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Coherent conditional probability in a fuzzy logic setting

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Abstract

Very recently, a (fuzzy modal) logic to reason about coherent conditional probability, in the sense of de Finetti, has been introduced by the authors. Under this approach, a conditional probability $\mu(\cdot \mid \cdot)$ is taken as a primitive notion defined over conditional events of the form "φ *given* ψ", $\varphi|\psi$ for short, where ψ is not the impossible event. The logic, called FCP(ŁΠ), exploits an idea already used by Hájek and colleagues to define a logic for (unconditional) probability in the framework of fuzzy logics. Namely, we take the probability of the conditional event "$\varphi|\psi$" as the truth-value of the (fuzzy) modal proposition $P(\varphi \mid \psi)$, read as "$\varphi|\psi$ is probable". The logic FCP(ŁΠ), which is built up over the many-valued logic ŁΠ$\frac{1}{2}$ (a logic which combines the well-known Łukasiewicz and Product fuzzy logics), was shown to be complete for modal theories with respect to the class of probabilistic Kripke structures induced by coherent conditional probabilities. Indeed, checking coherence of a (generalized) probability assessment to an arbitrary family of conditional events becomes tantamount to checking consistency of a suitably defined theory over the logic FCP(ŁΠ). In this paper we provide further results for the logic FCP(ŁΠ). In particular, we extend the previous completeness result by allowing the presence of non-modal formulas in the theories, which are used to describe logical relationships among events. This increases the knowledge modelling power of FCP(ŁΠ). Then, we improve the results concerning checking consistency of suitably defined theories in FCP(ŁΠ) to determine coherence by showing parallel results w.r.t. the notion of *generalized coherence* when dealing with imprecise assessments. Moreover we also show and discuss compactness results for our logic. Finally, FCP(ŁΠ) is shown to be a powerful tool for knowledge representation. Indeed, following ideas already investigated in the related literature, we show how FCP(ŁΠ) allows the definition of suitable notions of default rules which enjoy the core properties of nonmonotonic reasoning characterizing system **P** and **R**.

Keywords: Fuzzy Logics, Conditional Probability, Coherence, Generalized Coherence, Compactness, Default Reasoning.

1 Introduction: conditional probability and fuzzy logic

Probability theory is certainly the most well-known and deeply investigated formalism between those that aim at modelling reasoning under uncertainty. Such a research has had a remarkable influence also in the field of logic. Indeed, many logics which allow reasoning about probability have been proposed, some of them rather early. We may cite [3, 13, 14, 17, 18, 25, 30, 37–43] as some of the most relevant references. Besides, it is worth mentioning the recent book [27] by Halpern, where a deep investigation of uncertainty representations (not only probability) and uncertainty logics is presented. In general, all the above logical

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formalisms present some kind of probabilistic operators but all of them, with the exception of [18], are based on the classical two valued-logic.

An alternative treatment, originally proposed in [23] and further elaborated in [22] and in [20], allows the axiomatization of uncertainty measures in the framework of fuzzy logic. The basic idea is to consider, for each classical (two-valued) proposition φ, a (fuzzy) modal proposition $P\varphi$, which reads "φ is probable", and taking as truth-degree of $P\varphi$ the probability of φ. Then one can define theories about the $P\varphi$'s over a particular fuzzy logic including, as axioms, formulas corresponding to the basic postulates of probability theory. The advantage of such an approach, with respect to the previously mentioned ones, is that algebraic operations needed to compute with probabilities (or with any other uncertainty model) are embedded in the connectives of the many-valued logical framework, resulting in clear and elegant formalizations.

In reasoning with probability, a crucial issue concerns the notion of *conditional probability*. Traditionally, given a probability measure μ on an algebra of possible worlds W, if the agent observes that the actual world is in $A \subseteq W$, then the updated probability measure $\mu(\cdot \mid A)$, called conditional probability, is defined as $\mu(B \mid A) = \mu(B \cap A)/\mu(A)$, provided that $\mu(A) > 0$. If $\mu(A) = 0$ the conditional probability remains then undefined. This yields both philosophical and logical problems. For instance, in [20] where the logic FP(ŁΠ) is presented, conditional probability statements are handled by formulas $P(\varphi \mid \psi)$ which denote an abbreviation for $P\psi \to_{\Pi} P(\varphi \wedge \psi)$. Such a definition exploits the properties of Product logic implication $\to_{\Pi}$, whose truth function behaves like a truncated division:

$$e(\Phi \to_{\Pi} \Psi) = \begin{cases} 1, & \text{if } e(\Phi) \leq e(\Psi) \\ e(\Psi)/e(\Phi), & \text{otherwise.} \end{cases}$$

However, with such a logical modelling, whenever the probability of the conditioning event χ is 0, $P(\varphi \mid \chi)$ takes as truth-value 1. Therefore, this yields problems when dealing with zero probabilities. Two well-known proposals which aim at solving this problem consist in either adopting a non-standard probability approach (where events are measured on the hyper-real interval [0, 1] rather than on the usual real interval), or in taking conditional probability as a primitive notion. In the first case [26, 28, 33], the assignment of zero probability is only allowed to impossible events, while other events can take on an infinitesimal probability. This clearly permits to avoid situations in which the conditioning event has null probability. The second approach (that goes back to de Finetti, Rényi and Popper among others) considers conditional probability and conditional events as basic notions, not derived from the notion of unconditional probability, and provides adequate axioms. Coletti and Scozzafava's book [10] includes a rich elaboration of different issues of reasoning with *coherent* conditional probability, i.e. conditional probability in de Finetti's sense. We take from there the following definition.

Definition 1.1 ([10])
Let $\mathcal{G}$ be a Boolean algebra and let $\mathcal{B} \subseteq \mathcal{G}$ be closed with respect to finite unions (additive set). Let $\mathcal{B}^0 = \mathcal{B} \setminus \{\emptyset\}$. A *conditional probability* on the set $\mathcal{G} \times \mathcal{B}^0$ of conditional events, denoted as $E|H$, is a function $\mu : \mathcal{G} \times \mathcal{B}^0 \to [0, 1]$ satisfying the following axioms:

(1) $\mu(H \mid H) = 1$, for all $H \in \mathcal{B}^0$
(2) $\mu(\cdot \mid H)$ is a (finitely additive) probability on $\mathcal{G}$ for any given $H \in \mathcal{B}^0$
(3) $\mu(E \cap A \mid H) = \mu(E \mid H) \cdot \mu(A \mid E \cap H)$, for all $A \in \mathcal{G}$ and $E, H, E \cap H \in \mathcal{B}^0$.

