

A representation theorem for finite Gödel algebras with operators

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Abstract. In this paper we introduce and study *finite Gödel algebras with operators* (GAOs for short) and their dual frames. Taking into account that the category of finite Gödel algebras with homomorphisms is dually equivalent to the category of finite forests with order-preserving open maps, the dual relational frames of GAOs are *forest frames*: finite forests endowed with two binary (crisp) relations satisfying suitable properties. Our main result is a Jónsson-Tarski like representation theorem for these structures. In particular we show that every finite Gödel algebra with operators determines a unique forest frame whose set of subforests, endowed with suitably defined algebraic and modal operators, is a GAO isomorphic to the original one.

Keywords: Finite Gödel algebras; modal operators; finite forests; representation theorem.

1 Introduction

Fuzzy modal logic is an active and relatively recent area of research aimed at generalizing classical modal logic to the many-valued or fuzzy framework. This is usually done by considering a Kripke-style relational semantics in which both accessibility relations and evaluations of modal formulas (in each world) are allowed to take values in the real unit interval $[0, 1]$, instead of the classical two-valued set $\{0, 1\}$ (see [4, 5, 7] for instance).

In this contribution we put forward a new, algebraic-oriented perspective to the area of modal fuzzy logic, and in particular to Gödel modal logic by defining and studying the class of *finite Gödel algebras with operators* (GAOs for short). These structures are obtained by expanding the language of Gödel algebras (i.e. prelinear Heyting algebras) by means of two modal operators \diamond and \square equationally described by the same axioms used to define these operators in Boolean algebras with operators (BAOs), see [3].

Obviously, while in a BAO the operators \diamond and \square are inter-definable, this is not the general case in a GAO since the negation operator in a Gödel algebra

is not involutive. Hence, the equation $\diamond x = \neg \Box \neg x$ does not hold in general in a GAO.

In the same way the dual frames of BAOs are Kripke frames, the duality between finite Gödel algebras and finite forests (see [1]) leads us to introduce the dual structures of GAOs as triples $(\mathbf{F}, R_\diamond, R_\Box)$, where $\mathbf{F} = (F, \leq)$ is a finite forest, while R_\diamond and R_\Box are binary (crisp) relations on \mathbf{F} satisfying suitable conditions of (anti-)monotonicity in their first argument.

The main result of this paper is a Jónsson-Tarski like representation theorem for GAOs. In particular we will show how, starting from a Gödel algebra with operators $(\mathbf{A}, \diamond, \Box)$, one can uniquely define a forest frame $(\mathbf{F}, R_\diamond, R_\Box)$ such that $(\mathbf{A}, \diamond, \Box)$ is isomorphic to the GAO whose Gödel reduct is the algebra of subforests of \mathbf{F} and whose modal operators are defined from the binary relations R_\diamond and R_\Box .

Finally, we will discuss the effect of a stronger axiomatization for \diamond and \Box on the side of the corresponding forest frame. In particular we will see that the equations usually imposed on *positive modal algebras* [8, 6] allow for a simpler description of the forest frames needed in the representation theorem.

This paper is organized as follows. After this introduction, in the following Section 2 we will recall basic facts on finite Gödel algebras and finite forests. In Section 3 we will consider the case of Gödel algebras expanded by the modal operator \diamond , while Gödel algebras with a \Box operator will be studied in Section 4. Section 5 is dedicated to introduce Gödel algebras with both \diamond and \Box and also to discuss the effect of the stronger axiomatization for these modalities obtained by adding the equations of positive modal algebras. We will end this paper in Section 6 where we present our future work.

2 Finite Gödel algebras and forests

Gödel algebras, the algebraic semantics of infinite-valued Gödel logic [9], are idempotent, bounded, integral, commutative residuated lattices of the form $\mathbf{A} = (A, \wedge, \vee, \rightarrow, \perp, \top)$ satisfying the prelinearity equation: $(a \rightarrow b) \vee (b \rightarrow a) = \top$. In other words, Gödel algebras are *prelinear Heyting algebras*. If not unless specified, all algebras we will consider in this paper are finite.

Let \mathbf{A} be a Gödel algebra and denote by $F_{\mathbf{A}}$ the set of its prime filters, i.e., filters principally generated by the join-irreducible elements of \mathbf{A} . Unlike the case of boolean algebras, prime and maximal filters are not the same for Gödel algebras and indeed $F_{\mathbf{A}}$ can be ordered in a nontrivial way. In particular, if for $f_1, f_2 \in F_{\mathbf{A}}$ we define $f_1 \leq f_2$ iff (as prime filters) $f_1 \supseteq f_2$, $\mathbf{F}_{\mathbf{A}} = (F_{\mathbf{A}}, \leq)$ turns out to be a finite *forest*, i.e., a poset such that the downset of each element is totally ordered.

Finite forests play a crucial role in the theory of Gödel algebras. Indeed, let $\mathbf{F} = (F, \leq)$ be a finite forest, $S_{\mathbf{F}}$ be the set of all downward closed subsets of F (i.e., the *subforests* of \mathbf{F}) and consider the following operations on $S_{\mathbf{F}}$: for all $x, y \in F$,

1. $x \wedge y = x \cap y$ (the set-theoretic intersection);

2. $x \vee y = x \cup y$ (the set-theoretic union);
3. $x \rightarrow y = F \setminus \uparrow(x \setminus y)$ (where \setminus denotes the set-theoretical difference and for every $z \in F$, $\uparrow z = \{k \in F \mid k \geq z\}$).¹

The algebra $\mathbf{S}_F = (S_F, \wedge, \vee, \rightarrow, \emptyset, F)$ is a Gödel algebra [1, §4.2] and the following is a Stone-like representation theorem for these structures.

