma 3 and the previous Lemma 4, we get the following corollary, that can be seen as Glivenko-like theorem for the non-falsity preserving companions of SMTL logics.

Corollary 1. *(Glivenko theorem for SMTL) Let L be an axiomatic extension of SMTL. Then* $\varphi \models_{C_L^0} \psi$ *iff* $\neg\neg\varphi \vdash_L \neg\neg\psi$ *iff* $\varphi \vdash_{CL} \psi$.

Therefore, in order to get paraconsistent non-falsity preserving companions of SMTL logics, we turn our attention to expansions of such logics with an involutive negation $\sim$. Indeed, having an involutive negation in the logic makes the corresponding $\sim$-paraconsistent system not collapse into the classical case as we have seen above it happens with axiomatic extensions of SMTL logic. As it can be easily observed, the mapping h defined in the proof of Lemma 4, is no longer an homomorphism in the case the chain $\mathbf{A}$ has an involutive negation in its signature.

For the case of SBL and its extensions Gödel and Product logics, these expansions were defined in [22], and for the more general setting of axiomatic extensions of MTL in [23]. Following the latter, if L is an axiomatic extension of SMTL, then the logic $\mathrm{L}_\sim$ is obtained from L by adding the connective $\sim$ to the language of L, together with the following axioms, where $\Delta\varphi := \neg{\sim}\varphi$:

($\sim$1) $\varphi \leftrightarrow {\sim}{\sim}\varphi$
($\sim$2) $\Delta(\varphi \to \psi) \to ({\sim}\psi \to {\sim}\varphi)$
($\sim$3) $\neg\varphi \to {\sim}\varphi$

and the following inference rule: from φ derive $\Delta\varphi$. Note that in $\mathrm{L}_\sim$, $\neg\neg\varphi$ is logically equivalent to $\neg\Delta\neg\varphi$.

In [22] it was proved that $\mathrm{G}_\sim$ is complete with respect to a single matrix $C_{G_\sim} = \langle [0,1]_{G_\sim}, F_1\rangle$ over the standard $\mathrm{G}_\sim$-chain $[0,1]_{G_\sim} = ([0,1], \min, \max, *_G, \to_G, n, 0, 1)$, where $*_G = \min$, $\to_G$ is Gödel implication and $n(x) = 1 - x$, while $\Pi_\sim$ is complete w.r.t. the set of matrices $C_{\Pi_\sim} = \{\langle [0,1]_{\Pi,n}, F_1\rangle \mid n \text{ is a strong negation in } [0,1]\}$, where $[0,1]_{\Pi,n} = ([0,1], \min, \max, *_\Pi, \to_\Pi, n, 0, 1)$. Similarly, in [23], $\mathrm{SMTL}_\sim$ was proved to be complete w.r.t. the set of matrices $C_{\mathrm{SMTL}_\sim} = \{\langle [0,1]_{*,n}, F_1\rangle \mid * \text{ is a SMTL t-norm and } n \text{ is a strong negation in } [0,1]\}$.

Now, let $\mathrm{L}_\sim$ be the expansion with an involutive negation $\sim$ of an axiomatic extension L of SMTL, which we assume is complete with respect to a set of matrices $C_{L_\sim}$, and let us consider its corresponding set of matrices with filters $F_{(0}$, $C^0_{L_\sim} = \{\langle [0,1]_{*,n}, F_{(0}\rangle \mid \langle [0,1]_{*,n}, F_1\rangle \in C_{L_\sim}\}$.

Next lemma is the counterpart of Lemma 3 for expansions L of the SMTL logic with $\sim$ and shows the interpretation of non-falsity preserving logic $\models_{C^0_{L_\sim}}$ into the truth-preserving logic $\mathrm{L}_\sim$.

Lemma 5. *$\Gamma \models_{C^0_{L_\sim}} \psi$ iff $\{\neg\neg\varphi \mid \varphi \in \Gamma\} \vdash_{L_\sim} \neg\neg\psi$.*
Moreover, $\Gamma \cup \{\varphi\} \models_{C^0_{L_\sim}} \psi$ iff $\Gamma \models_{C^0_{L_\sim}} \neg\neg\varphi \to \neg\neg\psi$.

The key observation here is that $\mathrm{L}_\sim$ is in fact a Δ-core fuzzy logic since all axioms of Δ are provable, and hence $\mathrm{L}_\sim$ is an axiomatic expansion of L_Δ. Because of this, we can also interpret $\mathrm{L}_\sim$ into the non-falisty preserving consequence relation $\models_{C^0_{L_\sim}}$.

Lemma 6. *$\Gamma \vdash_{L_\sim} \psi$ iff $\{\Delta\varphi \mid \varphi \in \Gamma\} \models_{C^0_{L_\sim}} \Delta\psi$.*

As a consequence, we can make use of the results of the previous Subsection 3.1 regarding the non-falsity preserving companion of a Δ-core fuzzy logic.

We define now the non-falsity preserving companion of $\mathrm{L}_\sim$.

Definition 3. *The logic* nf-$L_\sim$ *is obtained by adding to the axioms of $L_\sim$ the following inference rules:*

- *Adjunction: (Adj)* $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- *Restricted Necessitation for Δ: (r-NΔ)* $\dfrac{\varphi}{\Delta\varphi}, \quad \text{if} \quad \vdash_{L_\sim} \varphi$
- *Restricted Modus Ponens: (r-MP)* $\dfrac{\varphi, \quad \vdash_{L_\sim} \varphi \to \psi}{\psi},$
- *Positive Modus Ponens: (Pos-MP)* $\dfrac{\varphi, \quad \neg\neg\varphi \to \neg\neg\psi}{\psi}$

Completeness for nf-$L_\sim$ directly follows as a corollary of Theorem 2.

Theorem 5. *nf-$L_\sim$ is sound and complete wrt the finitary logic defined by the set of matrices $C^0_{L_\sim}$.*

Thanks to Lemmas 5 and 6, we can observe that nf-$L_\sim$ will be finitary iff $L_\sim$ is so.

4 Logics of ‘acceptance’ and ‘tolerance’

The most distinctive feature of any system of fuzzy logic is that the notion of truth (and falsity) is a matter of degree. In a sense, in each interpretation e in a logic L, one can consider that every proposition φ is assigned a degree of being true, represented by the value $e(\varphi)$ and a degree of being false, represented by the value $e(\neg\varphi)$. Nonetheless, notice that the situation may be non-symmetric if the logic L is not involutive, that is, the degree in which $\neg\varphi$ is false, $e(\neg\neg\varphi)$, may be different from (bigger than, in fact) the degree in which φ is true, i.e. $e(\neg\neg\varphi) > e(\varphi)$.

Although the context is somewhat different, we take inspiration from [16] and consider that a proposition can be *accepted* (as if it were true) when it is “more true than false”. In other words, we propose to define that a proposition φ is *accepted* by a given interpretation e whenever $e(\varphi) > e(\neg\varphi)$, or conversely, to say that the interpretation e accepts φ. Then, the question we pose ourselves is which is the logic that captures reasoning with accepted propositions.

Definition 4. *Let L be a a finitary expansion of MTL which is complete with respect to a class $\mathcal{C}_L$ of standard L-algebras on $[0,1]$. Further let $[\mathbf{0},\mathbf{1}]_* \in \mathcal{C}_L$ be a standard L-algebra and let e be a $[\mathbf{0},\mathbf{1}]_*$-interpretation. We say that e* accepts *φ, written $e \models^{acc} \varphi$, whenever $e(\varphi) > e(\neg\varphi)$. The corresponding notion of consequence relation, denoted $\models^{acc}_L$, is accordingly defined as follows:*

$$\begin{array}{ll} \Gamma \models^{acc}_L \varphi \quad \textit{if,} & \textit{for any } [\mathbf{0},\mathbf{1}]_* \in \mathcal{C}_L \textit{ and any } [0,1]_*\textit{-evaluation } e, \\ & \textit{if } e \models^{acc} \psi \textit{ for any } \psi \in \Gamma, \textit{ then } e \models^{acc} \varphi \textit{ as well.} \end{array}$$

Let us show how this general definition instantiates in some noteworthy classes of MTL logics:

- If L is a SMTL logic, which means that the negation $\neg$ is Gödel negation on L-chains,[6] then $\models_L^{acc}$ is nothing but the non-falsity preserving logic nf-L, which in turn coincides with CPL, the classical propositional logic, see Lemma 4.
- In general, if L is a (non-SMTL) core fuzzy logic which is complete with respect a standard algebra $[0,1]_*$ whose negation has a fixpoint $0 < r < 1$, then $\models_L^{acc}$ coincides with the strict threshold logic $\models_L^{(r}$. The reason is because, for any evaluation e on $[0,1]_*$, $e(\varphi) > e(\neg\varphi)$ iff $e(\varphi) > r$.
- In particular, if L is Łukasiewicz logic Ł, which has 1/2 as negation fix point with its standard semantics, it turns out that $\models_{Ł}^{acc} = \models_{Ł}^{(1/2}$.
- Similarly, if L is Nilpotent Minimum logic NML, which has also 1/2 as negation fix point with its standard semantics, then $\models_{\mathrm{NM}}^{acc} = \models_{\mathrm{NM}}^{(1/2}$, and as it is shown in [25], $\models_{\mathrm{NM}}^{(1/2}$ coincides with the 3-valued Łukasiewicz logic $Ł_3$.

