On Finitely Valued Fuzzy Description Logics: The Łukasiewicz Case

Marco Cerami[1], Francesc Esteva[1], and Àngel García-Cerdaña[1,2]

[1] Artificial Intelligence Research Institute (IIIA - CSIC),
Campus de la UAB, 08193 Bellaterra, Catalonia, Spain
{cerami,esteva,angel}@iiia.csic.es
http://www.iiia.csic.es

[2] Departament de Lògica, Història i Filosofia de la Ciència,
Universitat de Barcelona, Montalegre 6, 08001 Barcelona, Catalonia, Spain
http://www.ub.edu

Abstract. Following the guidelines proposed by Hájek in [1], some proposals of research on Fuzzy Description Logics (FDLs) were given in [2]. One of them consists in the definition and development of a family of description languages, each one having as underlying fuzzy logic the expansion with an involutive negation and truth constants of the logic defined by a divisible finite t-norm. A general framework for finitely valued FDLs was presented in [3]. In the present paper we study the family of languages $\mathcal{ALC}_{\mathbf{Ł}_n^c}$ based on the finitely valued Łukasiewicz logics with truth constants. In addition, we provide an interpretation of these FDLs into fuzzy multi-modal systems. We also deal with the corresponding reasoning tasks and their relationships, and we report some results on decidability and computational complexity.

Keywords: Description Logics, Finitely Valued Description Logics, n-graded Łukasiewicz Description Logics.

1 Introduction

Description Logics (DLs) are knowledge representation languages particularly suited to specify ontologies, to create knowledge bases and to reason with them. A full reference manual of the field is [4]. The vocabulary of DLs consists of symbols for *individuals*, *concepts*, which denote sets of individuals, and *roles*, which denote binary relations among individuals. From atomic concepts and roles and by means of *constructors*, DL systems allow to build complex descriptions of both concepts and roles. These complex descriptions are used to describe a domain through a knowledge base (KB) containing the definitions of relevant domain concepts or some hierarchical relationships among them (*Terminological Box* or *TBox*) and a specification of properties of the domain instances (*Assertional Box* or *ABox*). One of the main issues of DLs is the fact that the semantics is given in a Tarski-style presentation and the statements in both *TBox* and *ABox* can be identified with formulas in first-order logic, and hence we can use reasoning to obtain implicit knowledge from the explicit knowledge in the KB.

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Nevertheless, the knowledge used in real applications is commonly imperfect and has to address situations of uncertainty, imprecision and vagueness. From a real world viewpoint, vague concepts like "patient with a high fever" and "person living near Paris" have to be considered. A natural generalization to cope with vague concepts and relations consists in interpreting DL concepts and roles as fuzzy sets and fuzzy relations, respectively. The initial proposals for Fuzzy Description Logics (FDLs) have been made (see [5,6,7]) mainly based on the earlier approaches to fuzzy logic. In recent times, fuzzy logics has evolved into what is known as Mathematical Fuzzy Logic (as a general reference for the field see [8]). The starting point is the book *Metamathematics of Fuzzy Logics* [9] where Hájek shows the connection between fuzzy logic systems and many-valued residuated logics based on continuous t-norms. Later on, in the paper *Making fuzzy description logic more general* [1], Hájek proposes to deal with FDLs taking as basis t-norm based fuzzy logics with the aim of enriching the expressive possibilities in FDLs. Following this line, we have developed the topic in [2,3]. Since real applications are mainly made using a finite number of values, we are interested in FDLs over finitely valued fuzzy logics. In the present paper we study FDLs based on finitely valued Łukasiewiz logics. In our proposal, description languages are restricted to constructors defined from logical connectives and the fuzzy versions of universal and existential quantifiers. We study the languages, reasoning tasks and their relationships, and decidability and complexity. Special mention is due to the modal translation section that, as far as we know, it is not already considered in the literature for the fuzzy case. Finally, let us mention the recent papers [10,11], which contain results related to our work.

2 The Finitely Valued Łukasiewicz Logic with Truth Constants

Given a positive integer $n \geq 2$, the algebra $\mathbf{Ł}_n$ is the structure $\langle L_n, \otimes, N, 0\rangle$, where $L_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$, $\otimes$ is the *Łukasiewicz t-norm* defined as $a \otimes b := \max\{0, a+b-1\}$, for each $a, b \in L_n$, and N is the negation associated to Łukasiewicz t-norm, defined as $N(a) = 1 - a$, for each $a \in L_n$.[1] Further operations are defined as follows:

$a \Rightarrow b$	$:=$	$N(a \otimes N(b))$	$\min\{0, 1-a+b\}$
$a \wedge b$	$:=$	$a \otimes (a \Rightarrow b)$	$\min\{a, b\}$
$a \vee b$	$:=$	$(a \Rightarrow b) \Rightarrow b$	$\max\{a, b\}$
$a \oplus b$	$:=$	$N(N(a) \otimes N(b))$	$\min\{1, a+b\}$
1	$:=$	$N(0)$	1

