The quest for the basic fuzzy logic

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1 Introduction

The present work intends to be both a survey and a position paper, conceived as a homage to Petr Hájek and his absolutely crucial contributions to Mathematical Fuzzy Logic (MFL). Our aim is to present some of the main developments in the area, starting with Hájek's seminal works and continuing with the contributions of many others, and we want to do it by taking the search of the basic fuzzy logic as the leitmotif. Indeed, as it will be apparent in the short historical account given later in this introduction, this search has been one of the main reasons for the development of new weaker systems of fuzzy logics and the necessary mathematical apparatus to deal with them. Hájek started the quest when he proposed his basic fuzzy logic BL, complete with respect to the semantics given by all continuous t-norms. Later weaker systems, complete with respect to broader (but still meaningful for fuzzy logics) semantics, have been introduced and disputed the throne of the basic fuzzy logic, such as MTL, UL or psMTL^r. We survey the development of these systems and how they have yielded systematical approaches to MFL. The chapter is also a position paper because we contribute to the quest with our own proposal of a basic fuzzy logic, a very weak system that we call SL^ℓ. Based on results appeared in previous works (mainly in [19, 15]) we introduce in technical details SL^ℓ and a framework based on this logic which allows to work in a uniform way with both propositional and first-order fuzzy logics. We present a wealth of results to illustrate the power and usefulness of our framework, which supports our thesis that, from a well-defined point of view, SL^ℓ can indeed be seen as the basic fuzzy logic.

The chapter is presented without proofs, because all the claims follow from results proved in previous works; we give all the necessary references. We think that the text can be used by beginners as an introduction to the main developments of MFL, putting the stress on Hájek's contributions, but also as a position paper to be critically considered and discussed by the experts in the area.

1.1 T-norm based fuzzy logics

Mathematical Fuzzy Logic (MFL) started as the study of logics based on particular continuous t-norms,[1] most prominently Łukasiewicz logic Ł, Gödel–Dummett logic G and Product logic Π. These logics are rendered in a language with the truth-constant $\overline{0}$ (falsum) and binary connectives $\to$ (implication) and & (fusion, residuated/strong conjunction). They are complete with respect

[1] For a more detailed historical account see [3].

to the *standard semantics*, which has the real-unit interval $[0,1]$ as the set of truth degree and interprets falsum $\bot$ by 0, fusion & by the corresponding t-norm, and the implication $\to$ by its residuum, which always exists for continuous t-norms. On the other hand, these systems are also complete with respect to an algebraic semantics (MV-algebras, Gödel algebras, and product algebras, respectively) and with respect to the subclass of their linearly ordered members, also known as (MV-/Gödel/product) *chains*. These three algebraic semantics are mutually incomparable superclasses of Boolean algebras, which amounts to say that Ł, G and Π are mutually incomparable subclassical logics. In fact, classical logic can be retrieved as axiomatic extension of any of these three systems obtained by adding the excluded middle axiom.

In this context, Petr Hájek introduced a natural question: is it possible to see Ł, G and Π (and, in general, any fuzzy logic with a continuous t-norm-based semantics) as extensions of the same fuzzy logic? In other words: is there *a basic fuzzy logic* underlying all (by then) known fuzzy logic systems? As an answer to this question, he introduced in his monograph [36, 37] a system, weaker than Ł, G and Π, which he named BL (for *basic logic*). This logic was given by means of a Hilbert-style calculus in the language $\mathcal{L} = \{\&, \to, \overline{0}\}$ of type $\langle 2, 2, 0\rangle$, with the deduction rule of *modus ponens* (MP)—from φ and $\varphi \to \psi$ infer ψ—and the following axioms (taking $\to$ as the least binding connective):

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \,\&\, \psi \to \varphi$
(A3)	$\varphi \,\&\, \psi \to \psi \,\&\, \varphi$
(A4)	$\varphi \,\&\, (\varphi \to \psi) \to \psi \,\&\, (\psi \to \varphi)$
(A5a)	$(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$
(A5b)	$(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7)	$\overline{0} \to \varphi$

Other connectives are introduced as follows:

$$\begin{aligned} \varphi \wedge \psi &= \varphi \,\&\, (\varphi \to \psi) & \neg\varphi &= \varphi \to \overline{0} \\ \varphi \vee \psi &= ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) & \overline{1} &= \neg\overline{0} \\ \varphi \leftrightarrow \psi &= (\varphi \to \psi) \wedge (\psi \to \varphi) \end{aligned}$$

Petr Hájek also introduced the corresponding algebraic semantics for his logic. A BL-*algebra* is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \to, 0, 1\rangle$ such that

- $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice
- $\langle A, \cdot, 1\rangle$ is a commutative monoid
- for each $a, b, c \in A$ we have

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a \to c \qquad \text{(residuation)}$$

$$(a \to b) \vee (b \to a) = 1 \qquad \text{(prelinearity)}$$

$$a \cdot (a \to b) = b \cdot (b \to a) \qquad \text{(divisibility)}$$

We say that a BL-algebra is:

- *Linearly ordered* (or BL-chain) if its lattice order is total.
- *Standard* if its lattice reduct is the real unit interval [0,1] ordered in the usual way.

Note that in a standard BL-algebra & is interpreted by a continuous t-norm and $\to$ by its residuum; and vice versa: each continuous t-norm fully determines its corresponding standard BL-algebra.

Hájek proved completeness of BL with respect to BL-algebras and BL-chains and conjectured that BL should be also complete with respect to the standard BL-algebras (i.e., the semantics given by all continuous t-norms). The conjecture was later proved true: Hájek himself showed the completeness by adding two additional axioms [36] which later were shown to be derivable in BL [11]. Therefore, BL could really be seen as a *basic fuzzy logic*. Indeed, it was a genuine *fuzzy* logic because it retained the defining property of fuzzy logics at that time: completeness with respect to a semantics based on continuous t-norms. And it was also *basic* in the following two senses:

1. it could not made be weaker without losing essential properties
2. it provided a base for the study of all fuzzy logics

The first item followed from the completeness of BL w.r.t. the semantics given by *all* continuous t-norms; thus, in a context of continuous t-norm based logics one could not possibly take a weaker system. The second meaning relied on the fact that the three main fuzzy logics (Ł, G, and Π) are all axiomatic extensions of BL and, in fact, the methods used by Hájek to introduce, algebraize, and study BL could be utilized for any other logic based on continuous t-norms. Actually, already in [37], Hájek developed a uniform mathematical theory for MFL. He considered all axiomatic extensions of BL (not just the three prominent ones) as fuzzy logics (he called them *schematic extensions*) and systematically studied their first-order extensions (inspired by Rasiowa's works; see [74]), extensions with modalities, complexity issues, etc.

Moreover, mainly thanks to the availability of good mathematical characterizations for continuous t-norms and BL-chains, BL has turned out to be a crucial logical system giving rise to an intense research with lots of nice results obtained by many authors (see e.g. [8]). For these reasons, we want to take on the occasion of the present tribute volume to Petr Hájek to propose that both BL logic and BL-algebras should rightfully be renamed after their creator as *Hájek logic* and *Hájek algebras* (HL and HL-algebras, for short).

Another strong reason supporting abandoning the name 'Basic Logic' is that the development of MFL has shown that HL was actually *not basic enough*. That is, HL was indeed a good basic logic for the initial framework in which it was formulated, but the active research area that Hájek helped creating with his monograph and his weakest logical system soon expanded its horizons to broader frameworks which demanded a revision of the basic logic. Therefore, Hájek had not settled but only initiated the quest for the basic fuzzy logic. The first step towards a broader point of view was taken by Esteva and Godo, who noticed that the necessary and sufficient condition for a t-norm to have a residuum is not continuity, but left-continuity. Inspired by this fact they introduced in [23] the logic MTL (shorthand for *Monoidal T-norm based Logic*) as an attempt to axiomatize the standard semantics given by all residuated t-norms. It was introduced by means of a Hilbert-style calculus in the language $\mathcal{L} = \{\&, \to, \wedge, \overline{0}\}$ of type $\langle 2, 2, 2, 0 \rangle$, ($\wedge$ is no longer a derived connective and has to be considered as primitive). This calculus is the same as the one for HL only the axiom (A4) is replaced by the following three weaker axioms:

(A4a) $\varphi \wedge \psi \to \varphi$
(A4b) $\varphi \wedge \psi \to \psi \wedge \varphi$
(A4c) $\varphi \,\&\, (\varphi \to \psi) \to \varphi \wedge \psi$

Similarly to the previous cases, Esteva and Godo introduced a broader class of algebraic structures, MTL-algebras (defined analogously to HL-algebras but without requiring the fulfilment of the divisibility condition) and proved that MTL is complete both w.r.t. the semantics given by all MTL-algebras and w.r.t. MTL-chains. Moreover, in [59], Jenei and Montagna indeed proved MTL to be complete with respect to the semantics given by all left-continuous (i.e. residuated) t-norms. Thus it was a better candidate than HL for a basic fuzzy logic, which again could be retrieved as a particular axiomatic extension of MTL.

1.2 Core fuzzy logics

In the broader framework for MFL promoted by Esteva and Godo, i.e. that of logics based on residuated t-norms, the system MTL indeed fulfilled our requirements for a basic fuzzy logic, as expressed in the previous subsection. It was a genuine fuzzy logic enjoying a standard completeness theorem w.r.t. a semantics based on left-continuous t-norms, it could not be made weaker without losing this property and all known fuzzy logics could be obtained as extensions of MTL, thus providing a good base for a new systematical study of MFL. In fact, Petr Hájek saluted MTL as the new basic fuzzy logic and defined (in a joint work with Petr Cintula [51]) a precise general framework taking MTL as the basic system and not restricting to its axiomatic extensions (i.e. logics in the same language as MTL) but rather to its axiomatic expansions (by allowing new additional connectives). In particular they introduced two classes of logics which, though not very broad from the general perspective of the whole logical landscape, are still large enough to cover the most studied fuzzy logics. Actually, the majority of papers on MFL study logics from these two classes; later they provided a useful framework for a general study of completeness of (propositional and first-order) fuzzy logics w.r.t. distinguished semantics in [13]; and the study of arithmetical complexity of first-order fuzzy logics in [67]. The rough idea was to capture, by simple syntactic means, logics that share many desirable properties with MTL.

Definition 1. *A logic* L *in a language* $\mathcal{L}$ *is a* core fuzzy logic *if:*

1. L *expands* MTL.

2. *For all* $\mathcal{L}$*-formulae* φ, ψ, χ *the following holds:*

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \chi', \tag{Cong}$$

   where χ' *is a formula resulting from* χ *by replacing some occurrences of its subformula* φ *by the formula* ψ*.*

3. L *has an axiomatic system with* modus ponens *as the only deduction rule.*[2]

Axiom schema	Name
$\neg\neg\varphi \to \varphi$	Involution (Inv)
$\neg\varphi \vee ((\varphi \to \varphi \,\&\, \psi) \to \psi)$	Cancellation (Can)
$\neg(\varphi \,\&\, \psi) \vee ((\psi \to \varphi \,\&\, \psi) \to \varphi)$	Weak Cancellation (WCan)
$\varphi \to \varphi \,\&\, \varphi$	Contraction (C)
$\varphi^{n-1} \to \varphi^n$	n-Contraction (C_n)
$\varphi \wedge \neg\varphi \to \overline{0}$	Pseudocomplementation (PC)
$\varphi \wedge \psi \to \varphi \,\&\, (\varphi \to \psi)$	Divisibility (Div)
$(\varphi \,\&\, \psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi \,\&\, \psi)$	Weak Nilpotent Minimum (WNM)
$\varphi \vee \neg\varphi$	Excluded Middle (EM)

Table 1: Some usual axiom schemata in fuzzy logics.

Therefore, core fuzzy logics are essentially well-behaved axiomatic expansions of MTL. Observe, that since MTL is a finitary logic[3] and we are only considering adding axioms, not rules, all core fuzzy logics remain finitary. Table 2 collects prominent members of the family of core fuzzy logics together with the axioms one needs to add to MTL to obtain them (see the definition of axioms in Table 1). An important logic, which does not fall under the scope of the previous definition, is the expansion of MTL with the Monteiro–Baaz projection connective $\triangle$ [1, 70]. This logic, denoted as $\mathrm{MTL}_\triangle$, is obtained by adding the unary connective $\triangle$ to the language, the rule of $\triangle$-*Necessitation* ($\mathrm{Nec}_\triangle$)—from φ infer $\triangle\varphi$—and the following axioms:

[2]The original definition of core fuzzy logics [51, Convention 1] required the validity of a variant of deduction theorem (see Theorem 3), but is shown equivalent with our definition in [51, Proposition 3]; analogously for $\triangle$-core fuzzy logics introduced in the next definition.

[3]This means that whenever $\Gamma \vdash_{\mathrm{MTL}} \varphi$, there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{MTL}} \varphi$.

Logic	Additional axiom schemata	References
SMTL	(PC)	[40]
ΠMTL	(Can)	[40]
WCMTL	(WCan)	[68]
IMTL	(Inv)	[23]
WNM	(WNM)	[23]
NM	(Inv) and (WNM)	[23]
C_nMTL	(C_n)	[10]
C_nIMTL	(Inv) and (C_n)	[10]
HL (BL)	(Div)	[37]
SHL (SBL)	(Div) and (PC)	[25]
Ł	(Div) and (Inv)	[37, 62]
Π	(Div) and (Can)	[52]
G	(C)	[37, 21, 35]
CPC	(EM)	

Table 2: Some axiomatic extensions of MTL obtained by adding the corresponding additional axiom schemata.

(△1) $\triangle\varphi \vee \neg\triangle\varphi$
(△2) $\triangle(\varphi \vee \psi) \to \triangle\varphi \vee \triangle\psi$
(△3) $\triangle\varphi \to \varphi$
(△4) $\triangle\varphi \to \triangle\triangle\varphi$
(△5) $\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$

Taking $\mathrm{MTL}_\triangle$ as an alternative basic logic, Hájek and Cintula defined another class of fuzzy logics, now with the △ connective:

Definition 2. *A logic* L *in a language* $\mathcal{L}$ *is a* △-core fuzzy logic *if:*

1. L *expands* $\mathrm{MTL}_\triangle$.
2. *For all* $\mathcal{L}$*-formulae* φ, ψ, χ *the following holds:*

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \chi', \qquad \text{(Cong)}$$

where χ' *is a formula resulting from* χ *by replacing some occurrences of its subformula* φ *by the formula* ψ.

3. L *has an axiomatic system with* modus ponens *and* ($Nec_\triangle$) *as the only deduction rules.*

Expansions of fuzzy logics with △ were already systematically studied by Petr Hájek in [37] and have since then been considered for most fuzzy logics, making the class of △-core fuzzy logics another largely populated useful class.

Other well-known fuzzy logics in expanded languages fall under the scope of the two classes we have introduced, such as logics with truth-constants for intermediate truth-values (a Petr Hájek's variant [37] of the Pavelka's extension of Łukasiewicz logic [73, 71] later studied in many works by other authors; see e.g. [76, 28] or a detailed survey in [26, Section 2]), logics $\mathrm{L}_\sim$ expanded with an extra involutive negation (again initiated by Petr Hájek et al [25] and followed by others; see e.g. [16, 25, 31, 55] or a detailed survey in [26, Section 4]), or logics combining conjunctions and implications corresponding to different t-norms; see e.g. [12, 57, 69, 27] or again a survey in [26, Section 5].

Core and △-core fuzzy logics are all finitary and well-behaved from several points of view. In particular, for every such logic L one can define in a natural way a corresponding class of algebraic structures, L-algebras, which provide a complete semantics as in the case of MTL or the previously mentioned logics. Moreover, these classes of algebras are always *varieties*, i.e. they can be presented in terms of equations or, equivalently, are closed under formation of homomorphic images, subalgebras and direct products. Another interesting property shared by the logics in these classes is the deduction theorem. Petr Hájek already proved deduction theorems for several fuzzy logics, including his basic logic, and also for his expansions with $\triangle$ in [37]. One can analogously obtain deduction theorems for all the logics in the classes just defined (in a local form for core fuzzy logics, global for $\triangle$-core):[4]

Theorem 3. *1. Let* L *be a core fuzzy logic in a language* $\mathcal{L}$*. For every* $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$*,* $\Gamma, \varphi \vdash_{\mathrm{L}} \psi$ *iff there is* $n \geq 0$ *such that* $\Gamma \vdash_{\mathrm{L}} \varphi^n \to \psi$*.*

2. Let L *be a core* $\triangle$*-fuzzy logic in a language* $\mathcal{L}$*. For every* $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$*,* $\Gamma, \varphi \vdash_{\mathrm{L}} \psi$ *iff* $\Gamma \vdash_{\mathrm{L}} \triangle\varphi \to \psi$*.*

We will give more precise details both about algebraization of logics and about deduction theorems in Section 2.

