A multi-adjoint lattice logic based on Gödel logic*

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Abstract. Multi-adjoint lattice logic (MLL) was introduced as a logic focused on capturing multi-adjoint algebras, which are general and flexible algebraic structures used, for example, as truth-values set in different formal tools to model data sets, such as, formal concept analysis, rough sets, and fuzzy relation equations. This paper enriches this logic by expanding it with two extra connectives, one associated with the Gödel implication and the one associated with the Baaz-Monteiro projection connective. As a consequence, the implication in MLL representing the ordering in the lattice becomes definable from these two operators, and vice versa.

Keywords: Bounded poset, multi-adjoint algebra, multi-adjoint lattice logic, fuzzy logic, Gödel fuzzy logic, Baaz-Monteiro operator

1 Introduction

This paper further explores and expands the multi-adjoint lattice logic (MLL) introduced in [4]. In this logic, the intended semantics is given by the class of order-right multi-adjoint algebras, and the axiomatization follows the philosophy of Hájek's Basic Fuzzy logic [10]. Due to the algebraic flexibility of multi-adjoint algebras, a selected implication connective $\to^d$ related to the ordering in the lattice was required to be introduced. In this paper we expand this logic with two new connectives, from which the selected order implication is definable.

More specifically, we add to MLL a new implication connective $\to$ with Gödel fuzzy logic semantics and Baaz-Monteiro's projection connective Δ, which is a particular case of a truth-stressing hedge [11]. The resulting new logic, that we

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denote MGL_Δ, is an extension of the Gödel logic with the Baaz-Monteiro operator G_Δ (see e.g. [10]), where the MLL order implication $\to^d$ becomes definable from Gödel implication and Baaz-Monteiro's operator, and vice versa. The new logic can also be related to a particular case of the logic $\mathrm{MLL}_{\vee-\imath t}$ presented in [4] in which these two connectives have been selected to play a significant role. Moreover, compared to MLL, an advantage of MGL_Δ is that it is shown to be complete with respect to the smaller class of *linear* order-right multi-adjoint algebras.

2 Preliminaries

2.1 Gödel logic

Here we provide some preliminaries on *Gödel logic* G and its expansion with Baaz-Monteiro operator Δ. To start with, the language of Gödel propositional logic is built as usual from a countable set of propositional variables V, the constant $\bot$ and the binary connectives $\wedge$ and $\to$. Disjunction and negation are defined as $\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$ and $\neg\varphi := \varphi \to \bot$, respectively, equivalence is defined as $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, and the constant $\top$ is taken as $\bot \to \bot$.

As a many-valued logic, Gödel logic is the axiomatic extension of Hájek's Basic Fuzzy Logic BL [10] (which is the logic of continuous t-norms and their residua [2]) by means of the contraction axiom (A7), see below. Then the following are the *axioms* of G:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $(\varphi \wedge \psi) \to \varphi$
(A3) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
(A4a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi)$
(A4b) $((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A5) $(\varphi \to \psi) \vee (\psi \to \varphi)$
(A6) $\bot \to \varphi$
(A7) $\varphi \to (\varphi \wedge \varphi)$

The *deduction rule* of G is modus ponens. The notion of *proof* for G, denoted as $\vdash_G$, is defined as usual from the above set of axioms and the inference rule.

Since the unique idempotent continuous t-norm is the minimum, this yields that Gödel logic is strongly complete with respect to its standard fuzzy semantics that interprets formulas over the structure $[0,1]_\mathrm{G} = ([0,1], \min, \Rightarrow_\mathrm{G}, 0, 1)$, called *standard* Gödel algebra. i.e. semantics defined by truth-evaluations of formulas e on $[0,1]$, where 1 is the only designated truth-value, such that $e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$, $e(\varphi \to \psi) = e(\varphi) \Rightarrow_\mathrm{G} e(\psi)$ and $e(\bot) = 0$, where $\Rightarrow_\mathrm{G}$ is the binary operation on $[0,1]$ defined as

$$x \Rightarrow_\mathrm{G} y = \begin{cases} 1, \text{ if } x \leq y \\ y, \text{ otherwise} \end{cases}$$

As a consequence, $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$ and $e(\neg\varphi) = e(\varphi) \Rightarrow_\mathrm{G} 0$. Given a set of *well-formed formulas* Γ, we write $\Gamma \models_\mathrm{G} \varphi$ to denote that φ is a *semantics consequence* of Γ, that is, for every evaluation e over $[0,1]_\mathrm{G}$, if $e(\gamma) = 1$ for every $\gamma \in \Gamma$ then $e(\varphi) = 1$. Then, the *strong standard completeness* for the logic G reads as follows: for every set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_G \varphi$ iff $\Gamma \models_G \varphi$.