Let us consider the set of formulas built from a countable set of *propositional variables* $\Phi = \{p_j : j \in J\}$ using the connectives & (strong conjunction), $\sim$ (involutive negation) and $\bar{0}$ (falsity truth constant). A *propositional evaluation* is a map $e : \Phi \to L_n$ which is extended to all $\langle \&, \sim, \bar{0}\rangle$-formulas by setting $e(\varphi \& \psi) = e(\varphi) \otimes e(\psi)$, $e(\sim\varphi) = N(e(\varphi))$, and $e(\bar{0}) = 0$. The n-valued

[1] In fact this is the unique involutive negation definable in L_n.

Łukasiewicz logic, which we denote by $\mathbf{\Lambda}(\mathbf{Ł}_n)$ is obtained by putting for all sets $\Gamma \cup \{\varphi\}$ of $\langle \&, \sim, \bar{0} \rangle$-formulas, $\Gamma \models_{\mathbf{Ł}_n} \varphi$ if, and only if, for every evaluation e, if $e[\Gamma] \subseteq \{1\}$, then $e(\varphi) = 1$. As defined connectives we have *implication*, *biconditional*, *additive conjunction*, *additive disjunction*, *strong disjunction*, and *true truth constant*: $\varphi \rightarrow \psi := \sim(\varphi \& \sim \psi)$, $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi)\&(\psi \rightarrow \varphi)$, $\varphi \wedge \psi := \varphi\&(\varphi \rightarrow \psi)$, $\varphi \vee \psi := \sim(\sim \varphi \wedge \sim \psi)$, $\varphi \veebar \psi := \sim(\sim \varphi \& \sim \psi)$, $\bar{1} := \sim \bar{0}$. It is well known that this logic is finitely axiomatizable having Modus Ponens as the unique inference rule (cf. [12, p.171]; see also [13]). The logic $\mathbf{\Lambda}(\mathbf{Ł}_n^c)$ is the expansion of $\mathbf{\Lambda}(\mathbf{Ł}_n)$ with truth constants. It is obtained by adding to the language n *canonical* constants: one truth constant $\bar{r}$ for each $r \in L_n \backslash \{0\}$; the semantics of the constant $\bar{r}$ is its canonical value r. $\mathbf{\Lambda}(\mathbf{Ł}_n^c)$ is finitely axiomatizable from an axiomatization of $\mathbf{\Lambda}(\mathbf{Ł}_n)$ by adding the so-called *book-keeping* axioms:

$$
\begin{array}{ll}
(bk_1) & \bar{r}\&\bar{s} \leftrightarrow \overline{r \otimes s} \\
(bk_2) & \sim \bar{r} \leftrightarrow \overline{N(r)}
\end{array}
$$

The predicate logic $\mathbf{\Lambda}(\mathbf{Ł}_n^c)\forall$ is defined from $\mathbf{\Lambda}(\mathbf{Ł}_n^c)$ as it is done for the fuzzy predicate logics introduced in [9, Chapter 5]. Let $\Sigma = \langle \mathcal{C}, \mathcal{P} \rangle$ be a first order signature (without functional symbols), $\mathcal{C}$ being a countable set of object constants and $\mathcal{P}$ a countable set of predicate symbols, each one with arity $k \geq 1$. An $\mathbf{Ł}_n^c$-*interpretation* for Σ is a tuple $\mathbf{M} = \langle M, \{a^{\mathbf{M}} : a \in \mathcal{C}\}, \{P^{\mathbf{M}} : P \in \mathcal{P}\} \rangle$, where 1) M is a non-empty set; 2) for each object constant $a \in \mathcal{C}$, $a^{\mathbf{M}}$ is an element of M; and 3) for each k-ary predicate symbol P, $P^{\mathbf{M}}$ is an n-graded k-ary relation defined on M, that is, a function $P^{\mathbf{M}} : M^k \rightarrow L_n$. Given an interpretation $\mathbf{M}$, a map v assigning an element $v(x) \in M$ to each variable x is called an *assignation of the variables in* $\mathbf{M}$. Given $\mathbf{M}$ and v, the *value of a term* t in $\mathbf{M}$, denoted by $\|t\|_{\mathbf{M},v}$, is defined as $v(x)$ when t is a variable x, and as $a^{\mathbf{M}}$ when t is a constant a. In order to emphasize that a formula α has its free variables in $\{x_1, \ldots, x_n\}$, we will denote it by $\alpha(x_1, \ldots, x_n)$. Let v be an assignation such that $v(x_1) = b_1, \ldots, v(x_n) = b_n$. The *truth value in* $\mathbf{M}$ *over* $\mathbf{Ł}_n^c$ *of the predicate formula* $\varphi(x_1, \ldots, x_n)$ *for the assignation* v, denoted by $\|\varphi\|_{\mathbf{M},v}$ or by $\|\varphi(b_1, \ldots, b_n)\|_{\mathbf{M}}$, is a value in L_n defined inductively as follows:

$$
\begin{array}{ll}
P^{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_k\|_{\mathbf{M},v}), & \text{if } \varphi = P(t_1, \ldots, t_k); \\
r, & \text{if } \varphi = \bar{r} \in \{\bar{0}, \bar{r}_1, \ldots, \bar{r}_{n-1}\}; \\
1 - \|\alpha\|_{\mathbf{M},v}, & \text{if } \varphi = \sim \alpha; \\
\|\alpha\|_{\mathbf{M},v} \otimes \|\beta\|_{\mathbf{M},v}, & \text{if } \varphi = \alpha\&\beta; \\
\inf\{\|\alpha(a, b_1, \ldots, b_n)\|_{\mathbf{M}} : a \in M\}, & \text{if } \varphi = (\forall x)\alpha(x, x_1, \ldots, x_n).
\end{array}
$$

A $\mathbf{Ł}_n^c$-interpretation $\mathbf{M}$ is an $\mathbf{Ł}_n^c$-*model*, or simply a *model*, of a set of formulas Γ if, for each $\varphi \in \Gamma$, and each assignation v, $\|\varphi\|_{\mathbf{M},v} = 1$. The logic $\mathbf{\Lambda}(\mathbf{Ł}_n^c)\forall$ is defined by a finite set of axioms. Moreover, we have the following result:[2]

Theorem 1. *The logic* $\mathbf{\Lambda}(\mathbf{Ł}_n^c)\forall$ *is strongly complete with respect to interpretations over* $\mathbf{Ł}_n^c$.

[2] A direct proof of this theorem is easy since the unique subdirectly irreducible algebra of the variety corresponding to $\mathbf{\Lambda}(\mathbf{Ł}_n^c)$ is $\mathbf{Ł}_n^c$.

3 The n-Valued Łukasiewicz Description Logics $\mathcal{ALC}_{\text{Ł}_n^c}$

Description Logics based on finitely valued Łukasiewicz logics are built in the same way as in the classical case, but now we need a set of constructors that corresponds to the logical symbols existing in the setting of the first order logics $\mathbf{\Lambda}(\text{Ł}_n^c)\forall$ (cf. [3]).[3] To do so we introduce some symbols for new propositional constructors: $\boxtimes$ for *strong intersection*; $\boxplus$ for *strong union*; $\sqsupset$ for *residuated implication*, and a *constant* $\mathfrak{r}$ for every $r \in L_n$. Moreover, in our setting, we have the classical $\sqcup$ and $\sqcap$ as defined constructors. It is worth pointing out that $\mathcal{ALC}$-like DLs based on n-valued Łukasiewicz logics are analogous to the classical $\mathcal{ALC}$ in the sense that the basic relations between connectives remain true:

- complementation is involutive,
- both pairs of weak and strong intersection and union are dual w.r.t. complementation,
- the universal and the existential quantifications are inter-definable by means of complementation,
- implication is definable from complementation and strong intersection or from complementation and strong union.

Notice that the above relations are not satisfied in other finitely valued t-norm based predicate fuzzy logics.

Definition 1 (The attributive languages $\mathcal{ALC}_{\text{Ł}_n^c}$). *Let us fix a description signature $\mathcal{D} = \langle N_I, N_A, N_R\rangle$, that is, a set of individual names N_I, a set N_A of concept names (the* atomic concepts*), and a set N_R of role names (the* atomic roles*). An $\langle \mathcal{ALC}_{\text{Ł}_n^c}, \mathcal{D}\rangle$-*description*, or simply an $\mathcal{ALC}_{\text{Ł}_n^c}$-*description*, is inductively defined in accordance with the following syntactic rules (we use the symbols C, C_1, C_2 as meta-variables for descriptions of concepts):*

$$
\begin{array}{lrll}
C, C_1, C_2 & \rightsquigarrow \quad A \mid & \textit{(atomic concept)} & \\
 & \bot \mid & \textit{(empty description)} & \\
 & \top \mid & \textit{(universal description)} & \\
 & \mathfrak{r} \mid & \textit{(constant description)} & \\
 & \neg C \mid & \textit{(strong complementary concept)} & (\mathcal{C}) \\
 & C_1 \boxplus C_2 \mid & \textit{(concept strong union)} & (\mathcal{U}) \\
 & C_1 \boxtimes C_2 \mid & \textit{(concept strong intersection)} & \\
 & \forall R.C \mid & \textit{(universal quantification)} & \\
 & \exists R.C \mid & \textit{(existential quantification)} & (\mathcal{E}) \\
 & R \mid & \textit{(atomic role)} &
\end{array}
$$

Further constructors are defined as follows:

$$
\begin{array}{ll}
C_1 \sqsupset C_2 := \neg(C_1 \boxtimes \neg C_2) & \textit{(residuated implication)} \\
C_1 \sqcap C_2 := C_1 \boxtimes (C_1 \sqsupset C_2) & \textit{(weak intersection)} \\
C_1 \sqcup C_2 := \neg(\neg C_1 \sqcap \neg C_2) & \textit{(weak union)}
\end{array}
$$

[3] In [3] a new hierarchy of attributive languages adapted to the behavior of the connectives in the fuzzy setting is proposed.