Two different logical treatments based respectively on the above two solutions have been recently proposed, both using a fuzzy logic approach as the one mentioned above. In fact, they both define probability logics over the fuzzy logic $Ł\Pi\frac{1}{2}$, which combines the well-known Łukasiewicz and Product fuzzy logics. In [15] Flaminio and Montagna introduced the logic FP(SŁΠ) whose models include non-standard probabilities. On the other hand, we introduced in [36] the logic FCP(ŁΠ) in whose models conditional probability is a primitive notion. FCP(ŁΠ) is equipped with a modal operator P directly defined over conditional events of the form $\varphi|\chi$. Unconditional probability, then, arises as non-primitive whenever the conditioning event is a (classical) tautology. The obvious reading of a statement like $P(\varphi \mid \chi)$ is "the conditional event "φ given χ" is probable". Similarly to the case mentioned above, the truth-value of $P(\varphi \mid \chi)$ is given by the conditional probability $\mu(\varphi \mid \chi)$. A completeness result of FCP(ŁΠ) with respect to a class of Kripke structures suitably equipped with a conditional probability is shown in [36]. Moreover, it is also shown that checking the *coherence* of an assessment to a family of conditional events, in the sense of de Finetti, Coletti and Scozzafava[1], is tantamount to checking consistency[2] of a suitably defined theory in FCP(ŁΠ).

In this paper, after this introduction, we provide in the next section the necessary background notions about the fuzzy logic $Ł\Pi\frac{1}{2}$. In the third and fourth sections we present the logic FCP(ŁΠ), we review its semantics, solving some technical problems in [36], and we enhance the completeness result given in [36]. Indeed, in that paper completeness was proved with respect to (finite) modal theories, i.e. theories only including modal (probabilistic) formulas. Although they are the most interesting kind of formulas, this clearly restricted the type of deductions allowed. In Section 4, we provide completeness results for general (finite) theories, i.e. theories including both modal and non-modal formulas, and adapted to the modified semantics. This will allow to represent logical relationships between events in the theories. In Section 5, we will be concerned with coherence of both precise and imprecise assessments of conditional probability. Starting from the result in [36] which makes explicit the link between coherence of rational assessments and theories in our logic, we prove that such a result can be generalized so as to deal with imprecise assessments of probability, i.e. all those situations in which we cannot provide but lower (or upper) bounds for the assessments. We also see how to capture the concepts of lower and upper coherent conditional probabilities presented in [10] under our framework. Moreover, in Section 6 we generalize a result obtained by Flaminio [16] on the compactness of our logic for coherent assessments, and we discuss a different approach to obtain similar compactness results. To conclude, in Section 7 we show that FCP(ŁΠ) is a powerful tool from the knowledge representation point of view. Indeed, many complex statements, both quantitative and qualitative, concerning conditional probabilities can be represented, as well as suitable notions of default rules which capture the core properties of nonmonotonic reasoning carved in system **P** and in some extension.

[1]Roughly speaking, an assessment to an arbitrary family of conditional events is called coherent when it can be extended to a whole conditional probability [10].

[2]Notice that this is just a formal equivalence with no aim of providing a study of computational complexity of the coherence test of conditional probabilities.

2 Preliminaries: the $Ł\Pi\frac{1}{2}$ logic

The language of the ŁΠ logic is built in the usual way from a countable set of propositional variables, three binary connectives $\rightarrow_L$ (Łukasiewicz implication), $\odot$ (Product conjunction) and $\rightarrow_\Pi$ (Product implication), and the truth constant $\bar{0}$. A truth-evaluation is a mapping e that assigns to every propositional variable a real number from the unit interval [0, 1] and extends to all formulas as follows:

$$\begin{array}{ll} e(\bar{0}) = 0, & e(\varphi \rightarrow_L \psi) = \min(1 - e(\varphi) + e(\psi), 1), \\ e(\varphi \odot \psi) = e(\varphi) \cdot e(\psi), & e(\varphi \rightarrow_\Pi \psi) = \begin{cases} 1, & \text{if } e(\varphi) \leq e(\psi) \\ e(\psi)/e(\varphi), & \text{otherwise} \end{cases}. \end{array}$$

The truth constant $\bar{1}$ is defined as $\varphi \rightarrow_L \varphi$. In this way we have $e(\bar{1}) = 1$ for any truth-evaluation e. Moreover, many other connectives can be defined from those introduced above:

$$\begin{array}{rll@{\qquad}rll} \neg_L\varphi & is & \varphi \rightarrow_L \bar{0}, & \neg_\Pi\varphi & is & \varphi \rightarrow_\Pi \bar{0}, \\ \varphi \wedge \psi & is & \varphi\&(\varphi \rightarrow_L \psi), & \varphi \vee \psi & is & \neg_L(\neg_L\varphi \wedge \neg_L\psi), \\ \varphi \oplus \psi & is & \neg_L\varphi \rightarrow_L \psi, & \varphi\&\psi & is & \neg_L(\neg_L\varphi \oplus \neg_L\psi), \\ \varphi \ominus \psi & is & \varphi\&\neg_L\psi, & \varphi \equiv \psi & is & (\varphi \rightarrow_L \psi)\&(\psi \rightarrow_L \varphi), \\ \Delta\varphi & is & \neg_\Pi\neg_L\varphi, & \nabla\varphi & is & \neg_\Pi\neg_\Pi\varphi, \end{array}$$

with the following interpretations:

$$\begin{array}{ll} e(\neg_L\varphi) = 1 - e(\varphi), & e(\neg_\Pi\varphi) = \begin{cases} 1, & \text{if } e(\varphi) = 0 \\ 0, & \text{otherwise} \end{cases}, \\ e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi)), & e(\varphi \vee \psi) = \max(e(\varphi), e(\psi)), \\ e(\varphi \oplus \psi) = \min(1, e(\varphi) + e(\psi)), & e(\varphi\&\psi) = \max(0, e(\varphi) + e(\psi) - 1), \\ e(\varphi \ominus \psi) = \max(0, e(\varphi) - e(\psi)), & e(\varphi \equiv \psi) = 1 - |e(\varphi) - e(\psi)|, \\ e(\Delta\varphi) = \begin{cases} 1, & \text{if } e(\varphi) = 1 \\ 0, & \text{otherwise} \end{cases}, & e(\nabla\varphi) = \begin{cases} 1, & \text{if } e(\varphi) > 0 \\ 0, & \text{otherwise} \end{cases}. \end{array}$$