Gödel logic can also be seen as the axiomatic extension of intuitionistic propositional logic by the prelinearity axiom (A5). Its algebraic semantics is therefore given by the variety of prelinear Heyting algebras, also known as Gödel algebras. A Gödel algebra is thus a (bounded, integral, commutative) residuated lattice $\mathbf{A} = (A, \wedge, \vee, *, \Rightarrow, 0, 1)$ such that the monoidal operation $*$ coincides with the lattice meet $\wedge$, and such that the prelinearity equation $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ is satisfied.

Gödel logic can be expanded with the Baaz-Monteiro projection connective Δ while preserving the strong standard completeness [1]. Standard truth-evaluations of Gödel logic are extended adding the clause

$$e(\Delta\varphi) = \begin{cases} 1, \text{ if } e(\varphi) = 1 \\ 0, \text{ otherwise} \end{cases}$$

Note that despite φ is many-valued, $\Delta\varphi$ is a two-valued formula that is to be understood as a kind of precisification of φ. Axioms and rules of this new logic, denoted as G_Δ, are those of Gödel logic G plus the following axioms for Δ:

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$
(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$
(Δ3) $\Delta\varphi \to \varphi$
(Δ4) $\Delta\varphi \to \Delta\Delta\varphi$
(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

and the Δ-*necessitation rule*: from φ derive $\Delta\varphi$.

So defined, G_Δ keeps being algebraizable and its equivalent algebraic semantics is given by the variety of G_Δ-algebras. Automatically, G_Δ is strongly complete with respect to the equational class of G_Δ-algebras which, in fact, is a semilinear variety, and hence G_Δ is also complete with respect to the class of linearly-ordered G_Δ-algebras. Moreover, in [10] it is proved that G_Δ still enjoys strong standard completeness: $\Gamma \vdash_{G_\Delta} \varphi$ iff $\Gamma \models_{G_\Delta} \varphi$, for any set Γ and formula φ, where $\vdash_{G_\Delta}$ and $\models_{G_\Delta}$ stand respectively for the notions of proof and semantic consequence for G_Δ defined in the natural way as for G.

2.2 Multi-adjoint lattice logic

Multi-adjoint lattice logic (MLL) was introduced in [4] as a many-valued propositional logic framework related to multi-adjoint algebras. In this section, we provide a brief summary with the main notions related to the syntax and semantics of MLL. The language $\mathcal{L}_{\mathfrak{A}_{\mathrm{ML}}}$ of MLL (set of well-formed formulas) is built in the usual way from a countable set of propositional symbols Π together with the set of binary connective symbols $\{\to^d, \wedge, \vee, \wedge_1, \to_1, \dots, \wedge_n, \to_n\}$ and the constant $\perp$.

Definition 1 (MLL axiomatization). *Given the language $\mathcal{L}_{\mathfrak{A}_{ML}}$, the* multi-adjoint lattice logic *(MLL) is defined from the following axioms:*

L1. $(\varphi \wedge \psi) \to^d \varphi$
L2. $(\varphi \wedge \psi) \to^d \psi$

L3. $(\chi \to^d \varphi) \to^d ((\chi \to^d \psi) \to^d (\chi \to^d (\varphi \wedge \psi)))$
L4. $\varphi \to^d (\varphi \vee \psi)$
L5. $\psi \to^d (\varphi \vee \psi)$
L6. $(\varphi \to^d \chi) \to^d ((\psi \to^d \chi) \to^d ((\varphi \vee \psi) \to^d \chi)))$
L7. $\varphi \to^d \varphi$
L8. $(\varphi \to^d \psi) \to^d ((\psi \to^d \chi) \to^d (\varphi \to^d \chi))$
L9. $(\psi \to^d \chi) \to^d ((\varphi \to^d \psi) \to^d (\varphi \to^d \chi))$
L10. $\bot \to^d \varphi$
L11. $\varphi \to^d (\bot \to^d \bot)$
M1. $(\varphi \to^d (\psi \to_i \chi)) \to^d ((\varphi \wedge_i \psi) \to^d \chi)$
M2. $((\varphi \wedge_i \psi) \to^d \chi) \to^d (\varphi \to^d (\psi \to_i \chi))$
M3. $(\psi \to^d \chi) \to^d ((\varphi \wedge_i \psi) \to^d (\varphi \wedge_i \chi))$

and modus ponens for the implication symbol $\to^d$ *as the only inference rule.*

Now, as for the semantics of MLL, we recall the notion of truth-evaluation of formulas with respect to a given bounded order-right multi-adjoint lattice.

