# Coherence in the aggregate: a betting method for belief functions on many-valued events

Tommaso Flaminio[1]

[1] *DiSTA - Department of Theoretical and Applied Sciences*
*University of Insubria, Varese, Italy*
email: tommaso.flaminio@uninsubria.it

Lluis Godo[2]

[2]*Artificial Intelligence Research Institute (IIIA - CSIC)*
*Campus UAB, Bellaterra 08193, Spain*
email: godo@iiia.csic.es

Hykel Hosni[3]

[3]*Centre for Philosophy of Natural and Social Science*
*London School of Economics, UK*
email: h.hosni@lse.ac.uk

**Abstract**

Betting methods, of which de Finetti's Dutch Book is by far the most well-known, are uncertainty modelling devices which accomplish a two-fold aim. Whilst providing an (operational) interpretation of the relevant measure of uncertainty, they also provide a formal definition of *coherence*. The main purpose of this paper is to put forward a betting method for *belief functions* on MV-algebras of many-valued events which allows us to isolate the corresponding coherence criterion, which we term *coherence in the aggregate*. Our framework generalises the classical Dutch Book method.

**KEYWORDS** *Belief functions, necessity measures, subjective probability, many-valued events, betting methods, de Finetti.*

## 1. Introduction and motivation

Betting methods, of which de Finetti's Dutch Book is by far the most well-known, are uncertainty modelling devices which accomplish a two-fold aim. Whilst providing an (operational) interpretation of the relevant measure of uncertainty, they also provide the formal setting to tell apart admissible from inadmissible quantifications of uncertainty. To emphasise the logical, rather than decision-theoretic, nature of this latter aspect, the term *coherence* is often used.[1] The main purpose of this paper is to put forward a betting method for *belief functions* (on many-valued events) which allows us to isolate the corresponding coherence criterion, which we term *coherence*

---

[1]De Finetti, who pioneered betting methods of the kind which will be of interest in this paper, used both the notion of "coherence" and that of "admissibility" depending on whether he wanted to emphasise the logical or decision-theoretic aspect of his analysis, respectively. Compare, for instance, Chapter 3 of [5] with [6].

*in the aggregate.* Since our setting builds on (and extends) de Finetti's method, we begin by recalling his own Dutch Book.

Consider two players, Bookmaker (**B**) and Gambler (**G**) and a finite set of *events of interest* $e_1 \ldots, e_k$ that can only be evaluated to either *true* or *false*. De Finetti's method is best described as an interactive, sequential choice problem (or game), in which the selection of an action, for each player, reveals the player's degree of belief in the corresponding outcome. At the first stage of the game, Bookmaker publishes a *book* $\beta$, i.e. a complete assigment of real numbers $\beta_i \in [0,1]$ to each event $e_i$. The real number $\beta_i$ is also referred to as the "betting odds" for $e_i$. Once the book has been published, Gambler chooses stakes $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$, one for each event $e_i$, and pays to **B** the amount $\sum_{i=1}^{n} \sigma_i \cdot \beta_i$ in Euros.[2] This makes the monies owed by **B** to **G** depend on a classical valuation (or possible world) $V$ which decides all the relevant events. That is to say, upon $V$ deciding the events of interest, Bookmaker must pay to Gambler $\sum_{i=1}^{n} \sigma_i \cdot V(e_i)$ Euros. Therefore, when all events are decided by some $V$, the total balance in $V$ for **B** is given by the expression:

$$\sum_{i=1}^{n} \sigma_i \cdot \beta_i - \sum_{i=1}^{n} \sigma_i \cdot V(e_i) = \sum_{i=1}^{n} \sigma_i \cdot (\beta_i - V(e_i)). \tag{1}$$

Clearly, if the result of the above expression (1) is positive, Bookmaker made a profit (in Euros) in $V$, whereas if it is negative, she made a loss in $V$. Since it is reasonable to assume that no Bookmaker would ever aim at losing money, de Finetti's criterion of coherence arises naturally from this setting.

DE FINETTI'S COHERENCE CRITERION. If $e_1, \ldots, e_n$ are events and $\beta$ is a book on them, then $\beta$ is *coherent* if and only if *it does not lead* **B** *to a sure loss*, that is to say, to a total balance for **B** which is negative in every possible world $V$.

De Finetti's celebrated *Dutch Book Theorem* states that a book $\beta$ is coherent if and only if $\beta$ coincides with the restriction to $\{e_1, \ldots, e_k\}$ of a finitely additive and normalised function $P$ mapping elements of the free Boolean algebra generated by the $e_i$'s to $[0,1]$. It is customary to say that $P$ is a probability measure *extending* $\beta$, or that $\beta$ *extends* to a (finitely additive) probability measure $P$.

A central feature of de Finetti's method is that a possible world $V$ decides completely and unambiguously the truth-value of the events of interest, that is to say, events are for de Finetti, modeled by the semantics of the classical propositional calculus[3]. A practical consequence of this assumption is that $V$ provides **B** and **G** with sufficient information about the (Boolean) events $e_i$'s to compute the value of the total balance in (1). However, it is natural to ask whether de Finetti's method can be extended to characterise coherent belief in those cases in which possible worlds *do not* determine completely whether events of interest are true of false.

Along this line, two generalisations have been proposed by Jaffray [19] and Mundici [25], respectively, to extend de Finetti's betting framework in two different ways. Jaffray investigated

---

[2]One central condition imposed by de Finetti on the game allows Gambler to choose negative stakes, thereby unilaterally imposing a payoff swap to Bookmaker, who is forced to accept. So if **G** puts a negative stake $-\sigma_i$ on event $e_i$, she is entitled to receive $\sigma_i \cdot \beta_i$ from **B**. This, and the remaining contractual conditions which underpin de Finetti's Dutch Book are fully analysed, in the language and notation of this paper, in [12]. There, we also emphasise the importance of the (implicit, in de Finetti's framework) assumption to the effect that, at the time of betting, **B** and **G** must be unaware of the truth values of the events involved in the game.

[3]We refer again to [12] for a more detailed analysis of this important point.

betting games where the information possessed by the agent at the time of resolving the uncertainty *may not determine completely* whether the events are true or false. Mundici, on the other hand, investigated betting games where the available information determines the truth value of all the events of interest, but considers a more general semantics than de Finetti's by allowing the *events of interest to be evaluated with degrees of truth between 0 and 1*.

Indeed, Jaffray's framework builds on the idea that if a given event $e$ (represented by a sentence of the classical propositional calculus) occurs, then every (non-contradictory event) which follows logically from $e$, also occurs. Jaffray's adaptation of de Finetti's betting method, which he terms a game under *partially resolving uncertainty*, mirrors rather closely the game recalled above. First **B** publishes a book $\beta : e_i \mapsto \beta_i$. Second **G** places stakes $\sigma_1, \ldots, \sigma_k$ on $e_1, \ldots, e_k$ at the betting odds written in $\beta$. Finally, **G** pays **B** for each $e_i$ the amount $\sigma_i \cdot \beta_i$ and **B** gains from **G** the amount $\sigma_i \cdot C_e(e_i)$, where $C_e(e_i) = 1$ if $e_i$ follows from $e$ (under classical propositional logic, i.e. if $\models e \rightarrow e_i$), and $C_e(e_i) = 0$ otherwise. Therefore, the total balance for **B** is given by

$$\sum_{i=1}^{k} \sigma_i(\beta_i - C_e(e_i)). \tag{2}$$

Jaffray calls a book $\beta$ *coherent under partially resolved uncertainty* if it does not lead **B** to incur a sure loss, i.e. if it is not the case that, for every fixed non-contradictory event $e$, $\sum_{i=1}^{n} \sigma_i(\beta_i - C_e(e_i)) < 0$. Finally he shows that this notion of coherence characterises Dempster-Shafer *belief functions* [30] (see Section 2.1) essentially in the same way probability measures are characterised by de Finetti's own notion of coherence:

**Theorem 1.1** ([19]). *A book $\beta$ under partially resolved uncertainty on events of interest $e_1, \ldots, e_k \in 2^W$ is coherent iff it can be extended to a belief function on the Boolean algebra $2^W$.*[4]

On the other hand, Mundici extends in [25] de Finetti's coherence criterion to formulas of the infinitely-valued Łukasiewicz calculus. In this setting events are represented by formulas which are evaluated by possible worlds into the real unit interval $[0, 1]$ (as opposed to the two element set $\{0, 1\}$) according to the semantics of Łukasiewicz logic. As in de Finetti's game **G** chooses stakes and pays **B**, for each $e_i$, $\sigma_i \cdot \beta_i$, while **B** receives from **G**, in the possible world $v$, $\sigma_i \cdot v(e_i)$, that is an amount proportional to the truth of $e_i$. Therefore, the total balance for **B** is given by

$$\sum_{i=1}^{k} \sigma_i(\beta_i - v(e_i)). \tag{3}$$

The notion of a coherent book is then defined exactly as in de Finetti's betting method: the book $\beta : e_i \mapsto \beta_i$ is said to be *state-coherent* if it does not lead **B** to incur a sure loss, i.e. if it is not the case that for every Łukasiewicz $[0, 1]$-truth evaluation $v$, $\sum_{i=1}^{n} \sigma_i(\beta_i - v(e_i)) < 0$. Mundici shows that his notion of state-coherence characterises *states* (see Section 2.2), i.e. normalised and finitely additive measures on *MV-algebras* (the algebraic counterpart of Łukasiewicz logic) in the same way de Finetti's coherence characterises finitely additive probabilities:

**Theorem 1.2** ([25]). *Let $A$ be an MV-algebra, and let $\{e_1, \ldots, e_n\} \subseteq A$. A book $\beta : e_j \mapsto \beta_j$ is state-coherent iff $\beta$ extends to a state on $A$.*

---

[4]Jaffray's original setting is slightly, but immaterially, different from our rendering since he takes Gambler's, rather than Bookmaker's point of view for the calculation of the total balance.

Thus states on MV-algebras can be seen to arise as a coherent quantification of uncertainty from a suitable extension of de Finetti's betting method in a many-valued framework.

Given these antecedents, the aim of this paper is to put forward an extension of de Finetti's criterion and operational interpretation of uncertainty on many-valued events. As a consequence, this paper explores methods for an operational quantification of the uncertainty of many-valued events whose evaluation is obtained aggregating (possibly inconsistent) *observations* provided by more or less reliable agents. The following fictitious example illustrates a decision-making problem in which such a generalised uncertainty quantification may arise in practice.

**Example 1.3.** The Global Health Agency (Agency, for short) is faced with the problem of choosing life-saving drugs against $k$ diseases, $D_1, \ldots, D_k$, which can turn into potential epidemics. On top of the uncertainty related to whether a certain epidemic will occur, Agency will be interested in the extent to which $D_i$ will spread across the globe. So, the *events* that Agency is facing are best seen as $e_i$: "the outbreak of $D_i$ will be aggressive". This can be captured by allowing each event $e_i$ be evaluated in the real unit interval $[0, 1]$, rather than, as usually done, in the binary set with values 0 (i.e. not aggressive) and 1 (i.e. aggressive).