The logic ŁΠ is defined Hilbert-style as the logical system whose axioms and rules are the following[3]:

(1) Axioms of Łukasiewicz Logic:
  (Ł1) $\varphi \rightarrow_L (\psi \rightarrow_L \varphi)$
  (Ł2) $(\varphi \rightarrow_L \psi) \rightarrow_L ((\psi \rightarrow_L \chi) \rightarrow_L (\varphi \rightarrow_L \chi))$
  (Ł3) $(\neg_L\varphi \rightarrow_L \neg_L\psi) \rightarrow_L (\psi \rightarrow_L \varphi)$
  (Ł4) $((\varphi \rightarrow_L \psi) \rightarrow_L \psi) \rightarrow_L ((\psi \rightarrow_L \varphi) \rightarrow_L \varphi)$
(2) Axioms of Product Logic[4]:
  (A1) $(\varphi \rightarrow_\Pi \psi) \rightarrow_\Pi ((\psi \rightarrow_\Pi \chi) \rightarrow_\Pi (\varphi \rightarrow_\Pi \chi))$
  (A2) $(\varphi \odot \psi) \rightarrow_\Pi \varphi$
  (A3) $(\varphi \odot \psi) \rightarrow_\Pi (\psi \odot \varphi)$

[3]This definition, proposed in [8], is actually a simplified version of the original definition of ŁΠ given in [11].
[4]Actually Product logic axioms also include axiom A7 $[\bar{0} \rightarrow_\Pi \varphi]$ which is redundant in ŁΠ.

(A4) $(\varphi \odot (\varphi \to_\Pi \psi)) \to_\Pi (\psi \odot (\psi \to_\Pi \varphi))$
(A5a) $(\varphi \to_\Pi (\psi \to_\Pi \chi)) \to_\Pi ((\varphi \odot \psi) \to_\Pi \chi)$
(A5b) $((\varphi \odot \psi) \to_\Pi \chi) \to_\Pi (\varphi \to_\Pi (\psi \to_\Pi \chi))$
(A6) $((\varphi \to_\Pi \psi) \to_\Pi \chi) \to_\Pi (((\psi \to_\Pi \varphi) \to_\Pi \chi) \to_\Pi \chi)$
(Π1) $\neg_\Pi\neg_\Pi\chi \to_\Pi (((\varphi \odot \chi) \to_\Pi (\psi \odot \chi)) \to_\Pi (\varphi \to_\Pi \psi))$
(Π2) $\varphi \wedge \neg_\Pi\varphi \to_\Pi \overline{0}$

(3) The following additional axioms relating Łukasiewicz and Product logic connectives:
(¬) $\neg_\Pi\varphi \to_L \neg_L\varphi$
(Δ) $\Delta(\varphi \to_L \psi) \equiv \Delta(\varphi \to_\Pi \psi)$
($L\Pi$) $\varphi \odot (\psi \ominus \chi) \equiv (\varphi \odot \psi) \ominus (\varphi \odot \chi)$

(4) Deduction rules of ŁΠ are modus ponens for $\to_L$ (modus ponens for $\to_\Pi$ is derivable), and necessitation for Δ: from φ derive $\Delta\varphi$.

The logic $Ł\Pi\frac{1}{2}$ is the logic obtained from ŁΠ by expanding the language with a propositional variable $\overline{\frac{1}{2}}$ and adding the axiom:

$$(Ł\Pi\frac{1}{2}) \quad \overline{\frac{1}{2}} \equiv \neg_L\overline{\frac{1}{2}}$$

Obviously, a truth-evaluation e for ŁΠ is easily extended to an evaluation for $Ł\Pi\frac{1}{2}$ by further requiring $e(\overline{\frac{1}{2}}) = \frac{1}{2}$.

From the above axiom systems, the notion of proof from a theory (a set of formulas) in both logics, denoted $\vdash_{L\Pi}$ and $\vdash_{L\Pi\frac{1}{2}}$ respectively, is defined as usual. Strong completeness of both logics for finite theories with respect to the given semantics has been proved in [11]. In what follows we will restrict ourselves to the logic $Ł\Pi\frac{1}{2}$.

Theorem 2.1
For any finite set of formulas T and any formula φ of $Ł\Pi\frac{1}{2}$, we have $T \vdash_{L\Pi\frac{1}{2}} \varphi$ iff $e(\varphi) = 1$ for each truth-evaluation e which is a model[5] of T.

As it is also shown in [11], for each rational $r \in [0, 1]$ a formula $\overline{r}$ is definable in $Ł\Pi\frac{1}{2}$ from the truth constant $\overline{\frac{1}{2}}$ and the connectives, so that $e(\overline{r}) = r$ for each evaluation e. Therefore, in the language of $Ł\Pi\frac{1}{2}$ we have a truth constant for each rational in [0, 1], and due to completeness of $Ł\Pi\frac{1}{2}$, the following book-keeping axioms for rational truth constants are provable:

$$\begin{array}{llll} (RL\Pi1) & \neg_L\overline{r} \equiv \overline{1-r}, & (RL\Pi2) & \overline{r} \to_L \overline{s} \equiv \overline{\min(1, 1-r+s)}, \\ (RL\Pi3) & \overline{r} \odot \overline{s} \equiv \overline{r \cdot s}, & (RL\Pi4) & \overline{r} \to_\Pi \overline{s} \equiv \overline{r \Rightarrow_P s}, \end{array}$$

where $r \Rightarrow_P s = 1$ if $r \le s$, $r \Rightarrow_P s = s/r$ otherwise.