1.3 Substructural logics as a framework for fuzzy logics

The quest for the basic fuzzy logic did not end with MTL (or $\mathrm{MTL}_{\triangle}$). Indeed, MTL has been further weakened in two different directions beyond the framework of core fuzzy logics:

(a) by dropping commutativity of conjunction Petr Hájek obtained a system, psMTL^r [42], which Jenei and Montagna proved to be complete with respect to the semantics on non-commutative residuated t-norms [60],

(b) by removing integrality (i.e. not requiring the neutral element of conjunction to be maximum of the order) Metcalfe and Montagna proposed the logic UL which is, in turn, complete with respect to left-continuous uninorms [64].

Hájek liked to describe this process of successive weakening of fuzzy logics by telling the joke of the crazy scientist that studied fleas by removing their legs one by one and checking whether they could still jump [47].[5] Namely, if HL was the original flea, it lost the 'divisibility leg' when it was substituted by MTL, and then psMTL^r and UL respectively lost the 'commutativity and the integrality leg' while retaining the ability to 'jump' (i.e., the completeness w.r.t. intended semantics based on reals).

These weaker fuzzy logics (and even MTL itself; see [24]) can be fruitfully studied in the context of substructural logics. Recall the bounded full Lambek logic FL,[6] a basic substructural logic which does not satisfy any of the usual three structural rules: exchange, weakening, and contraction. Although firstly presented by means of a Gentzen calculus, it can be given a Hilbert-style presentation and shown to be an algebraizable logic in the sense of [4] whose equivalent algebraic semantics is the variety of bounded pointed lattice-ordered residuated monoids (usually referred to as bounded pointed *residuated lattices* or FL-*algebras*). The main extensions of FL, obtained by adding some of the structural rules, correspond to subvarieties of residuated lattices satisfying corresponding extra algebraic properties (see e.g. [33]). Actually, many fuzzy logics have been shown to be axiomatic extensions of some of these prominent substructural logics by adding

[4] We need to recall the following inductively defined notation: $\varphi^0 = \overline{1}$, $\varphi^1 = \varphi$, and $\varphi^n = \varphi^{n-1} \,\&\, \varphi$.

[5] *A prominent biologist conducted a very important experiment. He trained a flea to jump upon giving her a verbal command ("Jump!"). In a first stage of the experiment he removed a flea's leg, told her to jump, and the flea jumped. So he wrote in his scientific notebook: "Upon removing one leg all flea organs function properly." So, he removed the second leg, asked the flea to jump, she obeyed, so he wrote again: "Upon removing the second leg all flea organs function properly." Thereafter he removed all the legs but one, the flea jumped when ordered, so he wrote again: "Upon removing the one but last leg all flea organs function properly." Then he removed the last leg. Told flea to jump, and nothing happened. He did not want to take a chance, so he repeated the experiment several times, and the legless flea never jumped. So he wrote the conclusion: "Upon removing the last leg the flea loses sense of hearing."*

[6] We use this notation for simplicity in this introduction, even though in the literature the symbol FL is usually used for the unbounded full Lambek logic whereas the bounded FL is denoted as $\mathrm{FL}_{\bot}$.

some axioms that enforce completeness with respect to some class of linearly ordered residuated lattices (or *chains*). For instance, Gödel–Dummett logic is the logic of linearly ordered Heyting algebras ($\mathrm{FL_{ewc}}$-chains), MTL is the logic $\mathrm{FL}^{\ell}_{\mathrm{ew}}$ of $\mathrm{FL_{ew}}$-chains,[7] UL is the logic $\mathrm{FL}^{\ell}_{\mathrm{e}}$ of $\mathrm{FL_e}$-chains, and psMTL^r is the logic $\mathrm{FL}^{\ell}_{\mathrm{w}}$ of $\mathrm{FL_w}$-chains.

This common feature, completeness with respect to their corresponding linearly ordered algebraic structures, has motivated the methodological paper [2] where the authors postulate that *fuzzy logics are the logics of chains*, in the sense that they are logics complete with respect to a semantics of chains. However, all the fuzzy logics mentioned so far do enjoy a stronger property: the *standard completeness theorem*, i.e. completeness with respect to a semantics of algebras defined on the real unit interval $[0, 1]$ which Petr Hájek and many others have considered to be the intended semantics for fuzzy logics. Following Hájek's flea joke, we could say that those fleas are fuzzy logics that *jump well* provided that they satisfy standard completeness. Actually, many authors implicitly (and sometimes even explicitly, e.g. in [64]) regard standard completeness as an essential requirement for fuzzy logics. It is, thus, reasonable to expect any candidate for the basic fuzzy logic to satisfy this stronger requirement. But, although they fulfill that, neither $\mathrm{FL}^{\ell}_{\mathrm{e}}$ nor $\mathrm{FL}^{\ell}_{\mathrm{w}}$ can be taken as basic because they are not comparable and hence do not satisfy our second meaning of *basic.* A reasonable candidate could be the logic FL^{ℓ} of FL-chains (a common generalization of $\mathrm{FL}^{\ell}_{\mathrm{e}}$ and $\mathrm{FL}^{\ell}_{\mathrm{w}}$). But, interestingly enough, this logic does not enjoy the standard completeness (as proved in [77]) and, therefore, we must discard it. Moreover, one can also argue that FL^{ℓ} is still *not basic enough* (in the first meaning) because it satisfies a remaining structural rule: associativity. Hence, in the context of substructural logics, it could still be made weaker by removing associativity.

There have actually been several studies on non-associative substructural logics, starting with the original Lambek non-associative calculus [61] (without lattice connectives), and followed (in the full language) e.g. by Buszkowski and Farulewski in [9]. Recently, a general algebraic framework to study fuzzy logics as a subfamily of (not necessarily associative) substructural logics has been developed in [19]. It is based on the logic SL, a non-associative version of the bounded Full Lambek calculus, introduced by Galatos and Ono in [34].[8] SL is formulated in the language $\mathcal{L}_{\mathrm{SL}} = \{\wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot\}$ (we also make use of the defined connectives $\top = \bot \to \bot$ and $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$) and axiomatized by means of the Hilbert-style calculus from [34, Figure 5] presented in Table 3.[9]

Moreover, Galatos and Ono proved that SL is an algebraizable logic whose equivalent algebraic semantics is the variety of bounded lattice-ordered residuated unital groupoids, where the monoidal structure of the previous logics has become just a groupoid on account of the lack of associativity (we will see more details in Section 2). Therefore, if we are looking for a logic complete with respect to chains in the non-associative context, it makes sense to consider, in a similar fashion as with FL and its extensions, the logic SL^{ℓ} as the logic of bounded linearly ordered residuated unital groupoids.

1.4 Goals and outline of the chapter

The main goal of this chapter is to propose SL^{ℓ} as a new basic fuzzy logic. The current stage of development in MFL requires a broader framework than that provided by core and $\triangle$-core fuzzy logics. This is witnessed by several works (some already mentioned) dealing with fuzzy logics weaker than MTL, e.g. [6, 32, 41, 42, 43, 54, 60, 63, 64, 77]. For this reason fuzzy logics have started being systematically studied in the context of (not necessarily associative) substructural logics in [19]. However, this work did not offer a basic fuzzy logic in its framework.

On the other hand, SL^{ℓ} has been introduced and studied as an axiomatic extension of SL in

[7] As notation convention (later precisely introduced in Definition 26) given a logic L, we denote by L^{ℓ} the logic of L-chains.

[8] Technically speaking, Galatos and Ono introduced an unbounded version of this logic and actually never named it. The name SL, standing for 'substructural logic', comes from [19].

[9] When writing formulae in this language we will assume that the increasing binding order of connectives is: first &, then $\{\wedge, \vee\}$, and finally $\{\to, \rightsquigarrow, \leftrightarrow\}$.

(R)	$\varphi \to \varphi$	(R$'$)	$\bar{1} \to (\varphi \to \varphi)$
(MP)	$\varphi, \varphi \to \psi \vdash \psi$	(Push)	$\varphi \to (\bar{1} \to \varphi)$
(Sf)	$\varphi \to \psi \vdash (\psi \to \chi) \to (\varphi \to \chi)$	($\bar{1}$)	$\bar{1}$
(Pf)	$\psi \to \chi \vdash (\varphi \to \psi) \to (\varphi \to \chi)$	(Bot)	$\bot \to \varphi$
(As)	$\varphi \vdash (\varphi \to \psi) \to \psi$	($\wedge$1)	$\varphi \wedge \psi \to \varphi$
(As$_{\ell\ell}$)	$\varphi \to ((\varphi \rightsquigarrow \psi) \to \psi)$	($\wedge$2)	$\varphi \wedge \psi \to \psi$
(Symm$_1$)	$\varphi \rightsquigarrow \psi \vdash \varphi \to \psi$	($\wedge$3)	$(\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$
(E$_{\rightsquigarrow 1}$)	$\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \rightsquigarrow \chi)$	($\vee$1)	$\varphi \to \varphi \vee \psi$
(Res$_1$)	$\psi \to (\varphi \to \chi) \vdash \varphi \,\&\, \psi \to \chi$	($\vee$2)	$\psi \to \varphi \vee \psi$
(Adj$_\&$)	$\varphi \to (\psi \to \psi \,\&\, \varphi)$	($\vee$3)	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
(Adj)	$\varphi, \psi \vdash \varphi \wedge \psi$	($\vee$3$_\rightsquigarrow$)	$(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \vee \psi \rightsquigarrow \chi)$

Table 3: Axiomatic system of SL.

the recent paper [15] where, among others, it has been shown to enjoy standard completeness. Based on these results we will defend here the thesis that SL^ℓ can serve as a basic fuzzy logic, good enough for the current needs of MFL. We will argue that is genuinely fuzzy and basic in both senses mentioned earlier in this introduction. To this end, we introduce a new class of logics containing core and $\triangle$-core fuzzy logics and much more: *core semilinear logics*. The adjective 'semilinear' in the name of this class refers to a notion introduced in [18] in order to capture the idea of fuzzy logics as logics of chains proposed in [2]. The idea is the following: if a logic has a reasonable implication $\to$ (which is the case of SL and many of its expansions like core and $\triangle$-core fuzzy logics) then its corresponding algebraic structures can be ordered in terms of the implication ($a \leq b$ iff $a \to b \geq 1$); the logic is said to be *semilinear* iff it is complete w.r.t. the class of algebras where the order just defined is total. Moreover, the definition of core semilinear logic, as we will see, is formally analogous to those of core and $\triangle$-core fuzzy logics. It is not difficult to check that the latter classes are contained in the former. Actually, the class of core semilinear logics provides a convenient intermediate level of generality, between that of core and $\triangle$-core fuzzy logics and that of finitary semilinear substructural logics introduced in [19], by fixing SL^ℓ (and, therefore, its language) as a common base and allowing for non-axiomatic extensions.

Outline of the chapter After this introduction that has presented the topic (historically and conceptually), the main logical systems, the classes of ($\triangle$-)core fuzzy logics, and the motivation for the forthcoming class of core semilinear logics,[10] Section 2 presents, in mathematical details, the necessary logical and algebraic framework for our approach, which mainly restricts to substructural logics understood as well-behaved expansions of the non-associative logic SL. Section 2.1 gives the basic notions, Section 2.2 shows deduction theorems for our logics, and Section 2.3 is devoted to generation of filters, algebraization, and completeness w.r.t. (finitely) subdirectly irreducible algebras. Section 3, as the central part of the paper, focuses on propositional core semilinear logics. After defining them, Section 3.1 shows several useful characterizations of semilinear logics and their axiomatizations, including a presentation of SL^ℓ as axiomatic extension of SL; Section 3.2 is a survey on completeness properties of core semilinear logics w.r.t. significant algebraic semantics, in particular we stress the standard completeness of SL^ℓ, making it a fuzzy logic in a genuine sense. Finally, Section 4 is devoted to first-order predicate counterparts of core semilinear logics, including $\mathrm{SL}^\ell\forall$, the first-order extension of SL^ℓ. Section 4.1 shows the axiomatization of these logics, Section 4.2 presents their semantics based on general and witnessed models,

[10] Although we have tried to make this paper reasonably self-contained, the obvious space limitations do not allow for an extensive presentation of all mentioned logical systems. For an up-to-date encyclopedical account of Mathematical Fuzzy Logic see [14].

and Section 4.3 focuses again on distinguished semantics, in particular stressing that $\mathrm{SL}^{\ell}\forall$ enjoys standard completeness too.

2 Logical framework

In order to deal with the classes of logics mentioned above, we need some flexibility as regards both propositional languages and logics. Therefore, for the sake of reference and in order to fix terminology in a convenient way for this chapter, we shall start with some standard general definitions and conventions.[11]

2.1 Basic syntax and semantics

In this chapter we consider logics as given by finitary Hilbert-style proof systems expanding that of SL (see Table 3 in the introduction). Following Hájek's methodology, we restrict to finitary systems as he did when proposing schematic extensions of HL as a systematical approach to MFL. This simplifies the presentation but it does not undermine the suitability of our proposed basic logic SL^{ℓ} (or its first-order counterpart) for the infinitary systems of fuzzy logic can still be retrieved as its extensions.

A *propositional language* $\mathcal{L}$ is a *countable* type, i.e. a function $ar\colon C_{\mathcal{L}} \to \mathsf{N}$, where $C_{\mathcal{L}}$ is a countable set of symbols called *connectives*, giving for each one its *arity*. Nullary connectives are also called *truth-constants*. We write $\langle c, n\rangle \in \mathcal{L}$ whenever $c \in C_{\mathcal{L}}$ and $ar(c) = n$. The basic language in this chapter is $\mathcal{L}_{\mathrm{SL}}$ with binary connectives $\wedge, \vee, \&, \to, \rightsquigarrow$ and truth-constants $\overline{0}, \overline{1}, \bot, \top$. Let *Var* be a fixed infinite countable set of symbols called *variables*. The set $Fm_{\mathcal{L}}$ of *formulae* in a propositional language $\mathcal{L}$ is the least set containing *Var* and closed under connectives of $\mathcal{L}$, i.e. for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$, $c(\varphi_1, \dots, \varphi_n)$ is a formula. $Fm_{\mathcal{L}}$ can be seen as the domain of the absolutely free algebra $\boldsymbol{Fm}_{\mathcal{L}}$ of type $\mathcal{L}$ and generators *Var*. An $\mathcal{L}$*-substitution* is an endomorphism on the algebra $\boldsymbol{Fm}_{\mathcal{L}}$, i.e. a mapping $\sigma\colon Fm_{\mathcal{L}} \to Fm_{\mathcal{L}}$, such that $\sigma(c(\varphi_1, \dots, \varphi_n)) = c(\sigma(\varphi_1), \dots, \sigma(\varphi_n))$ holds for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$. Since an $\mathcal{L}$-substitution is a mapping whose domain is a free $\mathcal{L}$-algebra, it is fully determined by its values on the generators (propositional variables).

An *axiomatic system* $\mathcal{AS}$ in a propositional language $\mathcal{L}$ is a pair $\langle Ax, R\rangle$ where Ax is set of formulae (the *axioms*) and R is a set of pairs $\langle \Gamma, \varphi\rangle$ (the *rules*) where Γ is a *finite* non-empty set of formulae and φ is a formula.[12] Moreover, both Ax and R are closed under arbitrary substitutions. Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, we say that φ *is provable from* Γ *in* $\mathcal{AS}$, in symbols $\Gamma \vdash_{\mathcal{AS}} \varphi$, if there exists a finite sequence of formulae $\langle \varphi_0, \dots, \varphi_n\rangle$ (a *proof*) such that:

- $\varphi_n = \varphi$, and
- for every $i \leq n$, either $\varphi_i \in \Gamma \cup Ax$ or there is $\langle \Delta, \varphi_i\rangle \in R$ such that $\Delta \subseteq \{\varphi_0, \dots, \varphi_{i-1}\}$.

Observe that the provability relation $\vdash_{\mathcal{AS}}$ is finitary, i.e., if $\Gamma \vdash_{\mathcal{AS}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathcal{AS}} \varphi$.

Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ be propositional languages and $\mathcal{AS}_i$ an axiomatic system in $\mathcal{L}_i$. We say that $\mathcal{AS}_2$ is an expansion of $\mathcal{AS}_1$ by axioms Ax and rules R if all its axioms (rules) are $\mathcal{L}_2$-substitutional instances of axioms (rules) of $\mathcal{AS}_1$ or formulae from Ax (rules from R).

Now we are ready to give our formal convention restricting logics to finitary expansions of SL with well-behaved connectives.

Convention 4. *A* logic L *in a language* $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{SL}}$ *is the provability relation given by an axiomatic system* $\mathcal{AS}$ *in* $\mathcal{L}$ *which is an expansion of that of* SL *(see Table 3) and for all* $\mathcal{L}$*-formulae* φ, ψ, χ *the following holds:*

$$\varphi \leftrightarrow \psi \vdash_{\mathcal{AS}} \chi \leftrightarrow \chi', \tag{Cong}$$

[11] The interested reader can complement the upcoming short presentation by consulting reference works on (Abstract) Algebraic Logic such as [4, 7, 14]. We deviate slightly from the standard treatment of some basic notions because we are tailoring them to the particular purposes of the present chapter.