The following are some interesting properties of $\models_L^{acc}$ for any L being an extension of MTL:

Lemma 7. *The following properties of $\models_L^{acc}$ hold:*

1. *Consistency:* $\models_L^{acc} \top$, $\not\models_L^{acc} \bot$
2. *Paracompleteness:* $\not\models_L^{acc} \varphi \vee \neg\varphi$*, unless L is a SMTL-logic*
3. *Explosiveness:* $\varphi, \neg\varphi \models_L^{acc} \bot$
4. *Closure under conjunction:*
   (i) $\varphi, \psi \models_L^{acc} \varphi \wedge \psi$
   (ii) $\varphi, \psi \models_L^{acc} \chi$ *iff* $\varphi \wedge \psi \models_L^{acc} \chi$
5. *If* $\models_L \varphi \to \psi$ *then* $\varphi \models_L^{acc} \psi$.

Proof. 1. Direct, since for all e, $1 = e(\top) > e(\bot) = 0$.

2. $\not\models_L^{acc} \varphi \vee \neg\varphi$ iff there exists e such that $x \vee \neg x \leq \neg x \vee \neg\neg x$, where $x = e(\varphi)$ and, since $x \leq \neg\neg x$, this always happens.
3. Clear, since if $x > \neg x$ and $\neg x > \neg\neg x$ then we would have $x > \neg\neg x$, which is in contradiction with the law $x \leq \neg\neg x$.
4. Both (i) and (ii) follow from observing that $x > \neg x$ and $y > \neg y$ iff $x \wedge y > \neg x \vee \neg y = \neg(x \wedge y)$
5. If $x \leq y$ and $x > \neg x$ then $\neg x \geq \neg y$, so we have $y \geq x > \neg x \geq \neg y$, hence $y > \neg y$.

□

Note that the above Item 5 admits two equivalent interesting readings, namely that:

(i) the restricted Modus Ponens rule

$$\frac{\varphi, \quad \vdash_L \varphi \to \psi}{\psi}$$

is sound for $\models_L^{acc}$, and that

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[6] That is, $\neg(0) = 1$ and $\neg(x) = 0$ for $x \neq 0$.

(ii) $\models_L^{\leq} \subseteq \models_L^{acc}$, since the condition $\models_L \varphi \to \psi$ is equivalent to $\varphi \models_L^{\leq} \psi$.

Moreover, $\models_L^{acc}$ is interpretable in $\models_L$, and hence in $\models_L^{(0}$ as well. Indeed the following equivalences hold:

$$\varphi \models_L^{acc} \psi \quad \text{iff} \quad \psi \to \neg\psi \models_L \varphi \to \neg\varphi \quad \text{iff} \quad \neg(\varphi \to \neg\varphi) \models_L^{(0} \neg(\psi \to \neg\psi) \tag{1}$$

The reason is that, for any evaluation e, the condition "$e(\varphi) > e(\neg\varphi)$ implies $e(\psi) > e(\neg\psi)$" is equivalent to "$e(\psi) \leq e(\neg\psi)$ implies $e(\varphi) \leq e(\neg\varphi)$".

4.1 The acceptance logic companion of a Δ-core fuzzy logic

The axiomatization of $\models_L^{acc}$ is rather simple in case the Δ connective is available. Indeed, let L be a Δ-core fuzzy logic such that the Baaz-Monteiro Δ operator is either primitive or definable, for instance MTL_Δ (primitive) or Gödel logic with involution $\mathrm{G}_\sim$ (definable). We further assume that L only has Modus Ponens and necessitation for Δ as inference rules, and that is complete with respect to a corresponding class of standard algebras $\mathcal{C}_L$.

In this logic we introduce the following definable connective:

$$\boxdot\varphi := \Delta(\neg\varphi \to \varphi) \wedge \neg\Delta(\varphi \to \neg\varphi).$$

Note that, for any interpretation e on any standard L-chain from $\mathcal{C}_L$, $e(\boxdot\varphi) = 1$ iff $e(\neg\varphi) < e(\varphi)$. In fact, the logic $\models_L^{acc}$ is mutually interpretable in $\models_L$ itself, since it clearly holds that.

$$\Gamma \models_L^{acc} \psi \quad \text{iff} \quad \boxdot\Gamma \models_L \boxdot\psi,$$

where $\boxdot\Gamma$ is a shorthand for $\{\boxdot\varphi \mid \varphi \in \Gamma\}$, and conversely, it is easy to check that

$$\Gamma \models_L \psi \quad \text{iff} \quad \Delta\Gamma \models_L^{acc} \Delta\psi$$

holds as well, where $\Delta\Gamma$ is a shorthand for $\{\Delta\varphi \mid \varphi \in \Gamma\}$. Indeed, for any interpretation e on any standard L-chain from $\mathcal{C}_L$, we have that $e(\Delta\varphi) > e(\neg\Delta(\varphi))$ iff $e(\varphi) = 1$.

It is easy to check that this mutual interpretability between $\models_L^{acc}$ and $\models_L$ preserves the property of being finitary.

Lemma 8. *If L is a Δ-core fuzzy logic, then $\models_L^{acc}$ is finitary iff $\models_L$ is so.*

From an axiomatic point of view, let us define the acceptance variant of L, denoted L^{acc}, as follows.

Definition 5. *Axioms and rules of L^{acc} are:*

- *Axioms of L*
- *Adjunction rule: (Adj)* $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- *Restricted Modus Ponens: (r-MP)* $\dfrac{\varphi, \quad \vdash_L \varphi \to \psi}{\psi}$

- *Restricted Δ-necessitation: (r-ΔNec)* $\dfrac{\vdash_L \varphi}{\Delta\varphi}$
- *$\boxdot$-Modus Ponens: ($\boxdot$-MP)* $\dfrac{\varphi, \quad \boxdot\varphi \to \boxdot\psi}{\psi}$

Theorem 6. *Let L be a Δ-core fuzzy logic . Then the system L^{acc} is a sound and complete axiomatization of the finitary $\models_L^{acc}$. That is, for any formulas $\varphi_1, ..., \varphi_n, \psi$, it holds*

$$\varphi_1, ..., \varphi_n \models_L^{acc} \psi \quad \text{ iff } \quad \varphi_1, ..., \varphi_n \vdash_{L^{acc}} \psi$$

Proof. Soundness is straightforward. As for completeness, assume $\varphi_1, ..., \varphi_n \models_L^{acc} \psi$. This is equivalent to assume $\boxdot\chi \models_L \boxdot\psi$, where $\chi = \varphi_1 \wedge ... \wedge \varphi_n$. By completness of L, this is equivalent to $\boxdot\chi \vdash_L \boxdot\psi$ and, by the Δ deduction theorem, to $\vdash_L \Delta(\boxdot\chi) \to \boxdot\psi$. Moreover, since $\boxdot\chi$ is a conjunction of Δ-formulas, which are Boolean, we have $\vdash_L \Delta(\boxdot\chi) \equiv \boxdot\chi$, hence we can safely assume $\vdash_L \boxdot\chi \to \boxdot\psi$.

Now, this means there is a proof $\langle \Pi_1, \ldots, \Pi_r = \boxdot\chi \to \boxdot\psi \rangle$, where all the Π_i's are theorems of L (either axioms of L or deduced from them using Modus Ponens or Δ-necessitation).

Therefore, since to obtain a proof in L^{acc} of ψ from $\varphi_1, ..., \varphi_n$, we only need to add: (i) a previous step to derive $\Pi_0 : \psi = \varphi_1 \wedge ... \wedge \varphi_n$ using the Adjunction rule, and (ii) a final step $\Pi_{r+1} = \psi$ where ψ is derived by the rule ($\boxdot$-IR)from χ and $\boxdot\chi \to \boxdot\psi$, which has been proved in step Π_r that is a theorem of L. Note that the inference rules of L used in the above proof steps Π_1, ..., Π_r in L, in fact to the restricted rule (r-MP) and (r-Δ) in the proof in L^{acc}. $\square$.

Notice that the unrestricted ($\boxdot$-MP) rule is sound. Indeed, $\boxdot\varphi \to \boxdot\psi$ is a Boolean formula (it can only take values 0 or 1), and an evaluation e accepts $\boxdot\varphi \to \boxdot\psi$ exactly when $e(\boxdot\varphi \to \boxdot\psi) = 1$, and this happens iff $e(\varphi) > e(\neg\varphi)$ implies $e(\psi) > e(\neg\psi)$. Therefore, in the above axiomatisation we need not restrict this time the ($\boxdot$-MP) rule.