3 A logic of conditional probability

In this section we describe the fuzzy modal logic FCP(ŁΠ) –FCP for Fuzzy Conditional Probability–, built up over the many-valued logic $Ł\Pi\frac{1}{2}$ described in the previous section. In what follows, given a set $D \subset \mathcal{L}$ of non-modal formulas, we will denote by $Con(D)$ the

[5]We say that an evaluation e is a *model* of a theory T whenever $e(\psi) = 1$ for each $\psi \in T$.

set of non-modulas formulas φ which follow from D in Classical propositional logic. Furthermore, $Taut(\mathcal{L}) \subset \mathcal{L}$ will denote the set of classical tautologies and $Sat(\mathcal{L}) \subset \mathcal{L}$ the set of (classically) satisfiable formulas. In other words, $Taut(\mathcal{L}) = Con(\emptyset)$ and $Sat(\mathcal{L}) = \{\varphi \mid \neg\varphi \notin Con(\emptyset)\}$.

The language of FCP(ŁΠ) is defined in two steps:

Non-modal formulas: they are built from a set V of propositional variables $\{p_1, p_2, \ldots.\ p_n, \ldots\}$ using the classical binary connectives $\wedge$ and $\neg$. Other connectives like $\vee$, $\rightarrow$ and $\leftrightarrow$ are defined from $\wedge$ and $\neg$ in the usual way. Non-modal formulas (we will also refer to them as Boolean propositions) will be denoted by lower case Greek letters φ, ψ, etc. The set of non-modal formulas will be denoted by $\mathcal{L}$.

Modal formulas: they are built from elementary modal formulas of the form $P(\varphi \mid \chi)$, where φ and χ are non-modal formulas with $\chi \in Sat(\mathcal{L})$, using the connectives of ŁΠ ($\rightarrow_L$, &, $\odot$, $\rightarrow_\Pi$, etc.) and the truth constants $\overline{r}$, for each rational $r \in [0, 1]$. We shall denote them by upper case Greek letters Φ, Ψ, etc. Notice that we do not allow nested modalities.

Definition 3.1
The axioms of the logic FCP(ŁΠ) are the following:

(1) The set $Taut(\mathcal{L})$ of tautologies of classical propositional logic
(2) Axioms of Ł$\Pi\frac{1}{2}$ for modal formulas
(3) Probabilistic modal axioms:

$$\begin{array}{ll}
\text{(FCP1)} & P(\varphi \rightarrow \psi \mid \chi) \rightarrow_L (P(\varphi \mid \chi) \rightarrow_L P(\psi \mid \chi)) \\
\text{(FCP2)} & P(\neg\varphi \mid \chi) \equiv \neg_L P(\varphi \mid \chi) \\
\text{(FCP3)} & P(\varphi \vee \psi \mid \chi) \equiv ((P(\varphi \mid \chi) \rightarrow_L P(\varphi \wedge \psi \mid \chi)) \rightarrow_L P(\psi \mid \chi) \\
\text{(FCP4)} & P(\varphi \wedge \psi \mid \chi) \equiv P(\psi \mid \varphi \wedge \chi) \odot P(\varphi \mid \chi) \\
\text{(FCP5)} & P(\chi \mid \chi)
\end{array}$$

Deduction rules of FCP(ŁΠ) are those of ŁΠ (i.e. *modus ponens* and *necessitation* for Δ), plus:
(4) *necessitation* for P: from φ derive $P(\varphi \mid \chi)$
(5) *substitution of equivalents* for the conditioning proposition:
from $\chi \leftrightarrow \chi'$, derive $P(\varphi \mid \chi) \equiv P(\varphi \mid \chi')$

Remark
The restriction imposed in the definition of elementary modal formulas that χ in a formula $P(\varphi \mid \chi)$ must belong to $Sat(\mathcal{L})$ is implicitly assumed in all the above axiom schemes and rules.

Due to the peculiar definition of the language, any theory (set of formulas) will be of the kind $\Gamma = D \cup T$, where D contains only non-modal formulas and T contains only modal formulas. Notice that in the above axioms and rules, there is no interplay between both kinds of formulas except for the inference rules of necessitation and substitution of equivalents, which allow the derivation of modal formulas from non-modal ones (but not vice-versa). Therefore, given an initial theory $\Gamma = D \cup T$, reasoning on non-modal formulas does not play an actual role in deductions from Γ, but it is just a way of generating new modal formulas to be considered with T. On the other hand, in proofs from Γ, we want to avoid the application of the above inference rules yielding modal formulas with conditioning events contradictory with D, since they would easily lead to inconsistencies. As an example, if $D = \{\neg p\}$, where p is a propositional variable, then from D one could derive $P(\neg p \mid p)$ by applying the necessitation rule, which is in clear contradiction with $P(p \mid p)$, an instance of axiom (FCP5). Therefore, we are led to define in FCP(ŁΠ) the notion of proof from a theory, written $\vdash_{FCP}$, in a non standard way, at least when the theory contains non-modal formulas. In what follows D denotes a propositional theory, T a modal theory, φ a non-modal formula and Φ a modal formula.

Definition 3.2
The proof relation $\vdash_{FCP}$ between sets of formulas and formulas is defined by:

(1) $D \cup T \vdash_{FCP} \varphi$ if $\varphi \in Con(D)$
(2) $T \vdash_{FCP} \Phi$ if Φ follows from T in the usual way from the above axioms and rules.
(3) $D \cup T \vdash_{FCP} \Phi$ if $T \cup D^P \vdash_{FCP} \Phi$,

where $D^P = \{P(\varphi \mid \chi) : \varphi \in Con(D),\ \ \neg\chi \notin Con(D)$ and χ appears as conditioning in subformulas of $\Phi\}$

Notice that the general lateral condition for all modal formulas that $\chi \in Sat(\mathcal{L})$, as well as the conditions $\varphi \in Con(D)$ and $\neg\chi \notin Con(D)$ for the consequence relation, are decidable, so the notion of proof is well-defined.