[12] Sometimes, especially when listing rules, we use the denotation $\Gamma \vdash \varphi$ rather than $\langle \Gamma, \varphi\rangle$.

where χ' is a formula resulting from χ by replacing some occurrences of its subformula φ by a formula ψ. In this case we say that $\mathcal{AS}$ is a presentation *of* L *(or that* L is axiomatized by *$\mathcal{AS}$) and write $\Gamma \vdash_{\mathrm{L}} \varphi$ instead of $\Gamma \vdash_{\mathcal{AS}} \varphi$.*

Remark 5. One can equivalently replace the condition (Cong) by the following:

$$\varphi \leftrightarrow \psi \vdash_{\mathcal{AS}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \leftrightarrow c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n) \qquad (\mathrm{Cong}_c^i)$$

for each $\langle c, n\rangle \in \mathcal{L}$ and each $0 \leq i < n$. Therefore, since this condition is already satisfied in SL by all its connectives, in order to check whether a particular expansion of SL is a logic in the sense just defined, it is enough to check (Cong_c^i) for all new connectives (this statement remains true if we replace SL by any other logic).

The notion of expansion can naturally be formulated for logics. Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ be propositional languages and L_i a logic in $\mathcal{L}_i$. We say that

- L_2 is the *expansion of* L_1 *by axioms Ax and rules R* if it is axiomatized by expanding some presentation of L_1 with axioms Ax and rules R.
- L_2 is an (axiomatic) *expansion of* L_1 if it is the expansion of L_1 by some axioms and rules (or just axioms respectively).

If $\mathcal{L}_1 = \mathcal{L}_2$, we use 'extension' instead of 'expansion'. Let $\mathcal{S}$ be a collection of extensions of a given logic L. We define the following two axiomatic systems and two logics:

$$\bigcap \mathcal{S} = \{\langle \Gamma, \varphi\rangle \mid \Gamma \vdash_{\mathrm{L}} \varphi \text{ for each } \mathrm{L} \in \mathcal{S}\} \qquad \bigwedge \mathcal{S} = \vdash_{\bigcap \mathcal{S}}$$

$$\bigcup \mathcal{S} = \{\langle \Gamma, \varphi\rangle \mid \Gamma \vdash_{\mathrm{L}} \varphi \text{ for some } \mathrm{L} \in \mathcal{S}\} \qquad \bigvee \mathcal{S} = \vdash_{\bigcup \mathcal{S}}$$

It is clear that $\bigwedge \mathcal{S}$ and $\bigvee \mathcal{S}$ are respectively the infimum and the supremum of $\mathcal{S}$ in the set of extensions of L ordered by inclusion. Therefore, the set of extensions of a given logic L always forms a complete lattice. Note that $\bigvee \mathcal{S}$ can be axiomatized by taking the union of arbitrary axiomatic systems for the logics in $\mathcal{S}$. Thus, in particular, if all logics in $\mathcal{S}$ are axiomatic extensions of L, then so is $\bigvee \mathcal{S}$. Therefore, the class of *axiomatic* extensions of L is a sub-join-semilattice of the lattice of all extensions of L. The axiomatization of meets is not so straightforward; at the end of Section 3.1 we will see how to deal with this problem in the restricted setting of core semilinear logics.

Some important axiomatic extensions of SL are obtained by adding the axioms $\mathrm{a}_1, \mathrm{a}_2, \mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}$ corresponding to structural rules (see Table 4).

a_1	$\varphi \,\&\, (\psi \,\&\, \chi) \to (\varphi \,\&\, \psi) \,\&\, \chi$	*re-associate to the left*
a_2	$(\varphi \,\&\, \psi) \,\&\, \chi \to \varphi \,\&\, (\psi \,\&\, \chi)$	*re-associate to the right*
e	$\varphi \,\&\, \psi \to \psi \,\&\, \varphi$	*exchange*
c	$\varphi \to \varphi \,\&\, \varphi$	*contraction*
i	$\psi \to (\varphi \to \psi)$	*left weakening*
o	$\overline{0} \to \varphi$	*right weakening*

Table 4: Axioms for structural rules.

Given any $S \subseteq \{\mathrm{a}_1, \mathrm{a}_2, \mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$, by SL_S we denote the axiomatic extension of SL by S. If $\{\mathrm{a}_1, \mathrm{a}_2\} \subseteq S$, then instead of them we write the symbol 'a'. Analogously if $\{\mathrm{i}, \mathrm{o}\} \subseteq S$, instead of them we write the symbol 'w'. Equivalent ways to formulate these axioms may be found e.g. in [19, Theorem 2.5.7]. SL_{a} is, in fact, the bounded full Lambek logic.

Now we introduce the basic algebraic notions that will allow to provide a semantics for our logics.

Definition 6. *A* bounded pointed lattice-ordered residuated unital groupoid*, or shortly just* SL-algebra*, is an algebra* $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 0, 1, \bot, \top\rangle$ *such that*

- $\langle A, \wedge, \vee, \bot, \top\rangle$ *is a bounded lattice*
- 1 *is the unit of* $\cdot$
- *for each* $a, b, c \in A$ *we have*

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a\backslash c \quad \text{iff} \quad a \leq c/b\,.$$

The class of all SL-algebras is a variety and it is denoted as $\mathbb{SL}$. Observe that the residuation condition together with the fact that 1 is a neutral element implies that for every SL-algebra $\boldsymbol{A}$ and each $a, b \in A$ we have

$$a \leq b \quad \text{iff} \quad 1 \leq a\backslash b \quad \text{iff} \quad 1 \leq b/a\,.$$

Given an SL-algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 0, 1, \bot, \top\rangle$, an $\boldsymbol{A}$*-evaluation* is a homomorphism from the algebra of formulae to $\boldsymbol{A}$ such that the connectives $\wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top$ are respectively interpreted by the functions $\wedge, \vee, \cdot, \backslash, /, 0, 1, \bot, \top$, i.e., a mapping from $Fm_{\mathcal{L}}$ to A such that $e(*) = *$ for $* \in \{0, 1, \bot, \top\}$ and $e(\varphi \circ \psi) = e(\varphi) \circ' e(\psi)$, where $\circ \in \{\wedge, \vee, \&, \to, \rightsquigarrow\}$ and $\circ'$ is the corresponding operation from $\{\wedge, \vee, \cdot, \backslash, /\}$. By means of this notion, we can give, more generally, the following definition for the algebraic counterpart of any logic.

Definition 7. *Let* L *be a logic in language* $\mathcal{L}$ *which is the expansion of* SL *by axioms* Ax *and rules* R*. An* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *is an* L*-algebra if*

- *its reduct* $\boldsymbol{A}_{\mathrm{SL}} = \langle A, \wedge, \vee, \cdot, \backslash, /, 0, 1, \bot, \top\rangle$ *is an* SL*-algebra,*
- *for every* $\varphi \in Ax$ *and every* $\boldsymbol{A}$*-evaluation* e*,* $e(\varphi) \geq 1$*,*
- *for every* $\langle \Gamma, \varphi\rangle \in R$ *and every* $\boldsymbol{A}$*-evaluation* e*, if* $e(\psi) \geq 1$ *for every* $\psi \in \Gamma$*, then* $e(\varphi) \geq 1$*.*

$\boldsymbol{A}$ *is a linearly ordered (or* L-chain*) if its lattice order is total. The class of all (linearly ordered)* L*-algebras is denoted by* $\mathbb{L}$ *(or* $\mathbb{L}_{lin}$ *respectively).*

Table 5 shows what conditions have to be added to SL-algebras, to obtain, for arbitrary $S \subseteq \{\mathrm{a}_1, \mathrm{a}_2, \mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$, the class of SL_S-algebras.

a_1	$x \cdot (y \cdot z) \leq (x \cdot y) \cdot z$
a_2	$(x \cdot y) \cdot z \leq x \cdot (y \cdot z)$
e	$x \cdot y = y \cdot x$
c	$x \leq x \cdot x$
i	$x \leq 1$
o	$0 \leq x$

Table 5: Equations defining important classes of SL-algebras.

The following completeness theorem follows from more general results (see Section 2.3 where we show more on the connection between logics and algebras) but can also be directly proved by means of the usual Lindenbaum–Tarski process. It shows how L-algebras really give an algebraic semantics for SL and its expansions.

Theorem 8. *Let* L *be a logic. Then for every set of formulae* Γ *and every formula* φ *the following are equivalent:*

1. $\Gamma \vdash_{\mathrm{L}} \varphi$*,*
2. *for every* $\boldsymbol{A} \in \mathbb{L}$ *and every* $\boldsymbol{A}$*-evaluation* e*, if* $e(\psi) \geq 1$ *for every* $\psi \in \Gamma$*, then* $e(\varphi) \geq 1$*.*

$(\mathrm{Adj}_{\&})$	$\varphi \to (\psi \to \psi \,\&\, \varphi)$	(Bot)	$\bot \to \varphi$
$(\mathrm{Adj}_{\&\rightsquigarrow})$	$\varphi \to (\psi \rightsquigarrow \varphi \,\&\, \psi)$	(Push)	$\varphi \to (\overline{1} \to \varphi)$
(Res')	$\psi \,\&\, (\varphi \,\&\, (\varphi \to (\psi \to \chi))) \to \chi$	(Pop)	$(\overline{1} \to \varphi) \to \varphi$
$(\mathrm{Res}'_{\rightsquigarrow})$	$(\varphi \,\&\, (\varphi \to (\psi \rightsquigarrow \chi))) \,\&\, \psi \to \chi$	$(\&\wedge)$	$(\varphi \wedge \overline{1}) \,\&\, (\psi \wedge \overline{1}) \to \varphi \wedge \psi$
(T')	$(\varphi \to (\varphi \,\&\, (\varphi \to \psi)) \,\&\, (\psi \to \chi)) \to (\varphi \to \chi)$		
$(\mathrm{T}'_{\rightsquigarrow})$	$(\varphi \rightsquigarrow ((\varphi \rightsquigarrow \psi) \,\&\, \varphi) \,\&\, (\psi \to \chi)) \to (\varphi \rightsquigarrow \chi)$		
$(\wedge 1)$	$\varphi \wedge \psi \to \varphi$	$(\vee 1)$	$\varphi \to \varphi \vee \psi$
$(\wedge 2)$	$\varphi \wedge \psi \to \psi$	$(\vee 2)$	$\psi \to \varphi \vee \psi$
$(\wedge 3)$	$(\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$	$(\vee 3)$	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
(MP)	$\varphi, \varphi \to \psi \vdash \psi$	$(\mathrm{Adj_u})$	$\varphi \vdash \varphi \wedge \overline{1}$
(α)	$\varphi \vdash \delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \varphi)$	(β)	$\varphi \vdash \delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \varphi)$
(α')	$\varphi \vdash \delta \,\&\, \varepsilon \to (\delta \,\&\, \varphi) \,\&\, \varepsilon$	(β')	$\varphi \vdash \delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \varphi)$

Table 6: New axiomatic system for SL.

2.2 Almost (MP)-based logics and deduction theorems

In the introduction we have formulated the usual deduction theorems for core and $\triangle$-core fuzzy logics (Theorem 3). In this section we show how this can be generalized to all logics in the present framework (expansions of SL) provided that the additional rules they satisfy are of a certain good form. Technically, this corresponds to the notion of almost (MP)-based logic that, as shown in [19, 15], essentially allows to repeat Hájek's old proof of deduction theorem now in this wide context. To this end, we introduce a few more syntactical notions. Let $\star$ be a new propositional variable not occurring in *Var*, which acts as placeholder for a special kind of substitutions. The notions depending on the set of variables (formula, substitution, logic, etc.) are augmented by the prefix $\star$- whenever they are construed over the set of variables $\mathit{Var} \cup \{\star\}$ and are left as they are if construed in the original set of variables *Var*. If φ and δ are $\star$-formulae, by $\delta(\varphi)$ we denote the formula obtained from δ when one replaces the occurrences of $\star$ by φ; note that if φ is a formula, then so is $\delta(\varphi)$ (i.e., $\star$ does not occur in $\delta(\varphi)$).

Definition 9. *Given a set of $\star$-formulae Γ, we define the sets of $\star$-formulae*

- Γ^* *as the smallest set containing $\star$ and $\delta(\gamma) \in \Gamma^*$ for each $\delta \in \Gamma$ and each $\gamma \in \Gamma^*$.*
- $\Pi(\Gamma)$ *as the smallest set of $\star$-formulae containing $\Gamma \cup \{\overline{1}\}$ and closed under &.*

We are now ready to give the formal definition of almost (MP)-based logic.

Definition 10. *Let* bDT *be a set of $\star$-formulae closed under all $\star$-substitutions σ such that $\sigma(\star) = \star$. A logic* L *is* almost (MP)-based w.r.t. the set of basic deduction terms bDT *if:*

- L *has a presentation where the only deduction rules are* modus ponens *and those from* $\{\langle \varphi, \gamma(\varphi)\rangle \mid \varphi \in Fm_{\mathcal{L}_{\mathrm{SL}}}, \gamma \in \mathrm{bDT}\}$*, and*
- *for each $\beta \in \mathrm{bDT}$ and each formulae φ, ψ, there exist $\beta_1, \beta_2 \in \mathrm{bDT}^*$ such that:*[13]

$$\vdash_{\mathrm{L}} \beta_1(\varphi \to \psi) \to (\beta_2(\varphi) \to \beta(\psi)).$$

L *is called* (MP)-based *if it admits the empty set as a set of basic deduction terms.*

[13] We deviate slightly from the original definition from [19], where β_1, β_2 were required to be in bDT, and follow that from [15] which has some technical advantages.

Logic L	$\mathrm{bDT_L}$
SL	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_w}$	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_e}$	$\{\alpha_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_{ew}}$	$\{\alpha_{\delta,\varepsilon}, \beta_{\delta,\varepsilon} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_a}$	$\{\lambda_\varepsilon, \rho_\varepsilon, \star \wedge \overline{1} \mid \varepsilon \text{ a formula}\}$
$\mathrm{SL_{aw}}$	$\{\lambda_\varepsilon, \rho_\varepsilon \mid \varepsilon \text{ a formula}\}$
$\mathrm{SL_{ae}}$	$\{\star \wedge \overline{1}\}$
$\mathrm{SL_{aew}}$	$\{\star\}$

Table 7: bDTs of prominent substructural logics.

SL can be shown to be indeed an almost (MP)-based logic. For this, of course, one needs to endow it with a convenient alternative presentation. Consider the axiomatic system from Table 6 and let us introduce a convenient notation for the terms appearing on the right-hand side of the rules (α), (α'), (β), and (β'). Given arbitrary formulae δ, ε, we define the following $\star$-formulae:

$$\alpha_{\delta,\varepsilon} = \delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \star) \qquad \beta_{\delta,\varepsilon} = \delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \star)$$

$$\alpha'_{\delta,\varepsilon} = \delta \,\&\, \varepsilon \to (\delta \,\&\, \star) \,\&\, \varepsilon \qquad \beta'_{\delta,\varepsilon} = \delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \star)$$

Note that the terms in the second line generalize the well-known notions of left and right conjugates used in associative logics:[14]

$$\lambda_\varepsilon = \varepsilon \to \star \,\&\, \varepsilon \qquad \rho_\varepsilon = \varepsilon \rightsquigarrow \varepsilon \,\&\, \star$$

In [15] it is proved that the axiomatic system from Table 6 is indeed a presentation of SL, therefore we can obtain the following result for SL and some of its notable axiomatic extension (it also shows how the sets of basic deduction terms, and so posteriorly the axioms systems, of these extensions can be simplified).

Theorem 11 ([15, Section 3.1])**.** *Let* $S \subseteq \{\mathrm{a}, \mathrm{e}, \mathrm{w}\}$. *Then any axiomatic extension of the logic* SL_S *is almost* (MP)*-based with respect to the corresponding set of basic deduction terms listed in Table 7.*

Theorem 12 (Local deduction theorem [15, Corollary 3.12])**.** *Let* L *be an almost* (MP)*-based logic with a set of basic deduction terms* bDT. *Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae the following holds:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \quad \textit{iff} \quad \Gamma \vdash_{\mathrm{L}} \gamma(\varphi) \to \psi \textit{ for some } \gamma \in \Pi(\mathrm{bDT}^*).$$

Therefore, we obtain a (parameterized or non-parameterized, depending on the presence of variables other than $\star$ in the set bDT) local deduction theorem for SL and its axiomatic extensions (sometimes with a simplified set bDT; see Table 7).