As for the evaluation of each event, the Agency is supposed to ask data about the actual pandemics to the individual national health organisations, say $a_1, \ldots, a_m$. These will provide $[0, 1]$-valued assessments $a_1(e_i), \ldots, a_m(e_i)$ of how aggressive is each disease $D_i$ in its country. Agency, in order to determine the evaluation for each event, and hence to determine the extent to which disease $D_i$ turned out to be aggressive, needs to aggregate these data according to some aggregation method **agg**. The aggregation methods considered in this paper will be based on determining a *reliability map* $\eta$ which assigns to each national health organisation $a_i$ a reliability degree $\eta(a_i)$. The idea behind this is that some countries might announce the outbreak of $D_i$ but could do so on the basis of very poor information, for instance because of inadequate sampling of the population. Other countries might do very accurate statistical sampling instead, thereby producing a completely reliable report. To cope with this we assume that Agency can determine the true reliability of the data produced by each country by running say a (completely reliable) statistical analysis on the sampling methods used by the national agencies.

By considering different aggregation methods, we provide an operational semantics for a wide class of uncertainty measures including belief functions on MV-algebras, plausibility functions on MV-algebras, but also belief functions on Boolean algebras and classical probability measures.

The remainder of the paper is organised as follows: In Section 2 we shall briefly recall the preliminary notions needed for the paper, namely belief functions on Boolean algebras, states of MV-algebras and belief functions on MV-algebras. Section 3 will be devoted to presenting our generalised betting framework. In the same section we shall characterize uncertainty measures on many-valued events by exploring the possible ways in which the aggregation map can be chosen. Refinements of the set of available reliability maps will be investigated in Section 4 where we show how particular classes of belief functions on MV-algebras can be described. In Section 5 we shall discuss on conclusions and future work.

The paper includes two appendices: Appendix A, provides the necessary notions and results about Łukasiewicz logic and MV-algebras. Appendix B collects the proofs of some technical results which, in the interest of readability, we refrained from including in the main body of the paper.

## 2. Preliminaries

### 2.1. Belief functions on classical events

We briefly recall in this section the main definitions of Dempster-Shafer belief functions [30] needed in the rest of the paper. Consider a finite set $W$ of mutually exclusive (and exhaustive) propositions of interest, and whose powerset $2^W$ represents all such propositions. We can think of $W$ as a *frame of discernment*, with elements $x \in W$ representing the lowest level of discernible information that can be dealt with.

A map $m : 2^W \to [0,1]$ is said to be a *basic belief assignment*, or a *mass assignment* whenever

$$m(\emptyset) = 0 \text{ and } \sum_{A \in 2^W} m(A) = 1.$$

Given such a mass assignment $m$ on $2^W$, for every $A \in 2^W$, the *belief of $A$* is defined as

$$Bel_m(A) = \sum_{B \subseteq A} m(B). \tag{4}$$

Notice that each mass assignment $m$ on $2^W$ is in fact a probability distribution on $2^W$ that naturally induces a probability measure $P_m$ on $2^{2^W}$. Consequently, the belief function $Bel_m$ defined from $m$ can be equivalently described as follows: for every $A \in 2^W$,

$$Bel_m(A) = P_m(\{B \in 2^W : B \subseteq A\}). \tag{5}$$

Therefore, identifying the set $\{B \in 2^W : B \subseteq A\}$ with its characteristic function defined by

$$\iota_A : B \in 2^W \mapsto \begin{cases} 1 & \text{if } B \subseteq A \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

it is easy to see that, for every $A \in 2^W$, and for every mass assignment $m : 2^W \to [0,1]$, we have

$$Bel_m(A) = P_m(\iota_A). \tag{7}$$

This characterization will be useful when discussing the extensions of belief functions on MV-algebras. Similarly useful to understand our generalised betting method is the rather obvious observation to the effect that for every $A \in 2^W$, $\iota_A$ can be regarded as a map evaluating the (strict) inclusion of $B$ into $A$, for every subset $B$ of $W$.

Finally, recall that a subset $A$ of $W$ such that $m(A) > 0$ is said to be a *focal element*. Every belief function is characterized by the value that $m$ takes over its focal elements, and therefore, the focal elements of a belief function $Bel_m$ carry all the relevant information about $Bel_m$.

### 2.2. States on MV-algebras

States on MV-algebras were introduced by Mundici in [24] as averaging processes for the infinitely-valued Łukasiewicz logic. A *state of an MV-algebra*[5] $\mathbf{M} = (M, \oplus, \neg, 0^M)$ is a map $s : M \to [0,1]$ satisfying the following:

---

[5]See Appendix A for details on MV-algebras.

(i)  $s(1^M) = 1$,

(ii)  $s(x \oplus y) = s(x) + s(y)$, for every $x, y \in M$ such that $x \odot y = 0^M$,

where $1^M = \neg 0^M$ and $x \odot y = \neg(\neg x \oplus \neg y)$. We denote by $\mathcal{S}(M)$ the subset of $[0,1]^M$ whose elements are the states of $M$.[6]

States play the same role on MV-algebra as probability measures do on Boolean algebras. Indeed, the two properties (i) and (ii) characterize each state on $M$ as a $[0,1]$-valued map that is normalised (i) and additive (ii) with respect to MV-algebraic operations. Moreover, it is easy to see that, for every MV-algebra $M$ and for every $s \in \mathcal{S}(M)$, the restriction of $s$ to the Boolean skeleton of $M$ (the largest Boolean algebra contained in $M$) is a finitely additive probability measure.

The following theorem independently proved by Kroupa [20, Theorem 28] and Panti [28, Proposition 1.1], provides an integral representation of states by Borel probability measures defined on the $\sigma$-algebra $\mathfrak{B}(Max(M))$ of Borel subsets of $Max(M)$, where $Max(M)$ is the *maximal spectral space of $M$* (see Appendix A for further details on $Max(M)$).[7]

**Theorem 2.1.** *For every MV-algebra $M$, there is a one-to-one correspondence between the class $\mathcal{S}(M)$ of states on $M$, and the regular Borel probability measures on $\mathfrak{B}(Max(M))$. In particular, for every state $s$ on $M$, there exists a unique regular Borel probability measure $\mu$ on $\mathfrak{B}(Max(M))$ such that for every $a \in M$,*

$$s(a) = \int_{Max(M)} a \, \mathrm{d}\mu. \tag{8}$$

**Remark 2.2.** It is worth noticing that, for MV-algebras of the form $[0,1]^Y$ ($Y$ can either be finite or infinite), their maximal spectral space $Max([0,1]^Y)$ coincides, up to bijections, with $Y$. As a matter of fact, for every $y \in Y$, the subset $\mathfrak{m}_y$ of $[0,1]^Y$ consisting on those functions $f : Y \to [0,1]$ such that $f(y) = 1$ is a maximal filter of $[0,1]^Y$. Furthermore, the map $y \in Y \mapsto \mathfrak{m}_y \in Max([0,1]^Y)$ is a bijection. In the remainder of this paper, we shall often use this remark.

When $M$ is an MV-algebra of functions of the kind $[0,1]^Y$, where $Y$ is countable set, the above representation theorem boils down to claiming that for any state $s$ on $M$ there exists a probability distribution $p : Y \to [0,1]$ such that, for any $f \in [0,1]^Y$,

$$s(f) = \sum_{y \in Y} f(y) \cdot p(y).$$

In other words, states on MV-algebras of $[0,1]$-valued functions are nothing else but Zadeh's probabilities of fuzzy events [35].

As we recalled in Section 1, generalising a previous result by Paris [29], Mundici [25] shows that states of MV-algebras can be seen to arise as a coherent quantification of uncertainty from a suitable extension of de Finetti's betting method. We now recall the theorem in more detail, as it will be useful in what follows.

---

[6]For the sake of a lighter notation, when no confusion is possible we will identify an MV-algebra **M** with its domaion $M$, and we will write 0 and 1 for $0^M$ and $1^M$ respectively.

[7]While Kroupa proved Theorem 2.1 in the case of semisimple MV-algebras, Panti showed that the hypothesis on the semisimplicity of the MV-algebra can be relaxed, since, for every MV-algebra $M$, there is a canonical bijection between the class $\mathcal{S}(M)$ of all the states on $M$, and the class $\mathcal{S}(M/Rad(M))$ of all the states on its most general semisimple quotient $M/Rad(M)$.

**Theorem 2.3** ([25]). *Let $M$ be an MV-algebra, let $\{e_1, \ldots, e_n\} \subseteq M$, and let $\beta : e_j \mapsto \beta_j$ be a book. Then the following are equivalent:*

(1) $\beta$ *is state-coherent,*

(2) $\beta$ *extends to a state on $M$.*

*2.3. Belief functions on MV-algebras of $[0,1]$-valued functions*

In the literature several attempts to extend belief functions on many-valued (i.e. *fuzzy*) events can be found. The first extensions of Dempster-Shafer theory to the general framework of fuzzy set theory was proposed by Zadeh in the context of information granularity and possibility theory [34] in the form of an expected conditional necessity, and by Smets who proposed in [31] to extend a classical belief function $Bel$ on $2^X$ to fuzzy subsets $A$ of $X$ as the lower expectation of the characteristic function of $A$ with respect to the class of probability measures lower bounded by $Bel$. After Zadeh and Smets, several further generalisations were proposed, depending on the way a measure of *inclusion among fuzzy sets* is used to define the belief functions of fuzzy events based on fuzzy evidence. Indeed, given a mass assignment $m$ for the bodies of evidence $\{A_1, A_2, \ldots\}$, and a measure $I(A \subseteq B)$ of inclusion among fuzzy sets, the belief of a fuzzy set $B$ can be defined in general by the value: $Bel(B) = \sum_i I(A_i \subseteq B) \cdot m(A_i)$. We refer the reader to [7, 18, 32, 33] for exhaustive surveys, and to [1] for another approach through fuzzy subsethood.

The set $[0,1]^W$ of fuzzy subsets of a set $W$ (mappings from $W$ into $[0,1]$) can be endowed with an MV-algebra structure by the pointwise extension of the MV-algebra operations in the standard MV-algebra $[0,1]_{MV}$ (see (2) and (4) in Example 5.1 in Appendix A). Belief functions were firstly generalised to this MV-algebraic setting by Kroupa [21] in the following way.