We close this subsection by emphazising that problem of axiomatising the acceptance logic $\models_L^{acc}$ in the case of L being an axiomatic extension of MTL or core fuzzy logic, i.e. without Δ in its language, is left open.

4.2 The dual paraconsistent companion: the tolerance logic companion

It is also very natural to consider for a proposition φ a dual notion of being acceptable: we define φ is *tolerated* when $\neg\varphi$ is not acceptable. In other words, in the setting of a core fuzzy logic L, we can say that an L-evaluation e tolerates φ when $e(\neg\varphi) \leq e(\neg\neg\varphi)$.[7] This leads to the following notion of tolerance-preserving logic.

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[7] Notice that this condition is somewhat weaker than the condition of requiring $e(\neg\varphi) \leq e(\varphi)$, that may also be a reasonable alternative in MTL logics that are non IMTL logics.

Definition 6. *$\varphi \models^{tol}_L \psi$ if, for any L-evaluation e, if $e(\neg\neg\varphi) \geq e(\neg\varphi)$ then $e(\neg\neg\psi) \geq e(\neg\psi)$.*

Obvioulsy, in case L is a IMTL-logic, the condition $e(\neg\neg\psi) \geq e(\neg\psi)$ in the above definition simplifies to $e(\psi) \geq e(\neg\psi)$.

It is very easy to check that $\models^{tol}$ is interpretable in both the acceptance logic $\models^{acc}$ and the logic L itself. Indeed, the following equivalences hold:

(i) $\varphi \models^{tol}_L \psi$ iff $\neg\psi \models^{acc}_L \neg\varphi$
(ii) $\Gamma \models^{tol}_L \psi$ iff $\{\neg\varphi \to \neg\neg\varphi \mid \varphi \in \Gamma\} \models_L \neg\psi \to \neg\neg\psi$

Indeed, as for (i), the condition "if $\neg\neg x \geq \neg x$ then $\neg\neg y \geq \neg y$" is equivalent to "if $\neg\neg y < \neg y$ then $\neg\neg x < \neg x$". And (ii) follows from (i) and (1). Moreover, we observe that if L is a IMTL-logic, then (i) also gives the interpretability of $\models^{acc}_L$ in $\models^{tol}$. And, by (ii), if $\models_L$ es finitary then $\models^{tol}_L$ is finitary as well.

The logic $\models^{tol}$ has similar properties as $\models^{acc}$, but there is a main difference: while $\models^{acc}$ is not paraconsistent (it is explosive), $\models^{tol}$ is paraconsistent.

Lemma 9. *The following properties of $\models^{tol}_L$ hold:*

1. *Consistency: $\models^{tol}_L \top$, $\not\models^{tol}_L \bot$*
2. *Excluded middle: $\models^{tol}_L \varphi \vee \neg\varphi$*
3. *Paraconsistency: $\varphi, \neg\varphi \not\models^{tol}_L \bot$, unless L is a SMTL-logic*
4. *Closure under conjunction:*
   (i) $\varphi, \psi \models^{tol}_L \varphi \wedge \psi$
   (ii) $\varphi, \psi \models^{tol}_L \chi$ iff $\varphi \wedge \psi \models^{tol}_L \chi$
5. *if $\models_L \varphi \to \psi$ then $\varphi \models^{tol}_L \psi$. (i.e. $\models^{\leq}_L \subseteq \models^{tol}_L$)*

Going to some particular cases, it is interesting to make the following observations, which are similar to those in the previous section regarding $\models^{acc}_L$:

- If L is a SMTL-logic, then both logics coincide, i.e. $\models^{tol}_L = \models^{acc}_L$. Indeed, for any $x \in [0,1]$, the condition $\neg\neg x \geq \neg x$ is equivalent to require $x > 0$ (since $\neg\neg x = 1$ if $x > 0$ and $\neg\neg 0 = 0$). Therefore, $\models^{tol}_L$ coincides with the non-falsity preserving logic *nf*-L, and in turn this is just classical propositional logic, so the exactly same situation with $\models^{acc}_L$ as we have seen in the previous subsection.
- If L is a IMTL logic which is complete w.r.t. a standard IMTL-algebra $[0,1]_L$, and the negation has a fix-point r, then $\models^{tol}_L$ is nothing but the non-strict threshold logic $\models^{r}_L$, since in $[0,1]_L$ the condition $\neg\neg x \geq \neg x$ turns out to be equivalent to $x \geq r$. Note that in the same case, as we have seen before, the logic $\models^{acc}_L$ corresponds to the strict threshold logic $\models^{(r}_L$.[8]

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[8] Smith defines in [33] a consequence relation $\models_S$ for a degree-based approach to vagueness that, leaving aside some language and semantic differences (his language is built only upon the connectives $\wedge$, $\vee$ and $\neg$), combines both strict and non-strict thresholds for $r = 0.5$, namely $\varphi \models_S \psi$ holds when, for every evaluation e, if $e(\varphi) > 0.5$ then $e(\psi) \geq 0.5$. A very similar notion of consequence relation, in a 3-valued logical setting, is the *Strict Tolerant logic* introduced by Cobreros et al. in [11], where it is also compared to Smith's in [11].

Finally, let us consider the case where L is a Δ-core logic. If we define the operator $\boxdot'$ as $\boxdot'\varphi := \Delta(\neg\varphi \to \neg\neg\varphi)$, then the logic L^{tol} resulting from replacing in Definition 5 the rule ($\boxdot$-IR) by the rule

$$(\boxdot'\text{-IR}) \quad \frac{\varphi, \quad \boxdot'(\varphi) \to \boxdot'(\psi)}{\psi},$$

is a sound and complete axiomatisation of $\models_L^{tol}$. The proof is analogous to that for L^{acc}. We note that in this case, similarly to the case of $\models_L^{acc}$, $\models_L^{tol}$ is finitary if, and only if, $\models_L$ is so.

5 Threshold-preserving logics

In previous sections we have studied two different variants of MTL logics L, namely the non-falsity preserving companion of L, $\models_L^{(0}$, and the corresponding acceptance and tolerance logics $\models_L^{acc}$ and $\models_L^{tol}$, respectively. As we have already mentioned earlier, the former are a particular case of threshold-preserving logics, where the lattice filter is $(0, 1]$, while the latter are also related to particular threshold-preserving logics of the kind $\models_L^{(s}$ and $\models_L^{s}$ for some values $0 < s < 1$ linked to the negation operators in L-standard algebras.[9]

In this section we focus our attention to logics preserving arbitrary (strict and non-strict) lower bounds of truth-values in general, in other words, logics whose semantic consequence relations are of the form $\models_L^{a}$ and $\models_L^{(a}$ for some positive value $a \in (0, 1]$, as defined in Section 2.3, for some logic L extension of MTL complete with respect to some class of standard L-algebras $\mathcal{C}_L$.

We start by stating some general but sufficient assumptions on L to guarantee a finitary axiomatisation of $\models_L^{a}$. More in detail, consider the following two conditions on L:

(C1) L satisfies a form of global Deduction Theorem in the sense that there exists a term t such that:

$$\Gamma \cup \{\varphi\} \vdash_L \psi \quad \text{iff} \quad \Gamma \vdash_L t(\varphi) \to \psi$$

(C2) The logic $\models_L^{a}$ is interpretable in L, that is, there exists a term r such that:

$$\varphi \models_a \psi \quad \text{iff} \quad r(\varphi) \models_L r(\psi)$$

Theorem 7. *Let L be an extension of MTL satisfying conditions (C1) and (C2). Then the calculus L_a defined syntactically by*

- *the axioms of L,*
- *the rules of L restricted to theorems of L,*

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[9] In fact, if L is complete with respect to a standard algebra $[\mathbf{0}, \mathbf{1}]_*$, then s is the value which splits the interval $[0, 1]$ into the subintervals of positive and negative elements, namely such that $(s, 1] = \{x \in [0, 1] \mid \neg x < x\}$ and $[0, s] = \{x \in [0, 1] \mid \neg x \geq x\}$.

- *the rule of Adjunction, and*
- *the restricted rule*

$$(R_{t,r})\qquad \frac{\varphi, \quad \vdash_L t(r(\varphi)) \to r(\psi)}{\psi}$$

is a sound and complete axiomatisation of the finitary $\models^a_L$.