Definition 2. *Let* $(L, \preceq)$ *be a lattice. An* bounded order-right multi-adjoint lattice *is an algebra* $\mathbf{L} = (L, \inf, \sup, 0, 1, \&_1, \swarrow^1, \dots, \&_n, \swarrow^n)$ *where* $(L, \inf, \sup, 0, 1)$ *is a bounded lattice, and each* $(\&_i, \swarrow^i)$ *is an order-right adjoint pair with respect to* L*, that is, satisfying the corresponding monotonic properties, and the adjoint property:* $x \&_i y \preceq z$ *iff* $x \preceq z \swarrow^i y$*, for all* $x, y, z \in L$*. Moreover, if this algebra also includes the binary operator* $\swarrow^d$ *on* L*, defined as* $z \swarrow^d y$ *is 1, if* $y = \inf\{y, z\}$*, and* 0 *otherwise, for all* $y, z \in L$*, then we say that* $\mathbf{L} = (L, \inf, \sup, 0, 1, \swarrow^d, \&_1, \swarrow^1, \dots, \&_n, \swarrow^n)$ *is a* d-bounded order-right multi-adjoint lattice.

These algebraic structures provides an extra level of flexibility in those frameworks where they are used [6–8, 12–14]. For example, in formal concept analysis [6], considering several adjoint pairs allows to associate different degrees of preference on the attributes/objects of a database. Finally, the notion of evaluation is introduced as usual.

Definition 3. *Let* $\mathbf{L} = (L, \inf, \sup, 0, 1, \swarrow^d, \&_1, \swarrow^1, \dots, \&_n, \swarrow^n)$ *be a d-bounded order-right multi-adjoint lattice. An* $\mathbf{L}$-evaluation *of formulas is a mapping* $e\colon \mathcal{L}_{\mathfrak{A}_{ML}} \to L$ *defined inductively as usual from the propositional variables of the language, interpreting the connectives by the operations of* $\mathbf{L}$*.*

The axiomatic system of MLL is sound with respect to these algebraic structures, i.e. for any evaluation on these algebras, the axioms are evaluated to 1 and modus ponens preserves validity. More details can be found in [4, 5].

3 Multi-adjoint Gödel logic

In this section, we introduce a multi-adjoint logic over Gödel logic with Δ, denoted as MGL_Δ. The language $\mathcal{L}_{\mathfrak{A}_{\mathrm{MGL}_\Delta}}$ of MGL_Δ is obtained by explanding that of MLL with two new symbols $\to$ and Δ, related to the logic G_Δ. In $\mathcal{L}_{\mathfrak{A}_{\mathrm{MGL}_\Delta}}$, we have the following definable connectives: $\top := \bot \to \bot$, $\neg\varphi := \varphi \to \bot$, and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, for all well-formed formulas φ and ψ. Actually, in this new language we will see that the order implication connective $\to^d$ becomes definable from the two newly introduced symbols $\to$ and Δ, namely as $\varphi \to^d \psi := \Delta(\varphi \to \psi)$.

The algebraic semantics for MGL_Δ will be given by the class of MGL_Δ-algebras introduced next.

Definition 4. *A MGL_Δ-algebra is a tuple $\mathbf{A} = (A, \inf, \sup, \to, \Delta, 0, 1, \&_1, \swarrow^1, \ldots, \&_n, \swarrow^n)$, where $(A, \inf, \sup, 0, 1, \&_1, \swarrow^1, \ldots, \&_n, \swarrow^n)$ is a bounded order-right multi-adjoint lattice, and $(A, \to, \inf, \sup, \Delta, 0, 1)$ is a G_Δ-algebra.*

Notice that, in this extended algebraic framework, the order implication $\to^d$ is definable in linear structures. Specifically, we have the following result.