Assume $W$ be finite, and for each element $a$ in the MV-algebra $[0,1]^W$, let the map $\hat{\rho}_a : 2^W \to [0,1]$ be defined by the following stipulation: for all $B \in 2^W$,

$$\hat{\rho}_a(B) = \begin{cases} \min_{w \in B} a(w) & \text{if } B \neq \emptyset, \\ 1 & \text{if } B = \emptyset. \end{cases} \tag{9}$$

It is clear that $\hat{\rho}_a$ generalises the map $\iota_A$ we discussed in Section 2.1 in the following sense: whenever $A \in 2^W$, then $\hat{\rho}_A = \iota_A$. Indeed, for every $A \in 2^W$, $\hat{\rho}_A(B) = 1$ if $B \subseteq A$, and $\hat{\rho}_A(B) = 0$, otherwise. Against this background Kroupa proposes to define belief functions on $[0,1]^W$ by replacing in (7) the maps $\iota_A$ by the maps $\hat{\rho}_a$ and the probabilities over $2^{2^W}$ by states on $[0,1]^{2^W}$.

**Definition 2.4** (Crisp focal belief function). Let $W$ be a finite nonempty set. Then a map $\hat{Bel} : [0,1]^W \to [0,1]$ is a *crisp-focal belief function*, if there exists a state $s : [0,1]^{(2^W)} \to [0,1]$ such that, for all $a \in [0,1]^W$

$$\hat{Bel}(a) = s(\hat{\rho}_a).$$

We call the maps $\hat{Bel}$ *crisp-focal* belief functions since the focal elements are (crisp) subsets of $2^W$. Indeed we have $s(\hat{\rho}_a) = \sum_{B \in 2^W} \hat{\rho}_a(B) \cdot \mu(B)$, where $\mu : 2^W \to [0,1]$ is the probability distribution on $2^W$ defined as $\mu(B) = s(\{B\})$, identifying $\{B\}$ with its characteristic function on $2^W$. Therefore it is clear, by construction, that those elements from $[0,1]^W$ with a positive probability can only be crisp subsets of $W$.

Kroupa's approach has been further generalised in [17] as follows.[8] For every finite set $W$, and for every element $a$ of the MV-algebra $[0,1]^W$, first define the map

$$\rho_a : b \in [0,1]^W \mapsto \min\{b(w) \Rightarrow a(w) \mid w \in W\}.[9]$$

The map $\rho_a$ generalises both the map $\iota_A$ introduced in (6) and the map $\hat{\rho}_a$ defined in (9). Indeed it is clear that the restriction of $\rho_a$ to $2^W$ coincides with $\hat{\rho}_a$, and for every crisp subset $A$ of $W$, the restriction of $\rho_A$ to $2^W$ coincides with $\iota_A$. Moreover, for every fixed $b \in [0,1]^W$ the map $N^b : [0,1]^W \to [0,1]$ defined by

$$N^b : a \in [0,1]^W \mapsto \rho_a(b) \in [0,1], \tag{10}$$

is a *homogeneous necessity measure* induced by the mapping $b$,[10] understood as a possibility distribution (see e.g. [15]). Moreover, $N^b$ is normalised (i.e. $N^b(\overline{0}) = 0$) if and only if so is $b$ (i.e. $\max_{w \in W} b(w) = 1$). An easy adaptation of [15, Theorem 3.3] shows that the following proposition holds.

**Proposition 2.5.** *(1) The class $\mathcal{N}([0,1]^W)$ of all necessity measures on $[0,1]^W$ coincides with the class*

$$\{\rho_{(\cdot)}(b) : f \in [0,1]^W \mapsto \rho_f(b) \mid b \in [0,1]^W\}.$$

*(2) The class $\mathcal{N}^\top([0,1]^W)$ of all normalised necessity measures on $[0,1]^W$ coincides with the class*

$$\{\rho_{(\cdot)}(b) : f \in [0,1]^W \mapsto \rho_f(b) \mid b \in [0,1]^W, \ \max_{w \in W} b(w) = 1\}.$$

For every finite set $W$ let $\mathcal{R}(W)$ be the MV-algebra generated by the set $\{\rho_a \mid a \in [0,1]^W\}$. The following holds.

**Proposition 2.6** ([14])**.** *For every finite set $W$, the algebra $\mathcal{R}(W)$ is a separating MV-algebra of continuous functions. That is, each $f \in \mathcal{R}(W)$ is a continuous map, and for each $w_1, w_2 \in W$ such that $w_1 \neq w_2$, there is an $f \in \mathcal{R}(W)$ such that $f(w_1) \neq f(w_2)$.*

Now we are ready to define belief functions on MV-algebras of fuzzy sets.

**Definition 2.7** ([17])**.** Let $W$ be a finite set. A map $Bel : [0,1]^W \to [0,1]$ is a *belief function* provided that there exists a state $s : \mathcal{R}(W) \to [0,1]$ such that, for every $a \in [0,1]^W$

$$Bel(a) = s(\rho_a)$$

The state $s$ is called the *state assignment* of $Bel$.

As pointed out in [17] (see also [13, 14]), since $\rho_0$ does not coincide in general with the zero-constant function $\overline{0}$, $Bel(\overline{0})$ cannot be ensured to equal 0. We call *normalised* each belief function $Bel$ on $[0,1]^W$ satisfying $Bel(\overline{0}) = 0$.

---

[8]We invite the interested reader to consult [14] for further details.

[9]Recall that $u \Rightarrow v = \min(1, 1 - u + v)$, for each $u, v \in [0,1]$.

[10]This means that the following properties hold: i) $N^b(a \wedge a') = \min\{N^b(a), N^b(a')\}$ and $N^b(\overline{r} \oplus a) = r \oplus N^b(a)$ for every $r \in [0,1]$. Here $u \oplus v = \min(1, u+v)$ and $\overline{r}$ denotes the constant function of value $r$.

If $W = \{w_1, \dots, w_n\}$ is a finite set, the Boolean algebra $2^W$ is clearly also finite, and hence any mass assignment $m : 2^W \to [0,1]$ obviously has only finitely many focal elements. Also in the framework of finite MV-algebras, the mass assignment can be easily defined. Indeed, every finite MV-algebra can be embedded into an MV-algebra of the form $(S_m)^W$ where $S_m = \{0, 1/m, \dots, (m-1)/m, 1\}$ and $W$ is a finite set, and every belief function $Bel$ on $(S_m)^W$ can be written as:

$$Bel(f) = \sum_{a \in (S_m)^W} \rho_a(f) \cdot \mu(\{a\}),$$

where $\mu$ is a uniquely determined probability measure on $2^{(S_m)^W}$ (cf. [14, Remark 4.10]). Hence, an element $a \in (S_m)^W$ is a focal element for $Bel$ if and only if $\mu(\{a\}) > 0$. (We will turn back on this, in Remark 4.4).

On the other hand, for a belief function defined on the MV-algebra $[0,1]^W$, Theorem 2.1 ensures that, if $\mu : \mathfrak{B}([0,1]^W) \to [0,1]$ is a regular Borel probability measure, the map $Bel : [0,1]^W \to [0,1]$, defined as

$$Bel(a) = \int_{[0,1]^W} \rho_a \, \mathrm{d}\mu,$$

is a belief function and conversely, that every belief function on $[0,1]^W$ arises in this way. Clearly, the MV-algebra $[0,1]^W$ has uncountably many elements, and hence we cannot find, in general, a *mass assignment* $\mu$ defined over $\mathfrak{B}([0,1]^W)$ which is supported by a set which is at most countable.[11] This observation leads to the following definition.

**Definition 2.8** ([14]). Let $\mathcal{K}$ be the set of all compact subsets of an MV-algebra of fuzzy sets $[0,1]^W$. For every regular Borel probability measure $\mu$ defined on $\mathfrak{B}([0,1]^W)$, we call the set

$$\mathrm{spt}\,\mu = \bigcap \{K | K \in \mathcal{K}, \mu(K) = 1\}$$

the *support of* $\mu$.

By Theorem 2.1 we can regard $\mathrm{spt}\,\mu$ as the support of the state assignment $s$ defined from $\mu$ via (8). In particular, the following holds:

$$Bel(a) = \int_{[0,1]^W} \rho_a \, \mathrm{d}\mu = \int_{\mathrm{spt}\,\mu} \rho_a \, \mathrm{d}\mu. \tag{11}$$

Therefore, the set of focal elements of a belief function $Bel$ on $[0,1]^W$ whose state assignment $s$ is represented by a regular Borel probability measure $\mu$ effectively coincides with $\mathrm{spt}\,\mu$. As a consequence we will freely speak of either the support or the set of focal elements of such belief functions with no risk of confusion arising.

As in the classical Dempster-Shafer theory, plausibility functions on fuzzy sets can also be defined by duality from a belief function. In other words, if $Bel$ is a belief function on $[0,1]^W$ (as in Definition 2.7), its dual *plausibility function* $Pl : [0,1]^W \to [0,1]$ is defined by the following stipulation: for every $a \in [0,1]^W$,

$$Pl(a) = 1 - Bel(\neg a) = 1 - s(\rho_{\neg a}) \tag{12}$$

--------

[11] If it happens to be that $\mu$ has a countable support (i.e. there is a countable subset $K \subset [0,1]^W$ such that $\mu(K) = 1$) then the belief function defined induced by $\mu$ takes the simplified form $Bel(f) = \sum_{a \in [0,1]^W} \rho_a(f) \cdot \mu(\{a\})$, for any $f \in [0,1]^W$.

where $s : \mathcal{R}(W) \to [0,1]$ is the state-assignment of *Bel*. Since states are self-dual (i.e. $s(\neg x) = 1 - s(x)$), the above expression reduces to $Pl(a) = s(\neg \rho_{\neg a})$, for all $a \in [0,1]^W$. Therefore, since by Proposition 2.5 the map $\rho_{(\cdot)}(b)$ is a necessity measure (and in particular it is normalised whenever $\max\{b(w) : w \in W\} = 1$), $\neg \rho_{\neg(\cdot)}(b)$ is a (normalised) possibility measure [15, Definition 3.2]. Moreover, every (normalised) possibility measure on $[0,1]^W$ can be obtained in this way.

## 3. A generalised betting framework

We now introduce a betting method which generalises both Jaffray and Mundici's frameworks recalled above. As anticipated and motivated in the introductory section of this paper, we generalise Jaffray's by considering many-valued events, and generalise Mundici's by considering several sources of information (each of distinct reliability) and allowing for (partly) unresolved uncertainty.

As usual, our two players will be labelled **B** (for Bookmaker) and **G** (for Gambler), and we fix a set of (many-valued) events of interest $\{e_1, \ldots, e_k\}$. As in the classical case, the game begins with **B** publishing a book $\beta$ which assigns, to each event $e_i$, a real number $\beta_i \in [0,1]$. Again, in full analogy with de Finetti's method, **B** and **G** agree that the *stakes* placed by **G** at the second stage of the game can either be positive or negative. In other words, for every $e_i$, **G** choses real numbers $\sigma_i$ and pays to **B** the amount $\sum_{i=1}^{k} \beta_i \cdot \sigma_i$ in Euros. This corresponds to the price that **B** accepts to pay to bet on the $e_i$'s.