The notion of *instance of a description* allows us to read description formulas of a given description signature $\mathcal{D}$ as predicate formulas of $\mathbf{\Lambda}(\text{Ł}_n^c)\forall$ as it is done in Definition 2 following Hajék's paper [1]. From this notion, we can define the truth value of a description formula as the truth value of a first order formula.

Definition 2 (Instance of a description). *Given a description signature* $\mathcal{D} = \langle N_I, N_A, N_R\rangle$*, we define the first order signature* $\Sigma_{\mathcal{D}} = \langle \mathcal{C}_{\mathcal{D}}, \mathcal{P}_{\mathcal{D}}\rangle$*, where* $\mathcal{C}_{\mathcal{D}} = N_I$ *and* $\mathcal{P}_{\mathcal{D}} = N_A \cup N_R$*. We read each individual name in* N_I *as an object constant, each atomic concept in* N_A *as a unary predicate symbol, and each atomic role in* N_R *as a binary predicate symbol. We define as* instances *of an* $\langle \mathcal{ALC}_{\text{Ł}_n}, \mathcal{D}\rangle$*-description the following formulas of* $\mathbf{\Lambda}(\text{Ł}_n^c)\forall$*:*

- *The* instance *of a truth constant is defined as* $\bar{0}$ *for* $\bot$*;* $\bar{1}$ *for* $\top$*; and* $\bar{r}$ *for* $\mathfrak{r}$*.*
- *Given a term* t *and a concept* D*, the* instance $D(t)$ *of* D *is defined as*

$$\begin{array}{ll} A(t) & \text{if } D \text{ is an atomic concept } A, \\ \sim C(t) & \text{if } D = \neg C, \\ C_1(t) \veebar C_2(t) & \text{if } D = C_1 \boxplus C_2, \\ C_1(t) \& C_2(t) & \text{if } D = C_1 \boxtimes C_2, \\ (\forall y)(R(t,y) \to C(y)) & \text{if } D = \forall R.C, \\ (\exists y)(R(t,y) \& C(y)) & \text{if } D = \exists R.C. \end{array}$$

- *An* instance *of an atomic role* R *is any atomic first order formula* $R(t_1, t_2)$*, where* t_1 *and* t_2 *are terms.*

We can define the consequence relation $\models_{\mathcal{ALC}_{\text{Ł}_n^c}}$ as the restriction of the consequence relation of the logic $\mathbf{\Lambda}(\text{Ł}_n^c)\forall$ to instances of $\mathcal{ALC}_{\text{Ł}_n^c}$-descriptions.

4 $\mathcal{ALC}_{\text{Ł}_n^c}$ and Modal Finite-Valued Łukasiewicz Logics

It is known that there is a translation between classical $\mathcal{ALC}$ and multi-modal logical systems (cf. [4, Chapter 4]). In this section we show that a similar translation is also possible between $\mathcal{ALC}_{\text{Ł}_n^c}$ and multi-modal finite-valued Łukasiewicz logics with truth constants.[4] The language of each one of these multi-modal systems, denoted by μ_m, is obtained by fixing a natural number m and expanding the language of $\mathbf{\Lambda}(\text{Ł}_n^c)$ with m unary connectives $\Box_1, \dots, \Box_m$ (m *necessity* operators).

Definition 3 (Kripke m-frames and m-models). *An* n-valued Kripke m-frame *is a tuple* $\mathfrak{F} = \langle W, R_1, \dots, R_m\rangle$*, where* W *is a non-empty set (the set of* worlds*) and* $R_1, \dots, R_m$ *are binary relations (the* accessibility relations*) valued in* L_n*. The Kripke frame is said to be* crisp *if the range of the relations* R_k *is included in* $\{0, 1\}$*. The class of all* n*-valued* m*-frames will be denoted by* Fr *and the class of crisp* m*-frames by* CFr*. A Kripke* $\langle \text{Ł}_n^c, m\rangle$-model *is a pair* $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$*, where* $\mathfrak{F}$ *is an* n*-valued Kripke* m*-frame and* V *is a* valuation *assigning*

[4] Modal finite-valued Łukasiewicz logics –with and without truth constants– have been studied in [14].

to each variable in $\Phi = \{p_j : j \in J\}$ and each world in W a value in L_n. The map V can be uniquely extended to a map, which we also denote by V, assigning an element of L_n to each pair formed by a μ_m-formula φ and a world w in such a way that:

- $V(\varphi \& \psi, w) = V(\varphi, w) \otimes V(\psi, w)$, $V({\sim}\varphi, w) = 1 - V(\varphi, w)$, $V(\bar{0}, w) = 0$;
- *for each canonical constant* $\bar{a}_i$, $i \in \{1, \ldots, n-1\}$, $V(\bar{a}_i, w) = a_i$;
- *for each* k, $1 \leq k \leq m$, $V(\Box_k\varphi, w) = \inf\{R_k(w, w') \Rightarrow V(\varphi, w') : w' \in W\}$.