Example 3.3
As an example of deduction, we show how to prove that conditional probability preserves classical equivalence, i.e. that $\varphi \leftrightarrow \psi \vdash_{FCP} P(\varphi \mid \chi) \equiv P(\psi \mid \chi)$, where $\chi \in Sat(\mathcal{L})$. From $D = \{\varphi \leftrightarrow \psi\}$, by definition of D^P, we get $P(\varphi \leftrightarrow \psi \mid \chi) \in D^P$, hence we derive it. Now, since $(\varphi \leftrightarrow \psi) \to (\varphi \to \psi)$ is a Boolean tautology, by necessitation we obtain $P((\varphi \leftrightarrow \psi) \to (\varphi \to \psi) \mid \chi)$. By applying FCP1 and modus ponens with $P(\varphi \leftrightarrow \psi \mid \chi)$ we derive $P(\varphi \to \psi \mid \chi)$, and again by FCP1 and modus ponens we get $P(\varphi|\chi) \to_L P(\psi|\chi)$. Similarly, starting from $(\varphi \leftrightarrow \psi) \to (\psi \to \varphi)$, we derive $P(\psi|\chi) \to_L P(\varphi|\chi)$ as well. Finally, by reasoning in ŁΠ we derive $(P(\varphi|\chi) \to_L P(\psi|\chi))\&(P(\psi|\chi) \to_L P(\varphi|\chi))$, hence we have shown $\varphi \leftrightarrow \psi \vdash_{FCP} P(\varphi \mid \chi) \equiv P(\psi \mid \chi)$.

The semantics for FCP(ŁΠ) is given by *conditional probability Kripke structures* $M = \langle W, \mathcal{U}, e, \mu\rangle$, where:

- W is a non-empty set of possible worlds.
- $e : V \times W \to \{0, 1\}$ provides for each world a *Boolean* (two-valued) evaluation of the propositional variables, that is, $e(p, w) \in \{0, 1\}$ for each propositional variable $p \in V$ and each world $w \in W$. A truth-evaluation $e(\cdot, w)$ is extended to Boolean propositions as usual. For a Boolean formula φ, we will write $[\varphi]_W = \{w \in W \mid e(\varphi, w) = 1\}$.
- $\mu : \mathcal{U} \times \mathcal{U}^0 \to [0, 1]$ is a conditional probability over $\mathcal{U} \times \mathcal{U}^0$, where $\mathcal{U}$ is a Boolean algebra of subsets of W[6], $\mathcal{U}^0 = \mathcal{U}\backslash\{\emptyset\}$, and such that $([\varphi]_W, [\chi]_W)$ is μ-measurable for all non-modal φ and χ (with $[\chi]_W \neq \emptyset$).
- $e(\cdot, w)$ is extended to elementary modal formulas by defining

$$e(P(\varphi \mid \chi), w) = \mu([\varphi]_W \mid [\chi]_W),\ \text{if } [\chi]_W \neq \emptyset$$

[6]Notice that in our definition the factors of the Cartesian product are the same Boolean algebra. This is clearly a special case of what is stated in Definition 1.1.

and we leave $e(P(\varphi \mid \chi), w)$ *undefined* otherwise[7]. Then e is extended to arbitrary modal formulas, when possible, according to $Ł\Pi\frac{1}{2}$ semantics, that is:

$$\begin{aligned}
e(\overline{r}, w) &= r, \\
e(\Phi \rightarrow_L \Psi, w) &= \min(1 - e(\Phi, w) + e(\Psi, w), 1), \\
e(\Phi \odot \Psi, w) &= e(\Phi, w) \cdot e(\Psi, w), \\
e(\Phi \rightarrow_\Pi \Psi, w) &= \begin{cases} 1, & \text{if } e(\Phi, w) \leq e(\Psi, w) \\ e(\Psi, w)/e(\Phi, w), & \text{otherwise} \end{cases}.
\end{aligned}$$

We call a Kripke structure $M = \langle W, \mathcal{U}, e, \mu \rangle$ *safe* for a formula Φ if $e(\Phi, w)$ is defined for every world w. Trivially, any Kripke structure is safe for all non-modal formulas. If Φ is modal and M is safe for it, then observe that the truth-evaluation $e(\Phi, w)$ depends only on the conditional probability measure μ and not on the particular world w. In this case, we will also write $e^M(\Phi)$ to denote $e(\Phi, w)$ for any $w \in W$.

If M is safe for Φ, then we say that M is a model for Φ, written $M \models \Phi$, if $e^M(\Phi) = 1$. If T is a set of formulas, we say that M is a model of T if M is safe for all formulas in T and $M \models \Phi$ for all $\Phi \in T$.

Remarks

(1) $M = \langle W, \mathcal{U}, e, \mu \rangle$ is safe for $P(\varphi \mid \chi)$ iff $[\chi]_W \neq \emptyset$ iff $M \not\models \neg\chi$
(2) $M = \langle W, \mathcal{U}, e, \mu \rangle$ is safe for a modal formula Φ iff M is so for every elementary modal subformula of Φ.

The notion of logical entailment relative to a class of structures $\mathcal{M}$, written $\models_{\mathcal{M}}$, is then defined as follows:

$$T \models_{\mathcal{M}} \Phi \text{ iff } M \models \Phi \text{ for each } M \in \mathcal{M} \text{ model of } T \text{ which is safe for } \Phi.$$

If $\mathcal{M}$ denotes the whole class of conditional probability Kripke structures we shall write $T \models_{FCP} \Phi$. When $\models_{\mathcal{M}} \Phi$ holds we will say that Φ is *valid* in $\mathcal{M}$, i.e. when Φ gets value 1 in all structures $M \in \mathcal{M}$ safe for Φ.

Remark
$\models_{\mathcal{M}} \Phi$ does not mean $e^M(\Phi) = 1$ in each structure $M \in \mathcal{M}$, but only in those structures which are safe for Φ.

Lemma 3.4
Axioms FCP1-FCP5 are valid in the class of conditional probability Kripke structures.

Proof. The proof is very similar to the one given in [22](8.4.5) for unconditional probability. Let $M = (W, \mathcal{U}, e, \mu)$ be a conditional probability structure which we will subsequently assume below to be safe for the different formulas corresponding to (instances of) the axioms.

[7]This possibility of having the evaluation of a modal formula as undefined, and its consequences, was missing in [36].