In the spirit of the Global Health Agency example above, our framework introduces the following main novelties with respect to de Finetti's:

(i) a finite set of *many-valued possible worlds* $W = \{w_1, \ldots, w_n\}$, to be interpreted as the set of possible (complete) *scenarios* regarding the events, i.e. scenarios which determine the truth-value (from $[0,1]$) in which every event holds.[In our running example, such scenarios determine the degree to which a certain desease turns out to be *aggressive*.]

(ii) a finite set $Ag$ of *informative agents* $\{a_1, \ldots, a_m\}$, each one notifying which is the resulting scenario according to its subjective point of view. [Again in our example, such agents coincide with the individual National Health Organisations.]

(iii) agents, as information sources, may be more or less reliable. [In our running example this depends on the fact that the measurement concerning the outbreak of a particular desease is intrinsically statistical and therefore it is susceptible of being more or less accurate.]

To operationalize this setting, we make the following working assumptions:

- we identify events with formulas of Łukasiewicz logic over a language built on a set of propositional variables $V$;

- we identify the set of possible worlds or scenarios with a subset of evaluations for formulas, i.e. $W \subseteq [0,1]^V$;

- we identify the information provided by each agent with the choice of one possible world or scenario from $W$, i.e. each agent $a \in A$ chooses one $w_a \in W$.

- we assume an oracle $\mathcal{O}$ to assign a reliability degree to each agent, i.e. $\mathcal{O}$ determines a reliability map $\eta : Ag \to [0,1]$, where $\eta(a) = 1$ means that agent $a$ is fully reliable, $0 < \eta(a) < 1$ means that $a$ is somewhat reliable (the higher the more reliable), and $\eta(a) = 0$

means that a is not reliable at all. We also assume that, amongst all agents, at least one is not completely unreliable, that is $\max_{a \in Ag} \eta(a) > 0$. The set of these reliability maps will be denoted $\Lambda^+(Ag)$.

Notice that, as far as betting in this framework is concerned, the first two assumptions above imply that events can only be distinguished by how possible worlds evaluate them, hence they can be represented as functions defined on $[0,1]^W$, namely one event $e$ can be understood as the function $w \mapsto w(e)$, for each $w \in W$.

A triple $\mathcal{E} = (Ag, \overline{w}, \eta)$, where $\overline{w} = \{w_a \in W \mid a \in Ag\}$ is a set of evaluations of events for each $a \in Ag$, will be called an *evaluating triple*.

All the above ingredients allow us to evaluate the events involved in the betting as a weighted aggregation of the information provided by the agents, where weights are related to reliability degrees. More concretely, we consider an *aggregation* method as a two-place function $\mathbf{agg}(\cdot, \cdot)$ such that, for each evaluating triplet $\mathcal{E}$ and for each event $e_i$, $\mathbf{agg}(\mathcal{E}, e_i) \in [0,1]$. We will assume that $\mathbf{agg}(\cdot, \cdot)$ satisfies some suitable properties that we leave unspecified for the moment. Whenever $\mathcal{E}$ is fixed, we shall denote by $\mathbf{agg}_{\mathcal{E}}(\cdot)$ the one-place map $\mathbf{agg}(\mathcal{E}, \cdot)$.

Finally, once an evaluation triplet $\mathcal{E}$ and the aggregation method $\mathbf{agg}$ are fixed and agreed by the bookmaker and the gambler, the specification of the betting framework is completed by determining that the amount gambler $\mathbf{G}$ receives from from bookmaker $\mathbf{B}$ for each event $e_i$ is proportional to the value $\mathbf{agg}_{\mathcal{E}}(e_i)$, the total amount being $\sum_{i=1}^{k} \sigma_i \cdot \mathbf{agg}_{\mathcal{E}}(e_i)$. Then the total amount for $\mathbf{B}$ is:

$$\sum_{i=1}^{k} \sigma_i \cdot (\beta_i - \mathbf{agg}_{\mathcal{E}}(e_i)). \tag{13}$$

Note that (13) generalises both (2) and (3). Indeed, if $Ag$ consists of only one agent $a$ and $\eta(a) = 1$, then we are in Mundici's betting framework. On the other hand, if $\eta(a) = 1$ for each $a \in Ag$ and $\mathbf{agg}(\mathcal{E}, e_i) = 1$ if $w_a(e_i) = 1$ for all $a \in Ag$, and $\mathbf{agg}(\mathcal{E}, e_i) = 0$ otherwise, then we essentially recover Jaffray's betting framework.[12]

Now we are ready to introduce the notion of coherence in our extended framework.

COHERENCE IN THE AGGREGATE CRITERION. We say that a book $\beta$ is *coherent in the aggregate* $\mathbf{agg}$, if it does not lead $\mathbf{B}$ to lose money independently of the evaluating triple $\mathcal{E}$.

In other words, $\beta$ is coherent in the aggregate (with respect to the aggregation $\mathbf{agg}$), if and only if, for every $\sigma_1, \dots, \sigma_k \in \mathbb{R}$, there exist an evaluating triple $\mathcal{E}$, such that

$$\sum_{i=1}^{k} \sigma_i (\beta_i - \mathbf{agg}_{\mathcal{E}}(e_i)) \geq 0. \tag{14}$$

Note that the underlying idea behind our notion of coherence in the aggregate is virtually identical to the seminal one which lead to de Finetti's original version of the Dutch Book: in a

---

[12]Actually, strictly speaking, in Jaffray's betting framework there is only one information source providing an (incomplete) description of the world, while in this particular scenario of our betting framework we have a set of informative agents, each one providing a *possible* complete description of the world. So, they are both equivalent forms of representing an incomplete information, or in Jaffray's terms, an "unresolved uncertainty" setting.

suitably specified betting problem, Bookmaker is incoherent if she exposes herself to the logical possibility of incurring sure loss. What our two-fold generalisation brings to the problem is a considerably refined understanding of what "the logical possibility of sure loss" means. Yet it is worth remarking that the formal adjustments do not require an essential modification of the concept of coherence. This, in our view, provides a solid foundation for the investigation of belief functions on many-valued events as *measures of rational belief* as opposed to a mathematically deep albeit purely formal exercise.

Going back to our main concern, the aggregate values $\mathbf{agg}_{\mathcal{E}}(e_i)$ can be obtained by several aggregation procedures, see e.g. [9]. In this paper we shall consider three relevant cases which we term *pessimistic*, *optimistic* and *average*, respectively. The terminology of pessimistic and optimistic attitudes and the way they are modelled (namely by generalised necessity and possibility measures) conform to the ones in use in possibilistic decision theory, see e.g. [8, 11, 10]. The average attitude, on the other hand, arises naturally in the context of the problem under investigation.

**Definition 3.1** (Aggregation). Let $\mathcal{E} = (Ag, \overline{w}, \eta)$ be an evaluating triple. For every event $e \in [0,1]^W$, we define the *optimistic*, the *pessimistic* and the *average* aggregation maps $\Pi_{\mathcal{E}}$, $N_{\mathcal{E}}$ and $M_{\mathcal{E}}$ respectively, as follows:

$$
\begin{aligned}
\Pi_{\mathcal{E}}(e) &= \max_{a \in Ag}\{\eta(a) \odot w_a(e)\}, \\
N_{\mathcal{E}}(e) &= \min_{a \in Ag}\{\eta(a) \Rightarrow w_a(e)\}, \\
M_{\mathcal{E}}(e) &= \left[\sum_{a \in Ag} \eta(a) \cdot w_a(e)\right] / \sum_{a \in Ag} \eta(a).
\end{aligned}
\tag{15}
$$

The next lemma shows that these three kinds of aggregation maps are in fact restrictions of possibility measures, necessity necessity measures and states on the MV-algebra of functions $[0,1]^W$.

**Lemma 3.2.** *For every evaluation tripe $\mathcal{E} = (Ag, \overline{w}, \eta)$ and for every class $C$ of events in $[0,1]^W$ the following properties hold:*

(i) $\Pi_{\mathcal{E}}$ *is the restriction on $C$ of a* possibility measure *on $[0,1]^W$. Conversely, for each possibility distribution $\pi : W \to [0,1]$ there exists an evaluating triple $\mathcal{E}$ such that $\Pi_{\pi}(e) = \max_{w \in W} \pi(w) \odot w(e) = \Pi_{\mathcal{E}}(e)$ for each $e \in C$.*

(ii) $N_{\mathcal{E}}$ *is the restriction on $C$ of a* necessity measure *on $[0,1]^W$. For each possibility distribution $\pi : W \to [0,1]$ there exists an evaluating triple $\mathcal{E}$ such that $N_{\pi}(e) = \max_{w \in W} \pi(w) \odot w(e) = N_{\mathcal{E}}(e)$ for each $e \in C$.*

(iii) $M_{\mathcal{E}}$ *is the restriction on $C$ of a* state *on $[0,1]^W$. Conversely, for each probability distribution $p : W \to [0,1]$ there exists an evaluating triple $\mathcal{E}$ such that $\sum_{w \in W} p(w) \cdot w(e) = M_{\mathcal{E}}$ for each $e \in C$.*

*Proof.* (i) Define the possibility distribution $\pi : W \to [0,1]$ as follows: $\pi(w) = \max\{\eta(a) \mid a \in Ag$ such that $w_a = w\}$. One can check then that, for every $e \in C$, $\Pi_{\mathcal{E}}(e) = \Pi_{\pi}(e) = \max_{w \in W} \pi(w) \odot w(e)$. Indeed, we have:

$$
\begin{aligned}
\Pi_{\pi}(e) &= \max_{w \in W} \pi(w) \odot w(e) \\
&= \max_{w \in W} \max\{\eta(a) \odot w(e) \mid a \in Ag, w = w_a\} \\
&= \max\{\eta(a) \odot w_a(e) \mid a \in Ag\} = \Pi_{\mathcal{E}}(e).
\end{aligned}
$$

To prove that for each possibility distribution $\pi : W \to [0,1]$ there exists an evaluating triple $(Ag, \overline{w}, \eta)$ such that $\Pi_\pi(e) = \Pi_\mathcal{E}(e)$ for each $e \in C$, it is enough to take $Ag = W$, $\eta = \pi$ and $w_a = a$ for each $a \in W$.

(ii) The proof of this item directly follows from (i) and recalling that possibility and necessity measures are dual, that is, for every $\pi : W \to [0,1]$ and for every $e \in C$, $\Pi_\pi(e) = 1 - N_\pi(1-e)$.