Proof. The following is a sketch of the proof:

(i) $\varphi \models^a_L \psi$ iff
(ii) $r(\varphi) \models_L r(\psi)$ iff –by condition (C2)
(iii) $\models_L t(r(\varphi)) \to r(\psi)$ iff –by condition (C1)
(iv) $\vdash_L t(r(\varphi)) \to r(\psi)$ iff –by completeness of L
(v) in L there is a proof $\langle \Pi_1, ..., \Pi_n \rangle$ where $\Pi_n = t(r(\varphi) \to r(\psi)$ iff
(vi) in L_a there is a proof $\langle \Pi_0, \Pi_1, ..., \Pi_n, \Pi_{n+1} \rangle$, where the steps $\Pi_1, ..., \Pi_n$ (with applications of the rules restricted to theorems) are as above and where $\Pi_0 = \varphi$ and $\Pi_{n+1} = \psi$, with Π_{n+1} obtained from Π_0 and Π_n by the application of the rule ($\text{R}_{t,r}$). ☐

5.1 The case of Łukasiewicz logics

We start this section by recalling some results from Bou [4] regarding the threshold logics $\models^a_{\text{Ł}}$ for $a \in (0, 1]$ and the logics $\models^{(a}_{\text{Ł}}$ for $a \in (0, 1)$, that show they are almost all different.

Proposition 3 ([4]). *The following conditions hold:*

- *For each* $a \in [0, 1) \cap \mathbb{Q}$, $\models^a_L \neq \models^{(a}_L$
- *For each* $a \in [0, 1) \setminus \mathbb{Q}$, $\models^a_L = \models^{(a}_L$
- *For each* $a, b \in (0, 1]$, *if* $a \neq b$ *then* $\models^a_L \neq \models^b_L$
- *For each* $a, b \in [0, 1)$, *if* $a \neq b$ *then* $\models^{(a}_L \neq \models^{(b}_L$

Therefore, there are a continuum of uncomparable threshold logics over the infinite-valued Łukasiewicz logic, all of them between the Łukasiewicz degree-preserving logic $\models^{\leq}_{\text{Ł}}$ and classical propositional calculus, see an approximate graphical representation in Fig. 1.

In fact, in [4] the problem of axiomatising these logics is left open, although, as it is mentioned also in [4], if $a \in [0, 1) \cap \mathbb{Q}$, then the logic $\models^a_{\text{Ł}}$ is interpretable in the 1-preserving logic thanks to the McNaughton terms. Indeed, in such a case, there is always a unary MacNaughton term f_a such that $f_a(x) = 1$ iff $x \geq a$. Therefore, it holds that $\varphi \vdash^a_{\text{Ł}} \psi$ iff $f_a(\varphi) \models_{\text{Ł}} f_a(\psi)$.

On the other hand, infinite-valued Łukasiewicz logic Ł does not fall in the scope of Theorem 7 since it does not satisfy condition (C1), namely as Ł does not have a form of a global deduction theorem. However, two particular instantiations of the above Theorem 7 are for the finite-valued Łukasiewicz logics[10] Ł_n

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[10] This case was partially studied in [14], here we provide more elegant axiomatisations.

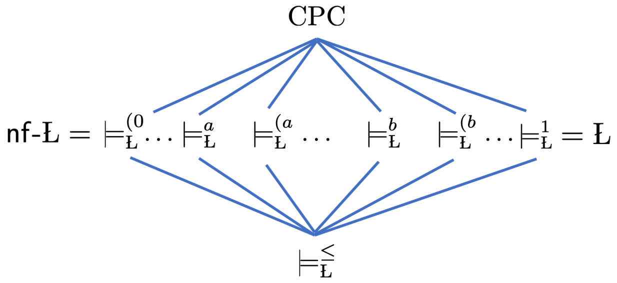

Fig. 1. Threshold-preserving Łukasiewicz logics with rational lower bounds

and for the infinite-valued Łukasiewicz logic Ł expanded with Baaz-Monteiro Δ operator $Ł_\Delta$.

Finite-valued Łukasiewicz logics $Ł_n$ are complete with respect to the matrices $\langle \mathbf{MV}_n, \{1\}\rangle$, where $\mathbf{MV}_n$ is the MV-chain over the n-element set $MV_n = \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$, and satisfies the above conditions (C1) and (C2). Namely, as is well-known, Baaz-Monteiro operator Δ is definable in $Ł_n$ as $\Delta\varphi := \varphi \& \overset{n}{\ldots} \& \varphi$ and $Ł_n$ enjoys a global deduction theorem: $\varphi \vdash_{Ł_n} \psi$ iff $\vdash_{Ł_n} \Delta\varphi \to \psi$. On the other hand, it is also well-known that, for every $a \in MV_n$, there is a McNaughton term $t_a(x)$ such that $t_a(x) = 1$ iff $x \geq a$. Therefore, it holds that $\varphi \vdash^{a}_{Ł_n} \psi \quad \text{iff} \quad r_a(\varphi) \models_{Ł_n} r_a(\psi)$. As a consequence, according to Theorem 7, the logic $Ł^a_n$ defined there provides a complete axiomatisation of the semantic consequence relation $\models^{a}_{Ł_n}$. In this case the restricted rule ($\mathrm{R}_{t,r}$) takes this form:

$$\frac{\varphi, \quad \vdash_{Ł_n} \Delta(r_a(\varphi)) \to r_a(\psi)}{\psi}.$$

Actually, thanks to the Δ operator, this rule could be replaced by this unrestricted form:

$$\frac{\varphi, \quad \Delta(r_a(\varphi)) \to \Delta(r_a(\psi))}{\psi}.$$

Note that for $n = 3$ and $a = 1/2$, the resulting logic $Ł^{1/2}_3$ provides an alternative axiomatisation (in the language of $Ł_n$) of the well-known D'Ottaviano and da Costa's paraconsistent logic J_3.

When we move to the case of infinite-valued Łukasiewicz logic Ł, condition (C2) keeps holding at least for every rational a thanks to the existence of McNaughton terms. But, as previously commented, (C1) fails since Ł does not have a global deduction theorem. To overcome this problem we can consider the logic $Ł_\Delta$, the expansion of Ł with the Δ operator, already axiomatised by Hájek in [26]. Then in $Ł_\Delta$ condition (C2) keeps holding for rational values a, while now condition (C1) is satisfied as well taking $t = \Delta$. Therefore, Theorem 7 can be applied to $Ł_\Delta$ to get axiomatisations of $Ł^a_\Delta$ for every rational a.

In particular, if we are interested on the logic to reason with *half-true* propositions, it is enough to instantiate Theorem 7 with $a = 1/2$, $t(\varphi) = \Delta\varphi$ and $r(\varphi) = \varphi \otimes \varphi$.

Note that, having Δ in the language, one can also consider the logic of strict thresholds $\models^{(a}_{\text{Ł}_\Delta}$. In fact, one easily check that $\varphi \models^{(a}_{\text{Ł}_\Delta} \psi$ iff $\neg\psi \models^{1-a}_{\text{Ł}_\Delta} \neg\varphi$. In the logic $\models^{(a}_{\text{Ł}_\Delta}$, the variant of the above rule $(\mathrm{R}_{t,r})$ that is at work is the following one:

$$\frac{\varphi, \quad \vdash_{\text{Ł}_\Delta} \Delta(r_{1-a}(\neg\psi)) \to r_{1-a}(\neg\varphi)}{\psi}.$$

$$\frac{\varphi, \quad \Delta(r_{1-a}(\neg\psi)) \to \Delta(r_{1-a}(\neg\varphi))}{\psi}.$$

In the particular case $a = 1/2$, the logic $\models^{(1/2}_{\text{Ł}}$ is interpretable in the non-falsity preserving variant of Ł, (and thus in the 1-preserving logic as well) since for any formulas φ, ψ it holds that:

$$\varphi \models^{(1/2}_{\text{Ł}} \psi \quad \text{iff} \quad \varphi^2 \models^{(0}_{\text{Ł}} \psi^2 \quad [\text{iff } \neg\psi^2 \models_{\text{Ł}} \neg\varphi^2].$$

These relationships are due to the fact that, for every φ and Ł-evaluation e, $e(\varphi) > 1/2$ iff $e(\varphi^2) > 0$.

In the remaining part of this section we consider the cases of three particular logics that also fall outside the scope of Theorem 7, namely, Nilpotent Minimum logic, Gödel logic and Product logic.

5.2 The case of Nilpotent Minimum logic

Nilpotent Minumum logic (NM logic for short) satisfies condition (C1) but not condition (C2) of Theorem 7, and hence it requires a specific consideration. In fact, the threshold logics $\models^{a}_{NM}$ and $\models^{(a}_{NM}$ have been already studied in [25], and hence, for the sake of being self-contained, we provide below a brief summary of those results.

We recall from Table 1 that Nilpotent Minimum logic (NML) is the axiomatic extension of IMTL with the axiom

$$(\mathrm{WNM}) \quad (\psi \,\&\, \varphi \to \bot) \vee (\psi \wedge \varphi \to \psi \,\&\, \varphi)$$

NML, as every axiomatic extension of MTL, is an algebraizable logic, being the variety of NM-algebras its equivalent algebraic semantics. Importantly, NML is also complete with respect to a single NM-algebra on the real unit interval $[0, 1]$, the so-called canonical standard algebra: $[\mathbf{0}, \mathbf{1}]_{\mathbf{NM}} = \langle [0, 1], *, \to, \wedge, \vee, 0, 1\rangle$, where

$$a * b = \begin{cases} 0, & \text{if } b \leq 1-a \\ \min\{a, b\}, & \text{otherwise} \end{cases}, \quad a \to b = \begin{cases} 1, & \text{if } a \leq b \\ \max\{1-a, b\} & \text{otherwise} \end{cases}.$$

As a consequence, we can restrict ourselves to interpretations on $[\mathbf{0}, \mathbf{1}]_{\mathbf{NM}}$ when considering the logics $\models^{a}_{NM}$ and $\models^{(a}_{NM}$.