Proposition 1. *If $(A, \inf, \sup, 0, 1)$ is a bounded linearly-ordered lattice, then the order implication $\swarrow^d$ and the pair composed of the Gödel implication $\swarrow$ and Δ are inter-definable, that is*

- *$z \swarrow^d y = \Delta(z \swarrow y)$, for all $y, z \in A$.*
- *$z \swarrow y = \sup\{z \swarrow^d y, z\}$ and $\Delta(z) = z \swarrow^d 1$, for all $y, z \in A$.*

The truth-evaluations of MGL_Δ-formulas in a MGL_Δ-algebra $\mathbf{A}$ are also defined as usual by mappings $e\colon \mathcal{L}_{\mathfrak{A}_{\mathrm{MGL}_\Delta}} \to \mathbf{A}$ respecting the interpretation rules of MLL and G_Δ.

Definition 5. *Given a set of formulas $\Gamma \cup \{\varphi\}$, φ is a semantic consequence of Γ, denoted as $\Gamma \models_{MGL_\Delta} \varphi$, whenever, for any MGL_Δ-algebra $\mathbf{A}$ and evaluation e on $\mathbf{A}$, if $e(\psi) = 1$ for every $\psi \in \Gamma$, then $e(\varphi) = 1$ as well.*

From a syntactical point of view, the following is the Hilbert-style definition of the logic MGL_Δ.

Definition 6. *Axioms and rules of MGL_Δ are those of MLL plus the axioms (A4a), (A4b), (A5) and $(\Delta 1) - (\Delta 5)$ from G_Δ. The inference rules are modus ponens for $\to$, and the Δ-necessitation rule.*

The corresponding notion of proof for MGL_Δ, denoted as $\vdash_{MGL_\Delta}$, is the usual one from the above axioms and inference rules. Note that modus ponens for $\to^d$ is now a derivable rule. Namely, from $\varphi \to^d \psi := \Delta(\varphi \to \psi)$ it follows $\varphi \to \psi$, by Axiom $(\Delta 3)$, and hence, by modus ponens for $\to$, we have $\varphi, \varphi \to^d \psi \vdash_{\mathrm{MGL}_\Delta} \psi$.

It is important to note that MGL_Δ can be seen as a strengthening (expansion) of a particular case of the logic $\mathrm{MLL}_{\vee - \mathfrak{u}}$ defined in [4], in which an adjoint triple (the Gödel one) and a truth-stressing hedge (the Baaz-Monteiro operator) have been fixed.

If we denote by LL^+ the logic given by axioms $L1 - L11$ plus (A4a), (A4b), (A5) and modus ponens, we observe that LL^+ is equivalent to Gödel logic itself because axioms (A6) and (A7) are theorems of the logic, so MGL_Δ is in fact an axiomatic expansion of G_Δ. Moreover, it can be shown that MGL_Δ is an algebraizable logic.

Proposition 2. *MGL_Δ is algebraizable.*

This follows from observing that the new connectives are well-behaved with respect to the logical equivalence [4]. Therefore, since MGL_Δ is an algebraizable axiomatic expansion of G_Δ, by general results on algebraic logic, see e.g. [3], MGL_Δ keeps being semilinear. In other words, the variety of MGL_Δ-algebras is generated by its linearly-ordered members. This means that to check whether an equation is valid in the whole variety, it is enough to check its validity in all the linearly-ordered MGL_Δ-algebras. Or equivalently, if an equation does not hold in the variety then there is a linearly ordered MGL_Δ-algebra where the equation does not hold either.

These observations immediately lead to present the following soundness and completeness results for MGL_Δ with respect to the class of linearly-ordered MGL_Δ-algebras.

Theorem 1. *For any set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{MLG_\Delta} \varphi$ iff for any linearly-ordered MGL_Δ-algebra $\mathbf{A}$ and evaluation e on $\mathbf{A}$, if $e(\psi) = 1$ for every $\psi \in \Gamma$, then $e(\varphi) = 1$ as well.*

4 Conclusions

In this paper we have extended the multi-adjoint lattice logic MLL with Gödel logic implication and the Δ operator obtaining completeness results with respect to linear multi-adjoint lattices. Future work will be devoted to study completeness with respect to multi-adjoint structures on the real unit interval $[0, 1]$. Moreover, we will apply the approach presented in this paper to real-world scenarios, such as those in Digital Forensic, taking advantage of the authors' participation in the DigForASP network [9].

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