(iii) The proof is a direct consequence of the representation theorem for states (Theorem 2.1) in the special case of finite $W$. In this case, in fact, all states are of the form $s(a) = \sum_{x \in W} a(x)p(x)$, where $p : W \to [0,1]$ is a probability distribution on W, i.e. $\sum_{x \in W} p(x) = 1$. To prove that for every probability distribution there exists an evaluation triple $\mathcal{E} = (Ag, \overline{w}, \eta)$ such that $M_\mathcal{E} = \sum_{w \in W} p(w) \cdot w(e)$ it is sufficient to settle $Ag = W$, $\eta = p$ and $w_a = w$ for every $a \in W$. $\qquad\square$

**Remark 3.3.** Actually, the way of building the possibility distribution $\pi$ in the proofs of items (i) and (ii) of the above Lemma 3.2 reflects the *disjunctive* nature of the information aggregation underlying the optimistic and pessimistic operators $\Pi_\mathcal{E}$ and $N_\mathcal{E}$ respectively. In fact, one can argue that two agents $a$ and $a'$, say both with equal reliability, reporting two different words $w_a \neq w_{a'}$ denote a contradiction in the information they provide, since worlds represent complete descriptions and hence if $w_a$ is supposed to hold for $a$, $w_{a'}$ cannot hold as well for $a$. However, in our betting metaphor we take the stand point that agents are providers of only *possible* scenarios, so in that case, agents $a$ and $a'$ are reporting that worlds $w_a$ and $w_{a'}$ are both possible scenarios, and they are possible to the extent they are considered reliable.

Actually, an equivalent alternative to the multi-agent betting framework for the case of optimistic and pessimistic aggregation could be considering the oracle $\mathcal{O}$ to provide an imprecise description of the world in the form a nested set of subsets of possible worlds $\emptyset \neq W_1 \subset W_2 \subset \ldots \subset W_n \subseteq W$ together with increasing confidence degrees $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n = 1$. Interpreting a values $\alpha_i$ as the necessity degrees with which the oracle believes the actual world is in the set $W_i$, the (epistemic) information provided by the oracle amounts to directly provide the possibility distribution $\pi$ on $W$ defined as $\pi(w) = \min\{1 - \alpha_i : 1 \leq i \leq n \text{ such that } w \notin W_i\}$. Indeed, using this betting scenario for the pessimistic aggregation, it becomes clearer that we are generalizing Jaffray's betting framework.[13]

It is easy to see that our assumption according to which at least one informative agent is not completely unreliable, i.e. that reliability maps are chosen in $\Lambda^+(Ag)$, forces the possibility distributions arising from the above Lemma 3.2 to belong to the class $\mathscr{P}^+(W) = \{\pi \in [0,1]^W \mid \max_{w \in W} \pi(w) > 0\}$.

We now introduce the three notions of coherence which arise in our generalised betting methods when instantiating the coherence in the aggregate considering criterion (13) with the tree-fold definition of aggregation given above. For each of them we will prove a characterisation result in terms of three corresponding generalised uncertainty measures for MV-algebras of events: plausibility functions, belief functions, and states.

**Definition 3.4** (Coherence in the aggregate)**.** Let $\{e_1, \ldots, e_k\}$ be events, and let $\beta : e_i \mapsto \beta_i$ be the book of interest. If for every $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$, there exist an evaluating triple $\mathcal{E}$ such that:

---

[13] We are thankful to one reviewer for pointing us this issue.

1. $\sum_{i=1}^{k} \sigma_i(\beta_i - N_{\mathcal{E}}(e_i)) \geq 0$, then the book $\beta$ is said to be *pessimistically coherent*,
2. $\sum_{i=1}^{k} \sigma_i(\beta_i - \Pi_{\mathcal{E}}(e_i)) \geq 0$, then the book $\beta$ is said to be *optimistically coherent*,
3. $\sum_{i=1}^{k} \sigma_i(\beta_i - M_{\mathcal{E}}(e_i)) \geq 0$, then the book $\beta$ is said to be *coherent in the average*,

where $N_{\mathcal{E}}, \Pi_{\mathcal{E}}, M_{\mathcal{E}}$ are defined as in (15) above.

**Remark 3.5.** If we let the evaluating triple $\mathcal{E}$ vary, then the MV-algebra generated by the functions $N_{\mathcal{E}}$ is exactly $\mathcal{R}(W)$ (recall Subsection 2.3).

**Lemma 3.6.** *Let $\{e_1, \ldots, e_k\}$ be events and $\beta : e_i \mapsto \beta_i$ be a book. The following are equivalent*

1. *There exists a belief function $Bel : [0,1]^W \to [0,1]$ such that $Bel(e_i) = \beta_i, i = 1, \ldots, k$.*
2. *for all $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$, there exists an MV-homomorphism $h \in \mathcal{H}(R(W), [0,1])$ such that*

$$\sum_{i=1}^{n} \sigma_i(\beta_i - h(\rho_{e_i})) \geq 0.$$

*Proof.* A map $Bel : [0,1]^W \to [0,1]$ is a belief function extending $\beta$ iff, by definition, there exists an state $s : \mathcal{R}(W) \to [0,1]$ such that, for all $i = 1, \ldots, k$, $s(\rho_{e_i}) = \beta_i$ iff, by Theorem 2.3, the book $\beta' : \rho_{e_i} \mapsto \beta_i$ is state coherent, iff, by definition, for all $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$, there exists an $h \in \mathcal{H}(\mathcal{R}(W), [0,1])$ such that $\sum_{i=1}^{n} \sigma_i(\beta_i - h(\rho_{e_i})) \geq 0$. Hence our claim is settled. $\square$

Finally, we are now in a position to prove the main result of this paper, which provides a three-fold characterisation of coherence arising from the betting method informally introduced in Section 1 and made precise in this section.

**Theorem 3.7** (Main Theorem)**.** *Let $\{e_1, \ldots, e_k\}$ be events and $\beta : e_i \mapsto \beta_i$ be a book Then the following conditions hold:*

1. *$\beta$ is pessimistically coherent iff there exists a belief function $Bel : [0,1]^W \to [0,1]$ such that $Bel(e_i) = \beta_i$, for $i = 1, \ldots, k$,*
2. *$\beta$ is optimistically coherent iff there exists a plausibility function $Pl : [0,1]^W \to [0,1]$ such that $Pl(e_i) = \beta_i$, for $i = 1, \ldots, k$,*
3. *$\beta$ is coherent in the average iff $\beta$ is state coherent.*

*Proof.* See Appendix B. $\square$

## 4. Refining coherence in the pessimistic case

The threefold criterion of coherence in the aggregate presupposes that an oracle $\mathcal{O}$ is choosing a reliability map among all the possible ones, i.e. in the whole class $\Lambda^+(Ag) = \{\eta : Ag \to [0,1] \mid$ there exists $a \in Ag, \eta(a) > 0\}$. However it may be desirable to capture the extra information that bookmaker and gambler may possess on the reliability of the informative agents. So, instead of belonging to the whole set $\Lambda^+(Ag)$, the reliability degrees attributed to the individual informative agents may be chosen among specified subsets. This section investigates how suitable restrictions to $\Lambda^+(Ag)$ give rise to interesting classes of belief functions.

Building again on the idea behind the Global Health Agency example of Section 1, it is quite natural to consider betting problems in which **B** and **G** agree to bet only when the reliability of informative agents is characterised by one of the following conditions:

- $\Lambda^N(Ag)$ is the set of reliability assignments $\eta$ on $Ag$ which corresponds to the assumption that at least one agent is completely reliable, in other words $\eta(a) = 1$ for some $a \in Ag$ and hence $\eta_a = 1$;

- $\Lambda^C(Ag)$ is the set of of reliability assignments $\eta$ on $Ag$ corresponding to the assumption that at least one agent is completely reliable, as in the previous case, and further that the others are completely unreliable, that is assignments satisfying $\eta(a) \in \{0,1\}$ for every $a \in Ag$.

- Finally, $\Lambda^D(Ag)$ is the set of of reliability assignments $\eta$ on $Ag$ arising from the assumption that there exists *exactly* one agent which is completely reliable, and the others are completely unreliable. This means, it consists of assignments $\eta$ such that there exists $a \in Ag$ with $\eta(a) = 1$, and $\eta(b) = 0$ for all $b \neq a$.

The next lemma is a direct consequence of the above definitions, and therefore we omit the proof. It shows how the restrictions of the set of possible reliability assignments to the above sets is mirrored by similar restrictions on the induced possibility distributions.

**Lemma 4.1.** *Let $\mathcal{E} = (Ag, \overline{w}, \eta)$ be an evaluation tripe, and let $\pi \in \mathscr{P}^+(W)$ as arising from Lemma 3.2, i.e. defined as $\pi(w) = \max\{\eta(a) \mid a \in Ag \text{ such that } w_a = w\}$, for every $w \in W$. The following hold:*

1. *if $\eta \in \Lambda^N(Ag)$ then $\pi$ is normalised, that is, there is a $w \in W$ such that $\pi(w) = 1$*
2. *if $\eta \in \Lambda^C(Ag)$ then $\pi$ is classical, that is $\pi(w) \in \{0,1\}$ for every $w \in W$,*
3. *if $\eta \in \Lambda^D(Ag)$ then $\pi$ is drastic, that is there is a unique $w_0 \in W$ such that $\pi(w_0) = 1$, and $\pi(w) = 0$ for all $w \neq w_0$.*

Finally, it is useful to have the following notation for relevant classes of possibility distributions on $W$:

- $\mathscr{N}(W) = \{\pi \in \mathscr{P}(W)^+ \mid \exists x \in W, \pi(x) = 1\}$, which we call the set of *normalised* possibility distributions.

- $\mathscr{C}(W) = \{\pi \in \mathscr{N}(W) \mid \forall x \in W, \pi(x) = 1 \text{ or } \pi(x) = 0\}$, which we call the set of *classical* (or *crisp*) possibility distributions, and finally

- $\mathscr{D}(W) = \{\pi \in \mathscr{C}(W) \mid \exists! x \in W, \pi(x) = 1\}$, which we call the class of *drastic* possibility distributions.

Obviously the following inclusions hold:

$$\mathscr{D}(W) \subseteq \mathscr{C}(W) \subseteq \mathscr{N}(W) \subseteq \mathscr{P}(W)^+.$$

We are now ready to formulate three refinements of the notion of pessimistic coherence each arising from the three kinds of restriction imposed on the classes of reliability assignments or possibility distributions.

**Definition 4.2** (Refined pessimistic coherence)**.** Let $\{e_1, \ldots, e_k\}$ be events, and let $\beta : e_i \mapsto \beta_i$ be a pessimistically coherent book. Then $\beta$ is said to be $\mathcal{N}$- *(resp. $\mathcal{C}$-, $\mathcal{D}$-) pessimistically coherent* if there exist an evaluating triple $\mathcal{E} = (Ag, \overline{w}, \eta)$ with $\eta \in \Lambda^N(Ag)$ (resp. $\eta \in \Lambda^C(Ag)$, $\eta \in \Lambda^D(Ag)$) such that $\sum_{i=1}^{n} \sigma_i(\beta_i - N_{\mathcal{E}}(e_i)) \geq 0$.