Note that since the algebra of values is finite, we have that this infimum is always a minimum. Therefore, we are sure that we can compute the value of formulas with $\Box_k$. For each operator $\Box_k$, an operator of *possibility* is defined as follows: $\Diamond_k\varphi := {\sim}\Box_k{\sim}\varphi$. According with this definition it is easy to see that:

$$V(\Diamond_k\varphi, w) = \sup\{R_k(w, w') \otimes V(\varphi, w') : w' \in W\}.$$

Definition 4 (Validity, the set $\mathbf{\Lambda}(\mathsf{K}, \text{Ł}_n^c)$). *Let $\mathfrak{M} = \langle W, r_1, \ldots, r_m, V\rangle$ be a $\langle \text{Ł}_n^c, m\rangle$-model. We will say that $w \in W$* satisfies *a formula φ in $\mathfrak{M}$ whenever $V(\varphi, w) = 1$; then we write $\mathfrak{M}, w \models^1 \varphi$. And we write $\mathfrak{M} \models^1 \varphi$ whenever $\mathfrak{M}, w \models^1 \varphi$ for every $w \in W$. Then we say that φ is* valid in $\mathfrak{M}$. *We say that φ is* valid in the frame $\mathfrak{F}$ *when φ is valid in any Kripke model based on $\mathfrak{F}$. Then we write $\mathfrak{F} \models^1 \varphi$. Given a class K of frames, we write $\mathsf{K} \models^1 \varphi$ to mean that φ is valid in all frames in this class. The set of all the formulas that are valid in all the frames of a class K will be denoted by $\mathbf{\Lambda}(\mathsf{K}, \text{Ł}_n^c)$.*[5]

Definition 5 (The standard translation in the Ł_n^c-valued framework). *Fix a positive natural number m and let us consider the propositional multimodal language with constants μ_m. Let $\Phi = \{p_j : j \in J\}$ be a countable set of propositional letters. Let $\mathcal{L}_{\mu_m}(\Phi)$ be the first order language which has a unary predicate P_j for each propositional letter $p_j \in \Phi$, and a binary relation symbol R_k for every necessity operator $\Box_k$ from μ_m. Let x be a first order variable. We define the* standard translation τ_x *from μ_m-formulas to $\mathcal{L}_{\mu_m}(\Phi)$-formulas as follows:*

$$\begin{aligned}
\tau_x(p_j) &= P_j(x), \text{ for each } j \in J,\\
\tau_x(\varphi \& \psi) &= \tau_x(\varphi) \,\&\, \tau_x(\psi),\\
\tau_x({\sim}\varphi) &= {\sim}\tau_x(\varphi),\\
\tau_x(\Box_k\varphi) &= (\forall y)(R_k(x, y) \to \tau_y(\varphi)),\ 1 \leq k \leq m,\\
\tau_x(\bar{0}) &= \bar{0},\\
\tau_x(\bar{a}_i) &= \bar{a}_i,\ 1 \leq i \leq n-1.
\end{aligned}$$

Proposition 1. *Let $\mathfrak{M} = \langle W, r_1, \ldots, r_m, V\rangle$ be a $\langle \text{Ł}_n^c, m\rangle$*-model. *From $\mathfrak{M}$ we define the $\mathcal{L}_{\mu_m}(\Phi)$-interpretation $\mathcal{I}_{\mathfrak{M}} = \langle W, (P_j^{\mathcal{I}_{\mathfrak{M}}})_{j \in J}, R_1^{\mathcal{I}_{\mathfrak{M}}}, \ldots, R_m^{\mathcal{I}_{\mathfrak{M}}}\rangle$, where $P_j^{\mathcal{I}_{\mathfrak{M}}} : W \to L_n$ such that $P_j^{\mathcal{I}_{\mathfrak{M}}}(w) = V(p_j, w)$, and $R_k^{\mathcal{I}_{\mathfrak{M}}} = r_k$. Then:*

1. *For every μ_m-formula φ and $w \in W$, $\|\tau_x(\varphi)(w)\|_{\mathcal{I}_{\mathfrak{M}}} = V(\varphi, w)$.*
2. *For every $w \in W$, $\mathfrak{M}, w \models^1 \varphi$ iff $\|\tau_x(\varphi)(w)\|_{\mathcal{I}_{\mathfrak{M}}} = 1$.*
3. *$\mathfrak{M} \models^1 \varphi$ iff $\|(\forall x)\,\tau_x(\varphi)\|_{\mathcal{I}_{\mathfrak{M}}} = 1$.*

[5] In [14, Section 4.2] finite axiomatizations for $\mathbf{\Lambda}(\mathsf{K}, \text{Ł}_n^c)$ are given when $\mathsf{K} \in \{\mathsf{Fr}, \mathsf{CFr}\}$.