2.3 Consequences of algebraization

Given a logic L in a language $\mathcal{L}$ and an $\mathcal{L}$-algebra $\boldsymbol{A}$, a set $F \subseteq A$ is an L-*filter* if for every set of formulae Γ and every formula φ such that $\Gamma \vdash_{\mathrm{L}} \varphi$ and every $\boldsymbol{A}$-evaluation e it holds: if $e[\Gamma] \subseteq F$,

[14] It is usual in the literature on algebraic study of substructural logics to find these terms defined in a slightly more complicated way: $\lambda_\varepsilon = (\varepsilon \to \star \,\&\, \varepsilon) \wedge \overline{1}$ and $\rho_\varepsilon = (\varepsilon \rightsquigarrow \varepsilon \,\&\, \star) \wedge \overline{1}$, although in the usual Hilbert-style axiomatizations of Full Lambek logic the simplified terms without $\wedge\overline{1}$ are used for the *product normality* rules. The reason for this more complicated form is to give algebraic terms which simultaneously cope with product normality rules and adjunction, whereas our formalism allows for a clearer distinction of their respective rôles.

then $e(\varphi) \in F$. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$. Since $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system (it clearly contains A and is closed under arbitrary intersections), one can define a notion of generated filter. Given $X \subseteq A$, the L-filter generated by X, denoted as $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X)$ is the least L-filter containing X (we omit the indexes when clear from the context). With this terminology one can also prove a semantical (or transferred) version of (parameterized) local deduction theorem; Theorem 12 is the particular case in which $\boldsymbol{A}$ is the algebra of formulae (observe that in this case $\varphi \in \mathrm{Fi}(\Gamma)$ iff $\Gamma \vdash_{\mathrm{L}} \varphi$). First we introduce two technical notions:

Definition 13. *Given a set of $\star$-formulae Γ, an* SL*-algebra $\boldsymbol{A}$, and a set $X \subseteq A$, we define*

- $\Gamma^{\boldsymbol{A}}$ *as the set of unary polynomials built using terms from Γ with coefficients from A and variable $\star$, i.e., $\{\delta(\star, a_1, \dots, a_n) \mid \delta(\star, p_1, \dots, p_n) \in \Gamma$ and $a_1, \dots, a_n \in A\}$.*
- $\Gamma^{\boldsymbol{A}}(X)$ *as the set $\{\delta^{\boldsymbol{A}}(x) \mid \delta(\star) \in \Gamma^{\boldsymbol{A}}$ and $x \in X\}$.*

Theorem 14 ([15, Theorem 3.11])**.** *Let* L *be an almost* (MP)*-based logic in a language $\mathcal{L}$ with a set of basic deduction terms* bDT*. Let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra and $X \cup \{x\} \subseteq A$. Then*

$$y \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X, x) \qquad \textit{iff} \qquad \gamma^{\boldsymbol{A}}(x)\backslash y \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) \textit{ for some } \gamma \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}.$$

On the other hand, Theorem 14 can be used to obtain a general form of the usual algebraic description of the filter generated by a set.

Corollary 15 ([15, Corollary 3.13])**.** *Let* L *be an almost* (MP)*-based logic with a set of basic deduction terms* bDT*. Let $\boldsymbol{A}$ be an* L*-algebra and $X \subseteq A$. Then*

$$\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = \{a \in A \mid a \geq x \text{ for some } x \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}(X)\}.$$

The algebraic completeness result we have seen above (Theorem 8) can be strengthened obtaining that SL is actually an algebraizable logic in the sense of [4] and $\mathbb{SL}$ is its equivalent algebraic semantics with translations $\rho(p \approx q) = p \leftrightarrow q$ and $\tau(p) = p \wedge \bar{1} \approx \bar{1}$. Indeed, if we consider formal equations in the language $\mathcal{L}_{\mathrm{SL}}$ as expressions of the form $\varphi \approx \psi$ where $\varphi, \psi \in Fm_{\mathcal{L}_{\mathrm{SL}}}$ and if $\models_{\mathbb{SL}}$ denotes the equational consequence with respect to the class $\mathbb{SL}$, it is easy to prove that:

1. $\Pi \models_{\mathbb{SL}} \varphi \approx \psi$ iff $\rho[\Pi] \vdash_{\mathrm{SL}} \rho(\varphi \approx \psi)$
2. $p \dashv\vdash_{\mathrm{SL}} \rho[\tau(p)]$

Actually, this result can be extended to every logic L and its corresponding class of algebras $\mathbb{L}$. If L is a logic in a language $\mathcal{L}$ which is the expansion of SL by axioms Ax and rules R, then L-algebras can also be described as the expansions of SL-algebras satisfying:

- the equation $\tau(\varphi)$ for each $\varphi \in Ax$
- the quasiequation $\tau(\varphi_1)$ and $\dots$ and $\tau(\varphi_n) \Rightarrow \tau(\varphi)$ for each $\langle\{\varphi_1, \dots, \varphi_n\}, \varphi\rangle \in R$.

Therefore, the class $\mathbb{L}$ is always a quasivariety and it is a variety if $R = \emptyset$, i.e. if L is an axiomatic expansion of SL (note that this condition is not necessary as demonstrated e.g. by the logic $\mathrm{MTL}_{\triangle}$). Conversely, given a quasivariety $\mathbb{L}$ of $\mathcal{L}$-algebras, one can always find a quasiequational base obtained by adding a set of equations E and a set of quasiequations Q to an equational base of $\mathbb{SL}$. Then $\mathbb{L}$ is the equivalent algebraic semantics of the logic obtained by adding the following to SL:

- $\rho(\varphi, \psi)$ for each $\varphi \approx \psi \in E$
- $\langle\{\rho(\varphi_1, \psi_1), \dots, \rho(\varphi_n, \psi_n)\}, \rho(\varphi, \psi)\rangle$ for each $(\varphi_1 \approx \psi_1)$ and $\dots$ and $(\varphi_n \approx \psi_n) \Rightarrow \varphi \approx \psi \in Q$.

Moreover, if we fix a language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{SL}}$ and a logic L in $\mathcal{L}$, translations between formulae and equations give a bijective correspondence between extensions of L and quasivarieties of L-algebras, and a bijective correspondence (its restriction) between axiomatic extensions of L and varieties of L-algebras. These bijections are, actually, dual lattice isomorphisms.

A logic is called *strongly algebraizable* if its corresponding quasivariety is actually a variety. Obviously, strongly algebraizable logics in $\mathcal{L}_{\mathrm{SL}}$ coincide with axiomatic extensions of SL.

Algebraizability also gives a strong correspondence between filters and (relative) congruences in L-algebras, which can be made explicit using the particular forms of the translations. Let $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ denote the lattice of congruences of $\boldsymbol{A}$ relative to $\mathbb{L}$, i.e. those giving a quotient in $\mathbb{L}$. If $\mathbb{L}$ is a variety, then $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ is precisely the lattice of all congruences of $\boldsymbol{A}$. The *Leibniz operator* $\Omega_{\boldsymbol{A}}$ is defined, for any $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, as $\Omega_{\boldsymbol{A}}(F) = \{\langle a, b\rangle \in A^2 \mid a\backslash b \in F \text{ and } b\backslash a \in F\}$. Now we can state a specific variant of a well-known theorem of abstract algebraic logic (see e.g. [20]), narrowed down to our setting.

Proposition 16. *Let* L *be a logic and* $\boldsymbol{A}$ *an* L*-algebra. The Leibniz operator* $\Omega_{\boldsymbol{A}}$ *is a lattice isomorphism from* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ *to* $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$*. Its inverse is the function that maps any* $\theta \in \boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ *to the filter* $\{a \in A \mid \langle a \wedge 1, 1\rangle \in \theta\}$.

Observe that the minimum filter is the one generated by the emptyset, $\mathrm{Fi}(\emptyset)$, and it must correspond to the identity congruence $Id_{\boldsymbol{A}}$. Therefore, using the previous proposition, we obtain that, on any L-algebra $\boldsymbol{A}$, $\mathrm{Fi}(\emptyset) = \{a \in A \mid a \geq 1\}$. This set is, of course, contained in any other filter. It is also worth noting that Proposition 16 and Corollary 15 give a description of the relative principal congruence generated by a pair of elements of a given algebra of an almost (MP)-based logic.

Finally, we focus on a restriction of the completeness theorem (Theorem 8) to a couple of subclasses of algebraic models that will play an important rôle when characterizing semilinearity in the next section: relatively (finitely) subdirectly irreducible algebras. Given a class of algebras $\mathbb{K}$ a algebra $\boldsymbol{A}$ is *(finitely) subdirectly irreducible relative to* $\mathbb{K}$ if for every (finite non-empty) subdirect representation α of $\boldsymbol{A}$ with a family $\{\boldsymbol{A}_i \mid i \in I\} \subseteq \mathbb{K}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. The class of all (finitely) subdirectly irreducible algebras relative to $\mathbb{K}$ is denoted as $\mathbb{K}_{\mathrm{R(F)SI}}$. Of course $\mathbb{K}_{\mathrm{RSI}} \subseteq \mathbb{K}_{\mathrm{RFSI}}$. Observe that the trivial algebra is in $\mathbb{K}_{\mathrm{RFSI}}$ but not in $\mathbb{K}_{\mathrm{RSI}}$. Again, the next theorem is a specific variant of a well-known fact of abstract algebraic logic.

Theorem 17. *Let* L *be a logic. Then for every set of formulae* Γ *and every formula* φ *the following are equivalent:*

1. $\Gamma \vdash_{\mathrm{L}} \varphi$,
2. *for every countable* $\boldsymbol{A} \in \mathbb{L}_{\mathrm{RSI}}$ *and every* $\boldsymbol{A}$*-evaluation* e*, if* $e(\psi) \geq 1$ *for every* $\psi \in \Gamma$*, then* $e(\varphi) \geq 1$.

3 Core semilinear logics

Let us start by recalling two notions mentioned in the introduction: first we give a formal semantical definition of the logic SL^ℓ (later in Theorem 28 we present some of its natural axiomatizations).

Definition 18. *The logic* SL^ℓ *is defined as follows for every set* Γ *of formulae and every formula* φ*:*

1. $\Gamma \vdash_{\mathrm{SL}^\ell} \varphi$ *if and only if*
2. $e(\varphi) \geq 1$ *for each* SL*-chain* $\boldsymbol{A}$ *and each* $\boldsymbol{A}$*-evaluation* e *such that* $e(\psi) \geq 1$ *for all* $\psi \in \Gamma$.

Remark 19. *Clearly* SL^ℓ *extends* SL *and from [19, Propositions 3.1.15. and 3.1.16.] follows that* SL^ℓ *is a logic in the sense of Convention 4 and that the classes of* SL^ℓ*-chains and* SL*-chains coincide.*

The second notion is that of core fuzzy logics formally defined in Definition 1. Let us reformulate this definition using the terminology introduced in the previous section (especially Convention 4 which stipulates that all logics satisfy the condition (Cong)):

Definition 20. *A logic* L *is a* core fuzzy logic *if it expands* MTL *by some set of axioms* Ax*.*

Let us now generalize this class in two aspects: first, we replace MTL by the (much) weaker logic SL^ℓ and, second, we include logics axiomatized by using extra rules provided that they satisfy a certain stability condition involving disjunction. As we shall soon see (in Theorem 22), these conditions are sufficient and necessary for an expansion of SL^ℓ to remain complete w.r.t. chains. Let us first introduce a convenient notation. Given a set Γ of formulae and a formula ψ, we denote by $\Gamma \vee \psi$ the set $\{\chi \vee \psi \mid \chi \in \Gamma\}$.

Definition 21. *A logic* L *is a* core semilinear logic *if it expands* SL^ℓ *by some sets of axioms* Ax *and rules* R *such that for each* $\langle \Gamma, \varphi \rangle \in R$ *and every formula* ψ *we have:*

$$\Gamma \vee \psi \vdash_{\mathrm{L}} \varphi \vee \psi.$$

Observe that if L is an expansion of a core semilinear logic by axioms Ax and rules R, then L is itself a core semilinear logic iff for each $\langle \Gamma, \varphi \rangle \in R$ we have $\Gamma \vee \psi \vdash_{\mathrm{L}} \varphi \vee \psi$. Thus in particular:

- Any *axiomatic* expansion of a core semilinear logic is a core semilinear logic.
- Any axiomatic expansion of SL is a core semilinear logic iff it expands SL^ℓ.

The first item justifies why Hájek considered all axiomatic extensions (*schematic extensions*) of HL in his framework for fuzzy logics, since they were all complete with respect to chains. Moreover, one can check that MTL is an extension of SL^ℓ; therefore MTL and all core fuzzy logics are core semilinear logics. Similarly, it is easy to show that $\triangle$-core fuzzy logics are core semilinear (note that we are adding only one rule $\langle \varphi, \triangle\varphi \rangle$ and we can easily prove that $\varphi \vee \psi \vdash_{\mathrm{L}} \triangle\varphi \vee \psi$ using axioms of $\mathrm{MTL}_\triangle$).

By restricting and re-elaborating results from the general theory presented in [19] (and by using some new results from [15, 17]), in this section we present several characterizations of core semilinear logics, some general methods to obtain their Hilbert-style axiomatizations, and a survey of their completeness results.

3.1 Characterizations and properties of core semilinear logics

The first characterization justifies the usage of the adjective 'semilinear'. This terminology comes from the theory of residuated lattices [72] where it denotes classes of algebras such that in all (relatively) subdirectly irreducible members the lattice order is linear.[15] Such property characterizes core semilinear logics as shown by conditions 3 and 4 of the following theorem. Moreover, as stated in 2, this is also equivalent with what we consider the main property of our logics: completeness with respect to the semantics given by chains.

Theorem 22 (Semilinearity)**.** *Let* L *be a logic. Then the following are equivalent:*

1. L *is a core semilinear logic.*
2. L *is complete w.r.t.* L*-chains, i.e. the following are equivalent for any set of formulae* $\Gamma \cup \{\varphi\}$*:*
   (a) $\Gamma \vdash_{\mathrm{L}} \varphi$
   (b) $e(\varphi) \geq 1$ *for each* L-chain $\boldsymbol{A}$ *and each* $\boldsymbol{A}$*-evaluation* e *such that* $e(\psi) \geq 1$ *for all* $\psi \in \Gamma$.
3. $\mathbb{L}_{\mathrm{RFSI}} = \mathbb{L}_{lin}$.

[15]This follows the tradition of Universal Algebra of calling a class of algebras 'semi$X$' whenever its subdirectly irreducible members have the property X; e.g. as in 'semisimple'.

4. $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{lin}$.

Proof. Logics satisfying the property 2 are called *weakly implicative semilinear logics* in [19]; thus we can use [19, Corollary 3.2.14.] to prove the equivalence of the first two properties (for L_1 being SL^ℓ and L_2 being L; we need to check the validity of three premises of that corollary: (a) SL^ℓ is a weakly implicative semilinear logic: directly from Definition 18 and its following remark, (b) $\vee$ is a protodisjunction: trivially satisfied, and (c) L proves ($\mathrm{MP}_\vee$): established in [19, Proposition 3.2.2.]).

The equivalence of the latter three claim follows from [19, Theorem 3.1.8]. □

Thus, as established in the proof of the theorem above, core semilinear logic are weakly implicative semilinear logics in the sense of [18, 19]. In fact, in the terminology of that paper, they are exactly algebraically implicative semilinear finitary expansions of SL^ℓ.

In order to formulate the syntactic characterization theorem for core semilinear logics (in terms of syntactic properties) we need to make use of special kinds of theories. Recall that a theory is a deductively closed set of formulae. We say that a theory T is

- *maximally consistent w.r.t. a formula* φ if $\varphi \notin T$ and for $\psi \notin T$ we have $T, \psi \vdash \varphi$
- *saturated* if it is maximally consistent w.r.t. some formula φ
- *linear*[16] if for each formulae φ and ψ we have $\varphi \to \psi \in T$ or $\psi \to \varphi \in T$
- *prime* if for each formulae φ and ψ we have $\varphi \in T$ or $\psi \in T$ whenever $\varphi \vee \psi \in T$.

Observe that theories are exactly the filters on the term algebra $\boldsymbol{Fm}_{\mathcal{L}}$. Thus it makes sense to generalize the above classes of theories to filters with '$T \vdash \varphi$' replaced by '$x \in \mathrm{Fi}(X)$'. This allows us to formulate the so-called *transferred* variants of the characterization theorem. Finally we will also need a special formula $(\varphi \to \psi) \vee (\psi \to \varphi)$, called *prelinearity* and usually denoted by ($\mathrm{P}_\vee$), which could be equivalently replaced in the formulation of the syntactic characterization theorem by any of the following two theorems of SL^ℓ (as shown in [19, Lemma 3.2.8]):

($\mathrm{lin}_\wedge$) $\quad (\varphi \wedge \psi \to \chi) \to (\varphi \to \chi) \vee (\psi \to \chi)$
($\mathrm{lin}_\vee$) $\quad (\chi \to \varphi \vee \psi) \to (\chi \to \varphi) \vee (\chi \to \psi)$.