For the sake of a simpler notation, in the remainder of this section we will rely heavily on Lemma 3.2 and Lemma 4.1 which allow us to identify evaluating triples with possibility distributions.

Let $\mathscr{S}(W)$ be a Borel subset of $\mathscr{P}(W)^+$. Note that $\mathscr{N}(W), \mathscr{C}(W)$, and $\mathscr{D}(W)$ are Borel subsets of $\mathscr{P}(W)^+$ since they are compact subsets of $[0,1]^W$. Then we will denote by $Bel^{\mathscr{S}}$ any belief function on $[0,1]^W$

$$Bel^{\mathscr{S}}(\cdot) = \int_{[0,1]^W} \rho_{(\cdot)} \, \mathrm{d}\mu^{\mathscr{S}} \tag{16}$$

whose state assignment has an integral representation given by a $\mu^{\mathscr{S}} : \mathfrak{B}([0,1]^W) \to [0,1]$ such that

$$\mathrm{spt}(\mu^{\mathscr{S}}) = \mathscr{S}(W).$$

Hence $\mu^{\mathscr{S}}(J) = 0$ for all $J \in \mathfrak{B}([0,1]^W)$ such that $J \cap \mathscr{S}(W) = \emptyset$.

**Lemma 4.3.** *Let $e_1, \ldots, e_k$ be events, $\beta : e_i \mapsto \beta_i$ a book and let $\mathscr{S} \in \{\mathscr{N}, \mathscr{C}, \mathscr{D}\}$. Then $\beta$ is $\mathscr{S}$-pessimistic coherent iff there is a $Bel^{\mathscr{S}}$ extending $\beta$.*

*Proof.* See Appendix B. $\qquad\qquad\square$

**Remark 4.4.** Suppose the events $e_1, \ldots, e_k$ in the previous Lemma 4.3 are functions from a finite set $W$ into $[0,1] \cap \mathbb{Q}$, the rational unit interval. In particular, for each $e_i$, let $d_i$ be the least common divisor of all the denominators of $e_i(x)$ (for $x \in W$), and let $m$ be the least common divisor of $d_1, \ldots, d_k$. Then, the $e_i$'s can be regarded, without loss of generality, as functions from $W$ into the finite MV-chain having domain $S_m = \{0, 1/m, \ldots, (m-1)/m, 1\}$. In this case, the set $\mathscr{P}(W)^+$ of all non-zero possibility distributions $\pi : W \to S_m$ is finite, and hence so is each $\mathscr{S}(W)$. Therefore, the belief function $Bel^{\mathscr{S}} : S_m^W \to [0,1]$ can be written as follows: for all $f \in S_m^W$,

$$Bel^{\mathscr{S}}(f) = \sum_{\pi \in S_m^W} \rho_f(\pi) \cdot \mu^{\mathscr{S}}(\{\pi\}).^{14}$$

Therefore, since $\mu^{\mathscr{S}}(\{\pi\}) = 0$ if $\pi \notin \mathscr{S}(W)$, it is reasonable to think at $\mathscr{S}(W)$ as the set to which the focal elements of $Bel^{\mathscr{S}}$ belong. Indeed, recalling Definition 2.8, it is easy to see that, in this particular case, $\mathscr{S}(W) \supseteq \mathrm{spt}(\mu^{\mathscr{S}})$. Obviously, the condition of being in $\mathscr{S}(W)$ is necessary but not sufficient for $\pi$ to be a focal element. In fact, it does not hold in general that $\mu^{\mathscr{S}}(\{\pi\}) > 0$ *iff* $\pi \in \mathscr{S}(W)$.

**Theorem 4.5.** *Let $e_1, \ldots, e_k$ be events and let $\beta : e_i \mapsto \beta_i$ be a book. Then:*

1. *$\beta$ is $\mathscr{N}$-pessimistically coherent iff there exists a normalised belief function $Bel : [0,1]^W \to [0,1]$ which extends $\beta$.*
2. *$\beta$ is $\mathscr{C}$-pessimistically coherent iff there exists a crisp-focal belief function $Bel : [0,1]^W \to [0,1]$ which extends $\beta$.*
3. *$\beta$ is $\mathscr{D}$-pessimistically coherent iff there exists a state $s : [0,1]^W \to [0,1]$ which extends $\beta$.*

*Proof.* For any $\mathscr{S} \in \{\mathscr{N}, \mathscr{C}, \mathscr{D}\}$, from Lemma 4.3 $\beta$ is $\mathscr{S}$-pessimistic coherent iff the belief function $Bel^{\mathscr{S}}(\cdot)$ defined as in (16) extends $\beta$.

We now consider each case in turn:

---

[14]We invite the reader to check [14, Remark 4.10] for further details.

1. Assume that $\mathscr{S} = \mathscr{N}$. Then, $\rho_{\overline{0}}^{\mathscr{N}} : \mathscr{P}(W)^+ \to [0,1]$ maps each $\pi \in \mathscr{P}(W)^+$ into 0 if $\pi \notin \mathscr{N}(W)$, and every $\pi \in \mathscr{N}(W)$ in

$$\inf\{\pi(w) \Rightarrow 0 \mid w \in W\} = \inf\{\neg\pi(w) \mid w \in W\}.$$

Since $\pi \in \mathscr{N}(W)$ iff there is an $w \in W$ such that $\pi(w) = 1$, $\rho_{\overline{0}}^{\mathscr{N}}$ is the zero constant function and hence $Bel^{\mathscr{N}}(\overline{0}) = s(\rho_{\overline{0}}^{\mathscr{N}}) = 0$. Hence $Bel^{\mathscr{N}}$ is normalised.

2. Assume $\mathscr{S} = \mathscr{C}$. Then, similarly to the above case, for every $f \in [0,1]^W$, $\rho_a^{\mathscr{C}}$ maps each $\pi \in \mathscr{P}(W)$ into 0 if $\pi \notin \mathscr{C}(W)$ and if $\pi \in \mathscr{C}(W)$ into,

$$\begin{aligned}
\rho_f^{\mathscr{C}}(\pi) &= \inf\{\pi(w) \Rightarrow f(w) \mid w \in W\} \\
&= \inf\{f(w) \mid \pi(w) = 1\} \\
&= \inf\{f(w) \mid w \in \pi\} \\
&= \hat{\rho}_f(\pi).
\end{aligned}$$

Then the claim follows from Definition 2.4.

3. If $\mathscr{S} = \mathscr{D}$ then, since for every $\pi \in \mathscr{D}(W)$ there exists exactly one $w_0 \in W$ for which $\pi(w_0) = 1$ and $\pi(w) = 0$ for all $w \neq w_0$, we have:

$$\rho_f^{\mathscr{D}}(\pi) = \inf\{\pi(w) \Rightarrow f(w) \mid w \in W\} = f(w_0).$$

Then, for all $f \in [0,1]^W$, identifying possibility distributions from $\mathscr{D}(W)$ and elements from $W$, we have that $\rho_f^{\mathscr{D}}(\cdot) = f(\cdot)$ and hence $Bel^{\mathscr{D}}(f) = s(\rho_f^{\mathscr{D}}) = s(f)$. $\qquad\square$

As a direct consequence of the above Theorem 4.5 we obtain an alternative semantics for Jaffray's coherence under partially resolving uncertainty we discussed in Section 2.1.

**Corollary 4.6.** *Let $e_1, \ldots, e_k$ be two-valued events and let $\beta : e_i \mapsto \beta_i$ be a book. Then $\beta$ is $\mathscr{C}(W)$-coherent iff $\beta$ is coherent under partially resolving uncertainty.*

*Proof.* From Theorem 1.1, we just need to show that $\beta$ is $\mathscr{C}$-pessimistically coherent iff there exists a classical belief function $Bel : 2^W \to [0,1]$ which extends $\beta$. From the above Theorem 4.5, $\beta$ is $\mathscr{C}$-pessimistic coherent iff the exists a crisp-focal belief function $Bel' : [0,1]^W \to [0,1]$ which extends $\beta$. Furthermore, since the events $e_1, \ldots, e_k$ are classical, then the map $Bel : 2^W \to [0,1]$ obtained by restricting $Bel'$ to the Boolean skeleton $2^W$ of $[0,1]^W$ is a classical belief function which extends $\beta$. Hence the claim is settled. $\qquad\square$

Finally let us remark that, although in this section we have focused on refinements of the notion of pessimistic coherence the associated classes of belief functions they characterise, we could formulate analogous refinements for the notion *optimistic* coherence and get their corresponding classes of plausibility functions. However, the strong duality between the notions of pessimistic and optimistic coherence on the one hand, and between belief and plausibility functions on the other hand, would turn this into a rather tedious and uninformative exercise.

## 5. Conclusions

We have put forward a rather general betting method which gives an interpretation and a notion of coherence *á la* de Finetti for a number of measures of uncertainty which are more expressive than subjective probability. In particular we have focussed on the many-valued extension of Dempster-Shafer belief functions and on the theory of states. In addition, suitable particularisations of the same method – essentially motivated by making various assumptions on the information provided by informative agents – led to characterising interesting subclasses of belief functions as well. It is therefore natural to ask whether our betting method is general enough to characterise other well-known measures of uncertainty, or more ambitiously, *all* interesting such measures.

Our initial investigation of this grand question suggests that possibility (and necessity) mesures, as well as imprecise probabilities, do not yield similar results. However further research is needed to understand whether such an ambitious result can be attained at all, or –equally interestingly– why such measures cannot be seen as arising from our betting method.

Possibility distributions, and possibility and necessity measures, can be further used to define a notion of *epistemic indeterminacy* for the events involved in the betting game. Indeed, given a possibility distribution $\pi$ with an associated possibility measure $\Pi$ and necessity measure $N$, we can define, for every event $e_i$, its degree of indeterminacy as

$$I_\pi(e_i) = \Pi_\pi(e_i) - N_\pi(e_i).$$

Hence we say that an event $e_i$ is undetermined if $I_\pi(e_i) = 1$ (i.e. $\Pi_\pi(e_i) = 1$ and $N_\pi(e_i) = 0$), while $e_i$ is determined, whenever $I_\pi(e_i) = 0$ (i.e. $\Pi_\pi(e_i) = N_\pi(e_i)$). Clearly the event $e_i$ is partially undetermined, whenever $0 < I_\pi(e_i) < 1$. Then we can construct a betting game on many-valued events in which Bookmaker is forced to call off bets on those events $e_i$ for which $I_\pi(e_i) = 0$ whilst paying to Gambler, on each other $e_j$, a monetary amount weighted by $I_\pi(e_j)$. In other words, a betting game in which, given a book $\beta : e_i \mapsto \beta(e_i) \in [0,1]$ and a possibility distribution $\pi \in \mathscr{P}(W)^+$, the total balance for **B** is calculated through

$$\sum_{i=1}^{k} (1 - I_\pi(e_i)) \cdot \sigma_i \cdot (\beta(e_i) - N_\pi(e_i)).$$

This clearly results in a *conditional* game similar to the one presented in [23] which however calculates the degree of indeterminacy in a different way. The investigation of this approach to conditional indeterminacy will have to be postponed for future work.