Given a description signature $\mathcal{D}$ with $N_A = \{A_1, A_2, \ldots\}$ as the set of atomic concepts, and $N_R = \{R_1, \ldots, R_m\}$ as the set of atomic roles, the corresponding language $\mathcal{ALC}_{\mathbf{L}_\mathbf{n}^\mathbf{c}}$ can be seen as a propositional language built from the concept names $A \in N_A$ (seen as propositional letters) using $\boxtimes$ as binary connective, the unary connective $\neg$, a unary connective denoted by $\forall R.$ for every $R \in N_R$, and the constants $\bot, \mathfrak{r}_1, \ldots, \mathfrak{r}_{n-1}$. We have an isomorphism f between the set of $\mathcal{ALC}_{\mathbf{L}_\mathbf{n}^\mathbf{c}}$-formulas built from the generators $\{A_1, A_2, \ldots\}$ and the set of μ_m-formulas generated by a set of propositional letters $\{p_j : j \in J\}$ with the same cardinality as N_A:

$$
\begin{aligned}
f(A_j) &= p_j, \text{ for each } j \in J,\\
f(C \boxtimes D) &= f(C) \& f(D),\\
f(\neg C) &= \sim f(C),\\
f(\forall R_k.C) &= \Box_k f(C),\ 1 \leq k \leq m,\\
f(\bot) &= \bar{0},\\
f(\mathfrak{r}_i) &= \bar{a}_i,\ 1 \leq i \leq n-1.
\end{aligned}
$$

This isomorphism is a preserving translation in the sense stated in the following proposition.

Proposition 2. *Let f be as above and let us consider the first order signature $\mathcal{L}(\mathcal{D}) = \langle (A_j)_{j\in J}, R_1, \ldots, R_m \rangle$ given by the description signature $\mathcal{D} = \langle N_A, N_R \rangle$. Let $\mathcal{I} = \langle W, (A_j^{\mathcal{I}})_{j\in J}, R_1^{\mathcal{I}}, \ldots, R_m^{\mathcal{I}} \rangle$ be an $\mathcal{L}(\mathcal{D})$-interpretation. From $\mathcal{I}$ we define a Kripke $\langle \mathbf{L}_n^c, m \rangle$-model $\mathfrak{M}_{\mathcal{I}} = \langle W, r_1^{\mathfrak{M}_{\mathcal{I}}}, \ldots, r_m^{\mathfrak{M}_{\mathcal{I}}}, V_{\mathfrak{M}_{\mathcal{I}}} \rangle$, where $r_j^{\mathfrak{M}_{\mathcal{I}}} = R_j^{\mathcal{I}}$, and $V_{\mathfrak{M}_{\mathcal{I}}} : \Phi \times W \to L_n$ such that $V_{\mathfrak{M}_{\mathcal{I}}}(p_j, w) = A_j^{\mathcal{I}}(w)$. Then:*

1. *For every concept C and every $w \in W$, $\|C(w)\|_{\mathcal{I}} = V_{\mathfrak{M}_{\mathcal{I}}}(f(C), w)$.*
2. *For every instance $C(x)$ and every $w \in W$, $\|C(w)\|_{\mathcal{I}} = 1$ iff $\mathfrak{M}_{\mathcal{I}}, w \models^1 f(C)$.*
3. *$\|(\forall x)C(x)\|_{\mathcal{I}} = 1$ iff $\mathfrak{M}_{\mathcal{I}}, \models^1 f(C)$.*

5 Reasoning

In this section we define firstly the graded axioms used to define knowledge bases for our n-graded DLs and after the equivalences between the corresponding reasoning tasks. Finally we report the state of the art of the research on the computational complexity of these reasoning tasks.

5.1 Knowledge Bases for $\mathcal{ALC}_{\mathbf{L}_n^\mathbf{c}}$

To define knowledge bases (KBs) for the description logics $\mathcal{ALC}_{\mathbf{L}_n^c}$, we need the notion of *evaluated formula.* Given $r \in L_n$, an evaluated formula of the logic $\mathbf{\Lambda}(\mathbf{L}_n^c)\forall$ is a formula of one of the forms $\bar{r} \to \varphi$, $\varphi \to \bar{r}$, or $\bar{r} \leftrightarrow \varphi$, where φ does not contain any occurrence of truth constants other than $\bar{0}$ or $\bar{1}$. In our framework, since we are interested in reasoning on partial truth of formulas, it seems reasonable to restrict ourselves to evaluated formulas for representing the knowledge contained in a knowledge base. Having truth constants in the language, we can handle graded inclusion axioms in addition to graded assertion axioms (see [2]), as usually done in FDLs (see [7,15]). Next we define these graded notions.