Theorem 23 (Syntactic characterization theorem)**.** *Let* L *be a logic. Then the following are equivalent:*

1. L *is a core semilinear logic.*
2. L *has the* Semilinearity Property, SLP, *i.e. for every set of formulae* $\Gamma \cup \{\varphi, \psi, \chi\}$ *the following rule holds*
$$\frac{\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}.$$
3. L *has the* Linear Extension Property, LEP, *i.e. for every theory* T *and a formula* φ *such that* $\varphi \notin T$, *there is a* linear *theory* $T' \supseteq T$ *such that* $\varphi \notin T'$.
4. *Saturated theories are linear.*
5. L *proves* ($\mathrm{P}_\vee$) *and has the* Proof by Cases Property, PCP, *i.e. for every set of formulae* $\Gamma \cup \{\varphi, \psi, \chi\}$ *holds*
$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi}.$$
6. L *proves* ($\mathrm{P}_\vee$) *and has an axiomatic system* $\langle Ax, R\rangle$ *such that for each* $\langle \Gamma, \varphi\rangle \in R$ *we have:*
$$\Gamma \vee \psi \vdash_{\mathrm{L}} \varphi \vee \psi.$$

[16] Petr Hájek in [37] called this kind of theories 'complete'. However, after recent developments (see e.g. [19]) we prefer the more descriptive terminology used here.

7. L *proves* $(\mathrm{P}_\vee)$ *and has the* Prime Extension Property*,* PEP*, i.e. for every theory* T *and a formula* φ *such that* $\varphi \notin T$*, there is a* prime *theory* $T' \supseteq T$ *such that* $\varphi \notin T'$*.*

8. L *proves* $(\mathrm{P}_\vee)$ *and its saturated theories are prime.*

Proof. The equivalence of the first three claims follows from [19, Theorem 3.1.8.]. To prove 1 implies 4 observe that saturated theories are finitely $\cap$-irreducible (by [19, Proposition 2.3.7.]) and so by the same theorem as before such theories are linear. Conversely, if saturated theories are linear, then the Abstract Lindenbaum Lemma ([19, Lemma 2.3.8.]) clearly implies LEP (i.e. claim 3).

The equivalence of 1 and 5 is established using [19, Theorem 3.2.4.] (after observing that any logic L proves $(\mathrm{MP}_\vee)$ as established in [19, Proposition 3.2.2.]); the equivalence of 5 and 6 follows from [19, Theorem 2.7.15.]. We use [19, Theorem 2.7.23.] to directly prove that 5 is equivalent with 7 and 7 implies 8. Finally, by using a similar reasoning as in the proof of 4 implies 3, we complete the whole proof by showing 8 implies 7. ☐

Remark 24. *Most of these characterizations are inspired by the original ideas behind Hájek's proof of completeness of* HL *and its schematic extensions in [37]: actually in Lemma 2.3.15. he gives a direct proof transferred* PEP *(the third line of the following theorem) and in Lemma 2.4.2 he proves* LEP *by proving* SLP *first (without naming it).*

Observe that item 6 is a minor reformulation of Definition 21 which provides an easy way to check whether a logic is core semilinear without having to prove that it extends SL^ℓ. The next theorem collects the transferred versions of the characterizations.

Theorem 25. *Let* L *be a core semilinear logic and* $\boldsymbol{A}$ *an* L*-algebra. Then:*

1. *For each set* $X \cup \{a, b\} \subseteq A$ *the following holds:*

$$\mathrm{Fi}(X, a \to b) \cap \mathrm{Fi}(X, b \to a) = \mathrm{Fi}(X) \qquad \mathrm{Fi}(X, a) \cap \mathrm{Fi}(X, b) = \mathrm{Fi}(X, a \vee b).$$

2. *Linear and prime filters coincide and contain the set of saturated filters.*

3. *For each filter* $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ *and each* $a \in A$ *such that* $a \notin F$*, there is a* linear/prime *filter* $F' \supseteq F$ *such that* $a \notin F'$*.*

4. *The lattice of* L*-filters is distributive.*

5. *The lattice of relative* $\mathbb{L}$*-congruences is distributive.*

6. *The* $\{\vee, \wedge\}$*-reduct of* $\boldsymbol{A}$ *is a distributive lattice.*

Proof. We will freely use all equivalent characterizations of core semilinear logics established before.

1. [19, Theorems 2.7.18. and 3.1.8.].

2. [19, Theorems 2.7.23. and 3.1.8.] and [19, Proposition 2.3.7.].

3. [19, Theorem 2.7.23.] and the previous claim.

4. [19, Theorem 2.7.20.].

5. The previous claim and Proposition 16.

6. [19, Theorem 3.2.12.].

☐

The soundness of the following crucial definition (i.e., its compliance with Convention 4) is established by [19, Proposition 3.1.16.].

Definition 26. *For a logic* L *we define the logic* L^ℓ *as the least core semilinear logic extending* L.

The following two theorems give useful, semantical and syntactical, descriptions of L^ℓ. The first one is very general and, besides providing a semantical characterization of L^ℓ as the logic of L-chains, it shows how to extend any axiomatization of L into an axiomatization of L^ℓ. Roughly speaking, it adds prelinearity and the $\vee$-form of all rules (cf. the syntactic characterization theorem 23). Note that Petr Hájek also obtained some logics in these ways: e.g. he showed that G was in fact the logic of linearly ordered Heyting algebras or defined psMTL^r in [42] as the logic psMTL-chains.

Theorem 27. *Let* L *be a logic. Then:*

- L^ℓ*-chains coincide with* L*-chains and the class* $\mathbb{L}^\ell$ *of* L^ℓ*-algebras is exactly the quasivariety generated by* $\mathbb{L}_{lin}$.
- *If* L *is axiomatized by axioms* Ax *and rules* R*, then* L^ℓ *is the extension of* L *obtained by adding the axiom* $(\mathrm{P}_\vee)$ *and the set of rules* $\{\langle \Gamma \vee \psi, \varphi \vee \psi\rangle \mid \langle \Gamma, \varphi\rangle \in R\}$.
- *If* L *is obtained as the expansion of some core semilinear logic by adding axioms* Ax *and rules* R*, then* L^ℓ *is the extension of* L *obtained by adding* $\{\langle \Gamma \vee \psi, \varphi \vee \psi\rangle \mid \langle \Gamma, \varphi\rangle \in R\}$.

Proof. The first claim follows from [19, Proposition 3.1.15.]; the second follows from [19, Proposition 3.2.9.] and [19, Theorem 2.7.27.]. The third one is obvious. □

The problem of the axiomatization provided by this theorem is that it adds new rules to the logic. We show that in the case of almost (MP)-based logics L^ℓ is actually an *axiomatic* extension of L by adding *variations* of the prelinearity axiom. We present two variants, B and C, of this axiomatization because they generalize two different usual formulations appearing in the literature; for comparison we also add a presentation A resulting from the direct application of the previous theorem. Note that this theorem can be used to axiomatize the two logics mentioned above and studied by Petr Hájek: G and psMTL^r.

Theorem 28 ([15, Theorem 4.29])**.** *Let* L *be an almost* (MP)*-based logic with a set* bDT *of basic deductive terms. Then* L^ℓ *is axiomatized by adding to* L *any of the following:*

A $\quad (\varphi \to \psi) \vee (\psi \to \varphi)$
$\quad (\varphi \to \psi) \vee \chi, \varphi \vee \chi \vdash \psi \vee \chi$
$\quad \varphi \vee \psi \vdash \gamma(\varphi) \vee \psi$*, for every* $\gamma \in \mathrm{bDT}$

B $\quad ((\varphi \to \psi) \wedge \overline{1}) \vee \gamma((\psi \to \varphi) \wedge \overline{1})$*, for every* $\gamma \in \mathrm{bDT} \cup \{\star\}$

C $\quad (\varphi \vee \psi \to \psi) \vee \gamma(\varphi \vee \psi \to \varphi)$*, for every* $\gamma \in \mathrm{bDT} \cup \{\star \wedge \overline{1}\}$.

Table 8 collects axiomatizations of important semilinear substructural logics obtained as axiomatization B from Theorem 28. We present them in the form of axiom schemata, sometimes altered a little (in an equivalent way) for simplicity or to obtain some form known from the literature (see [15] for details).

As mentioned in Section 2.2, finding nice Hilbert-style presentations for meets in the lattice of extensions of a given logic (in particular, showing that the meet of axiomatic extensions is itself an axiomatic extension of the base logic) is not straightforward. The following theorem gives a presentation for meets of extensions of a given core semilinear logic by capitalizing on the presence of the disjunction connective.

Theorem 29. *Let* L_1 *and* L_2 *be semilinear extensions of a core semilinear logic* L *defined by the sets of axioms* Ax_i *and rules* R_i*. Then* $\mathrm{L}_1 \cap \mathrm{L}_2$ *is the extension of* L *obtained by adding*

- *the set of axioms* $\{\varphi \vee \psi \mid \varphi \in Ax_1 \text{ and } \psi \in Ax_2\}$ *and*
- *the union of the following three sets of rules:*

Logic L	additional axioms needed to axiomatize L^ℓ
SL	$((\varphi \to \psi) \wedge \overline{1}) \vee \gamma((\psi \to \varphi) \wedge \overline{1})$, for every $\gamma \in \{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}\}$
$\mathrm{SL_w}$	$(\varphi \to \psi) \vee \gamma(\psi \to \varphi)$, for every $\gamma \in \{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}\}$
$\mathrm{SL_e}$	$\alpha_{\delta,\varepsilon}((\varphi \to \psi) \wedge \overline{1}) \vee \beta_{\delta',\varepsilon'}((\psi \to \varphi) \wedge \overline{1})$
$\mathrm{SL_{ew}}$	$\alpha_{\delta,\varepsilon}(\varphi \to \psi) \vee \beta_{\delta',\varepsilon'}(\psi \to \varphi)$
$\mathrm{SL_a}$	$(\lambda_\varepsilon(\varphi \to \psi) \wedge \overline{1}) \vee (\rho_{\varepsilon'}(\psi \to \varphi) \wedge \overline{1})$
$\mathrm{SL_{ae}}$	$((\varphi \to \psi) \wedge \overline{1}) \vee ((\psi \to \varphi) \wedge \overline{1})$
$\mathrm{SL_{aew}}$	$(\varphi \to \psi) \vee (\psi \to \varphi)$

Table 8: Axiomatization of L^ℓ for prominent substructural logics.

– $\langle \Gamma \vee \chi, \varphi \vee \psi \vee \chi \rangle \mid \langle \Gamma, \varphi \rangle \in R_1,\ \psi \in Ax_2,$ *and* χ *a formula*$\}$
– $\langle \Gamma \vee \chi, \varphi \vee \psi \vee \chi \rangle \mid \langle \Gamma, \varphi \rangle \in R_2,\ \psi \in Ax_1,$ *and* χ *a formula*$\}$
– $\langle (\Gamma_1 \cup \Gamma_2) \vee \chi, \varphi_1 \vee \varphi_2 \vee \chi \rangle \mid \langle \Gamma_1, \varphi_1 \rangle \in R_1,\ \langle \Gamma_2, \varphi_2 \rangle \in R_2,$ *and* χ *a formula*$\}$

Proof. It follows from [17, Theorem 5.10.]. □

Corollary 30. *The intersection of a finite family of core semilinear logics in the same language is a core semilinear logic.*

Corollary 31. *Let* L *be a logic. Then the class of core semilinear extensions of* L *is a sublattice of the lattice of extensions of* L^ℓ*. Furthermore, the class of core semilinear* axiomatic *extensions of* L *is a principal filter in the lattice of axiomatic extensions of* L^ℓ*.*

3.2 Completeness results

We devote this subsection to completeness theorems for core semilinear logics. As discussed in the introduction, a crucial guideline for Petr Hájek and others when studying new fuzzy logics was to find logical systems complete with respect to a semantics of algebras defined on the real unit interval $[0,1]$. This kind of completeness results have been known as *standard completeness theorems*, although this terminology is not univocally defined. Indeed, by *standard* semantics one means the semantics that, due to some design choices, is considered to be the *intended* one for the logic. In some cases it consists of all algebras defined over $[0,1]$ (e.g. for HL, SHL, MTL, SMTL, or IMTL); in other cases it consists of algebras with a fixed interpretation using particular operations (e.g. for Ł, G or Π where one interprets & as the corresponding t-norm [37], or for logics with an additional involutive negation $\sim$ where one interprets it as $1-x$ [25]). In all the examples taken from ($\triangle$-)core fuzzy logics, the standard semantics is based on left-continuous t-norms and their residua. Later on, the introduction of weaker systems brought forth an analogous relaxation for the corresponding algebraic structures on $[0,1]$, such as residuated uninorms (for UL) or residuated non-commutative t-norms (for psMTL^r). Recently, when considering a standard semantics for SL^ℓ in [15], even associativity has been dropped giving rise to residuated unital groupoids on $[0,1]$.

Some other works have however focused on other kinds of semantics for fuzzy logics, besides the real-valued one. It is the case of rational-chain semantics, hyperreal-chain semantics or finite-chain semantics (see e.g. [13, 29, 30, 67]) where instead of $[0,1]$ one respectively takes the rational unit interval, any hyperral interval or any finite linearly ordered set as the domain for the intended models. A systematical study of the corresponding completeness properties is better presented in the following general formulation.

Definition 32. *Let* L *be a core semilinear logic and* $\mathbb{K} \subseteq \mathbb{L}_{lin}$*. We say that* L *has the* Strong $\mathbb{K}$-Completeness*,* S$\mathbb{K}$C *for short, when for every set of formulae* $\Gamma \cup \{\varphi\}$*: the following are equivalent:*

1. $\Gamma \vdash_{\mathrm{L}} \varphi$*,*

2. *for every* $\boldsymbol{A} \in \mathbb{K}$ *and every* $\boldsymbol{A}$*-evaluation* e*, if* $e(\psi) \geq 1$ *for every* $\psi \in \Gamma$*, then* $e(\varphi) \geq 1$.

If the equivalence above holds for finite Γ *(or only for* $\Gamma = \emptyset$*) we speak about* Finite Strong $\mathbb{K}$-Completeness *(or* $\mathbb{K}$-Completeness*, respectively). The Finite Strong* $\mathbb{K}$*-Completeness is denoted* FS$\mathbb{K}$C *whereas the* $\mathbb{K}$*-Completeness is denoted* $\mathbb{K}$C.

It is easy to show that the failure of completeness properties is inherited by conservative expansions (recall that a logic L_2 in a language $\mathcal{L}_2$ is a *conservative* expansion of a logic L_1 in a language $\mathcal{L}_1$ if $\mathcal{L}_1 \subseteq \mathcal{L}_2$ and for each set of $\mathcal{L}_1$-formulae $\Gamma \cup \{\varphi\}$ we have that $\Gamma \vdash_{\mathrm{L}_2} \varphi$ iff $\Gamma \vdash_{\mathrm{L}_1} \varphi$).

Proposition 33 ([19, Proposition 3.4.14.])**.** *Let* L' *be a conservative expansion of* L*,* $\mathbb{K}'$ *a class of* L'*-chains and* $\mathbb{K}$ *the class of their* L*-reducts. If* L' *enjoys the* $\mathbb{K}'$C*, then* L *enjoys the* $\mathbb{K}$C*. The analogous claim holds for* FS$\mathbb{K}'$C *and* S$\mathbb{K}'$C.

We recall now several algebraic characterizations of completeness properties from [13, 19]. In what follows we will use the following operators on classes of algebras of the same type:

- $\mathbf{S}(\mathbb{K})$ is the class of subalgebras of members in $\mathbb{K}$,
- $\mathbf{I}(\mathbb{K})$ is the class of algebras isomorphic to a member in $\mathbb{K}$,
- $\mathbf{H}(\mathbb{K})$ is the class of homomorphic images of members in $\mathbb{K}$,
- $\mathbf{P}(\mathbb{K})$ is the class of direct products of members in $\mathbb{K}$,
- $\mathbf{P}_{fin}(\mathbb{K})$ is the class of finite direct products of members in $\mathbb{K}$,
- $\mathbf{P}_{\mathrm{U}}(\mathbb{K})$ is the class of ultraproducts of members in $\mathbb{K}$,
- $\mathbf{P}_{\sigma\text{-}f}(\mathbb{K})$ is the class of reduced products of members in $\mathbb{K}$ over countably complete filters (i.e. filters closed under countable intersections),
- $\mathbf{V}(\mathbb{K})$ is the variety generated by $\mathbb{K}$, i.e., $\mathbf{V}(\mathbb{K}) = \mathbf{HSP}(\mathbb{K})$,
- $\mathbf{Q}(\mathbb{K})$ is the quasivariety generated by $\mathbb{K}$, i.e., $\mathbf{Q}(\mathbb{K}) = \mathbf{ISPP}_{\mathrm{U}}(\mathbb{K})$.