Then it is easy to check that the validity of each axiom in M amounts to a corresponding property of μ: (FCP3), (FCP4) and (FCP5) directly correspond to the three axioms of conditional probability given in Definition 1.1. Actually, the validity of (FCP3) amounts to the additivity of $\mu(\cdot \mid [\chi]_W)$. The cases of (FCP4) and (FCP5) are obvious. The case of (FCP2) is also a consequence of the additivity of $\mu(\cdot \mid [\chi]_W)$. As for (FCP1), if we simply write $\mu(\varphi \mid \psi)$ for $\mu([\varphi]_W \mid [\psi]_W)$ and $x \Rightarrow y$ for $\min(1 - x + y, 1)$, it amounts to check $\mu(\varphi \to \psi \mid \chi) \leq \mu(\varphi \mid \chi) \Rightarrow \mu(\psi \mid \chi)$. Put $\mu(\varphi \wedge \psi \mid \chi) = a$, $\mu(\varphi \wedge \neg\psi \mid \chi) = b$ and $\mu(\neg\varphi \wedge \psi \mid \chi) = c$. Then we have $\mu(\varphi \to \psi \mid \chi) = 1 - b \leq \min(1 - b + c, 1) = \min(1 - (b + a) + (c + a), 1) = \mu(\varphi \mid \chi) \Rightarrow \mu(\psi \mid \chi)$. ∎

Lemma 3.5
The FCP(ŁΠ) inference rules preserve validity in a model.

Proof. We need to check that the rule of substitution of equivalents and the necessitation rule for P preserve validity in a model. Namely, let $M = (W, \mathcal{U}, e, \mu)$ be such that $M \models \chi \leftrightarrow \chi'$ and M is safe for $P(\varphi \mid \chi)$ and $P(\varphi \mid \chi')$. Then, $[\chi]_W = [\chi']_W \neq \emptyset$ and hence obviously $e(P(\varphi \mid \chi), w) = e(P(\varphi \mid \chi'), w)$ for all $w \in W$, that is, $M \models P(\varphi \mid \chi) \equiv P(\varphi \mid \chi')$.

As for the necessitation rule, if we assume $M \models \varphi$ and M is safe for $P(\varphi \mid \chi)$, then $[\varphi]_W = W$ and $[\chi]_W \neq \emptyset$, hence $e(P(\varphi \mid \chi), w) = 1$, that is $M \models P(\varphi \mid \chi)$. ∎

The two preceeding lemmas are the basis for the following soundness result.

Proposition 3.5 (**Soundness**)
The logic FCP(ŁΠ) is sound with respect to the class of conditional probability Kripke structures, i.e. if $\Gamma \vdash_{FCP} \Phi$ then $\Gamma \models_{FCP} \Phi$.

Proof. Assume $\Gamma \vdash_{FCP} \Phi$ and recall Definition 3.2. If Φ is non-modal it is obvious, thus assume Φ is modal. Now, let us assume Γ to be modal. Then, by lemmas 3.4 and 3.5, we also have $\Gamma \models_{FCP} \Phi$. Finally, let $\Gamma = D \cup T$ where D is non-modal and T modal. Let $M = (W, \mathcal{U}, e, \mu)$ be such that $M \models D \cup T$ and M is safe for Φ, we have to show that $M \models \Phi$. Since M is safe for Φ, it means that $[\chi]_W \neq \emptyset$ for every χ in atomic modal formulas $P(\cdot \mid \chi)$ appearing in Φ. On the other hand, since $M \models D$, then $[\psi]_W = W$ for every $\psi \in Con(D)$. This means that $M \models D^P$, hence $M \models T \cup D^P$. But now $T \cup D^P$ is a modal theory, hence $M \models \Phi$ as well. ∎

4 Extended completeness for FCP(ŁΠ)

The completeness result for FCP(ŁΠ) shown in [36] only considers (finite) modal theories, that is, theories involving only probabilistic formulas. However it is worth considering theories also including non-modal formulas since they can allow us to take into account logical representations of the relationships between events. For instance, if φ and ψ represent incompatible events, we may want to include in our probabilistic theory the non-modal formula $\neg(\varphi \wedge \psi)$, or if the event represented by ψ is *included* in φ then we may need to include the formula $\psi \to \varphi$.

Let $D \subset \mathcal{L}$ be any given non-modal (propositional) theory (possibly empty). For any $\varphi, \psi \in \mathcal{L}$, define $\varphi \sim_D \psi$ iff $\varphi \leftrightarrow \psi$ follows from D in classical propositional logic, i.e. if $\varphi \leftrightarrow \psi \in Con(D)$. The relation $\sim_D$ is an equivalence relation in $\mathcal{L}$ and $[\varphi]_D$ will denote the equivalence class of φ. Obviously, the quotient set $\mathcal{L}/_{\sim_D}$ forms a Boolean algebra which is isomorphic to a subalgebra $\mathbf{B}(\Omega_D)$ of the power set of the set Ω_D of Boolean interpretations of the crisp language $\mathcal{L}$ which are model of D[8]. For each $\varphi \in \mathcal{L}$, we shall identify the equivalence class $[\varphi]_D$ with the set $\{\omega \in \Omega_D \mid \omega(\varphi) = 1\} \in \mathbf{B}(\Omega_D)$ of models of D that make φ true. We shall denote by $\mathcal{CP}(D)$ the set of conditional probabilities over $\mathcal{L}/_{\sim_D} \times (\mathcal{L}/_{\sim_D} \setminus [\bot])$ or, equivalently, on $\mathbf{B}(\Omega_D) \times \mathbf{B}(\Omega_D)^0$.

Notice that each conditional probability $\mu \in \mathcal{CP}(D)$ induces a conditional probability Kripke structure $\langle \Omega_D, \mathbf{B}(\Omega_D), e_\mu, \mu\rangle$ where $e_\mu(p, \omega) = \omega(p) \in \{0, 1\}$ for each $\omega \in \Omega_D$ and each propositional variable p. We shall denote by $\mathcal{M}_D$ the class of probabilistic Kripke structures which are models of D, by $\mathcal{CP}(D)$ the class of conditional probabilities defined on $\mathbf{B}(\Omega_D) \times \mathbf{B}(\Omega_D)^0$, and by $\mathcal{CPS}(D)$ the class of probabilistic Kripke models $\{(\Omega_D, \mathbf{B}(\Omega_D), e_\mu, \mu) \mid \mu \in \mathcal{CP}(D)\}$. Obviously, $\mathcal{CPS}(D) \subset \mathcal{M}_D$.

Abusing the language, we will say that a conditional probability $\mu \in \mathcal{CP}(D)$ is a *model* of a modal theory T whenever the induced Kripke structure $\langle \Omega_D, \mathbf{B}(\Omega_D), e_\mu, \mu\rangle$ is a model of T (obviously $\langle \Omega_D, \mathbf{B}(\Omega_D), e_\mu, \mu\rangle$ is a model of D as well).