Let C, D be concepts without occurrences of any truth constant other than $\bot$ or $\top$, R be an atomic role and a, b be constant objects. Finally let $r \in L_n$.

A *graded concept inclusion axiom* is an expression of the form $\langle C \sqsubseteq D, \bar{r}\rangle$, whose corresponding evaluated first order sentence is $\bar{r} \to (\forall x)(C(x) \to D(x))$.

An *graded equivalence axiom* is an expression of the form $\langle C \equiv D, \bar{r}\rangle$, whose corresponding evaluated first order sentence is $\bar{r} \to (\forall x)(C(x) \leftrightarrow D(x))$.

A *graded concept assertion axiom* (or *graded assertion*) is an expression of the form $\langle C(a), \bar{r}\rangle$, whose corresponding evaluated first order sentence is $\bar{r} \to C(a)$.

Finally, *graded role assertion axioms* is an expression of the form $\langle R(a, b), \bar{r}\rangle$. Its corresponding evaluated first order sentence is $\bar{r} \to R(a, b)$.

A *TBox* for a graded DL language is a finite set of graded concept inclusion axioms. An *ABox* is a finite set of graded concept and role assertion axioms. A *knowledge base* (KB) is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{A}\rangle$, where the first component is a *TBox* and the second one is an *ABox*.

5.2 Reasoning Tasks in $\mathcal{ALC}_{\text{Ł}_n^c}$

Among the reasoning tasks that can be defined in a multi-valued framework we can find the usual ones, i.e, KB consistency, concept satisfiability and subsumption with respect to a (possibly empty) KB and entailment of an assertion axiom from a (possibly empty) KB (see [16]). In this framework we can define the following graded notions:

Definition 1 (Satisfiability). *A concept C is* satisfiable *w.r.t. a knowledge base $\mathcal{K}$ in a degree greater or equal than r iff there is an Ł_n^c-model $\mathbf{M}$ of $\mathcal{K}$, and an individual $a \in M$ such that $\|C(a)\|_{\mathbf{M}} \geq r$. In particular, C is* positively satisfiable *when $r = \frac{1}{n}$ (strictly greater than 0) and* 1-satisfiable *when $r = 1$.*

Definition 2 (Subsumption). *A concept C is* subsumed *by a concept D in a degree greater or equal than r w.r.t. a KB $\mathcal{K}$ iff, for every Ł_n^c-model $\mathbf{M}$ of $\mathcal{K}$, it holds that $\|(\forall x)(C(x) \to D(x))\|_{\mathbf{M}} \geq r$. In case C is subsumed by D in a degree greater or equal than 1, we will simply say that C is subsumed by D.*

In our language we only need the notions of 1 and positive satisfiability since, thanks to the expressive power given by the presence of the truth constants in the language, all other graded notions can be reduced to them in $\mathcal{ALC}_{\text{Ł}_n^c}$.

Proposition 1. *Let $\mathcal{K}$ be a (possibly empty) knowledge base, C, D be $\mathcal{ALC}_{\text{Ł}_n^c}$-concepts and $r \in L_n$, then the following equivalences hold:*

1. *C is satisfiable w.r.t. $\mathcal{K}$ in a degree greater or equal than r iff the concept $\bar{r} \sqsupset C$ is 1-satisfiable w.r.t. $\mathcal{K}$.*
2. *C is subsumed by D w.r.t. $\mathcal{K}$ in a degree greater or equal than r iff concept $\bar{r} \boxtimes C$ is subsumed by D in degree 1.*

3. *A concept C is subsumed in degree* 1 *by a concept D w.r.t. a knowledge base $\mathcal{K}$ iff $\mathcal{K} \cup \{C \boxtimes \sim D\}$ is not positively satisfiable.*

Notice that this last result is true only for the Łukasiewicz case and it is not achievable in the general framework of (finite) t-norm based FDLs presented in [2,3], where negation is not necessarily involutive.

5.3 Decidability and Complexity Issues

From results in [1] it is easy to prove that concept satisfiability for $\mathcal{ALC}$-like FDLs based on finite t-norms is a decidable problem. The method used in that paper, based on a recursive reduction to propositional satisfiability, can be also used to obtain decidability for the ABox consistency and the concept satisfiability w.r.t. an ABox. Moreover, in [11] it has been proved that concept satisfiability w.r.t. a general TBox for FDLs over finite lattices is EXPTIME-complete. From these results, we can easily obtain the following decidability results and complexity bounds for our finite-valued Łukasiewicz FDLs:

- TBox consistency is EXPTIME-complete.
- Entailment of an assertion from an ABox is decidable.
- Entailment of an inclusion axiom from a TBox is EXPTIME-complete.