Let us fix a core semilinear logic L and a class of L-chains $\mathbb{K}$. We present several characterizations of the general completeness properties. The first one relates them respectively with generation of the class of algebras as a variety, a quasivariety and a generalized quasivariety, respectively.

Theorem 34 ([19, Theorem 3.4.3.])**.**

1. L *has the* $\mathbb{K}$C *if, and only if,* $\mathbf{H}(\mathbb{L}) = \mathbf{V}(\mathbb{K})$.
2. L *has the* FS$\mathbb{K}$C *if, and only if,* $\mathbb{L} = \mathbf{Q}(\mathbb{K})$.
3. L *has the* S$\mathbb{K}$C *if, and only if,* $\mathbb{L} = \mathbf{ISP}_{\sigma\text{-}f}(\mathbb{K})$.

The completeness properties of L can be also characterized in terms of (finitely) subdirectly irreducible algebras relative to $\mathbb{L}$. Recall that, by Theorem 22, finitely subdirectly irreducible L-algebras relative to $\mathbb{L}$ coincide with the class of L-chains, i.e., we have $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{\mathrm{RFSI}} = \mathbb{L}_{lin}$. Given a class of algebras $\mathbb{M}$, the class of its nontrivial members is denoted $\mathbb{M}^{+}$. Similarly, $\mathbb{M}^{\sigma}$ stands for countable members of $\mathbb{M}$.

Theorem 35. *We have the following chains of equivalences:*

1. L *has the* $\mathbb{K}$C *iff* $\mathbb{L}_{lin} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$ *iff* $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$.
2. L *has the* FS$\mathbb{K}$C *iff* $\mathbb{L}^{+}_{lin} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$ *iff* $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$.
3. L *has the* S$\mathbb{K}$C *iff* $\mathbb{L}^{\sigma+}_{lin} \subseteq \mathbf{IS}(\mathbb{K})$ *iff* $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{IS}(\mathbb{K})$.

Proof. The first claim is a consequence of Dziobiak's result [22] showing that for a congruence distributive quasivariety $\mathbb{Q}$ and a class $\mathbb{M} \subseteq \mathbb{Q}$ we have $\mathbf{V}(\mathbb{M}) \cap \mathbb{Q}_{\mathrm{RFSI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{M})$. Indeed, let $\mathbb{Q} = \mathbb{L}$ and $\mathbb{M} = \mathbb{K}$. Then we obtain

$$\mathbf{V}(\mathbb{K}) \cap \mathbb{L}_{lin} = \mathbf{V}(\mathbb{K}) \cap \mathbb{L}_{\mathrm{RFSI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})\,.$$

Now assume that L has the $\mathbb{K}$C. Then by Theorem 34 we have $\mathbf{H}(\mathbb{L}) = \mathbf{V}(\mathbb{K})$. Consequently, $\mathbb{L}_{lin} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$ because $\mathbb{L}_{lin} \subseteq \mathbf{H}(\mathbb{L})$. Further, it is obvious that $\mathbb{L}_{lin} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$ implies $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$ since $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{lin}$. Finally, suppose that $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$. By Theorem 34 it is sufficient to show that $\mathbf{H}(\mathbb{L}) = \mathbf{V}(\mathbb{K})$. Since $\mathbf{V}(\mathbb{L}) = \mathbf{H}(\mathbb{L})$ and $\mathbb{K} \subseteq \mathbb{L}$, we always have $\mathbf{V}(\mathbb{K}) \subseteq \mathbf{H}(\mathbb{L})$. Conversely, L is strongly complete w.r.t. $\mathbb{L}^{\sigma}_{\mathrm{RSI}}$ by Theorem 17. Thus by Theorem 34 we have $\mathbf{H}(\mathbb{L}) = \mathbf{V}(\mathbb{L}^{\sigma}_{\mathrm{RSI}})$. Consequently, by our assumption we obtain

$$\mathbf{H}(\mathbb{L}) = \mathbf{V}(\mathbb{L}^{\sigma}_{\mathrm{RSI}}) \subseteq \mathbf{V}(\mathbf{HSP}_{\mathrm{U}}(\mathbb{K})) = \mathbf{V}(\mathbb{K})\,.$$

The first equivalence of the second claim is proved in [19, Theorem 3.4.11.]. To prove the remaining one, one can argue similarly as above. Since $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbb{L}^{+}_{lin}$, $\mathbb{L}^{+}_{lin} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$ implies $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$. Conversely, assume that $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$. Again using Theorems 17 and 34, we obtain

$$\mathbb{L} = \mathbf{Q}(\mathbb{L}^{\sigma}_{\mathrm{RSI}}) \subseteq \mathbf{Q}(\mathbf{ISP}_{\mathrm{U}}(\mathbb{K})) = \mathbf{Q}(\mathbb{K}) \subseteq \mathbb{L}\,.$$

Thus L enjoys FS$\mathbb{K}$C by Theorem 34.

The last claim is proved in [19, Theorem 3.4.6.]. □

Corollary 36. *If* L *enjoys* FS$\mathbb{K}$C*, then it enjoys the* $\mathrm{S}\mathbf{P}_{\mathrm{U}}(\mathbb{K})$C *as well.*

Alternatively, for logics with finitely many propositional connectives, an algebraic property equivalent to finite strong $\mathbb{K}$-completeness is expressed in terms of partial embeddings. This was, in fact, the property used by Hájek and others to prove standard completeness of HL.

Definition 37. *Given two algebras* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *of the same language* $\mathcal{L}$*, we say that a finite subset* X *of* $\boldsymbol{A}$ *is partially embeddable into* $\boldsymbol{B}$ *if there is a one-to-one mapping* $f\colon X \to B$ *such that for each* $\langle c, n\rangle \in \mathcal{L}$ *and each* $a_1, \ldots, a_n \in X$ *satisfying* $c^{\boldsymbol{A}}(a_1, \ldots, a_n) \in X$*,* $f(c^{\boldsymbol{A}}(a_1, \ldots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \ldots, f(a_n))$*.*

A class $\mathbb{K}$ *of algebras is* partially embeddable into *a class* $\mathbb{K}'$ *if every finite subset of every member of* $\mathbb{K}$ *is partially embeddable into a member of* $\mathbb{K}'$*.*

Theorem 38. *Assume that the language of* L *is finite. Then the following are equivalent:*

1. L *has the* FS$\mathbb{K}$C*.*
2. *The class* $\mathbb{L}^{\sigma}_{\mathrm{RSI}}$ *is partially embeddable into* $\mathbb{K}$*.*
3. *The class* $\mathbb{L}^{+}_{lin}$ *is partially embeddable into* $\mathbb{K}$*.*
4. $\mathbb{L}$ *is partially embeddable into* $\mathbf{P}_{fin}(\mathbb{K})$*.*

Proof. The equivalence of the first three claims is proved in [19, Theorem 3.4.8.].

(3)⇒(4): Let $\boldsymbol{A} \in \mathbb{L}$ and $X \subseteq A$ a finite subset. By a well-known fact from universal algebra, every algebra $\boldsymbol{C}$ in a quasivariety $\mathbb{Q}$ is a subdirect product of subdirectly irreducible algebras relative to $\mathbb{Q}$. Since $\mathbb{L}$ is a quasivariety, it follows that $\boldsymbol{A}$ can be viewed as a subdirect product of a family $\{\boldsymbol{A}_i \in \mathbb{L}_{\mathrm{RSI}} \mid i \in I\}$. Since X is finite, it suffices to consider only finitely many $\boldsymbol{A}_i$'s in order to separate elements of X. Thus X is partially embeddable into a finite direct product of some subdirectly irreducible algebras relative to $\mathbb{L}$. Since $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{lin}$ by Theorem 22, X is partially embeddable into $\mathbf{P}_{fin}(\mathbb{K})$ by (3).

(4)⇒(1): Assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$ for a finite set Γ of formulae. By Theorem 17 there is a counter-model $\boldsymbol{A} \in \mathbb{L}_{\mathrm{RSI}}$. By (4) we have also a counter-model $\boldsymbol{B} \in \mathbf{P}_{fin}(\mathbb{K})$. Since $\boldsymbol{B}$ is a direct product of members from $\mathbb{K}$, one of them actually has to be a counter-model as well. □

Remark 39. Notice that the implications from 2 or 3 to 1 hold also for infinite languages, whereas the converse ones do not (as shown in [13, Example 3.10]).

Let us now deal with particular notable semantics. We consider first the class of all finite L-chains, denoted by $\mathcal{F}$. The following characterization of S$\mathcal{F}$C is a direct consequence of Theorem 35 together with the downward Löwenheim–Skolem Theorem.

Theorem 40 ([19, Theorems 3.4.16.])**.** *The following are equivalent:*

1. L *enjoys the* S$\mathcal{F}$C*.*
2. *All* L*-chains are finite.*
3. *There exists* $n \in \mathsf{N}$ *such each* L*-chain has at most* n *elements.*
4. *There exists* $n \in \mathsf{N}$ *such that* $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$*.*

Corollary 41. *For any core semilinear logic* L *and a natural number* n*, the axiomatic extension* $\mathrm{L}_{\leq n}$ *obtained by adding the schema* $\bigvee_{i<n}(x_i \to x_{i+1})$ *is a semilinear logic which is strongly complete with respect the* L*-chains of length less than or equal to* n*.*

Next we show that the properties of FS$\mathcal{F}$C and $\mathcal{F}$C have purely algebraic characterizations in terms of basic notions studied in universal algebra. We say that a class of algebras $\mathbb{M}$ has:

- the *finite embeddability property* (FEP) if $\mathbb{M}$ is partially embeddable into the class of its finite members,
- the *(strong) finite model property* ((S)FMP) if every (quasi-)identity that fails to hold in $\mathbb{M}$ can be refuted in a finite member of $\mathbb{M}$.

The next theorem follows from Theorem 38 but can be also seen as an instance of a purely universal-algebraic result from [5] (after replacing the first claim by an equivalent algebraic formulation using Theorem 34).

Theorem 42. *The following are equivalent:*

1. L *enjoys the* FS$\mathcal{F}$C*.*
2. $\mathbb{L}$ *enjoys the* SFMP*.*
3. $\mathbb{L}$ *enjoys the* FEP*.*

Finally, the algebraic characterization of $\mathcal{F}$C is not much of use because it involves free algebras whose structure is usually quite complex, but we include it for the sake of completeness.

Theorem 43. *The following are equivalent:*

1. L *enjoys the* $\mathcal{F}$C*.*
2. $\mathbb{L}$ *enjoys the* FMP*.*
3. *The class of finitely generated* $\mathbb{L}$*-free algebras is partially embeddable into the class of finite members of* $\mathbb{L}$*.*

Proof. (1)⇒(2): The first claim clearly implies the second one due to algebraizability of L.

(2)⇒(3): Assume that $\mathbb{L}$ has the FMP. Let $\boldsymbol{F}$ be a finitely generated $\mathbb{L}$-free algebra and $X \subseteq F$ a finite subset. We will construct a partial embedding

$$f\colon X \to \prod_{\substack{x,y \in X \\ x \neq y}} A_{x,y}\,,$$

where $\boldsymbol{A}_{x,y}$ are going to be finite members of $\mathbb{L}$. Let $x, y \in X$ such that $x \neq y$. Since $\boldsymbol{F}$ is free, x, y can be viewed as equivalence classes of terms. Consider any term t_x belonging to x and similarly any term t_y from y. Then the identity $t_x \approx t_y$ does not hold in $\mathbb{L}$ because $x \neq y$. By FMP there is a finite algebra $\boldsymbol{A}_{x,y} \in \mathbb{L}$ where $t_x \approx t_y$ can be refuted. Since $\boldsymbol{F}$ is free, we have a surjective homomorphism $f_{x,y}\colon F \to A_{x,y}$ such that $f_{x,y}(x) \neq f_{x,y}(y)$. The collection of homomorphisms $f_{x,y}$ induces a homomorphism

$$g\colon F \to \prod_{\substack{x,y\in X\\ x\neq y}} A_{x,y}$$

defined by $g(z) = \langle f_{x,y}(z)\rangle_{x,y\in X, x\neq y}$ whose restriction to X gives the desired partial embedding f.

(3)⇒(1): Let φ be a formula which is not a theorem of L, i.e., $\nvdash_{\mathrm{L}} \varphi$. By algebraizability the identity $\overline{1} \approx \overline{1} \wedge \varphi$ does not hold in $\mathbb{L}$. Consequently, $\overline{1} \approx \overline{1} \wedge \varphi$ does not hold in a finitely generated $\mathbb{L}$-free algebra $\boldsymbol{F}$. Since $\boldsymbol{F}$ is partially embeddable into a finite member $\boldsymbol{A} \in \mathbb{L}$, $\overline{1} \approx \overline{1} \wedge \varphi$ does not hold in $\boldsymbol{A}$. By a well-known fact from universal algebra, every algebra $\boldsymbol{C}$ in a quasivariety $\mathbb{Q}$ is a subdirect product of subdirectly irreducible algebras relative to $\mathbb{Q}$ which are homomorphic images of $\boldsymbol{C}$. Since $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{lin}$, $\boldsymbol{A}$ is a subdirect product of chains $\boldsymbol{B}_i$, $i \in I$, which are homomorphic images of $\boldsymbol{A}$. Thus $\boldsymbol{B}_i$'s have to be finite as well. Consequently, $\overline{1} \approx \overline{1} \wedge \varphi$ does not hold in $\prod_{i\in I} \boldsymbol{B}_i$ and therefore $\overline{1} \approx \overline{1} \wedge \varphi$ can be refuted in one of the components $\boldsymbol{B}_i$'s. □

The completeness properties w.r.t. the class $\mathcal{F}$ of finite L-chains are usually used in order to show decidability of theorems and finite consequence of L. More precisely, if L is finitely axiomatizable then $\mathcal{F}$C implies decidability of the set $\{\varphi \mid\ \vdash_{\mathrm{L}} \varphi\}$ and FS$\mathcal{F}$C implies decidability of $\{\langle \Gamma, \varphi\rangle \mid \Gamma \vdash_{\mathrm{L}} \varphi,\ \Gamma \text{ finite}\}$. Table 9 lists several known results on completeness properties w.r.t. $\mathcal{F}$ (see [56, 78] and references thereof).

Logic	S$\mathcal{F}$C	FS$\mathcal{F}$C	$\mathcal{F}$C
$\mathrm{SL}^{\ell}_{\mathrm{S}}$, for each $\mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i},\mathrm{o}\}$	No	Yes	Yes
$\mathrm{SL}^{\ell}_{\mathrm{aw}}$	No	Yes	Yes
MTL, IMTL, SMTL	No	Yes	Yes
UL, WCMTL, ΠMTL, Π	No	No	No
HL, SHL, Ł	No	Yes	Yes
G, WNM, NM, C_nMTL, C_nIMTL	No	Yes	Yes
CPC	Yes	Yes	Yes

Table 9: Finite strong completeness w.r.t. $\mathcal{F}$ for some core semilinear logics.

We now consider the semantics given by chains defined over the rational and the real unit interval. We present both notions together because their completeness properties are much related.[17]

Definition 44. *The class $\mathcal{R} \subseteq \mathbb{L}_{lin}$ is defined as: $\boldsymbol{A} \in \mathcal{R}$ if the domain of $\boldsymbol{A}$ is the real unit interval $[0,1]$ and $\leq_{\boldsymbol{A}}$ is the usual order on reals. The class $\mathcal{Q} \subseteq \mathbb{L}_{lin}$ is analogously defined requiring the rational unit interval as domain.*

Theorem 45 ([19, Theorem 3.4.19.])**.**

1. *L has the FS$\mathcal{Q}$C iff it has the S$\mathcal{Q}$C.*
2. *If L has the $\mathcal{R}$C, then it has the $\mathcal{Q}$C.*
3. *If L has the FS$\mathcal{R}$C, then it has the S$\mathcal{Q}$C.*

[17] Another closely related semantics is that of hyperreal or non-standard reals proposed as a semantics for fuzzy logics in [30]. See [13] for results linking hyperreal completeness with real and rational completeness.

Observe that the completeness properties with respect to $\mathcal{Q}$ are, in fact, equivalent to completeness properties with respect to the whole class of densely ordered chains. Indeed, when we have an evaluation over a densely ordered linear model providing a counterexample to some derivation, we can apply the downward Löwenheim–Skolem Theorem to the (countable) subalgebra generated by the image of all formulae by the evaluation and obtain a rational countermodel.