Lemma 1 ([1, Theorem 4.2.1]). *Every Gödel algebra \mathbf{A} is isomorphic to \mathbf{S}_{F_A} through the map $r : \mathbf{A} \rightarrow \mathbf{S}_{F_A}$*

$$r : a \in A \mapsto \{f \in F_A \mid a \in f\}.$$

Example 1. Let free_1 be the 1-generated free Gödel algebra (Fig. 1). Its prime filters, which are all principally generated as upsets of its join-irreducible elements, are $f_1 = \{y \in \text{free}_1 \mid y \geq x\} = \{x, x \vee \neg x, \neg\neg x, \top\}$, $f_2 = \{y \in \text{free}_1 \mid y \geq \neg x\} = \{\neg x, x \vee \neg x, \top\}$, and $f_3 = \{y \in \text{free}_1 \mid y \geq \neg\neg x\} = \{\neg\neg x, \top\}$. The forest $\mathbf{F}_{\text{free}_1}$ is obtained by ordering $\{f_1, f_2, f_3\}$ by reverse inclusion.

Let us consider the set $S_{\mathbf{F}_{\text{free}_1}}$ of subforests of $\mathbf{F}_{\text{free}_1}$:

$$S_{\mathbf{F}_{\text{free}_1}} = \{\emptyset, F_{\text{free}_1}, \{f_2\}, \{f_1\}, \{f_2, f_1\}, \{f_3, f_1\}\}$$

with operations $\wedge, \vee, \rightarrow$ as in (1-3) above. Lemma 1 shows that algebra $\mathbf{S}_{\mathbf{F}_{\text{free}_1}}$ is a Gödel algebra which is isomorphic to free_1 .

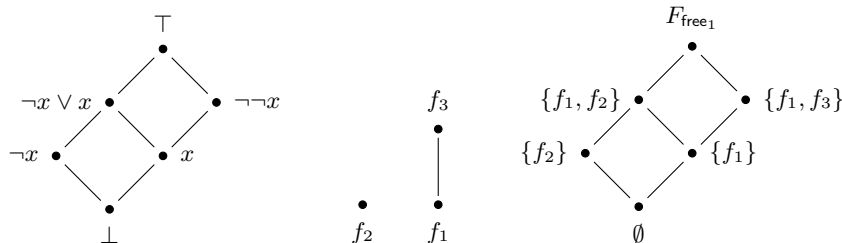


Fig. 1. From left to right: The Hasse diagram of the free Gödel algebra over one generator free_1 , the forest $\mathbf{F}_{\text{free}_1}$ of its prime filters, and the Hasse diagram of its isomorphic copy $\mathbf{S}_{\mathbf{F}_{\text{free}_1}}$

3 Gödel algebras with \diamond -operators

Definition 1. A \diamond -Gödel algebra is a pair (\mathbf{A}, \diamond) where \mathbf{A} is a Gödel algebra and $\diamond : A \rightarrow A$ satisfies the following equations:

¹ Without danger of confusion, and thanks to the following result, we will not distinguish the symbols of a Gödel algebra \mathbf{A} from those of \mathbf{S}_F

- ($\diamond 1$) $\diamond(\perp) = \perp$;
($\diamond 2$) $\diamond(a \vee b) = \diamond a \vee \diamond b$.

Definition 2. A \diamond -forest frame is a pair (\mathbf{F}, R) where $\mathbf{F} = (F, \leq)$ is a finite forest and $R \subseteq F \times F$ satisfies the following condition:

- (A) for all $x, y, z \in F$, if $y \leq x$ and $R(x, z)$, then $R(y, z)$ ².

For every \diamond -forest frame (\mathbf{F}, R) , let $\mathbf{S}_{\mathbf{F}}$ be defined as in the previous section and consider the map $\delta_R : \mathbf{S}_{\mathbf{F}} \rightarrow \mathbf{S}_{\mathbf{F}}$ such that, for every $a \in \mathbf{S}_{\mathbf{F}}$

$$\delta_R(a) = \{y \in F \mid \exists z \in a, R(y, z)\}. \quad (1)$$

Notice that, for all $a \in \mathbf{S}_{\mathbf{F}}$, $\delta_R(a) \in \mathbf{S}_{\mathbf{F}}$, i.e., $\delta_R(a)$ is a subforest of \mathbf{F} . Indeed if $x \in \delta_R(a)$ then there exists $z \in a$ such that $R(x, z)$. Let $y \leq x$ in \mathbf{F} . Then (A) of Definition 2 implies $R(y, z)$ as well, that is $y \in \delta_R(a)$ and hence $\delta_R(a)$ is downward closed. Further, the following properties hold.

Proposition 1. For every \diamond -forest frame (\mathbf{F}, R) let $\delta_R : \mathbf{S}_{\mathbf{F}} \rightarrow \mathbf{S}_{\mathbf{F}}$ be defined as in (1). Then:

1. $\delta_R(\perp) = \perp$;
2. For all $b \in \mathbf{S}_{\mathbf{F}}$, $\delta_R(b) = \bigcup \{\delta_R(a) \mid a \leq b \text{ and } a \text{ is join-irreducible}\}$.

Proof. (1) The bottom element of $\mathbf{S}_{\mathbf{F}}$ is the empty forest, whence $\emptyset = \{y \in F \mid \exists z \in \emptyset, R(y, z)\} = \delta_R(\perp)$.

(2) The claim is trivial if b is join irreducible. Thus, let $b = a_1 \vee \dots \vee a_m$ with the a_i 's being join irreducible. Therefore, $\delta_R(b) = \{y \in F \mid \exists z \in b, R(y, z)\} = \{y \in F \mid \exists z \in a_1 \vee \dots \vee a_m, R(y, z)\} = \{y \in F \mid \exists z \in a_1 \cup \dots \cup a_m, R(y, z)\} = \bigcup_{i=1}^m (\{y \in F \mid \exists z \in a_i, R(y, z)\}) = \bigcup_{i=1}^m \delta_R(a_i)$. \square

Lemma 2. For each \diamond -forest frame (\mathbf{F}, R) , $(\mathbf{S}_{\mathbf{F}}, \delta_R)$ is a \diamond -Gödel algebra.