A basic result from [25] is that many of these logics collapse for different values of a. Indeed, we have the following result, cf [25, Prop. 1].

Proposition 4. *[25, Prop. 1] We have the following properties:*

1. $\models^{a}_{NM}$, $\models^{(a}_{NM}$ *and* $\models_{NM}$ *are the same logic for all* $a \in (1/2, 1)$,
2. $\models^{a}_{NM}$, $\models^{(a}_{NM}$ *and* $\models^{(0}_{NM}$ *are the same logic for all* $a \in (0, 1/2)$,
3. $\models^{(1/2}_{NM}$ *and* $\models_{Ł_3}$ *are the same logic,*
4. $\models^{1/2}_{NM}$ *and* $\models^{1/2}_{Ł_3}$ *are the same logic.*

where $Ł_3$ *denotes the 3-valued Łukasiewicz logic.*

Therefore, we only have four different threshold NM logics: the truth-preserving logic NML, the non-falsity preserving logic nf-NML, the 3-valued Łukasiewicz logic $Ł_3$ and the Jaskowski logic J_3 (which is equivalent to $\models^{1/2}_{Ł_3}$), all of them with known axiomatisations, in particular the one for nf-NML can be obtained by from Definition 8 when taking L = NML. Figure 2 provides a graphical representation of these logics and their relations.

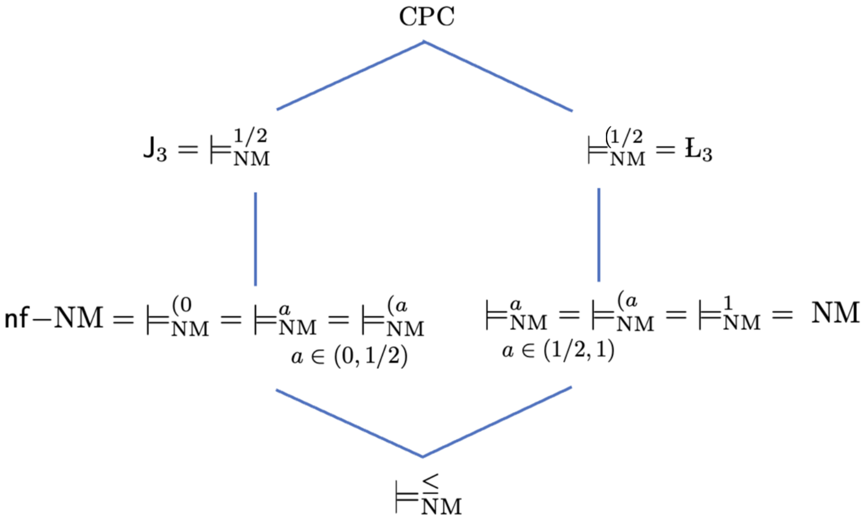

Fig. 2. Threshold-preserving NM logics

5.3 The cases of Gödel and Product logics

In this final section we consider the cases of Gödel and Product logics. These logics also fall outside the scope of Theorem 7, and hence they require a specific consideration.

The case of Gödel logic. The analysis of Gödel logic turns out to be very simple. Gödel logic can be seen as the axiomatic extension of MTL with the axiom (Con), see Table 1. In fact, Gödel logic is standard complete with respect to the single matrix $M_1 = \langle [\mathbf{0}, \mathbf{1}]_{\mathbf{G}}, \{1\} \rangle$, where $[\mathbf{0}, \mathbf{1}]_{\mathbf{G}}$ denotes the standard Gödel algebra $([0,1], \min, \max, *_G, \to_G, 0, 1)$, with $*_G = \min$ and $\to_G$ is its residuum.

For $a \in (0,1)$, let us denote by $\models^a_G$ and $\models^{(a}_G$ the logics defined by the logical matrices $M^a = \langle [0,1]_G, [a,1] \rangle$ and $M^{(a} = \langle [0,1]_G, (a,1] \rangle$ respectively. We will also denote the logic $\models^1_G$ simply as $\models_G$.

As is well-known, a distinctive characteristic of Gödel logic is that, for any $a \in [0,1]$, the mapping $g^a : [0,1] \to [0,1]$ defined by $g^a(x) = x$ for $x \in [0,a)$ and $g^a(x) = 1$ for $x \in [a,1]$ is morphism of Gödel-algebras, analogously with the mapping $g^{(a} : [0,1] \to [0,1]$ defined by $g^a(x) = x$ for $x \in [0,a]$ and $g^a(x) = 1$ for $x \in (a,1]$. Note that, in particular, $g^{(0}$ maps $[0,1]$ into $\{0,1\}$. These well-known facts allow us to prove the following result, see also the left-hand lattice of logics in Figure 3.

Proposition 5. *$\models^{(0}_G$ coincides with classical logic, while for any $a \in (0,1]$, $\models^a_G$ $= \models^{(a}_G = \models_G$.*

The case of Product logic. Product logic is defined as the axiomatic extension of MTL with axioms (Div) and (C), see Table 1. Product logic is standard complete with respect to the single matrix $M_1 = \langle [\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}, \{1\} \rangle$, where $[\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}$ denotes the standard product algebra $([0,1], \min, \max, *_\Pi, \to_\Pi, 0, 1)$, with $*_\Pi$ being the product t-norm and $\to_\Pi$ is its residuum.

For $a \in (0,1)$, let us denote by $\models^a_\Pi$ and $\models^{(a}_\Pi$ denote the logics defined by the logical matrices $M^a = \langle [0,1]_\Pi, [a,1] \rangle$ and $M^{(a} = \langle [0,1]_\Pi, (a,1] \rangle$ respectively. We will also denote the logic $\models^1_\Pi$ simply as $\models_\Pi$.

In the following we will make use of a known result about automorphisms of the standard product algebra $[\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}$. Namely, let $\alpha \in \mathbb{R}^+$ and define the mapping $h^\alpha : [0,1] \to [0,1]$ by $h(x) = x^\alpha$. Then h^α is an automorphism of $[\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}$, see [31].[11] This means that, for $\otimes \in \{\min, \max, *_\Pi, \to_\Pi\}$ and and every α, $h^\alpha(e(\varphi \otimes \psi)) = h^\alpha(e(\varphi)) \otimes h^\alpha(e(\psi))$, for any formulas φ, ψ, every $[0,1]_\Pi$-evaluation e.

Now we can prove a series of results that we will allow to completely characterise all the $\models^a_\Pi$ and $\models^{(a}_\Pi$ logics.

Lemma 10. *For any $a, b \in (0,1)$, $\models^a_\Pi = \models^b_\Pi$.*

Proof. Indeed, assume $\varphi \not\models^a_\Pi \psi$. Then there exists an evaluation e such that $e(\varphi) \geq a$ and $e(\psi) < a$. We know there exists $\alpha \in \mathbb{R}^+$ such that $a^\alpha = b$. Then, if we let $e' = h^\alpha \circ e$, we have that $e'(\varphi) \geq a^\alpha = b$ and $e'(\psi) < a^\alpha = b$, hence $\varphi \not\models^b_\Pi \psi$.

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[11] In fact all the automorphisms of $[\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}$ are of form h^α.

Lemma 11. *The following conditions hold:*
(i) $\models_{\Pi}^{(0}$ *is classical propositional calculus*
(ii) for any $a \in (0,1)$, $\models_{\Pi}^{a} = \models_{\Pi}^{(a}$.

Proof. (i) is a direct consequence of the fact that the mapping $k : [0,1] \to \{0,1|\}$ such that $h(0) = 0$ and $h(x) = 1$ for $x \in (0,1]$ is a morphism of product algebras,

As for (ii), first assume $\varphi \not\models_{\Pi}^{a} \psi$, and hence there is e such that $e(\varphi) \geq a > e(\psi)$. Let $\alpha < 1$ such that $(e(\psi))^{\alpha} = a$, then $(e(\varphi))^{\alpha} \geq a^{\alpha} > a = (e(\psi))^{\alpha}$. Therefore, for $e' = h^{\alpha} \circ e$, we have $e'(\varphi) > a \geq e'(\psi)$, and hence $\varphi \not\models_{\Pi}^{(a} \psi$.