## References

[1] L. Biacino. Fuzzy subsethood and belief functions of fuzzy events. *Fuzzy Sets and Systems* 158(1): 38–49, 2007.

[2] D. Butnariu, E. P. Klement. *Triangular Norm Based Measures and Games with Fuzzy Coalitions*. Kluwer, Dordrecht, 1993.

[3] C. C. Chang, Algebraic Analysis of Many-valued Logics. *Trans. Am. Math. Soc.* 88, 467–490, 1958.

[4] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, *Algebraic Foundations of Many-valued Reasoning.* Kluwer, Dordrecht, 2000.

[5] B. de Finetti, *Theory of Probability*, Vol.1, Wiley, New York, 1974.

[6] B. de Finetti, *Probability, Induction and Statistics* , Wiley, New York, 1972.

[7] D. Dubois, H. Prade. Evidence Measures Based on Fuzzy Information. *Automatica* 21(5), 547–562, 1985.

[8] D. Dubois, H. Prade. Possibility theory as a basis for qualitative decision theory. Proc. of the 14th Inter. Joint Conf. on Artificial Intelligence (IJCAI'95), Montral, Canada, Aug. 2025, 1924-1930, 1995.

[9] D. Dubois, H. Prade. On the use of aggregation operations in information fusion processes. *Fuzzy Sets and Systems* 142, 143–161, 2004.

[10] D. Dubois, L. Godo, H. Prade, and A. Zapico. On the possibilistic- based decision model: From decision under uncertainty to case-based decision. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 7(6): 631–670, 1999.

[11] D. Dubois, H. Prade, R. Sabbadin, Decision-theoretic foundations of possibility theory. *European J. Oper. Res.* 128, 459–478, 2001.

[12] T. Flaminio, L. Godo, H. Hosni. On the logical structure of de Finetti's notion of event. *Journal of Applied Logic* 12 (3), 279–301, 2104.

[13] T. Flaminio, L. Godo, T. Kroupa. Combination and Soft-Normalization of Belief Functions on MV-Algebras. MDAI 2012: 23-34, 2012.

[14] T. Flaminio, L. Godo, T. Kroupa. Belief functions on MV-algebras of fuzzy sets: an overview. In *Non-Additive Measures, theory and applications.* V. Torra, Y. Narukawa, M. Sugeno (ed.). Studies in Fuzziness and Soft Computing Volume 310 - Springer, pp. 173-200, 2014.

[15] T. Flaminio, L. Godo, E. Marchioni. On the logical formalization of possibilistic counterpart of states over n-valued Łukasiewicz events. J. Logic Comput. 21(3):447–464, 2011.

[16] T. Flaminio, L. Godo, E. Marchioni. Belief Functions on MV-Algebras of Fuzzy Events Based on Fuzzy Evidence. In Proceedings of ECSQARU 2011. Lecture Notes in Artificial Intelligence 6717, Weiru Liu (Ed.):628–639, 2011.

[17] T. Flaminio, L. Godo, E. Marchioni. A logic for belief functions on MV-algebras. International Journal of Approximate Reasoning, 54(4): 491–512, 2013.

[18] C. Hwang and M. Yang. Generalization of Belief and Plausibility Functions to Fuzzy Sets Based on the Sugeno Integral. *International Journal of Intelligent Systems* 22: 1215–1228, 2007.

[19] J-Y. Jaffray. Coherent bets under partially resolving uncertainty and belief functions. Theory and Decision 26, 99–105, 1989.

[20] T. Kroupa. Every state on semisimple MV-algebra is integral. Fuzzy Sets and Systems, 157:2771–2782, 2006.

[21] T. Kroupa, From Probabilities to Belief Functions on MV-algebras, in Combining Soft Computing and Statistical Methods in Data Analysis, volume 77 of Advances in Soft Computing: 387–394. Springer-Verlag, 2010.

[22] T. Kroupa, Extension of Belief Functions to Infinite-valued Events. Soft Computing 16 (11), pp. 1851–1861, 2012.

[23] F. Montagna. Partially Undetermined Many-Valued Events and Their Conditional Probability. *Journal of Philosophical Logic* 41:563–593, 2012.

[24] D. Mundici, Averaging the truth-value in Łukasiewicz logic. *Studia Logica* 55 (1):113–127, 1995.

[25] D. Mundici, Bookmaking over infinite-valued events. *International Journal of Approximate Reasoning* 43: 223–240, 2006.

[26] D. Mundici, *Advanced Łukasiewicz calculus and MV-algebras.* Trends in Logic, vol 35. Springer, 2011.

[27] M. Navara. Triangular norms and measures of fuzzy sets. In E.P. Klement and R. Mesiar, editors, *Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms*, 345–390. Elsevier, 2005.

[28] G. Panti. Invariant measures on free MV-algebras. *Communications in Algebra* 36(8), 2849–2861, 2008.

[29] J. B. Paris. A note on the Dutch Book method, Revised version of a paper of the same title which appeared in The Proceedings of the Second Internat. Symp. on Imprecise Probabilities and their Applications, ISIPTA01, Ithaca, New York, 2001.

[30] G. Shafer. *A Mathematical Theory of Evidence.* Princeton University Press, Princeton 1976.

[31] P. Smets. The Degree of Belief in a Fuzzy Event. *Information Sciences*, 25: 1–19, 1981.

[32] J. Yen. Generalizing the Dempster-Shafer theory to fuzzy sets. *IEEE Transactions on Systems, Man, and Cybernetics*, 20(3): 559–570, 1990.

[33] J. Yen. Computing Generalized Belief Functions for Continuous Fuzzy Sets. *Int. J. Approx. Reasoning* 6: 1–31,

1992.

[34] L. A. Zadeh. Fuzzy sets and information granularity. In *Advances in Fuzzy Sets Theory and Applications*, (M. Gupta et al. eds), North Holland: 3–18, 1979.

[35] L.A. Zadeh. Probability measures of fuzzy events. *J. Math. Anal. Appl.* 23, 421–427, 1968.

## Appendix A: Łukasiewicz logic and MV-algebras

Infinitely-valued Łukasiewicz logic is one of the most well known and studied fuzzy logics, whose algebraic semantics is provided by the class of the so-called MV-algebras [3, 4, 26]. For present purposes, by many-valued events we refer to elements of particular MV-algebras of fuzzy sets. This Appendix collects some basic facts about MV-algebras and the generalisation of the notion of (finitely additive) probabilities to the MV-algebraic realm.

An MV-algebra is a structure $M = (M, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following equations:

**(MV1)** $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

**(MV2)** $x \oplus y = y \oplus x$,

**(MV3)** $x \oplus 0 = x$,

**(MV4)** $\neg\neg x = x$,

**(MV5)** $x \oplus \neg 0 = \neg 0$,

**(MV6)** $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Further (definable) operations can be defined from $\oplus, \neg$ and 0 In particular: $x \Rightarrow y = \neg x \oplus y$, $x \odot y = \neg(\neg x \oplus \neg y)$; $x \vee y = \neg(\neg x \oplus y) \oplus y$; $x \wedge y = \neg(\neg x \vee \neg y)$; $1 = \neg 0$. In any MV-algebra $M$, a partial order is definable by the following stipulation: for all $x, y \in M$, $x \leq y$ if and only if $x \Rightarrow y = 1$.

Let $M$ be an MV-algebra. Then a non-empty subset $\mathfrak{f}$ of $M$ is said to be a *filter* of $M$ iff: (i) $1 \in \mathfrak{f}$, (ii) if $x, y \in \mathfrak{f}$, then $x \odot y \in \mathfrak{f}$, and (iii) if $x \in \mathfrak{f}$ and $y \geq x$, then $y \in \mathfrak{f}$. A filter $\mathfrak{f}$ of an MV-algebra $M$ is said to be *proper*, if $\mathfrak{f} \neq M$. A filter $\mathfrak{m}$ is said to be a *maximal filter* (or an *ultrafilter*) whenever for any proper filter $\mathfrak{f}$ such that $\mathfrak{f} \supseteq \mathfrak{m}$, either $\mathfrak{f} = M$, or $\mathfrak{f} = \mathfrak{m}$. The set of all ultrafilters of an MV-algebra $M$ will be henceforth denoted by $Max(M)$, or, when there is no danger of confusion, simply by $Max$. For every MV-algebra $M$, the set $Max(M)$ is non-empty and it can be endowed with a compact Hausdorff topology, the so-called *spectral topology*: for an arbitrary filter $\mathfrak{f}$ of $M$, any set of the form $O_{\mathfrak{f}} = \{\mathfrak{m} \in Max(M) : \mathfrak{m} \not\supseteq \mathfrak{f}\}$ is open in this topology.

The intersection of all the maximal filters of an MV-algebra $M$ is called the *radical* of $M$ and it is usually written $Rad(M)$. An MV-algebra $M$ is *semisimple* whenever $Rad(M) = \{1\}$. It is well-known (see [4] for instance) that the congruences lattice and the filters lattice of any MV-algebra $M$ are mutually isomorphic, via the isomorphism which associates to every congruence[15] $\theta$ the filter $\mathfrak{f}_{\theta} = \{x \in M \mid (x, 1) \in \theta\}$.

**Example 5.1.** The following are four relevant examples of MV-algebras:

---

[15] A congruence $\theta$ in a MV-algebra $M$ is an equivalence relation on $M$ respecting the operations, i.e. if $(x, y) \in \theta$ then $(\neg x, \neg y) \in \theta$, and if $(x, y), (x', y') \in \theta$ then $(x \oplus x', y \oplus y') \in \theta$.

(1) Every Boolean algebra is an MV-algebra, and moreover for every MV-algebra $M$, the set $B(M) = \{x \in M : x \oplus x = x\}$ of its idempotent elements is the domain of the largest Boolean subalgebra of $M$. The algebra having $B(M)$ as universe is usually called the *Boolean skeleton* of $M$.

(2) Define on the real unit interval $[0, 1]$ the operations $\oplus$ and $\neg$ as follows: for all $x, y \in [0, 1]$,

$$x \oplus y = \min\{1, x + y\}, \text{ and } \neg x = 1 - x.$$

Then the structure $[0, 1]_{MV} = ([0, 1], \oplus, \neg, 0)$ is an MV-algebra. The MV-algebra $[0, 1]_{MV}$ is generic for the variety of MV-algebras (i.e. it generates the whole variety) and is usually called the *standard* MV-algebra. In equivalent terms, Łukasiewicz logic is complete with respect to the semantics defined by the standard MV-algebra.