Proof. From Lemma 1, $\mathbf{S}_{\mathbf{F}}$ is a Gödel algebra. Equation ($\diamond 1$) holds because of Proposition 1(1). Further, if $a, b \in \mathbf{S}_{\mathbf{F}}$, by Proposition 1(2), $\delta_R(a \vee b) = \delta_R(a) \cup \delta_R(b) = \delta_R(a) \vee \delta_R(b)$ by definition of δ_R . Thus, δ_R satisfies ($\diamond 2$). \square

Now, let (\mathbf{A}, \diamond) be a \diamond -Gödel algebra, let $\mathbf{F}_{\mathbf{A}}$ be as in Section 2 and define $Q_{\diamond} \subseteq F_{\mathbf{A}} \times F_{\mathbf{A}}$ as follows: for all $f_1, f_2 \in F_{\mathbf{A}}$,

$$Q_{\diamond}(f_1, f_2) \text{ iff } \diamond(f_2) \subseteq f_1, \quad (2)$$

where, for every filter f , $\diamond(f) = \{\diamond x \mid x \in f\}$. Then the following holds.

Lemma 3. For each \diamond -Gödel algebra (\mathbf{A}, \diamond) , $(\mathbf{F}_{\mathbf{A}}, Q_{\diamond})$ is a \diamond -forest frame.

Proof. It is enough to prove that the condition (A) of Definition 2 holds. Let $f_1, f_2, f_3 \in F_{\mathbf{A}}$ and assume $Q_{\diamond}(f_1, f_3)$ (i.e., $\diamond(f_3) \subseteq f_1$) and $f_1 \geq f_2$, meaning that, as prime filters, $f_1 \subseteq f_2$. Then, $\diamond(f_3) \subseteq f_1 \subseteq f_2$ and hence $Q_{\diamond}(f_2, f_3)$.

² Along this paper we will adopt the notation $R(x, y)$ to denote that the pair (x, y) belongs to the relation R .

Now, our aim is to extend the isomorphism r of Lemma 1 to the case of \diamond -Gödel algebras. Let hence (\mathbf{A}, \diamond) be a \diamond -Gödel algebra and define, for every $a \in A$,

$$r(\diamond(a)) = \{f \in F_{\mathbf{A}} \mid \diamond(a) \in f\}. \quad (3)$$

Theorem 1. *For every \diamond -Gödel algebra (\mathbf{A}, \diamond) , the map $r : (\mathbf{A}, \diamond) \rightarrow (\mathbf{S}_{F_{\mathbf{A}}}, \delta_{Q_{\diamond}})$ is an isomorphism. In particular, for all $a \in A$,*

$$r(\diamond(a)) = \delta_{Q_{\diamond}}(r(a)). \quad (4)$$

Proof. We proved in Lemma 1 that $\mathbf{S}_{F_{\mathbf{A}}}$ is a Gödel algebra and the map $r : \mathbf{A} \rightarrow \mathbf{S}_{F_{\mathbf{A}}}$ is a Gödel isomorphism. Thus, it remains to show that (4) holds. First of all notice that it is sufficient to prove it for the case of a being a join-irreducible element of \mathbf{A} . Indeed, assume that (4) holds for join irreducible elements and let b be not join irreducible. Then b can be displayed as $b = a_1 \vee \dots \vee a_k$, where the a_i 's are join irreducible. By ($\diamond 2$), $\diamond(b) = \diamond(a_1) \vee \dots \vee \diamond(a_k)$. Therefore, since r is a Gödel algebra isomorphism,

$$r(\diamond(b)) = r(\diamond(a_1)) \vee \dots \vee r(\diamond(a_k)).$$

By assumption, $r(\diamond a_i) = \delta_{Q_{\diamond}}(r(a_i))$ for all $i = 1, \dots, k$. Thus, $r(\diamond(b)) = \delta_{Q_{\diamond}}(a_1) \vee \dots \vee \delta_{Q_{\diamond}}(a_k)$ which equals $\delta_{Q_{\diamond}}(b)$ by Proposition 1(2).

Let hence a be join irreducible. By Lemma 1, we have:

$$\begin{aligned} \delta_{Q_{\diamond}}(r(a)) &= \{f \in F_{\mathbf{A}} \mid \exists g \in r(a), Q_{\diamond}(f, g)\} \\ &= \{f \in F_{\mathbf{A}} \mid \exists g \in F_{\mathbf{A}}, (a \in g \ \& \ Q_{\diamond}(f, g))\} \\ &= \{f \in F_{\mathbf{A}} \mid \exists g \in F_{\mathbf{A}}, (a \in g \ \& \ \diamond(g) \subseteq f)\} \end{aligned}$$

Therefore, if $f \in \delta_{Q_{\diamond}}(r(a))$, $\diamond(a) \in f$ and hence $f \in r(\diamond(a))$.

To prove the other inclusion we have to show that if $f' \in r(\diamond(a))$, there exists an $f \in F_{\mathbf{A}}$ such that $a \in f$ and $\diamond(f) \subseteq f'$. Since a is join irreducible, the filter $f_a = \{b \in A \mid b \geq a\}$ is prime. Let us prove that $\diamond(f_a) \subseteq f'$.

Claim. $\diamond(f_a) \subseteq f_{\diamond(a)} = \{x \in A \mid x \geq \diamond(a)\}$.

As a matter of fact, if $z \in \diamond(f_a)$, then there exists $b \geq a$ such that $z = \diamond(b)$. Since \diamond is monotone, $\diamond(b) \geq \diamond(a)$, whence $z = \diamond(b) \in f_{\diamond(a)}$.

Claim. For all $f' \in r(\diamond(a))$, $f_{\diamond(a)} \subseteq f'$.