Conversely, assume $\varphi \not\models_{\Pi}^{(a} \psi$. Then there is e such that $e(\varphi) > a \geq e(\psi)$. Let $\alpha > 1$ be such that $(e(\varphi))^{\alpha} = a$ and hence we have $a = (e(\varphi))^{\alpha} > (e(\psi))^{\alpha}$. This means that $e'(\varphi) \geq a > e'(\psi)$, where $e' = h^{\alpha} \circ e$, that is, $\varphi \not\models_{\Pi}^{a} \psi$.

Lemma 12. *For any* $a \in (0,1)$, $\models_{\Pi} \varphi$ *iff* $\models_{\Pi}^{a} \varphi$.

Proof. It is clear that if $\models_{\Pi} \varphi$ then $\models_{\Pi}^{a} \varphi$. Conversely, assume $\not\models_{\Pi} \varphi$. Then there is e such that $e(\varphi) < 1$. Let $b = e(\varphi)$. Then $\not\models_{\Pi}^{a} \varphi$ for all a such that $b < a < 1$. Hence, by (i), $\not\models_{\Pi}^{a} \varphi$ for any $a \in (0,1)$ as well.

Lemma 13. *For any* $a \in (0,1)$, $\varphi \models_{\Pi}^{a} \psi$ *iff* $\models_{\Pi}^{a} \varphi \to \psi$.

Proof. Assume $\varphi \not\models_{\Pi}^{a} \psi$. Then there is a such that $e(\varphi) \geq a$ and $e(\psi) < a$. It follows that $1 > e(\varphi \to \psi) = e(\psi)/e(\varphi)$, and thus there is α such that $(e(\psi)/e(\varphi))^{\alpha} < a$, that is, $e'(\varphi \to \psi) < a$, where $e' = h^{\alpha} \circ e$. Thus, $\not\models_{\Pi}^{a} \varphi \to \psi$.

Conversely, assume $\not\models_{\Pi}^{a} \varphi \to \psi$. Then there is e such that $e(\varphi \to \psi) < a$, hence $e(\psi)/e(\varphi) < a$, that is, $e(\psi) < a \cdot e(\varphi)$. Let α such that $(e(\varphi))^{\alpha} = a$. Then we have $(e(\psi))^{\alpha} < a^{\alpha} \cdot (e(\varphi))^{\alpha} = a^{\alpha} \cdot a < a$. Hence, we have $e'(\varphi) = a$ while $e'(\psi) < a$, where again $e' = h^{\alpha} \circ e$. Therefore, $\varphi \not\models_{\Pi}^{a} \psi$.

The intersection of all the logics $\models_{\Pi}^{b}$ for all $b \in (0,1]$ is usually known as the degree-preserving companion of $\models_{\Pi}$, and denoted as $\models_{\Pi}^{\leq}$. Then as a consequence of Lemma 10 we have the following corollary.

Corollary 2. $\models_{\Pi}^{\leq} = \models_{\Pi} \cap \models_{\Pi}^{a}$ *for any* $a \in (0,1)$.

Next is the final result we can show.

Lemma 14. *For every* $a \in (0,1)$, $\models_{\Pi}^{a} = \models_{\Pi}^{\leq}$.

Proof. The inclusion $\models_{\Pi}^{\leq} \subseteq \models_{\Pi}^{a}$ is clear. Assume $\varphi \not\models_{\Pi}^{\leq} \psi$. Then there exists e such that $e(\varphi) > e(\psi)$. Let α such that $(e(\varphi))^{\alpha} = a$, hence $(e(\psi))^{\alpha} < a$. Therefore, if $e' = h^{\alpha} \circ e$, then $e'(\varphi) = a > e'(\psi)$, and thus $\varphi \not\models_{\Pi}^{a} \psi$. Therefore $\models_{\Pi}^{a} \subseteq \models_{\Pi}^{\leq}$.

The following is the summary of the above results, which is also graphically shown in the right-hand lattice of logics in Figure 3.

Theorem 8. $\models_{\Pi}^{(0}$ *is classical logic, while for any* $a \in (0,1]$, $\models_{\Pi}^{a} = \models_{\Pi}^{(a} \subsetneq \models_{\Pi}$

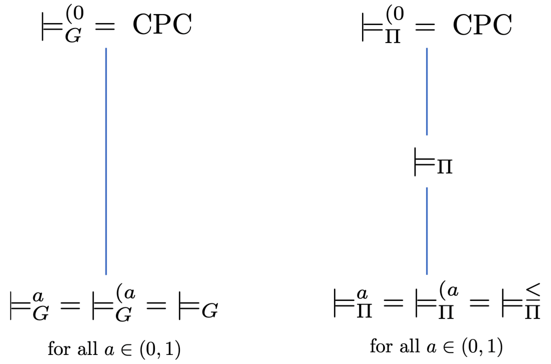

Fig. 3. Lattices of threshold logics $\models_L^a$ and $\models_L^{(a}$ for L = G (left) and L = Π (right).

5.4 The case of ordinal sums

In this final section, we focus on how to axiomatize the threshold logics $\models_L^a$, where L is the logic of a standard algebra $[0,1]_*$ which is (isomorphic) to an ordinal sum of standard BL-chains $\mathbf{A} = \bigoplus_{\mathbf{i}\in\mathbf{I}}[\mathbf{0},\mathbf{1}]_{*_\mathbf{i}}$, where $(I,\leq)$ is an ordered index set. The solution for the general case is out of the scope of this paper, but we will consider several particular cases that are representative enough to provide a general idea of how to approach the problem and its limitations as well.

A first comment is that, in the case of L being a core-fuzzy logic (so without having the Δ operator), if we fix $a \in [0,1]$ corresponding (by the above mentioned isomorphism) to a value a' in the j-component $[0,1]_{*_j}$ (with $j \in I$), then the matrix logic $\models_L^a = \langle \bigoplus_{i\in I}[0,1]_{*_i}, F_{a'}\rangle$ collapses to the logic $\models_{L'}^a = \langle \bigoplus_{i\leq j}[0,1]_{*_i}, F_{a'} \cap [0,1]_{*_j}\rangle$. In other words, when looking at the threshold logic $\models_L^a$, we can safely assume that a belongs to the last component of the ordinal sum defining L.

This is due to the fact that, for any $j \in I$, $F_j = \bigoplus_{i>j}[0,1]_{*_i}$ is an implicative filter of $\mathbf{A} = \bigoplus_{\mathbf{i}\in\mathbf{I}}[\mathbf{0},\mathbf{1}]_{*_\mathbf{i}}$ and, considering the induced congruence relation $\equiv_{F_j}$, it easily follows that $\mathbf{A}_{/\equiv_{\mathbf{F_j}}}$ is isomorphic to the reduced ordinal sum $\mathbf{A}[j] = \bigoplus_{i\leq j}[0,1]_{*_i}$.

Proposition 6. *For any* $a \in [0,1]_{*_j}$, $\langle \mathbf{A}, F_a\rangle = \langle \mathbf{A}[j], F_a \cap [0,1]_{*_j}\rangle$.

Proof. If $\varphi \not\models_{L'}^a \psi$ then there is an L'-evaluation e such that $e(\varphi) \geq a$ and $e(\psi) < a$. Since e is also an L-evaluation, we have that $\varphi \not\models_L^a \psi$ as well.

Conversely, suppose $\varphi \not\models_L^a \psi$, so that there is an L-evaluation e, $e(\varphi) \geq a$ and $e(\psi) < a$. Let $e' = h(e)$, where $h : A \to A/_{\equiv_F}$ is the epimorphism defined as $a \mapsto [a] = \{b \mid a \to b, b \to a \in F\}$. Note that $[a] = \{b \in A[j] \mid b \geq a\}$ if $a \in A[j]$ and $[a] = 1$ otherwise. Since $\mathbf{A}/_{\equiv_\mathbf{F}}$ is isomorphic to $\bigoplus_{\mathbf{i}\leq\mathbf{j}}[\mathbf{0},\mathbf{1}]_{*_\mathbf{i}}$, identifying $[a]$

with a, then e' is in fact a L'-evaluation such that $e'(\varphi) \geq a$ and $e'(\psi) < a$, so that $\varphi \not\models^a_{L'} \psi$. $\square$

Remark 1. if L is a Δ-core fuzzy logic, the above reduction does not hold since then $[0,1]_{L_1} \oplus ... \oplus [0,1]_{L_n}$ is simple, and hence the only non-trivial filter is $F = \{1\}$.

For the purpose of providing some examples, we consider next some non-complex particular cases.