Given the above notions, we now prove the probabilistic completeness of FCP(ŁΠ) with respect to finite arbitrary theories, hence extending the result given in [36] for modal theories.

Theorem 4.1 (**Extended finite probabilistic completeness of FCP(ŁΠ)**)
Let T be a finite modal theory over FCP(ŁΠ), D a finite propositional theory and Φ a modal formula with the following constraint: any modal formula $P(\varphi \mid \chi)$ appearing (as subformula) in $T \cup \{\Phi\}$ is such that $\neg\chi \notin Con(D)$. Then $T \cup D \vdash_{FCP} \Phi$ iff $e_\mu(\Phi) = 1$ for each conditional probability $\mu \in \mathcal{CP}(D)$ model of T.

Proof. Soundness is clear (see Proposition 3.6). For completeness, the proof below is an adaptation of the proof of [36, Th.2], which in turn follows [20, 22]. The basic idea consists in transforming modal theories over FCP(ŁΠ) into theories over $Ł\Pi\frac{1}{2}$ and then taking advantage of the $Ł\Pi\frac{1}{2}$-completeness.

Define a theory over $Ł\Pi\frac{1}{2}$, called $\mathcal{F}$, as follows:

(1) take as new propositional variables, variables of the form $f_{\varphi|\chi}$, where φ and χ are classical propositions from $\mathcal{L}$ and $\chi \in Sat(\mathcal{L})$
(2) take as axioms of the theory the following ones, for each φ, ψ and χ:
  ($\mathcal{F}1$) $f_{\varphi|\chi}$, for $\varphi \in Con(D)$ and χ such that $\neg\chi \notin Con(D)$.
  ($\mathcal{F}2$) $f_{\varphi|\chi} \equiv f_{\varphi|\chi'}$, for any $\chi, \chi' \in$ such that $\neg\chi, \neg\chi' \notin Con(D)$, $\chi \leftrightarrow \chi' \in Con(D)$
  ($\mathcal{F}3$) $f_{\varphi\to\psi|\chi} \to_L (f_{\varphi|\chi} \to_L f_{\psi|\chi})$,
  ($\mathcal{F}4$) $f_{\neg\varphi|\chi} \equiv \neg_L f_{\varphi|\chi}$,
  ($\mathcal{F}5$) $f_{\varphi\vee\psi|\chi} \equiv [(f_{\varphi|\chi} \to_L f_{\varphi\wedge\psi|\chi}) \to_L f_{\psi|\chi}]$,
  ($\mathcal{F}6$) $f_{\varphi\wedge\psi|\chi} \equiv f_{\psi|\varphi\wedge\chi} \odot f_{\varphi|\chi}$.
  ($\mathcal{F}7$) $f_{\varphi|\varphi}$,
  where in all formulas of the kind $f_{\varphi|\chi}$, it is assumed that $\chi \in Sat(\mathcal{L})$.

Define a mapping $*$ from FCP(ŁΠ) modal formulas to $Ł\Pi\frac{1}{2}$-formulas as follows:

(1) $(P(\varphi \mid \chi))^* = f_{\varphi|\chi}$
(2) $\overline{r}^* = \overline{r}$
(3) $(\Phi \circ \Psi)^* = \Phi^* \circ \Psi^*$, for $\circ \in \{\&, \to_L, \odot, \to_\Pi\}$

[8]Actually, $\mathbf{B}(\Omega_D) = \{\{\omega \in \Omega_D \mid \omega(\varphi) = 1\} \mid \varphi \in \mathcal{L}\}$. Needless to say, if the language has only finitely many propositional variables then the algebra $\mathbf{B}(\Omega_D)$ is just the whole power set of Ω_D, otherwise it is a strict subalgebra.

Let us denote by T^* and $(D^P)^*$ the set of all translated formulas from T and D^P respectively.

Then, by the construction of $\mathcal{F}$ and $(D^P)^*$, one can easily check that for any Φ,

$$T \cup D \vdash_{FCP} \Phi \text{ iff } T^* \cup \mathcal{F} \cup (D^P)^* \vdash_{Ł\Pi\frac{1}{2}} \Phi^*. \tag{1}$$

Notice that in a proof from $T^* \cup \mathcal{F} \cup (D^P)^*$ the use of instances of $(\mathcal{F}1)$ and $(\mathcal{F}2)$ corresponds to the use of inference rules of necessitation for P and of substitution of equivalents in FCP(ŁΠ), while instances of $(\mathcal{F}3) - (\mathcal{F}7)$ obviously correspond to axioms (FCP1) - (FCP5) respectively.

Now we prove that the semantical analogue of (1) also holds, that is,

$$T \cup D \models_{FCP} \Phi \text{ iff } T^* \cup \mathcal{F} \cup (D^P)^* \models_{Ł\Pi\frac{1}{2}} \Phi^*. \tag{2}$$

Assume $T^* \cup \mathcal{F} \cup (D^P)^* \not\models_{Ł\Pi\frac{1}{2}} \Phi^*$. This means that there exists an $Ł\Pi\frac{1}{2}$-evaluation e which is model of $T^* \cup \mathcal{F} \cup (D^P)^*$ such that $e(\Phi^*) < 1$. We show that there is a Kripke structure $M_e = (\Omega_D, \mathbf{B}(\Omega_D), e', \mu_e)$ which is a model of $T \cup D$, safe for Φ and $M_e \not\models \Phi$. Define:

- $\mu_e : \mathbf{B}(\Omega_D) \times \mathbf{B}(\Omega_D)^0 \to [0, 1]$ as follows:

$$\mu_e([\varphi]_{\Omega_D} \mid [\chi]_{\Omega_D}) = e(f_{\varphi|\chi}).$$

  for each $\varphi, \chi \in \mathcal{L}$ such that $\neg\chi \notin Con(D)$ (hence $[\chi]_{\Omega_D} \neq \emptyset$).
- $e'(p, w) = w(p)$ for each propositional variable p

Since by hypothesis e is a model of $\mathcal{F}$, it is easy to see that μ_e is indeed a conditional probability. Moreover, so defined, M_e is clearly a model of D, it is safe for all formulas of $T \cup \{\Phi\}$ (because of the precondition in the theorem) and moreover, by construction, $e'(\Psi, w) = e(\Psi^*)$ for any modal formula $\Psi \in T \cup \{\Phi\}$, hence $e'(\Psi, w) = 1$ for all $\Psi \in T$ and $e(\Phi, w) < 1$. Thus, $M_e \models T$ but $M_e \not\models \Phi$.