Indeed, if $x \in f_{\diamond(a)}$, then $x \geq \diamond(a)$ and hence $x \in f'$ because $\diamond(a) \in f'$ and f' is upward closed.

By the above claims, for all $f' \in r(\diamond(a))$, $\diamond(f_a) \subseteq f'$, whence

$$r(\diamond(a)) \subseteq \delta_{Q_{\diamond}}(r(a)).$$

Thus, for all a , $r(\diamond(a)) = \delta_{Q_{\diamond}}(r(a))$ which settles the claim. \square

Example 2. Let \mathbf{free}_1 be as in Example 1 and let $\diamond : \mathbf{free}_1 \rightarrow \mathbf{free}_1$ be the following map:

$$\begin{aligned}\diamond(\perp) &= \perp; \diamond(x) = \neg x; \diamond(\neg x) = \neg x \vee x; \diamond(\neg x \vee x) = \neg x \vee x; \\ \diamond(\neg\neg x) &= \top; \diamond(\top) = \top.\end{aligned}$$

It is easy to check that \diamond satisfies $(\diamond 1)$ and $(\diamond 2)$ of Definition 1 and hence $(\mathbf{free}_1, \diamond)$ is a \diamond -Gödel algebra.

Let $\mathbf{F}_{\mathbf{free}_1}$ be the dual forest of \mathbf{free}_1 as in Example 1 and let us compute Q_\diamond according to (2). First: $\diamond(f_1) = \{\neg x, x \vee \neg x, \top\}$; $\diamond(f_2) = \{x \vee \neg x, \top\}$ and $\diamond(f_3) = \{\top\}$. Therefore, (see Figure 2)

$$Q_\diamond = \{(f_1, f_2), (f_1, f_3), (f_2, f_2), (f_2, f_1), (f_2, f_3), (f_3, f_3)\}.$$

The relation Q_\diamond satisfies the property (A) of Definition 2. Indeed, $f_1 \leq f_3$, and for all $f \in F_{\mathbf{free}_1}$, if $Q_\diamond(f_3, f)$ then $Q_\diamond(f_1, f)$. Therefore $(F_{\mathbf{free}_1}, Q_\diamond)$ is a \diamond -forest frame.

Finally, let $\mathbf{S}_{\mathbf{F}_{\mathbf{free}_1}}$ be the isomorphic copy of \mathbf{free}_1 as in Example 1 and let $\delta_{Q_\diamond} : \mathbf{S}_{\mathbf{F}_{\mathbf{free}_1}} \rightarrow \mathbf{S}_{\mathbf{F}_{\mathbf{free}_1}}$ be as in (1):

$$\begin{aligned}\delta_{Q_\diamond}(\emptyset) &= \{f \in F_{\mathbf{free}_1} \mid \exists g \in \emptyset, Q_\diamond(f, g)\} = \emptyset; \\ \delta_{Q_\diamond}(\{f_1\}) &= \{f \in F_{\mathbf{free}_1} \mid \exists g \in \{f_1\}, Q_\diamond(f, g)\} = \{f_2\}; \\ \delta_{Q_\diamond}(\{f_2\}) &= \{f \in F_{\mathbf{free}_1} \mid \exists g \in \{f_2\}, Q_\diamond(f, g)\} = \{f_1, f_2\}; \\ \delta_{Q_\diamond}(\{f_1, f_2\}) &= \{f \in F_{\mathbf{free}_1} \mid \exists g \in \{f_1, f_2\}, Q_\diamond(f, g)\} = \{f_1, f_2\}; \\ \delta_{Q_\diamond}(\{f_1, f_3\}) &= \{f \in F_{\mathbf{free}_1} \mid \exists g \in \{f_1, f_3\}, Q_\diamond(f, g)\} = \{f_1, f_2, f_3\} = F_{\mathbf{free}_1}; \\ \delta_{Q_\diamond}(F_{\mathbf{free}_1}) &= \{f \in F_{\mathbf{free}_1} \mid \exists g \in F_{\mathbf{free}_1}, Q_\diamond(f, g)\} = F_{\mathbf{free}_1}.\end{aligned}$$

Therefore, $(\mathbf{free}_1, \diamond)$ and $(\mathbf{S}_{\mathbf{F}_{\mathbf{free}_1}}, \delta_{Q_\diamond})$ are isomorphic \diamond -Gödel algebras.

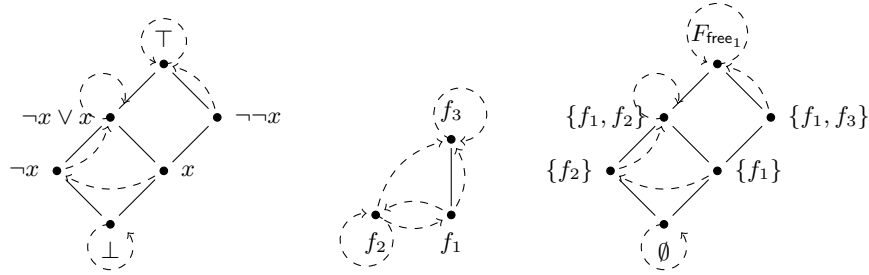


Fig. 2. From left to right: The Hasse diagram of the free Gödel algebra over one generator \mathbf{free}_1 and a \diamond operator (dotted arrows); the forest $\mathbf{F}_{\mathbf{free}_1}$ of its prime filters and the relation Q_\diamond (dotted arrows); the Hasse diagram of its isomorphic copy $\mathbf{S}_{\mathbf{F}_{\mathbf{free}_1}}$ endowed with the operator δ_{Q_\diamond} (dotted arrows).