1. $\mathbf{L} = \text{Ł} \oplus \text{Ł}$

Let L be the logic complete w.r.t. the standard algebra that is the ordinal sum of two standard MV-algebras $\mathbf{A} = [\mathbf{0},\mathbf{1}]_{\text{Ł}} \oplus [\mathbf{0},\mathbf{1}]_{\text{Ł}}$. Now consider the matrix logic $(\mathbf{A}, F)$ with a lattice filter $F \subseteq {}^2[0,1]$ being a subset of the domain of the second summand algebra, that we denote by ${}^2[0,1]_{\text{Ł}}$ to distinguish from the first one ${}^1[0,1]_{\text{Ł}}$. Then we have the following cases:

1. If $F = {}^2[0,1]$ then, by Proposition 6, it is clear that $\langle \mathbf{A}, F\rangle = \text{Ł}$.
2. If $F = {}^2(0,1]$, it is not clear whether the logic $\models_L^{(0^2}$ can be interpreted in $\models_L$, or conversely, unless we expand expand L with Baaz-Monteiro operator Δ. In such a case, it is easy to check that, for any $x \in [0,1] \oplus [0,1]$, $x > 0^2$ iff $(\neg\neg x) \wedge (\Delta(x \to x * x) \to \Delta x) = 1$. Indeed, $\neg\neg x = 1$ if, and only if, x belongs to the second component of the ordinal sum, while the additional condition $\Delta(x \to x * x) \to \Delta x = 1$ ensures that the only idempotent in F is $x = 1$. Then, in the context of the logic L_Δ, its companion logic $\models_{L_\Delta}^{(0^2}$ can be axiomatised with the following set of axioms and rules:
    - axioms of L_Δ,
    - rule of Adjunction,
    - restricted necessitation for Δ,
    - restricted Modus Ponens rule, and
    - S-Modus Ponens (S-MP) $\dfrac{\varphi, \quad S\varphi \to S\psi}{\psi}$

    where $S\varphi := (\Delta\neg\neg\varphi) \wedge (\Delta(\varphi \to \varphi * \varphi) \to \Delta\varphi)$.
3. If $F = {}^2[a,1]_{\text{Ł}}$ for some rational $a \in (0,1)$, then let r_a a McNaughton term which is increasing and expressed in the sublanguage $\{\wedge, \vee, *, \to\}$ such that $r_a(x) = 1$ iff $x \geq a$. Then, $\models^a_L$ is interpretable in $\models_L$ as follows: $\varphi \models^a_L \psi$ iff $\neg\neg\varphi \wedge r_a(\varphi) \models_L \neg\neg\psi \wedge r_a(\psi)$, since in $\mathbf{A}$ we have $\neg\neg x = 1$ iff x is in the second summand ${}^2[0,1]$, and this is guaranteed if $x \geq a$.
    Moreover, in this case, if we expand L with Baaz-Monteiro operator Δ, by slightly adapting Theorem 7, we can axiomatise $\models^a_{L_\Delta}$ in the following way. In fact, note that the condition $\neg\neg\varphi \wedge r_a(\varphi) \models_{L_\Delta} \neg\neg\psi \wedge r_a(\psi)$ is equivalent to $\models_{L_\Delta} (\neg\neg\varphi \wedge \Delta(r_a(\varphi))) \to (\neg\neg\psi \wedge r_a(\psi))$. Then, the system defined by:
    - the axioms of L_Δ,

- the rule of Adjunction,
- restricted necessitation for $\varDelta$,
- the restricted Modus Ponens rule, and
- the restricted rule

$$(\mathrm{R}_a^{\text{Ł}\oplus\text{Ł}}) \quad \frac{\varphi, \quad (\varDelta\neg\neg\varphi \wedge \varDelta(r_a(\varphi))) \to \varDelta(\neg\neg\psi \wedge r_a(\psi))}{\psi}$$

is a sound and complete axiomatisation of the finitary $\models_{L_\varDelta}^a$.

2. $\mathbf{L} = \text{Ł} \oplus \varPi$

Let L be the logic complete w.r.t. the standard algebra that is the ordinal sum of a standard MV-algebra and a standard produc algebra $\mathbf{A} = [\mathbf{0},\mathbf{1}]_{\text{Ł}} \oplus [\mathbf{0},\mathbf{1}]_{\boldsymbol{\Pi}}$. In this case, we have to consider again a lattice filter $F \subseteq {}^2[0,1]$ of the domain of the second summand algebra. Then we have the following cases:

1. If $F = [0,1]$, then $\langle \text{Ł} \oplus \varPi, F\rangle = \text{Ł}$.
2. if $F = (0,1]_{\varPi}$, then $\langle \text{Ł} \oplus \varPi, F\rangle = \text{Ł} \oplus \mathbf{2}$, where $\mathbf{2}$ is the two-element Boolean algebra.
3. If $F = [a,1]$ for some $a \in (0,1)$, then in this case the logic $\models_L^a$, of the matrix $\langle \text{Ł} \oplus \varPi, F\rangle$, is interpretable in $\models_L$ as follows:

   $$\varphi \models_L^a \psi \text{ iff } \neg\neg\varphi \wedge ((\varphi \to \varphi * \varphi) \to \varphi) \models_L \varphi \to \psi.$$

   Note that condition is saying that if the value of φ is in the second summand and it is cancellative (which characterises the fact that $x > 0^2$), then the value of ψ is greater or equal than that of φ. In turn, this latter condition, according to Lemma 14, amounts to require that $\varphi \models_\varPi^a \psi$. Note that the case $F = (a,1]$ yields exactly the same logic (see Lemma 11).
   As for the axiomatisation, and similarly to above, we can only provide it if we first expand L with $\varDelta$. In fact, the system defined by:

   - the axioms of $\mathrm{L}_\varDelta$,
   - the rule of Adjunction,
   - restricted necessitation for $\varDelta$,
   - the restricted Modus Ponens rule, and
   - the restricted rule

   $$(\mathrm{R}_a^{\text{Ł}\oplus\varPi}) \quad \frac{\varphi, \quad \varDelta(\neg\neg\varphi \wedge ((\varphi \to \varphi * \varphi) \to \varphi)) \to \varDelta(\varphi \to \psi)}{\psi}$$

   is a sound and complete axiomatisation of both the finitary logics $\models_L^a$ and $\models_L^{(a}$, for any $a \in (0,1)$.

3. $\mathbf{L} = \mathbf{L_1} \oplus \ldots \oplus \mathbf{L}_n$, with $\mathbf{L}_i \in \{\text{Ł}, \varPi\}$

Let L be the logic of the standard algebra $\mathbf{A} = \mathbf{A}_1 \oplus \ldots \oplus \mathbf{A}_n$ where, for each i, either $\mathbf{A}_i \approx [\mathbf{0}, \mathbf{1}]_{\text{Ł}}$ or $\mathbf{A}_i \approx [\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}$. Further let $\mathbf{A}_\varDelta$ the expansion of $\mathbf{A}$ with the operator $\varDelta$.

For simplicity, assume that, for each $i \in \{0, \ldots, n-1\}$, $A_i = [a_i, a_{i+1}]$, where $0 = a_0 < a_1 < \ldots < a_{n-1} < a_n = 1$. Let $p_1, \ldots, p_n$ be n pairwise propositional variables and consider the following compound formula in the expanded language with $\varDelta$:

$$\varSigma = \bigwedge_{i=1}^{n} \varDelta(p_i \to p_i * p_i) \ \wedge \ \bigwedge_{i=1}^{n-1} \neg\varDelta((p_i \to p_{i+1}) \wedge \neg(p_{i+1} \to p_i))$$

Note that, for every $\mathbf{A}$-evaluation e, $e(\varSigma) = 1$ iff $e(p_i) = a_i$, for each $i = 1, \ldots, n$. Let now $a \in A_k$ with $k < n$ and consider the two following cases regarding $\mathbf{A}_k$:

- $\mathbf{A_k} \approx [\mathbf{0}, \mathbf{1}]_{\text{Ł}}$. In this case we have:

  $$\varphi \models^a_{\mathrm{L}_\varDelta} \psi \quad \text{iff} \quad \varSigma, \varDelta(p_k \to \varphi), \varDelta(r_a(\varphi) \vee p_{k+1} \to \varphi) \models_{\mathrm{L}_\varDelta} r_a(\psi) \vee (p_{k+1} \to \psi)$$

  Since $\varSigma$ is propositional combination of subformulas with $\varDelta$, we can apply the $\varDelta$-deduction theorem and get the equivalent condition:

  $$\varphi \models^a_{\mathrm{L}_\varDelta} \psi \quad \text{iff} \quad \models_{\mathrm{L}_\varDelta} \varPhi^k_{\text{Ł}} \to r_a(\psi) \vee (p_{k+1} \to \psi),$$

  where $\varPhi^k_{\text{Ł}} = \varSigma \wedge \varDelta(p_k \to \varphi) \wedge \varDelta(r_a(\varphi) \vee p_{k+1} \to \varphi)$.
- $\mathbf{A_k} \approx [\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}$. In this case we have:

  $$\varphi \models^a_{\mathrm{L}_\varDelta} \psi \quad \text{iff} \quad \varSigma, \varDelta(p_k \to \varphi) \models_{\mathrm{L}_\varDelta} (\varphi \to \psi) \vee (p_{k+1} \to \psi)$$

  or equivalently,

  $$\varphi \models^a_{\mathrm{L}_\varDelta} \psi \quad \text{iff} \quad \models_{\mathrm{L}_\varDelta} \varPhi^k_{\varPi} \to (\varphi \to \psi) \vee (p_{k+1} \to \psi),$$

  where $\varPhi^k_{\varPi} = \varSigma \wedge \varDelta(p_k \to \varphi)$,

Finally, to get a sound and complete axiomatization of the logics $\models^a_{L_\varDelta}$ and $\models^{(a}_{L_\varDelta}$ we only need to consider the appropriate replacements of the rule $(\mathrm{R}_{t,r})$ in the two previous schemes, namely by the rules

$$(\mathrm{R}^L_{\text{Ł}_k,a}) \quad \frac{\varphi, \quad \varPhi^k_{\text{Ł}} \to \varDelta(r_a(\psi) \vee (p_{k+1} \to \psi))}{\psi}$$

and

$$(\mathrm{R}^L_{\varPi_k,a}) \quad \frac{\varphi, \quad \varPhi^k_{\varPi} \to \varDelta((\varphi \to \psi) \vee (p_{k+1} \to \psi))}{\psi}$$

respectively.