Strong rational completeness also admits a proof-theoretic characterization in terms of the Density Property:

Theorem 46. *The following are equivalent:*

1. L *has the* SQC.
2. L *has the* Density Property DP, *i.e. if for any set of formulae* $\Gamma \cup \{\varphi, \psi, \chi\}$ *and any variable* p *not occurring in* $\Gamma \cup \{\varphi, \psi, \chi\}$ *the following meta-rule holds:*

$$\frac{\Gamma \vdash_{\mathrm{L}} (\varphi \to p) \vee (p \to \psi) \vee \chi}{\Gamma \vdash_{\mathrm{L}} (\varphi \to \psi) \vee \chi}.$$

3. L *is the intersection of all its extensions satisfying the* DP.

Proof. The equivalence of 1 and 2 follows from [19, Theorem 3.3.8.]. The equivalence with 3 follows from [19, Theorem 3.3.13.].

□

The last claim gives some insight into an approach used in the fuzzy logic literature to prove completeness w.r.t. the semantics of densely ordered chains (e.g. in [64] for the logic UL). Indeed, in this approach one starts from a suitable proof-theoretic description of a logic L, which then is extended into a proof-system for the intersection of all extensions of L satisfying the DP just by adding DP as a rule (in the proof-theoretic sense, not as we understand rules here). This rule is then shown to be eliminable (using analogs of the well-known cut-elimination techniques), i.e., the condition 3 is met and hence the original logic is complete w.r.t. its densely ordered chains (of course, our general theory is not helpful in this last step, because here one needs to use specific properties of the logic in question).

Many works in the literature of MFL focus on the study of these completeness properties. Besides the historical papers devoted to particular logical systems, there are more systematic approaches dealing with the study of these properties such as [13, 56]. Table 10 collects the results for some prominent core semilinear logics. Unlike FL^{ℓ}, the weakest core semilinear logic SL^{ℓ} does enjoy all these completeness properties, as proved in [15]. In particular, if one considers residuated groupoids defined over $[0, 1]$ as its intended semantics, then SL^{ℓ} enjoys standard completeness in the strong version, and hence, can be regarded as a genuine fuzzy logic as much as HL, MTL or UL. On the other hand, it can arguably be seen as a basic logic in the meanings described in the introduction. Indeed, the class of core semilinear logics is based in this logic and provides a useful framework covering virtually all the work done nowadays in MFL; moreover in the context of substructural logics complete w.r.t. chains could not be made weaker. We have therefore defended the rôle of SL^{ℓ} as basic fuzzy logic in the framework of propositional logics. In the last part of the chapter we argue that this is also the case at the first-order level.

4 First-order core semilinear logics

In this section we present the theory of first-order core semilinear logics. The presentation, definitions, and results of the first two subsections closely follow the fifth section of [19] simplified to our setting of core semilinear logics. The third subsection generalizes results of [13] (proved there for core *fuzzy* logics) and shortly surveys the undecidability results treated in detail in [53].

Logic	$\mathcal{R}$C	FS$\mathcal{R}$C	S$\mathcal{R}$C	$\mathcal{Q}$C	FS$\mathcal{Q}$C = S$\mathcal{Q}$C
$\mathrm{SL}^{\ell}_{\mathrm{S}}$, for each $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{SL}^{\ell}_{\mathrm{a}}$, $\mathrm{SL}^{\ell}_{\mathrm{ac}}$	No	No	No	No	No
$\mathrm{UL} = \mathrm{SL}^{\ell}_{\mathrm{ae}}$, $\mathrm{SL}^{\ell}_{\mathrm{aw}}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{MTL} = \mathrm{SL}^{\ell}_{\mathrm{aew}}$, IMTL, SMTL	Yes	Yes	Yes	Yes	Yes
WCMTL, ΠMTL	Yes	Yes	No	Yes	Yes
HL, SHL, Ł, Π	Yes	Yes	No	Yes	Yes
G, WNM, NM, C_nMTL, C_nIMTL	Yes	Yes	Yes	Yes	Yes
CPC	No	No	No	No	No

Table 10: Real and rational completeness for some core semilinear logics.

4.1 Syntax

In the following let L be a fixed core semilinear logic in a propositional language $\mathcal{L}$. The language of a first-order extension of L is defined in the same way as in classical first-order logic. In order to fix the notation and terminology we give an explicit definition:

Definition 47. *A* predicate language $\mathcal{P}$ *is a triple* $\langle Pred_{\mathcal{P}}, Func_{\mathcal{P}}, Ar_{\mathcal{P}}\rangle$, *where* $Pred_{\mathcal{P}}$ *is a non-empty set of* predicate symbols, $Func_{\mathcal{P}}$ *is a set (disjoint with* $Pred_{\mathcal{P}}$*) of* function symbols, *and* $Ar_{\mathcal{P}}$ *is the* arity function, *assigning to each predicate or function symbol a natural number called the* arity *of the symbol. The function symbols* F *with* $Ar_{\mathcal{P}}(F) = 0$ *are called* object *or* individual constants. *The predicates symbols* P *for which* $Ar_{\mathcal{P}}(P) = 0$ *are called* truth constants.[18]

$\mathcal{P}$-terms and (atomic) $\mathcal{P}$-formulae of a given predicate language are defined as in classical logic (note that the notion of formula also depends on propositional connectives in $\mathcal{L}$). A $\mathcal{P}$*-theory* is a set of $\mathcal{P}$-formulae. The notions of free occurrence of a variable, substitutability, open formula, and closed formula (or, synonymously, *sentence*) are defined in the same way as in classical logic. Unlike in classical logic, in fuzzy logics without involutive negation the quantifiers $\forall$ and $\exists$ are not mutually definable, so the primitive language of L$\forall$ has to contain both of them.

There are several variants of the first-order extension of a propositional fuzzy logic L that can be defined. Following Hájek's original approach in [37] and his developments in [48, 49, 50], here we restrict ourselves to logics of models over linearly ordered algebras (see Subsection 4.2) and introduce the first-order logics L$\forall$ and L$\forall^{\mathrm{w}}$ (respectively, complete w.r.t. all models or w.r.t. witnessed models). The axiomatic systems of the logics L$\forall$ and L$\forall^{\mathrm{w}}$ are defined as follows:

Definition 48. *Let* $\mathcal{P}$ *be a predicate language. The logic* L$\forall$ *in language* $\mathcal{P}$ *has the following axioms:*[19]

- (P) The axioms of L
- ($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where the $\mathcal{P}$-term t is substitutable for x in φ
- ($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where the $\mathcal{P}$-term t is substitutable for x in φ
- ($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, where x is not free in χ
- ($\exists$2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, where x is not free in χ
- ($\forall$3) $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, where x is not free in χ.

The deduction rules of L$\forall$ *are those of* L *plus the rule of* generalization:

(Gen) $\langle \varphi, (\forall x)\varphi\rangle$.

[18]The rôles of nullary predicates of $\mathcal{P}$ and nullary connectives of $\mathcal{L}$ are analogous, even though the values of the former are only fixed under a given interpretation of the predicate language, while the values of the latter are fixed under all such interpretations. The ambiguity of the term *truth constant* is thus a harmless abuse of language.

[19]When we speak about axioms or deduction rules of a propositional logic, we actually consider them with $\mathcal{P}$-formulae substituted for propositional variables.

The logic $\mathrm{L}\forall^{\mathrm{w}}$ *is the extension of* $\mathrm{L}\forall$ *by the axioms:*

(C∀) $(\exists x)(\varphi(x) \to (\forall y)\varphi(y))$

(C∃) $(\exists x)((\exists y)\varphi(y) \to \varphi(x))$.

The notions of proof and provability are defined for first-order core semilinear logics in the same way as in first-order classical logic. The fact that the formula φ *is provable in* $\mathrm{L}\forall$ *from a theory* T *will be denoted by* $T \vdash_{\mathrm{L}\forall} \varphi$*, and analogously for* $\mathrm{L}\forall^{\mathrm{w}}$*; in a fixed context we can write just* $T \vdash \varphi$*.*

Helena Rasiowa gave a first general theory of first-order non-classical logics in [74] based on her notion of propositional implicative logic. The presentation of her first-order logics, which we denote $\mathrm{L}\forall^{\mathrm{m}}$, omitted the axiom (∀3).[20] The superscript 'm' stands for 'minimal', because $\mathrm{L}\forall^{\mathrm{m}}$ is, in a sense, the weakest first-order extension of L. Indeed, $\mathrm{L}\forall^{\mathrm{m}}$ is sound and complete w.r.t. first-order models built over *arbitrary* L-algebras. However, the axioms of $\mathrm{L}\forall^{\mathrm{m}}$ are not strong enough to ensure the completeness w.r.t. first-order models (see the next subsection for technical details) over *linearly ordered* L-algebras —i.e., the usual chain completeness theorem, which we have presented as an essential common trait of all core semilinear logics. That is the reason why Hájek needed to add the axiom (∀3) in his presentation of first-order fuzzy logics. This axiom is valid in all models over L-chains (though not generally in models over arbitrary L-algebras, see e.g. [19, Example 4.1.18]) and ensures the chain completeness theorem for the resulting logic $\mathrm{L}\forall$.[21] This makes $\mathrm{L}\forall$ a natural choice for *the* first-order extension of a given core *semilinear* logic L. Consequently, we denote this first-order logic as $\mathrm{L}\forall$ with no superscript (though in some works the more systematic denotation $\mathrm{L}\forall^{\ell}$ is used). Finally let us note that in the context of MFL, the logics $\mathrm{L}\forall^{\mathrm{m}}$ were rediscovered by Petr Hájek in [38], where he denoted them by $\mathrm{L}\forall^{-}$.

Let us list some important theorems that are provable in all logics $\mathrm{L}\forall$. For their proofs in MTL or HL see [23, 37]; their proofs in a weaker setting can be found in [19].

Theorem 49 ([19, Propositions 4.2.5. and 4.3.2])**.** *Let* $\mathcal{P}$ *be a predicate language,* φ, ψ, χ $\mathcal{P}$*-formulae,* x *a variable not free in* χ*, and* x' *a variable not occurring in* φ*. The following* $\mathcal{P}$*-formulae are theorems of* $\mathrm{L}\forall$:

(T∀1) $\chi \leftrightarrow (\forall x)\chi$

(T∀2) $(\exists x)\chi \leftrightarrow \chi$

(T∀3) $(\forall x)\varphi(x) \leftrightarrow (\forall x')\varphi(x')$

(T∀4) $(\exists x)\varphi(x) \leftrightarrow (\exists x')\varphi(x')$

(T∀5) $(\forall x)(\forall y)\varphi \leftrightarrow (\forall y)(\forall x)\varphi$

(T∀6) $(\exists x)(\exists y)\varphi \leftrightarrow (\exists y)(\exists x)\varphi$

(T∀7) $(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$

(T∀8) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$

(T∀9) $(\chi \to (\forall x)\varphi) \leftrightarrow (\forall x)(\chi \to \varphi)$

(T∀10) $((\exists x)\varphi \to \chi) \leftrightarrow (\forall x)(\varphi \to \chi)$

(T∀11) $(\exists x)(\chi \to \varphi) \to (\chi \to (\exists x)\varphi)$

(T∀12) $(\exists x)(\varphi \to \chi) \to ((\forall x)\varphi \to \chi)$

(T∀13) $(\forall x)(\varphi \wedge \psi) \leftrightarrow (\forall x)\varphi \wedge (\forall x)\psi$

(T∀14) $(\exists x)(\varphi \vee \psi) \leftrightarrow (\exists x)\varphi \vee (\exists x)\psi$

(T∀15) $(\forall x)(\varphi \vee \chi) \leftrightarrow (\forall x)\varphi \vee \chi$

(T∀16) $(\exists x)(\varphi \wedge \chi) \leftrightarrow (\exists x)\varphi \wedge \chi$

(T∀17) $(\exists x)(\varphi \,\&\, \chi) \leftrightarrow (\exists x)\varphi \,\&\, \chi$

(T∀18) $(\exists x)(\varphi^n) \leftrightarrow ((\exists x)\varphi)^n$

(T∀19) $(\exists x)\varphi \to \neg(\forall x)\neg\varphi$

(T∀20) $\neg(\exists x)\varphi \leftrightarrow (\forall x)\neg\varphi$.

Remark 50. The converse implication of (T∀19) is provable in Ł∀, IMTL∀, or NM∀, i.e., in logics where $\neg$ is involutive (i.e. proves $\neg\neg\varphi \to \varphi$). Thus in such logics the existential quantifier is definable and the axioms (∃1) and (∃2) become redundant. Actually, for this claim to hold, the presence of an arbitrary unary connective $\sim$ such that $\varphi \to \psi \vdash_{\mathrm{L}} {\sim}\psi \to {\sim}\varphi$ and $\vdash_{\mathrm{L}} \varphi \leftrightarrow {\sim}{\sim}\varphi$. is sufficient (which could be either the 'natural' logical negation given by implication, or a new primitive connective added in logics $\mathrm{L}_{\sim}$).

[20] Actually her axiomatization omitted also the generalization rule, and the axioms (∀2) and (∃2) were replaced by the corresponding rules. However it can be shown that in the context of core semilinear logics her axiomatization and ours (without (∀3)) are equivalent.

[21] This fact was first observed for Gödel logic by Horn in [58].

The provability of the converse implications of (T∀11) or (T∀12) is equivalent to provability of (C∃) or (C∀) resp., i.e., if L∀ proves them, then $\mathrm{L}\forall = \mathrm{L}\forall^{\mathrm{w}}$. This is the case of Łukasiewicz logic; product logic proves (C∃) (and so the converse of (T∀11)) but not (C∀), and Gödel logic proves neither. Finally it is worth noting that the axiom (∀3) is redundant in the axiomatization of Ł∀ and therefore $\text{Ł}\forall^{\mathrm{m}} = \text{Ł}\forall = \text{Ł}\forall^{\mathrm{w}}$.

Some syntactic metatheorems valid in propositional core semilinear logics hold analogously for their first-order logics:

Theorem 51 ([19, Theorems 4.2.6. and 4.2.8., Corollary 4.2.11., and Theorem 4.3.9.]). *Let* L *be a core semilinear logic,* $\vdash$ *be either* $\vdash_{\mathrm{L}\forall}$ *or* $\vdash_{\mathrm{L}\forall^{\mathrm{w}}}$*, and* $\mathcal{P}$ *be a predicate language. Then the following holds for each* $\mathcal{P}$*-theory* T*,* $\mathcal{P}$*-sentences* φ, ψ*, and* $\mathcal{P}$*-formula* χ*:*

1. *The* intersubstitutability*:*
$$\varphi \leftrightarrow \psi \vdash \chi \leftrightarrow \chi',$$
where χ' *is a* $\mathcal{P}$*-formula obtained from* χ *by replacing some occurrences of* φ *by* ψ*.*

2. *The* constants theorem*:*
$$T \vdash \chi(c) \qquad \textit{iff} \qquad T \vdash \chi(x),$$
for any constant c not occurring in $T \cup \{\chi\}$*.*

3. *The* proof by cases *property:*
$$\frac{T, \varphi \vdash \chi \qquad\qquad T, \psi \vdash \chi}{T, \varphi \vee \psi \vdash \chi}$$

4. *The* semilinearity *property:*
$$\frac{T, \varphi \to \psi \vdash \chi \qquad\qquad T, \psi \to \varphi \vdash \chi}{T \vdash \chi}$$

If, furthermore, L *is almost* (MP)*-based with a set of basic deductive terms* bDT*, we can add:*

5. *The* local deduction theorem:
$$T, \varphi \vdash \chi \qquad \textit{iff} \qquad T \vdash \delta(\varphi) \to \chi \quad \textit{for some } \delta \in \Pi(\mathrm{bDT}^*)_{\mathcal{P}}.$$ [22]

Petr Hájek in [37] (or in [51]) used the local deduction theorems for schematic extensions of HL (for (△-)core fuzzy logics resp.) to show the semilinearity property (even though only in [51] it is formulated explicitly), which in turn is a crucial prerequisite for proving the completeness theorem.

4.2 General and witnessed semantics

In this subsection we shall introduce the (witnessed) semantics of predicate fuzzy logics, corresponding to the axiomatic systems L∀ and $\mathrm{L}\forall^{\mathrm{w}}$ respectively. To simplify the formulation of upcoming definitions let us fix: a core semilinear logic L in a propositional language $\mathcal{L}$, a predicate language $\mathcal{P} = \langle \mathit{Pred}, \mathit{Func}, \mathit{Ar} \rangle$, and an L-chain $\boldsymbol{B}$.

Definition 52. *A* $\boldsymbol{B}$-structure $\mathbf{M}$ *for the predicate language* $\mathcal{P}$ *has the form:* $\mathbf{M} = \langle M, (P_{\mathbf{M}})_{P \in \mathit{Pred}}, (F_{\mathbf{M}})_{F \in \mathit{Func}} \rangle$*, where* M *is a non-empty domain; for each* n*-ary predicate symbol* $P \in \mathit{Pred}$*,* $P_{\mathbf{M}}$ *is an* n*-ary fuzzy relation on* M*, i.e., a function* $M^n \to B$ *(identified with an element of* B *if* $n = 0$*); for each* n*-ary function symbol* $F \in \mathit{Func}$*,* $F_{\mathbf{M}}$ *is a function* $M^n \to M$ *(identified with an element of* M *if* $n = 0$*).*

[22] By $\Pi(\mathrm{bDT}^*)_{\mathcal{P}}$ we denote the set of formulae resulting from any $\star$-formula from $\Pi(\mathrm{bDT}^*)$ by replacing all its propositional variables other than $\star$ by arbitrary $\mathcal{P}$-sentences.