4 Gödel algebras with \Box -operators

Definition 3. A \Box -Gödel algebra is a pair (\mathbf{A}, \Box) such that \mathbf{A} is a Gödel algebra and $\Box : A \rightarrow A$ satisfies the following equalities:

- ($\Box 1$) $\Box(\top) = \top$;
- ($\Box 2$) $\Box(a \wedge b) = \Box a \wedge \Box b$.

Definition 4. A \Box -forest frame is a pair (\mathbf{F}, R) where $\mathbf{F} = (F, \leq)$ is a finite forest and $R \subseteq F \times F$ satisfies the following condition:

- (M) for all $x, y, z \in F$, if $x \leq y$ and $R(x, z)$, then $R(y, z)$.

For every \Box -forest frame (\mathbf{F}, R) , let $\beta_R : S_{\mathbf{F}} \rightarrow S_{\mathbf{F}}$ be defined as follows: for all $a \in S_{\mathbf{F}}$,

$$\beta_R(a) = \{y \in F \mid \forall z \in F, (R(y, z) \Rightarrow z \in a)\}. \quad (5)$$

For all $a \in S_{\mathbf{F}}$, $\beta_R(a)$ is a subforest of \mathbf{F} . Indeed, if $x \in \beta_R(a)$ then $\forall z \in F$, $(R(x, z) \Rightarrow z \in a)$. Let $y \leq x$. Thus, for all $z \in F$ either $R(y, z)$ is false (and in this case the condition $R(y, z) \Rightarrow z \in a$ is trivially true), or $R(y, z)$ is true in which case $R(x, z)$ is true as well, because of (M), and hence $z \in a$. Thus $y \in \beta_R(a)$.

Proposition 2. The following properties hold:

1. $\beta_R(\top) = \top$;
2. For all $b \in A_{\mathbf{F}}$, $\beta_R(b) = \bigcup(\{\beta_R(a) \mid a \leq b \text{ and } a \text{ is join irreducible}\})$.

Proof. (1) Recall from Section 2 that the top element of $S_{\mathbf{F}}$ is F . Thus, $\beta_R(\top) = \beta_R(F) = \{y \in F \mid \forall z \in F, (R(y, z) \Rightarrow z \in F)\}$. Obviously, the condition $(R(y, z) \Rightarrow z \in F)$ is true for all $z \in F$ and hence $\beta_R(F) = F$.

(2) Skipping the trivial case in which b is join irreducible, let $b = a_1 \vee \dots \vee a_m$ with the a_i 's join irreducible. Remember that in classical logic, for every finite k , $x \Rightarrow (\exists i \in \{1, \dots, k\}(y_i)) = \exists i \in \{1, \dots, k\} (x \Rightarrow y_i)$, hence

$$\begin{aligned} \beta_R(b) &= \{y \in F \mid \forall z \in F, (R(y, z) \Rightarrow z \in b)\} \\ &= \{y \in F \mid \forall z \in F, (R(y, z) \Rightarrow z \in \bigvee_{i=1}^m a_i)\} \\ &= \{y \in F \mid \forall z \in F, (R(y, z) \Rightarrow (\exists i \in \{1, \dots, m\} (z \in a_i)))\} \\ &= \{y \in F \mid \forall z \in F, \exists i \in \{1, \dots, m\} (R(y, z) \Rightarrow z \in a_i)\} \\ &= \bigcup_{i=1}^m \{y \in F \mid \forall z \in F, (R(y, z) \Rightarrow z \in a_i)\} \\ &= \bigcup_{i=1}^m \beta_R(a_i). \end{aligned}$$

The claim is hence settled. \square

Lemma 4. For every \Box -forest frame (\mathbf{F}, R) , $(S_{\mathbf{F}}, \beta_R)$ is a \Box -Gödel algebra.

Proof. We already showed that $\beta_R(\top) = \top$. If $a, b \in S_{\mathbf{F}}$ and recalling that, as subforests of F , $a \wedge b = a \cap b$, one has

$$\begin{aligned} \beta_R(a \wedge b) &= \{y \in F \mid \forall z \in F, (R(y, z) \Rightarrow z \in a \wedge b)\} \\ &= \{y \in F \mid \forall z \in F, (R(y, z) \Rightarrow z \in a \cap b)\} \\ &= \{y \in F \mid \forall z \in F, (R(y, z) \Rightarrow z \in a)\} \cap \\ &\quad \{y \in F \mid \forall z \in F, (R(y, z) \Rightarrow z \in b)\} \\ &= \beta_R(a) \cap \beta_R(b) \\ &= \beta_R(a) \wedge \beta_R(b). \end{aligned}$$

□

Let (\mathbf{A}, \square) be a \square -Gödel algebra and define Q_{\square} on $F_{\mathbf{A}} \times F_{\mathbf{A}}$ as follows: for each $f_1, f_2 \in F_{\mathbf{A}}$,

$$Q_{\square}(f_1, f_2) \text{ iff } \square^{-1}(f_1) \subseteq f_2, \quad (6)$$

where, for every filter f , $\square^{-1}(f) = \{x \in A \mid \square(x) \in f\}$.

Lemma 5. *For every \square -Gödel algebra (\mathbf{A}, \square) , $(\mathbf{F}_{\mathbf{A}}, Q_{\square})$ is a \square -forest frame.*

Proof. Let $f_1, f_2, f_3 \in F_{\mathbf{A}}$. If $f_1 \leq f_2$ in the order of $\mathbf{F}_{\mathbf{A}}$, then $f_1 \supseteq f_2$ as prime filters, whence if $\square^{-1}(f_1) \subseteq f_3$ then $\square^{-1}(f_2) \subseteq f_3$. Therefore, if $Q_{\square}(f_1, f_3)$, then $Q_{\square}(f_2, f_3)$ which settles the claim.