5.5 Some final remarks on dualising threshold-preserving IMTL logics

In this section we have been concerned on general threshold-preserving logics $\models_L^a$ and $\models_L^{(a}$, where L is a MTL-logic (a core or a Δ-core fuzzy logic) defined by lattice filters defined by both closed and open upper subintervals $[a, 1]$ and $(a, 1]$ respectively of the real unit interval $[0, 1]$. In the case L is a IMTL, i.e. when the (residual) negation $\neg$ is involutive, it is very easy to observe that there are strong mutual interpretations between some of these logics.

Lemma 15. *If L is an IMTL logic which is complete w.r.t. a set of matrices* $\{\langle [\mathbf{0}, \mathbf{1}]_{*_i}, \{1\}\rangle \mid *_i$ *is a IMTL t-norm s.t.* $\neg_i x = 1 - x\}_{i \in I}$ *then the following condition holds: for all* $a \in (0, 1]$,

$$\varphi \models_L^a \psi \quad \textit{iff} \quad \neg\psi \models_L^{(1-a} \neg\varphi.$$

Besides, one can also notice a different kind of direct non-standard relations between upper interval-preserving logics like $\models_L^a$ or $\models_L^{(a}$ with logics that preserve lower intervals. Indeed, it is trivial to observe that, by contraposition, the condition "if $x \geq a$ then $y \geq a$" is equivalent to "if $y < a$ then $x < a$", for any $a \in (0, 1]$. We could argue that the latter condition can be related to a sort of non-standard logic *preserving partial degrees of non-truth*. In fact, for the sake of simplicity, let us focus in the case $a = 1$ and consider the following notion of a consequence-like relation $\models_L^{1)}$.

Definition 7. *For any finite set of formulas* $\Gamma \cup \{\psi\}$ *define* $\Gamma \models_L^{1)} \psi$ *iff, for any L-evaluation e,* $e(\varphi) < 1$ *for all* $\varphi \in \Gamma$ *implies* $e(\psi) < 1$.

Then it is direct to check that $\models_L^{1)}$, that we can call it *non-truth*-preserving companion of L, can be straightforwardly interpreted in the standard 1-preserving logic $\models_L$ as follows:

$$\varphi_1, ..., \varphi_n \models_L^{1)} \psi \quad \text{iff} \quad \varphi_1 \vee ... \vee \varphi_n \models_L^{1)} \psi \quad \text{iff} \quad \psi \models_L \varphi_1 \vee ... \vee \varphi_n.$$

In particular, $\bot \models_L^{1)} \psi \quad \text{iff} \quad \psi \models_L \bot$, that is, the theorems of $\models_L^{1)}$ are the contradictions of $\models_L$, that is, the negations of the theorems of $\models_L$.

Thus, the finitary *non-truth*-preserving relation $\models_L^{1)}$ basically amounts to revert the $\models_L$ consequence relation once the (finite) sets of premises in $\models_L^{1)}$ are transformed into disjunctions of consequents in $\models_L$. Note that, conversely, $\models_L$ can also be interpreted in $\models_L^{1)}$ as follows:

$$\varphi_1, ..., \varphi_n \models_L \psi \quad \text{iff} \quad \psi \models_L^{1)} \varphi_1 \wedge ... \wedge \varphi_n.$$

It is interesting to observe that this sort of duality regarding the semantical relations $\models_L^{1)}$ and $\models_L$ carries over to the syntactical level as well. We will not go into details but it is not difficult to check that one can axiomatize $\models_L^{1)}$ in a completely dual form as we have done it for the case of the non-falsity preserving companion of a IMTL logic $\models_L^{(0}$.

Definition 8. *Let L be a IMTL logic. Its non-truth preserving companion* nt-L *is defined by the following system of axioms and inference rules:*

- *Axioms:* $\neg\varphi$*, if* φ *is an axiom of* L
- *Rule of dual Adjunction: (d-Adj)* $\dfrac{\varphi, \quad \psi}{\varphi \vee \psi}$ $\dfrac{\varphi, \quad \vdash_L \psi \to \varphi}{\psi}$,
- *Conjunctive Reverse Modus Ponens:* (MP^{cr}) $\dfrac{\psi \wedge \chi}{\varphi \wedge (\varphi \to \psi) \wedge \chi}$

With a proof similar (and simpler) to the one for nf-L in Section 3.2, one can finally check that the logic nf-L is sound and complete w.r.t. $\models_L^{1)}$.

It is worth mentioning that a somewhat related formalisms are Skura's refutation systems, see e.g. [32]. Although for instance the rule (rd-MP) is also a corner stone in these systems, there is a notable difference. If we consider the case of L being classical logic, then the theorems of $\models_L^{1)}$ are the negations of the classical logic theorems, while the theorems in Skura's systems are the non-theorems of classical logic, i.e. formulas that can be falsified at least by some evaluation.

6 Conclusions

In this paper we have been concerned with the definition and axiomatisation of a number of companions of MTL logics that in their consequence relations implement truth preservation conditions alternative to the full truth (represented by the truth-value 1) preservation which is the usual one in systems of Mathematical Fuzzy logic. In particular, we have first considered *non-falsity preserving* logics, that is, logics to reason with non-false propositions. These logics are clearly paraconsistent but also uncomparable to the corresponding 1-preserving logics (unless they collapse to classical logic). Then we have introduced the logics of *acceptable* and *tolerable* propositions, in which the truth-preservation condition also involves the truth-value of the negated propositions. In fact, a proposition φ is here called acceptable when the truth value of φ is strictly greater than its negation $\neg\varphi$, and dually, φ is tolerable when it is not the case that $\neg\varphi$ is acceptable. Both notions are in fact related to logics of matrices with lattice filters of the form $(b, 1]$ and $[b, 1]$ for some value b determined by the negation operation in the algebra of the matrices. Finally, we have considered *threshold* companions of MTL logics L, that is logics of matrices with principal lattice filters of the form $[a, 1]$ and with strict filters $(a, 1]$ for values $a \in (0, 1)$, denoted $\models_L^a$ and $\models_L^{(a}$, and have restricted our attention in particular to several families of notable logics, with and without Δ, like Łukasiewicz logics, Nilpotent Minimum logic, Gödel logic, Product logic and logics of ordinal sums of Łukasiewicz and Product components. Note that the axiomatisation of the Łukasiewicz threshold logics $\models_{\mathrm{Ł}}^a$ and $\models_{\mathrm{Ł}}^{(a}$ is an open problem, as it was left in [4].

It is worth noticing that in the axiomatization of different kinds of companions of a given core or Δ-core fuzzy logic we have provided along the paper, we have always restricted ourselves to axiomatising finitary consequence relations. This means that the defined axiomatic systems are finite-strong complete (i.e. complete for deductions from a finite set of premises). It is an interesting open question to check in which cases we also obtain systems that are strongly complete (i.e. complete for deductions from an arbitrary set of premises). We know very little about this question, but in some cases, thanks to the interpretation of some companion logics into the corresponding truth-preserving logics, we can say something. For instance, thanks to the existence of McNaughton terms, one can check that nf-Ł does not have strong completeness, while our conjecture is that nf-NM is indeed strongly complete. On the other hand, if L is a Δ-core fuzzy logic, we can always interpret L into the different companion logics we have considered, and thus if L is non-finitary this will also be the case of the companion logic. If we further have the converse interpretation as well, as we have seen in the case of the acceptance and tolerance logics, the status of the companion logic coincides with the one of their corresponding truth-preserving logics. We leave the problem of having a full picture of the status of the companion logics with respect to the property of strong completeness for future research.

On the other hand, while the study and characterisation of non-falsity preserving companions of MTL logics is already quite exhaustive, the general study of threshold-preserving companions still allows for further developments. In future work we aim at completing this study, and consider possible applications of these logics to graded logical systems for reasoning under uncertainty or with preferences.

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