(3) Fix a $k \in \mathbb{N}$, and let $F(k)$ be the set of all the McNaughton functions (cf. [4]) from the hypercube $[0, 1]^k$ into $[0, 1]$. In other words, $F(k)$ is the set of all those functions $f : [0, 1]^k \to [0, 1]$ which are continuous, piecewise linear and such that each piece has integer coefficients. The following pointwise operations defined on $F(k)$,

$$(f \oplus g)(x) = \min\{1, f(x) + g(x)\}, \text{ and } (\neg f)(x) = 1 - f(x),$$

make the structure $\mathcal{F}(k) = (F(k), \oplus, \neg, \overline{0})$ into an MV-algebra, where $\overline{0}$ clearly denotes the function constantly equal to 0. Actually, $\mathcal{F}(k)$ is the free MV-algebra over $k$ generators.

(4) Let $W$ be a non-empty set, and let $[0, 1]^W$ the set of all functions from $W$ into $[0, 1]$, endowed with operations defined by the pointwise application of those in $[0, 1]_{MV}$. The structure $[0, 1]^W$ is clearly MV-algebra. Every MV-subalgebra of $[0, 1]^W$ is called an *MV-clan* (cf. [2, 27]).

It is worth noticing that in $[0, 1]_{MV}$, the standard interpretation of the lattice operations of $\wedge$ and $\vee$, is respectively in terms of min and max. As a consequence of this observation we will freely alternate between the notations $\wedge$ and min on the one hand, and $\vee$ and max on the other hand.

A semisimple MV-algebra $[0, 1]^W$ is said to be *separating* provided that for each $w_1 \neq w_2 \in W$, there exists a $f \in [0, 1]^W$ such that $f(w_1) \neq f(w_2)$. The following holds.

**Theorem 5.2** ([4])**.** *Let $W$ be a compact Hausdorff space and $M$ be a separating MV-subalgebra of the algebra $C(W)$ of continuous functions from $W$ to $[0, 1]$. Then there exists a one-one correspondence between the points of $W$ and the class $\mathcal{H}(M, [0, 1]_{MV})$ of homomorphisms of $M$ in the standard MV-algebra $[0, 1]_{MV}$.*

## Appendix B: Proofs

*Proof of Theorem 3.7*

*Proof.* 1. The book $\beta$ is pessimistically coherent iff for every $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$ there exists $\mathcal{E} = (Ag, \eta, \overline{w})$ such that $\sum_{i=1}^k \sigma_i(\beta_i - N_{\mathcal{E}}(e_i)) \geq 0$. By Lemma 3.2, there is a possibility distribution $\pi : W \to [0, 1]$ such that $N_{\mathcal{E}}(\cdot) = N_\pi(\cdot)$ over $\{e_1, \ldots, e_k\}$. Now, by Proposition 2.5 (1), $N_{\mathcal{E}}(\cdot) =$

$\rho_{(\cdot)}(\pi)$ over $\{e_1, \ldots, e_k\}$, and hence, $\beta$ is pessimistically coherent iff for every $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$ there exists a $\pi \in [0,1]^W$ such that $\sum_{i=1}^{k} \sigma_i(\beta_i - \rho_{e_i}(\pi)) \geq 0$. By Theorem 5.2 (see Appendix A) and since $\mathcal{R}(W)$ is an MV-subalgebra of $[0,1]^{[0,1]^W}$ of continuous functions, $\sum_{i=1}^{k} \sigma_i(\beta_i - \rho_{e_i}(\pi)) \geq 0$ iff there exists an $h \in \mathcal{H}(\mathcal{R}(W), [0,1]_{MV})$ such that

$$\sum_{i=1}^{k} \sigma_i(\beta_i - h(\rho_{e_i})) \geq 0,$$

Finally, by Lemma 3.6, the latter holds iff $\beta$ extends to a belief function $Bel : [0,1]^W \to [0,1]$.

2. By an analogous argument and [15, Definition 3.2], $\beta$ is optimistically coherent iff there exists a state $s : \mathcal{R}(W) \to [0,1]$ such that, for all $i = 1, \ldots, k$, $\beta_i = s(1 - \rho_{\neg e_i})$ iff there exists a plausibility function $Pl : [0,1]^W \to [0,1]$ extending $\beta$.

3. Assume $\beta$ to be coherent in the average. Then, following the lines of the previous proofs, we can find a state $s : [0,1]^{\mathscr{P}(W)^+} \to [0,1]$ such that, for all $i = 1, \ldots, k$, $\beta_i = s(M^{(\cdot)}(e_i))$. Let us call $\gamma_s : [0,1]^W \to [0,1]$ the map such that, for all $f \in [0,1]^W$, $\gamma_s(f) = s(M_{(\cdot)}(f))$. Then it is left to show that $\gamma_s$ is a state. Lemma 3.2 (3) shows that $M_{(\cdot)}(1) = 1$ and, if $f, g \in [0,1]^W$ and $f \odot g = 0$, then, for every $\pi \in \mathscr{P}(W)^+$, $M_\pi(f \oplus g) = M_\pi(f) + M_\pi(g) = M_\pi(f) \oplus M_\pi(g)$. Then, $M_{(\cdot)}(f \oplus g) = M_{(\cdot)}(f) + M_{(\cdot)}(g)$. Hence, we also have that $M_{(\cdot)}(f) \odot M_{(\cdot)}(g) = 0$. Therefore,

$$\begin{aligned} \gamma_s(f \oplus g) &= s(M_{(\cdot)}(f \oplus)g) \\ &= s(M_{(\cdot)}(f) \oplus M_{(\cdot)}(g)) \\ &= s(M_{(\cdot)}(f)) + s(M_{(\cdot)}(g)) \\ &= \gamma_s(f) + \gamma_s(g) \end{aligned}$$

and hence $\gamma_s$ is a state and $\beta$ is state-coherent via Theorem 2.3.

Conversely, assume $\beta$ to be state-coherent. Then there exists an state $s$ on $[0,1]^X$ such that $s(e_i) = \beta_i$ for each $i = 1, \ldots, k$. Since we assume $W$ finite, let $p$ be its corresponding probability distribution on $W$, i.e. such that $s(f) = \sum_{w \in W} f(w) \cdot p(w)$. Now let us consider $\mathcal{E}$ putting $Ag = W$, $\eta = p$ and $w_a = a$ for each $a \in Ag$. It is readily to see that $\beta_i = s(e_i) = \sum_{w \in W} e_i(w) \cdot p(w) = \sum_{a \in Ag} w_a(e_i) \cdot \eta(a) = M_\mathcal{E}(e_i)$. Therefore we have that $\sum_{i=1}^{n} \sigma_i(\beta_i - M_\mathcal{E}(e_i)) = 0$, and the book is coherent in the average. $\qquad \square$

*Proof of Lemma 4.3*
*Proof.* The book $\beta$ is $\mathscr{S}$-pessimistic coherent iff, by definition, for all $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$ there exists an evaluating triple $(Ag, \eta, \overline{w})$ such that $\sum_{i=1}^{k} \sigma_i(\beta_i - \rho_{e_i}(\pi)) \geq 0$. By Lemmas 4.1, there is a $\pi \in \mathscr{S}(W)$ such that $\sum_{i=1}^{k} \sigma_i(\beta_i - \rho_{e_i}(\pi)) \geq 0$. Let us define, for every $f \in [0,1]^W$, the map $\rho_f^{\mathscr{S}} : \mathscr{P}(W)^+ \to [0,1]$ in the following way: for all $\pi \in \mathscr{P}(W)^+$,

$$\rho_f^{\mathscr{S}}(\pi) = \begin{cases} \rho_f(\pi) & \text{if } \pi \in \mathscr{S}(W) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\beta$ is $\mathscr{S}$-pessimistic coherent iff

$$\sum_{i=1}^{k} \sigma_i(\beta_i - \rho_{e_i}^{\mathscr{S}}(\pi)) \geq 0. \tag{17}$$

Let $\mathcal{R}^{\mathscr{S}}$ be the MV-algebra $[0,1]^{\mathscr{S}(W)}$. The following is immediate to check.

**Claim 1.** *For all $\pi \in \mathscr{S}(W)$, the map $h_\pi : f \in \mathcal{R}^{\mathscr{S}} \mapsto f(\pi)$ is an MV-homomorphism of $\mathcal{R}^{\mathscr{S}}$ into $[0,1]_{MV}$.*

*Proof.* (of Claim 1) Clearly the map $h_\pi$ maps $\mathcal{R}^{\mathscr{S}}$ into $[0,1]_{MV}$. Moreover, $h_\pi$ is an MV-homomorphism. In fact, if $f, g \in \mathcal{R}^{\mathscr{S}}$, then $h_\pi(f \oplus g) = (f \oplus g)(\pi) = f(\pi) \oplus g(\pi) = h_\pi(f) \oplus h_\pi(g)$. Similarly $h_\pi(0) = 0$ and $h_\pi(\neg f) = 1 - h_\pi(f)$. $\square$

Claim 1 and equation (17) show that $\beta$ is $\mathscr{S}$-pessimistic coherent iff for all $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$, there is a homomorphism $h : \mathcal{R}^{\mathscr{S}} \to [0,1]_{MV}$ such that

$$\sum_{i=1}^{k} \sigma_i(\beta_i - h(\rho_{e_i}^{\mathscr{S}})) \geq 0,$$

iff, from Theorem 2.3, there exists a state $\hat{s} : \mathcal{R}^{\mathscr{S}} \to [0,1]$ extending $\beta$. Notice that, if we denote by $\hat{\mu} : \mathfrak{B}(\mathscr{S}(W)) \to [0,1]$ that unique regular Borel probability measure which characterizes $\hat{s}$ through Theorem 2.1, then $\hat{\mu}$ uniquely extends to a $\mu^{\mathscr{S}} : \mathfrak{B}([0,1]^W) \to [0,1]$ such that $\mathrm{spt}(\mu^{\mathscr{S}}) = \mathscr{S}(W)$ setting, for all $J \in \mathfrak{B}([0,1]^W)$,

$$\mu^{\mathscr{S}}(J) = \hat{\mu}(J \cap \mathscr{S}(W)).$$

Notice that, since $J, \mathscr{S}(W) \in \mathfrak{B}([0,1]^W)$, then $J \cap \mathscr{S}(W) \in \mathfrak{B}(\mathscr{S}(W))$ and hence $\mu^{\mathscr{S}}$ is well defined. Moreover, for all $i = 1, \ldots, k$,

$$\beta_i = \hat{s}(\rho_{e_i}^{\mathscr{S}}) = \int_{\mathscr{S}(W)} \rho_{e_i}^{\mathscr{S}} \, \mathrm{d}\hat{\mu} = \int_{[0,1]^W} \rho_{e_i}^{\mathscr{S}} \, \mathrm{d}\mu^{\mathscr{S}} = \int_{[0,1]^W} \rho_{e_i} \, \mathrm{d}\mu^{\mathscr{S}} = Bel^{\mathscr{S}}(e_i).$$

Hence our claim is settled. $\square$