Concept satisfiability can be seen as concept satisfiability w.r.t. the empty TBox, thus obtaining EXPTIME upper bound for this problem. Nevertheless, in [17] has been proved that finite-valued Łukasiewicz modal logic is PSPACE-complete and, since this logic can be seen as a notational variant of $\mathcal{ALC}_{\mathbf{Ł}_n^c}$, we obtain PSPACE-completeness of the concept satisfiability problem in $\mathcal{ALC}_{\mathbf{Ł}_n^c}$.

Another remarkable result is the one reported in [18], where a reduction from finitely valued fuzzy $\mathcal{ALCH}$ to classical $\mathcal{ALCH}$ is provided. Since, however, the reduction is not polynomial, it can only be used to obtain decidability of its reasoning tasks, but not to obtain their computational complexity. Finally, let us mention [10] where the authors show how to reason with a fuzzy extension of the description language $\mathcal{SROIQ}$ under finitely valued Łukasiewicz logics. They show that it is decidable by presenting a reasoning preserving procedure to obtain a crisp representation of the logics.

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References

1. Hájek, P.: Making fuzzy description logic more general. Fuzzy Sets and Systems 154(1), 1–15 (2005)

2. García-Cerdaña, À., Armengol, E., Esteva, F.: Fuzzy Description Logics and t-norm based fuzzy logics. International Journal of Approximate Reasoning 51(6), 632–655 (2010)
3. Cerami, M., García-Cerdaña, À., Esteva, F.: From classical description logic to n-graded fuzzy description logic. In: Proceedings of the FUZZ-IEEE 2010. WCCI 2010 IEEE World Congress on Computational Intelligence, pp. 1506–1513 (2010)
4. Baader, F., Calvanese, D., McGuinness, D.L., Nardi, D., Patel-Schneider, P.F. (eds.): The Description Logic Handbook: Theory, Implementation, and Applications. Cambridge University Press, New York (2003)
5. Yen, J.: Generalizing Term Subsumption Languages to Fuzzy Logic. In: Proc. of the 12th IJCAI, Sidney, Australia, pp. 472–477 (1991)
6. Tresp, C.B., Molitor, R.: A Description Logic for Vague Knowledge. Technical Report RWTH-LTCS Report 98-01. Aachen University of Technology (1998)
7. Straccia, U.: Reasoning within Fuzzy Description Logics. Journal of Artificial Intelligence Research 14, 137–166 (2001)
8. Cintula, P., Hájek, P., Noguera, C. (eds.): Handbook of Mathematical Fuzzy Logic. Studies in Logic, vol. 1,2. College Publications, London (2011)
9. Hájek, P.: Metamathematics of Fuzzy Logic. Trends in Logic. Studia Logica Library, vol. 4. Kluwer Academic Publishers, Dordrecht (1998)
10. Bobillo, F., Straccia, U.: Reasoning with the Finitely Many-valued Łukasiewicz Fuzzy Description Logic SROIQ. Information Sciences 181, 758–778 (2011)
11. Borgwardt, S., Peñaloza, R.: Description logics over lattices with multi-valued ontologies. In: Proceedings of the Twenty-Second International Conference on Artificial Intelligence, pp. 768–773 (2011)
12. Cignoli, R., D'Ottaviano, I., Mundici, D.: Algebraic Foundations of Many-Valued Reasoning. Trends in Logic—Studia Logica Library, vol. 7. Kluwer Academic Publishers, Dordrecht (2000)
13. Tuziak, R.: An Axiomatization of the Finite-Valued Łukasiewicz Calculus. Studia Logica 47(1), 49–55 (1988)
14. Bou, F., Esteva, F., Godo, L., Rodríguez, R.: On the Minimum Many-Valued Modal Logic over a Finite Residuated Lattice. Journal of Logic and Computation 21(5), 739–790 (2011)
15. Straccia, U., Bobillo, F.: Mixed Integer Programming, General Concept Inclusions and Fuzzy Description Logics. Mathware and Soft Computing 14(3), 247–259 (2007)
16. Baader, F., Horrocks, I., Sattler, U.: Description logics. In: van Harmelin, F., Lifshitz, V., Porter, B. (eds.) Handbook of Knowledge Representation, pp. 135–179. Elsevier (2008)
17. Bou, F., Cerami, M., Esteva, F.: Finite-valued Łukasiewicz Modal Logic is PSPACE-complete. In: Proceedings of the IJCAI 2011, pp. 774–779 (2011)
18. Bobillo, F., Straccia, U.: Finite fuzzy description logics: A crisp representation for finite fuzzy $\mathcal{ALCH}$. In: Bobillo, F., et al. (eds.) Proceedings of the 6th ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2010). CEUR Workshop Proceedings, vol. 654, pp. 61–72 (November 2010)