Let $\mathbf{M}$ *be a* $\boldsymbol{B}$*-structure for* $\mathcal{P}$*. An* $\mathbf{M}$-evaluation *of the object variables is a mapping* v *which assigns an element from* M *to each object variable. Let* v *be an* $\mathbf{M}$*-evaluation,* x *a variable, and* $a \in M$*. Then by* $\mathrm{v}[x{\mapsto}a]$ *we denote the* $\mathbf{M}$*-evaluation such that* $\mathrm{v}[x{\mapsto}a](x) = a$ *and* $\mathrm{v}[x{\mapsto}a](y) = \mathrm{v}(y)$ *for each object variable* y *different from* x*.*

Let $\mathbf{M}$ *be a* $\boldsymbol{B}$*-structure for* $\mathcal{P}$ *and* v *an* $\mathbf{M}$*-evaluation. We define the* values *of terms and the* truth values *of formulae in* $\mathbf{M}$ *for an evaluation* v *recursively as follows noting that in the last two clauses, if the infimum or supremum does not exist, then the corresponding value is taken to be undefined, and in all clauses, if one of the arguments is undefined, then the result is undefined:*

$$\begin{aligned}
\|x\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \mathrm{v}(x) \\
\|F(t_1,\dots,t_n)\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= F_{\mathbf{M}}(\|t_1\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}},\dots,\|t_n\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}) \quad \textit{for } F \in \textit{Func} \\
\|P(t_1,\dots,t_n)\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= P_{\mathbf{M}}(\|t_1\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}},\dots,\|t_n\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}) \quad \textit{for } P \in \textit{Pred} \\
\|c(\varphi_1,\dots,\varphi_n)\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= c_{\boldsymbol{B}}(\|\varphi_1\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}},\dots,\|\varphi_n\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}) \quad \textit{for } c \in \mathcal{L} \\
\|(\forall x)\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \inf\{\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}[x\mapsto a]} \mid a \in M\} \\
\|(\exists x)\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \sup\{\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}[x\mapsto a]} \mid a \in M\}.
\end{aligned}$$

We say that the $\boldsymbol{B}$*-structure* $\mathbf{M}$ *is*

- safe *if* $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}$ *is defined for each* $\mathcal{P}$*-formula* φ *and each* $\mathbf{M}$*-evaluation* v*,*
- witnessed *if for each* $\mathcal{P}$*-formula* φ *we have:*

$$\begin{aligned}
\|(\forall x)\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \min\{\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}[x\mapsto a]} \mid a \in M\} \\
\|(\exists x)\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \max\{\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}[x\mapsto a]} \mid a \in M\}.
\end{aligned}$$

Note that each witnessed structure is safe. To simplify the upcoming definitions and theorems we write $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ if $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} \geq_{\boldsymbol{B}} 1$ for each $\mathbf{M}$-evaluation v.

Definition 53. *Let* $\mathbf{M}$ *be a* $\boldsymbol{B}$*-structure for* $\mathcal{P}$ *and* T *a* $\mathcal{P}$*-theory. Then* $\mathbf{M}$ *is called a* $\boldsymbol{B}$-model of T *if it is safe and* $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ *for each* $\varphi \in T$*.*

Observe that models are safe structures by definition. As obviously each safe $\boldsymbol{B}$-structure is a $\boldsymbol{B}$-model of the empty theory, we shall use the term *model* for both models and safe structures in the rest of the text.[23] By a slight abuse of language we use the term model also for the pair $\langle \boldsymbol{B}, \mathbf{M}\rangle$.

The following completeness theorems show that the syntactic presentations introduced above succeed in capturing the intended general chain-semantics for first-order fuzzy logics.[24]

Theorem 54 ([19, Theorems 4.4.10 and 4.3.5.])**.** *Let* L *be a core semilinear logic,* $\mathcal{P}$ *a predicate language,* T *a* $\mathcal{P}$*-theory, and* φ *a* $\mathcal{P}$*-formula. Then the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$*.*
- $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ *for each* L*-chain* $\boldsymbol{B}$ *and each model* $\langle \boldsymbol{B}, \mathbf{M}\rangle$ *of the theory* T*.*

Theorem 55 ([19, Theorem 4.5.12. and Example 4.5.3.])**.** *Let* L *be an axiomatic expansion of* $\mathrm{FL}^{\forall}_{\mathrm{e}}$*,* $\mathcal{P}$ *a predicate language,* T *a* $\mathcal{P}$*-theory, and* φ *a* $\mathcal{P}$*-formula. Then the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall^{\mathrm{w}}} \varphi$*.*
- $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ *for each* L*-chain* $\boldsymbol{B}$ *and each* witnessed *model* $\langle \boldsymbol{B}, \mathbf{M}\rangle$ *of the theory* T*.*

[23] In the literature the term 'ℓ-model' is sometimes used instead to stress that $\boldsymbol{B}$ is linearly ordered.

[24] For the proofs of Theorems 54 and 55 for core and $\triangle$-core fuzzy logics see [51]. Instances of Theorem 54 for various core semilinear logics were originally proved separately (usually for countable predicate languages only): the proofs for prominent core fuzzy logics can be found in [23, 37].

4.3 Standard semantics

Already in the pioneering works of Petr Hájek, as in the case of propositional fuzzy logics, the general chain completeness we have just seen was not considered sufficient and, in fact, a crucial item in his agenda was again the search for standard completeness theorems with respect to distinguished classes of models. In order to survey the corresponding results in our framework, in this section we restrict ourselves to *countable* predicate languages. We shall say that $\langle \boldsymbol{B}, \mathbf{M}\rangle$ is a $\mathbb{K}$-*model* (of T) for some $\mathbb{K} \subseteq \mathbb{L}_{lin}$ if $\langle \boldsymbol{B}, \mathbf{M}\rangle$ is a model (of T) and $\boldsymbol{B} \in \mathbb{K}$.

Definition 56. *Let* L *be a core semilinear logic and* $\mathbb{K} \subseteq \mathbb{L}_{lin}$*. We say that* L∀ *enjoys enjoys* (finite) strong $\mathbb{K}$-completeness S$\mathbb{K}$C (FS$\mathbb{K}$C resp.) *if for each countable predicate language* $\mathcal{P}$*,* $\mathcal{P}$*-formula* φ*, and (finite)* $\mathcal{P}$*-theory* T *holds:*

$$T \vdash_{\mathrm{L}\forall} \varphi \quad \textit{iff} \quad \langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi \textit{ for each } \mathbb{K}\textit{-model } \langle \boldsymbol{B}, \mathbf{M}\rangle \textit{ of the theory } T$$

We say that L∀ *enjoys* $\mathbb{K}$-completeness $\mathbb{K}$C *if the equivalence holds for* $T = \emptyset$*.*

All these properties are stronger than their corresponding ones for propositional logics:

Theorem 57. *Let* L *be a core semilinear logic and* $\mathbb{K}$ *a class of* L*-chains. If* L∀ *has the* S$\mathbb{K}$C *(*FS$\mathbb{K}$C *or* $\mathbb{K}$C *respectively), then* L *has the* S$\mathbb{K}$C *(*FS$\mathbb{K}$C *or* $\mathbb{K}$C *respectively).*

As in the propositional case (Theorem 35), strong $\mathbb{K}$-completeness is related to an embedding property, although it is a stronger one requiring preservation of existing suprema and infima:

Definition 58. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be two algebras of the same type with (defined) lattice operations. We say that an embedding* $f\colon \boldsymbol{A} \to \boldsymbol{B}$ *is a* σ-embedding *if* $f(\sup C) = \sup f[C]$ *(whenever* $\sup C$ *exists) and* $f(\inf D) = \inf f[D]$ *(whenever* $\inf D$ *exists) for each countable* $C, D \subseteq A$*.*

Theorem 59. *Let* L *be a core semilinear logic. If every countable* L*-chain* $\boldsymbol{A}$ *can be* σ*-embedded into some* L*-chain* $\boldsymbol{B} \in \mathbb{K}$*, then* L∀ *enjoys* S$\mathbb{K}$C*.*

The proof of this theorem is almost straightforward. Unlike in the propositional case, the existence of σ-embeddings is not a necessary condition, as shown in [13, Theorem 5.38], but nevertheless it is the usual method for proving these results. It also has an interesting corollary for completeness w.r.t. finite chains (recall the characterizations of S$\mathcal{F}$C in Theorem 40).

Corollary 60. *Let* L *be a core semilinear logic. Then* L *enjoys the* S$\mathcal{F}$C *iff* L∀ *enjoys the* S$\mathcal{F}$C*.*

It is obvious that every $\boldsymbol{B}$-structure over a finite L-chain is necessarily witnessed. Thus we have the following proposition which can be used in order to disprove the $\mathcal{F}$C for many logics.

Proposition 61. *Let* L *be a core semilinear logic such that* L∀ *enjoys* $\mathcal{F}$C*. Then* L∀ = L∀[w]*.*

For instance, the examples in [37, Lemma 5.3.6] can be used to show that (C∃) is unprovable in G∀ and (C∀) is unprovable both in G∀ and in Π∀. We can also easily show that $\nvdash_{\mathrm{NM}\forall}$ (C∀). Thus these logics do not enjoy $\mathcal{F}$C.

Table 11 collects results on real, rational and finite-chain completeness of prominent core semilinear logics. Their proofs are scattered in the literature; see [37, 23] for the proofs for the main core fuzzy logics, and [13] for more information and detailed references. For logics weaker than MTL∀ the negative results are derived from the corresponding failure of completeness at the propositional level, while the positive ones are justified by observing that the embeddings built to prove completeness for the underlying propositional logic are actually σ-embeddings. Observe that the family of logics in the first row of the table include $\mathrm{SL}^{\ell}\forall$, which enjoys both strong rational and real (i.e. standard) completeness. This fact was not stated before in the literature because the only paper dealing with SL^{ℓ} [15] concentrated only on propositional logics, but it is not difficult to check that for SL^{ℓ} the obtained embedding is also actually a σ-embedding, i.e., we obtain:

Logic	$\mathcal{R}$C	FS$\mathcal{R}$C, S$\mathcal{R}$C	$\mathcal{Q}$C, FS$\mathcal{Q}$C, S$\mathcal{Q}$C	$\mathcal{F}$C, FS$\mathcal{F}$C, S$\mathcal{F}$C
$\mathrm{SL}^{\ell}_{\mathrm{S}}\forall$, for each $\mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i},\mathrm{o}\}$	Yes	Yes	Yes	No
$\mathrm{SL}^{\ell}_{\mathrm{a}}\forall$	No	No	No	No
$\mathrm{SL}^{\ell}_{\mathrm{aw}}\forall$	Yes	Yes	Yes	No
MTL∀, IMTL∀, SMTL∀	Yes	Yes	Yes	No
WCMTL∀, ΠMTL∀	?	No	?	No
HL∀, SHL∀	No	No	No	No
Ł∀, Π∀	No	No	Yes	No
G∀, WNM∀, NM∀	Yes	Yes	Yes	No
$\mathrm{C_n}$MTL∀, $\mathrm{C_n}$IMTL∀	Yes	Yes	Yes	No
CPC∀	No	No	No	Yes

Table 11: Completeness properties for some first-order core semilinear logics.

Theorem 62. *Let* $\mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i},\mathrm{o}\}$*. Then for each countable predicate language* $\mathcal{P}$*,* $\mathcal{P}$*-formula* φ*, and* $\mathcal{P}$*-theory* T *holds:*

$$T \vdash_{\mathrm{SL}^{\ell}_{\mathrm{S}}\forall} \varphi \quad \textit{iff} \quad \langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi \textit{ for each } \mathcal{R}\textit{-model } \langle \boldsymbol{B}, \mathbf{M}\rangle \textit{ of the theory } T.$$

It is worth adding that for all core fuzzy logics appearing in the table the same results hold for their expansions with △; moreover $\mathrm{G}_{\sim}$ behaves like G, while $\mathrm{SHL}_{\sim}$ and ŁΠ behave like HL. Observe the rather surprising behavior of continuous t-norm based logics regarding the rational-chain semantics: while Ł∀, Π∀, and G∀ enjoy the S$\mathcal{Q}$C, the logics HL∀ and SHL∀ do not even have $\mathcal{Q}$C (see [13] for the proofs of these facts). Petr Hájek already gave in [37] an important hint towards the failure of rational completeness in HL∀ and SHL∀: he found a formula, $(\forall x)(\chi \mathbin{\&} \varphi) \to (\chi \mathbin{\&} (\forall x)\varphi)$, which holds in every model on a densely ordered HL-chain but (as proved later) it is not a tautology of any of those two logics; therefore it makes sense to extend them with this axiom and, by doing so, one obtains new first-order logics complete with respect to all models over rational HL-chains or, respectively, SHL-chains (again, see [13]).

Finding particular examples of formulae witnessing failure of $\mathcal{R}$C of a given logic is not in easy task. Even though some examples were found by Petr Hájek for some particular cases [45], a usual method of proving this is to show that the set of standard tautologies (see next definition) is not recursively enumerable, and therefore it cannot coincide with the set of its theorems. Determining the position in the arithmetical hierarchy (see e.g. [75]) of prominent sets of formulae (such as the tautologies of a given logic) is an important field of study in mathematical logic with major contributions done by Petr Hájek. Here we just briefly mention a few results related to fuzzy logics: for a full treatment of the arithmetical complexity of first-order (△-)fuzzy logics see [53]. Other references on the topic are [46, 39, 44, 65, 66, 67].

First let us introduce some prominent sets of formulae given by a core semilinear logic L:

Definition 63. *We say that a sentence* φ *is*

- *A* general *(resp.,* standard*)* tautology *of* L∀ *if* $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ *for each (*$\mathcal{R}$*-)model* $\langle \boldsymbol{B}, \mathbf{M}\rangle$*.*
- Generally *(resp.,* standardly*)* satisfiable *in* L∀ *if* $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ *for some (*$\mathcal{R}$*-)model* $\langle \boldsymbol{B}, \mathbf{M}\rangle$*.*

The sets of general and standard tautologies and generally and standardly satisfiable sentences are denoted, respectively, by genTAUT*,* stTAUT*,* genSAT*, and* stSAT*.*

For illustration, let us state the results for four predicate logics: HL∀, Ł∀, G∀, and Π∀. For each of them, the set of general tautologies is $\boldsymbol{\Sigma_1}$-complete (thus they are recursively axiomatizable, but undecidable) and the set of generally satisfiable formulae is $\boldsymbol{\Pi_1}$-complete. For the arithmetical complexity of their standard semantics see Table 12 (where "-c" stands for "-complete" and "NA" for "non-arithmetical"). It can be seen that as far as standard semantics is concerned, the four logics differ drastically with respect to their degree of undecidability.

	G∀	Ł∀	Π∀	HL∀
stTAUT	$\mathbf{\Sigma_1}$-c	$\mathbf{\Pi_2}$-c	NA	NA
stSAT	$\mathbf{\Pi_1}$-c	$\mathbf{\Pi_1}$-c	NA	NA

Table 12: Arithmetical complexity of standard semantics.

5 Conclusions

In the last section we have presented a general approach to first-order fuzzy logics based again on the logic SL^ℓ. We have obtained a broad class of first-order systems built over core semilinear logics, shown their axiomatization and their completeness with respect to models over chains. Moreover, we have surveyed their completeness results with respect to distinguished semantics and obtained, in particular, that the weakest first-order fuzzy logic of our framework, $\mathrm{SL}^\ell\forall$, enjoys the standard completeness theorem. Therefore, our flea still jumps (and jumps very well, even in the first-order case!) and we can arguably say that the quest for the basic fuzzy logic initiated by Petr Hájek so far seems to culminate with SL^ℓ. Indeed, both for propositional and first-order predicate logics, SL^ℓ provides a good ground level to build broad families of logics containing all the important particular systems of fuzzy logic: propositional fuzzy logics are captured inside the class of core semilinear logics, while first-order fuzzy logics are obtained as extensions of the logic L∀ built over a core semilinear logic L. Moreover, SL^ℓ is the weakest possible logic one could take in the context of substructural logics in a language with lattice connectives, a conjunction which is not required to satisfy any property corresponding to the usual structural rules and its left and right residua. We do not know whether Mathematical Fuzzy Logic will require an even weaker system to serve as the basic fuzzy logic in the future. Only time will tell. What we can say is that, at the moment, we do not see any remaining legs to be pulled.

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