The following result is the analogous of Theorem 1 in the case of \square -Gödel algebras where r is the map of Lemma 1 which extends to all elements of a \square -Gödel algebra (\mathbf{A}, \square) by the following stipulation:

$$r(\square(a)) = \{f \in F_{\mathbf{A}} \mid \square(a) \in f\}. \quad (7)$$

Theorem 2. *For every finite G_{\square} -algebra (\mathbf{A}, \square) , the map $r : (\mathbf{A}, \square) \rightarrow (\mathbf{S}_{\mathbf{F}_{\mathbf{A}}}, \beta_{Q_{\square}})$ defined as above is an isomorphism. In particular, for all $a \in A$,*

$$r(\square(a)) = \beta_{Q_{\square}}(r(a)). \quad (8)$$

Proof. Let us start proving that for all $a \in A$, $\beta_{Q_{\square}}(r(a)) \subseteq r(\square(a))$. By definition,

$$\begin{aligned} \beta_{Q_{\square}}(r(a)) &= \{f \in F_{\mathbf{A}} \mid \forall g \in F_{\mathbf{A}} (Q_{\square}(f, g) \Rightarrow g \in r(a))\} \\ &= \{f \in F_{\mathbf{A}} \mid \forall g \in F_{\mathbf{A}} (\square^{-1}(f) \subseteq g \Rightarrow a \in g)\}. \end{aligned}$$

Let $f \in \beta_{Q_{\square}}(r(a))$ and assume, by way of contradiction, that $f \not\subseteq r(\square(a))$, that is to say, $a \notin \square^{-1}(f)$. Notice that this assumption forces $a \neq \top$.

Claim. $\square^{-1}(f)$ is a filter of \mathbf{A} .

As a matter of facts, $\top \in \square^{-1}(f)$ because $\top \in f$ and $\square\top = \top$. Further, if $a, b \in \square^{-1}(f)$, then $\square a \in f$ and $\square b \in f$, whence $\square(a) \wedge \square(b) \in f$ since f is a filter. Hence $\square(a \wedge b) \in f$ by $(\square 2)$ showing that $\square^{-1}(f)$ is \wedge -closed. Finally, if $a \in \square^{-1}(f)$ and $b \geq a$, then by the monotonicity of \square , $\square(b) \geq \square(a)$, hence $\square(b) \in f$ because f is upward closed. Therefore, $\square^{-1}(f)$ is a filter of \mathbf{A} .

Going back to the proof of Theorem 2, if $a \in \Box^{-1}(f)$ and since $a \neq \top$, by [9, Lemma 2.3.15], there exists a prime filter p of \mathbf{A} such that $p \supseteq \Box^{-1}(f)$ and $a \notin p$. On the other hand, $Q_\Box(f, p)$ because p extends $\Box^{-1}(f)$ and $a \notin p$. Thus, $f \notin \beta_{Q_\Box}(r(a))$ and a contradiction has been reached.

For the other inclusion, we have to prove that if $\Box(a) \in f$, then for all $g \in F_{\mathbf{A}}$, $Q_\Box(f, g) \Rightarrow a \in g$. If $\Box(a) \in f$, then $a \in \Box^{-1}(f)$. Therefore, for all $g \in F_{\mathbf{A}}$, if $Q_\Box(f, g)$, then $\Box^{-1}(f) \subseteq g$ and hence $a \in g$ which settles the claim. \square

Example 3. As in the previous Examples 1 and 2 let \mathbf{free}_1 the free, 1-generated Gödel algebra and consider the map $\Box : \mathbf{free}_1 \rightarrow \mathbf{free}_1$ defined as follows (dashed arrows in the leftmost picture of Figure 3):

$$\Box(\perp) = \perp; \Box(x) = x; \Box(\neg x) = \perp; \Box(\neg x \vee x) = x; \Box(\neg \neg x) = \neg \neg x; \Box(\top) = \top.$$

That operation makes (\mathbf{free}_1, \Box) into a \Box -Gödel algebra.

For the reader convenience, let us compute $\Box^{-1}(f)$ (for $f \in F_{\mathbf{free}_1}$): Adopting the same notation of the previous examples,

$$\Box^{-1}(f_1) = \{\top\}; \Box^{-1}(f_2) = f_2; \Box^{-1}(f_3) = f_3.$$

Therefore, by (6), $Q_\Box \subseteq F_{\mathbf{free}_1} \times F_{\mathbf{free}_1}$ is the following relation (check Figure 3, central picture):

$$Q_\Box = \{(f_1, f_1), (f_2, f_2), (f_3, f_3), (f_2, f_1), (f_2, f_3), (f_3, f_1)\}.$$

Notice that $(F_{\mathbf{free}_1}, Q_\Box)$ is a \Box -forest frame. Indeed, $f_1 \leq f_3$ and for all $f \in F_{\mathbf{free}_1}$, $Q_\Box(f_1, f) \leq Q_\Box(f_3, f)$.

Finally, let $\mathbf{S}_{\mathbf{free}_1}$ be the isomorphic copy of \mathbf{free}_1 as in Example 1 and let us define β_{Q_\Box} as above, i.e., for all $a \in S_{\mathbf{free}_1}$,

$$\beta_{Q_\Box}(a) = \{f \in F_{\mathbf{free}_1} \mid \text{for all } g \in F_{\mathbf{free}_1}, \text{ if } Q_\Box(f, g) \text{ then } g \in a\}.$$

The computation is tedious and we will only show $\beta_{Q_\Box}(\{f_1, f_2\})$. The remaining cases are left to the reader.

$$\beta_{Q_\Box}(\{f_1, f_2\}) = \{f \in F_{\mathbf{free}_1} \mid \text{for all } g \in F_{\mathbf{free}_1}, \text{ if } Q_\Box(f, g) \text{ then } g \in \{f_1, f_2\}\}.$$

Let us enter a case distinction:

- $f_1 \in \beta_{Q_\Box}(\{f_1, f_2\})$. Let g be arbitrary in $F_{\mathbf{free}_1}$. In particular, if $g = f_1$, then $Q_\Box(f_1, f_1)$ and $f_1 \in \{f_1, f_2\}$; if $g = f_2$, we have $Q_\Box(f_1, f_2)$ and again $f_2 \in \{f_1, f_2\}$; if $g = f_3$, $(f_1, f_3) \notin Q_\Box$ whence we conclude that $f_1 \in \beta_{Q_\Box}(\{f_1, f_2\})$.
- $f_2 \in \beta_{Q_\Box}(\{f_1, f_2\})$. Again we distinguish the following cases: for $g = f_1$ or $g = f_2$, $Q_\Box(f_2, g)$ and $g \in \{f_1, f_2\}$; if $g = f_3$, $Q_\Box(f_2, f_3)$ but $f_3 \notin \{f_1, f_2\}$, whence $f_2 \notin \beta_{Q_\Box}(\{f_1, f_2\})$.
- $f_3 \in \beta_{Q_\Box}(\{f_1, f_2\})$. Notice immediately that for $g = f_3$ one has $Q_\Box(f_3, f_3)$ but $f_3 \notin \{f_1, f_2\}$, whence $f_3 \notin \beta_{Q_\Box}(\{f_1, f_2\})$.

Therefore, $\beta_{Q_\Box}(\{f_1, f_2\}) = \{f_1\}$ (see Figure 3, dashed arrows in the rightmost picture, for the remaining cases).

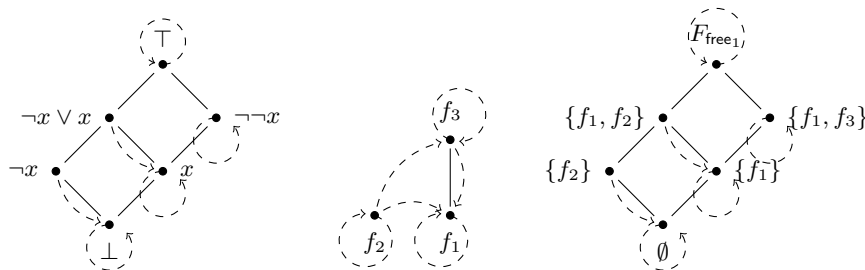


Fig. 3. From left to right: The Hasse diagram of the free Gödel algebra over one generator free_1 and a \square operator (dotted arrows); the forest $\mathbf{F}_{\text{free}_1}$ of its prime filters and the relation Q_\square (dotted arrows); the Hasse diagram of its isomorphic copy $\mathbf{S}_{\mathbf{F}_{\text{free}_1}}$ endowed with the operator β_{Q_\square} (dotted arrows).

5 Gödel algebras with \diamond and \square operators

The notions of results provided in the previous sections immediately give us the following

Definition 5. A Gödel algebra with operators (GAO for short) is a triple $(\mathbf{A}, \diamond, \square)$ where \mathbf{A} is a Gödel algebra, \diamond and \square are unary operators of A which satisfy the equations $(\diamond 1)$ - $(\diamond 2)$ and $(\square 1)$ - $(\square 2)$ of Definitions 1 and 3 respectively.

Let us observe that the equations for \diamond and \square are *minimal* in the sense that $(\diamond 1)$ - $(\diamond 2)$ and $(\square 1)$ - $(\square 2)$ are the weakest requirements we may ask the modal operators to satisfy, taking into account that, since the negation operator in Gödel algebras is not involutive, \diamond and \square are not inter-definable as in the classical setting. A similar remark concerning the minimality of those equations, but in the more general setting of *Heyting algebras with operators*, can be found in [10].

This remark leads us to the following notion of *frame* for GAOs which, not surprisingly, includes both that of \diamond - and \square -forest frame.

Definition 6. A forest frame is a triple $(\mathbf{F}, R_\diamond, R_\square)$ such that (\mathbf{F}, R_\diamond) is a \diamond -forest frame and (\mathbf{F}, R_\square) is a \square -forest frame.

Given a GAO $(\mathbf{A}, \diamond, \square)$ and following exactly the same constructions and results described in the previous Sections 3 and 4, it is immediate to show that, indeed, $(\mathbf{F}_\mathbf{A}, Q_\diamond, Q_\square)$ is a forest frame and, vice versa, given any forest frame $(\mathbf{F}, R_\diamond, R_\square)$, the algebra $(\mathbf{S}_\mathbf{F}, \delta_{R_\diamond}, \beta_{R_\square})$ is a GAO. The following result, which is an immediate consequence of Theorem 1 and Theorem 2, is a Jónsson-Tarski like representation for GAOs.

Theorem 3. Let $(\mathbf{A}, \diamond, \square)$ be a GAO. The map $r : (\mathbf{A}, \diamond, \square) \rightarrow (\mathbf{A}_{\mathbf{F}_\mathbf{A}}, \delta_{Q_\diamond}, \beta_{Q_\square})$, where δ_{Q_\diamond} and β_{Q_\square} are defined by equations (3) and (7), is an isomorphism. In particular, for all $a \in A$,

$$r(\diamond(a)) = \delta_{Q_\diamond}(r(a)) \text{ and } r(\square(a)) = \beta_{Q_\square}(r(a)). \quad (9)$$

Following [8, 6], let us consider the following equations:

$$(D1) \quad \Box(a \vee b) \leq \Box a \vee \Diamond b;$$

$$(D2) \quad \Box a \wedge \Diamond b \leq \Diamond(a \wedge b).$$

For every Gödel algebra \mathbf{A} , let us denote by \mathbf{A}^- its $\{\rightarrow, \neg\}$ -free reduct. Then, if $(\mathbf{A}, \Diamond, \Box)$ satisfies (D1) and (D2), $(\mathbf{A}^-, \Diamond, \Box)$ is a *positive modal algebra* in the sense of [8, 6]. Since the set of prime filters of \mathbf{A} and that of \mathbf{A}^- coincide, $\mathbf{F}_{\mathbf{A}^-} = \mathbf{F}_{\mathbf{A}}$ and, following [6], let us define $R_{\mathbf{A}} \subseteq F_{\mathbf{A}^-} \times F_{\mathbf{A}^-}$ as follows: for all $f_1, f_2 \in F_{\mathbf{A}}$,

$$R_{\mathbf{A}}(f_1, f_2) \text{ iff } \Box^{-1}(f_1) \subseteq f_2 \subseteq \Diamond^{-1}(f_1).$$

Observing that $f_2 \subseteq \Diamond^{-1}(f_1)$ iff $\Diamond(f_2) \subseteq f_1$, by [6, Lemma 2.1(1)], we have that $R_{\mathbf{A}} = Q_{\Diamond} \cap Q_{\Box}$, where Q_{\Diamond} and Q_{\Box} are defined as in (2) and (6) respectively.

