Some categorical equivalences involving Gödel algebras

S. Aguzzoli* T. Flaminio† E. Marchioni‡

Abstract

The aim of this work is to investigate three categorical equivalences of Gödel algebras (cf [2]): the first one involves *Idempotent Involutive Uninorm* (IIU in symbols)-algebras (cf. [4]), the second is between Gödel algebras and the subcategory of Nilpotent Minimum algebras NM^+ of those algebras whose involutive negation has a fix point, and finally the third one is the subcategory of those Nilpotent Minimum algebras NM^- whose negation has not a fix point.

Recall that a IIU-algebra is a bounded commutative residuated lattices $\langle A, *, \to, \leq, \mathbf{e}, \bot, \top \rangle$ satisfying $(x \to \mathbf{e}) \to \mathbf{e} = x$ (for all $x \in A$), and $x*x = x$ for all $x \in A$. To simplify the notation, we write $\neg x$ instead of $x \to \mathbf{e}$. The standard example of IIU-algebra is the system $\langle [0,1], *, \to, \frac{1}{2}, 0, 1 \rangle$, where for every $x, y \in [0,1]$,

$$x * y = \begin{cases} \max\{x, y\} & \text{if } x + y > 1 \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

and

$$x \to y = \begin{cases} \max\{1 - x, y\} & \text{if } x \leq y \\ \min\{1 - x, y\} & \text{otherwise.} \end{cases}$$

A NM-algebra is a any algebra in the signature $\langle \odot, \to, \wedge, \vee, \bot, \top \rangle$ of type $(2, 2, 2, 2, 0, 0)$. The variety of NM-algebras is generated by the *standard* NM-algebra $\langle [0,1], \odot, \Rightarrow, 0, 1 \rangle$, where for all $x, y \in [0,1]$,

$$x \odot y = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \max\{1 - x, y\} & \text{otherwise.} \end{cases}$$

*Dipartimento di Scienze dell'Informazione Via Comelico 39-41, 20135 Milano, Italy. `email: aguzzoli@dsi.unimi.it`

†Artificial Intelligence Research Institute (IIIA - CSIC), Campus Universitat Autonoma de Barcelona, 08193 Bellaterra, Spain. `email: tommaso@iiia.csic.es`

‡Artificial Intelligence Research Institute (IIIA - CSIC), Campus Universitat Autonoma de Barcelona, 08193 Bellaterra, Spain.`email: enrico@iiia.csic.es`

The *negation* of any NM-algebra is defined as $\neg x = x \Rightarrow 0$, and the equation $\neg\neg x = x$ holds in any NM-algebra. An algebra A in the signature of NM, is said to be a NM^--algebra if the following is satisfied:

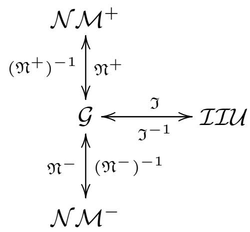
$$\neg((\neg(x \odot x) \odot \neg(x \odot x)) = (\neg(\neg x \odot \neg x)) \odot (\neg(\neg x \odot \neg x)).$$

Consider the signature of NM-algebras, extended by a fresh symbol for a constant $\mathbf{f}$. In this extended signature, we say that an algebra A is an NM^+-algebra if A satisfies the fix point equation:

$$\neg\mathbf{f} = \mathbf{f}.$$

We respectively denote by $\mathcal{G}$, $\mathcal{IIU}$, $\mathcal{NM}^+$, and $\mathcal{NM}^-$ the categories whose objects are Gödel, IIU, NM^+, and NM^--algebras, and having homomorphisms as morphisms. Functors between the subdiretly irreducible elements of any of the above categories can be defined by adapting the Jenei [3] constructions of connected and disconnected rotations (to respectively define subdirectly irreducible $\mathcal{NM}^+$ and $\mathcal{NM}^-$ algebras by subdirectly irreducible Gödel algebras), and an analogous rotation-like construction to define a subdirectly irreducible IIU-algebra, by a subdirectly irreducible Gödel algebra. On the other way round, a Gödel algebra can always be defined by restricting a IIU, NM^+, or NM^--algebra on the domain.

The following diagram summarizes the main equivalences:



Once the functor $\mathfrak{I}$, $\mathfrak{N}^+$ and $\mathfrak{N}^-$ are defined on subdirectly irreducible algebras, and on morphisms accordingly to the chosen rotation-like construction, the equivalence follows by the subdirect representation theorem.

Our investigation is now about the following directions:

(i) We firstly explore if the above introduced functors preserve basic logical and algebraic properties we know hold for $\mathcal{G}$, $\mathcal{NM}^+$, $\mathcal{NM}^-$, and $\mathcal{IIU}$.

(ii) Then we want to establish if the proposed categorical equivalence allows to define additional structure to IIU, NM^+ and NM^--algebras. In particular we are interested in showing what *states on Gödel algebras* (cf. [1]) correspond once the functors $\mathfrak{I}$, $\mathfrak{N}^+$ and $\mathfrak{N}^-$ are applied. Moreover we address the problem of showing whether de Finetti's theorem for states on Gödel algebras extends to IIU, NM^+ and NM^--algebras as well.

References

[1] S. Aguzzoli, B. Gerla, E. Marra. De Finetti's No-Dutch-Book Criterion for Gödel Logic. Studia Logica 90, 25–41, 2008.

[2] P. Hájek. Metamathematics of Fuzzy Logic. Kluwer, 1998.

[3] S. Jenei. On the structure of rotation-invariant semigroups. Archive for Mathematical Logic, 42, 489–514, 2003.

[4] E. Marchioni. Amalgamation through quantifier elimination for varieties of commutative residuated lattices. Submitted.