Now, let $\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}}$ be the Gödel algebra of subforests of $\mathbf{F}_{\mathbf{A}^-}$ and define $\delta_{R_{\mathbf{A}}}$ and $\beta_{R_{\mathbf{A}}}$ on $\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}}$ by (1) and (5) respectively. Then the following is an immediate consequence of [6, Theorem 2.2] (see also [8, Theorem 8.1]).

Proposition 3. *Let $(\mathbf{A}, \Diamond, \Box)$ be a GAO which satisfies (D1) and (D2). Then its $\{\rightarrow, \neg\}$ -free reduct $(\mathbf{A}^-, \Diamond, \Box)$ and the positive algebra $((\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}})^-, \delta_{R_{\mathbf{A}}}, \beta_{R_{\mathbf{A}}})$ are isomorphic (as positive modal algebras).*

Clearly, $\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}} = \mathbf{S}_{\mathbf{F}_{\mathbf{A}}}$. Now, it is not difficult to extend the above result to GAOs satisfying (D1) and (D2) by expanding the algebra $((\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}})^-, \delta_{R_{\mathbf{A}}}, \beta_{R_{\mathbf{A}}})$ by the operator \rightarrow defined as in Section 2: for all $x, y \in \mathbf{S}_{\mathbf{F}_{\mathbf{A}}}$,

$$x \rightarrow y = F_{\mathbf{A}} \setminus \uparrow(x \setminus y).$$

Then, $(\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}})^-$ plus \rightarrow and \neg (defined as usual by $\neg x = x \rightarrow \emptyset$) is a Gödel algebra isomorphic to $\mathbf{S}_{\mathbf{F}_{\mathbf{A}}}$. Thus, the following holds.

Theorem 4. *Every GAO $(\mathbf{A}, \Diamond, \Box)$ satisfying (D1) and (D2) is isomorphic to $(\mathbf{S}_{\mathbf{F}_{\mathbf{A}}}, \delta_{R_{\mathbf{A}}}, \beta_{R_{\mathbf{A}}})$ (as Gödel algebras with operators).*

6 Conclusion and future work

In the present paper we have introduced finite Gödel algebras with modal operators and their dual forest frames. Our main result is a Jónsson-Tarski like representation theorem for these structures. Further, we have introduced a proper subclass of Gödel algebras with operators, and we have shown for them a simplified representation which uses, on the dual side of forest frames, only one accessibility relation. It is important to notice that, in contrast with [5] where the authors consider Kripke frames for Gödel modal logic with a unique $[0, 1]$ -valued accessibility relation, the dual frames of our Gödel algebras with operators have two crisp accessibility relations. This latter observation offers, in our opinion, a fresh new perspective on the semantic approach to fuzzy modal logics which deserves to be further investigated.

As for future work we plan to address the following questions:

- (1) To extend the results of this paper to the whole class of Gödel algebras. In this direction we will investigate an extension of Theorem 3 for *general* Gödel algebras with operators. In order to achieve this goal we will take into account that the prime spectrum of a Gödel algebra forms an Esakia space whose underline poset is a forest (see [11, Theorem 2.4]).
- (2) The whole class of Gödel algebras with operators forms a variety which determines the equivalent algebraic semantics of a Gödel modal logic. This logic, denoted by $\mathbf{G}_{\Box\Diamond}$, can be regarded as the axiomatic extension of intuitionistic modal logic $\mathbf{IntK}_{\Box\Diamond}$ [12] by the prelinearity axiom $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$. Our main plans in this direction are to show that $\mathbf{G}_{\Box\Diamond}$ has the finite model property and to compare $\mathbf{G}_{\Box\Diamond}$ with the other approaches to Gödel modal logic existing in the literature, in particular with that of [5]. In this paper the authors introduce a logic with both \Box and \Diamond operators, stronger than $\mathbf{G}_{\Box\Diamond}^3$, that is shown to be complete with respect to the class of Kripke models over the standard Gödel algebra (on the unit real interval $[0, 1]$) where both the accessibility relation and formulas are evaluated on $[0, 1]$.
- (3) Finite Nilpotent Minimum (NM) algebras with (or without) a negation fixpoint are dually equivalent to the category of finite forests (and hence categorically equivalent to finite Gödel algebras) [1, Proposition 4.5.4 and §4.5] and [2, Corollary 4.10]. In particular, the connected (disconnected, respectively) rotation of the $\{\perp\}$ -free reduct of any finite, directly indecomposable Gödel algebra \mathbf{A} is a finite, directly indecomposable, NM-algebra with (without) negation fixpoint and each directly indecomposable NM-algebra with (without) negation fixpoint arises in this way (see [1, §4.5] and references therein). Taking into account this structural description, we plan to extend the analysis reported in this paper to the classes of NM-algebras with, or without, negation fixpoint.

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³ In particular it includes Fisher-Servi connecting axioms $\Diamond(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Diamond\psi)$ and $(\Diamond\varphi \rightarrow \Box\varphi) \rightarrow \Box(\varphi \rightarrow \psi)$.

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