Conversely, assume $T \cup D \not\models_{FCP} \Phi$, that is, assume there is a conditional probability Kripke structure $M = (W, \mathcal{U}, e, \mu)$ which is a model of $T \cup D$ (hence safe for T), safe for Φ but $M \not\models \Phi$. Thus, M is also a model of D^P since for each $P(\varphi \mid \chi) \in D^P$, $[\varphi]_W = W$ and $[\chi]_W \neq \emptyset$ and hence $\mu([\varphi]_W \mid [\chi]_W) = 1$. We show that there also exists an $Ł\Pi\frac{1}{2}$-evaluation v_M model of $T^* \cup \mathcal{F} \cup (D^P)^*$ such that $v_M(\Psi^*) = e(\Psi, w)$ for each modal formula Ψ and each $w \in W$. To do this, take an arbitrary $w \in W$, and define:

$$v_M(p) = e(p, w),$$
$$v_M(f_{\varphi|\chi}) = \begin{cases} e(P(\varphi \mid \chi)) = \mu([\varphi]_W \mid [\chi]_W), & \text{if } [\chi]_W \neq \emptyset \\ \textit{arbitrary}, & \text{otherwise} \end{cases}.$$

Clearly $v_M(\varphi) = 1$ for each $\varphi \in D$, and v_M is a model of axioms $\mathcal{F}1 - \mathcal{F}7$ since μ is a conditional probability. Finally one can easily check that for each modal formula $\Psi \in T \cup D^P$ we have $v_M(\Psi^*) = e(\Psi, w)$ since this value is defined (M is safe for $T \cup \{\Phi\}$, hence also for D^P), and moreover it only depends on μ. Therefore $v_M(\Psi^*) = 1$ for every $\Psi^* \in T^* \cup \mathcal{F} \cup (D^P)^*$ but $v_M(\Phi^*) < 1$, as desired. Hence we have proved the equivalence (2).

From (1) and (2), to prove the theorem it remains to show that

$$T^* \cup \mathcal{F} \cup (D^P)^* \vdash_{L\Pi\frac{1}{2}} \Phi^* \textit{ iff } T^* \cup \mathcal{F} \cup (D^P)^* \models_{L\Pi\frac{1}{2}} \Phi^*.$$

Note that Ł$\Pi\frac{1}{2}$ is strongly complete but only for finite theories. Here the initial theories T and D are finite, so is T^*. However $\mathcal{F}$ contains infinitely many instances of axioms $\mathcal{F}1 - \mathcal{F}7$ and $(D^P)^*$ also contains infinitely many formulas since $Con(D)$ is so. Nonetheless one can prove that such infinitely many formulas can be safely replaced by only finitely many, by using propositional normal forms, following the lines of [22, 8.4.12].

Indeed, take n propositional variables $p_1, \ldots, p_n$ containing at least all variables in $T \cup D$. For any formula φ built from these propositional variables, take the corresponding disjunctive normal form $(\varphi)_{dnf}$. Notice that there are only finitely many different such formulas. Then, when translating a modal formula Φ into Φ^*, we replace each atom $f_{\varphi|\chi}$ by $f_{(\varphi)_{dnf}|(\chi)_{dnf}}$ to obtain its normal translation Φ^*_{dnf}. The theory T^*_{dnf} is the (finite) set of all Ψ^*_{dnf} with $\Psi \in T$, and $(D^P)^*_{dnf}$ is the (finite) set of all Ψ^*_{dnf} with $\Psi \in D^P$. The theory $\mathcal{F}_{dnf}$ is the *finite* set of instances of axioms $\mathcal{F}1 - \mathcal{F}7$ for disjunctive normal forms of Boolean formulas built from the propositional variables $p_1, \ldots, p_n$. We can now prove the following equivalences:

(i) $T^* \cup \mathcal{F} \cup (D^P)^* \vdash_{L\Pi\frac{1}{2}} \Phi^*$ iff $T^*_{dnf} \cup \mathcal{F}_{dnf} \cup (D^P)^*_{dnf} \vdash_{L\Pi\frac{1}{2}} \Phi^*_{dnf}$.
(ii) $T^* \cup \mathcal{F} \cup (D^P)^* \models_{L\Pi\frac{1}{2}} \Phi^*$ iff $T^*_{dnf} \cup \mathcal{F}_{dnf} \cup (D^P)^*_{dnf} \models_{L\Pi\frac{1}{2}} \Phi^*_{dnf}$.

The proof of (i) and (ii) is similar to that provided in [22, 8.4.13]. Finally, we obtain the following chain of equivalences:

$T \cup D \vdash_{FCP} \Phi$	iff	$T^* \cup \mathcal{F} \cup (D^P)^* \vdash_{L\Pi\frac{1}{2}} \Phi^*$	by (1)
	iff	$T^*_{dnf} \cup \mathcal{F}_{dnf} \cup (D^P)^*_{dnf} \vdash_{L\Pi\frac{1}{2}} \Phi^*_{dnf}$	by (i) above
	iff	$T^*_{dnf} \cup \mathcal{F}_{dnf} \cup (D^P)^*_{dnf} \models_{L\Pi\frac{1}{2}} \Phi^*_{dnf}$	by Ł$\Pi\frac{1}{2}$ – finite strong compl.
	iff	$T^* \cup \mathcal{F} \cup (D^P)^* \models_{L\Pi\frac{1}{2}} \Phi^*$	by (ii) above
	iff	$T \cup D \models_{FCP} \Phi$	by (2).

This completes the proof of the theorem. ■

Similarly to the case of modal theories (see [36]), we have the following interesting types of deduction. If T is a finite (modal) conditional theory over FCP(ŁΠ), D is a propositional (non-modal) theory, and φ and χ are non-modal formulas, with $\neg\chi \notin Con(D)$, then we have:

(i) $T \cup D \vdash_{FCP} \bar{r} \to P(\varphi \mid \chi)$ iff $\mu(\varphi \mid \chi) \geq r$, for each conditional probability $\mu \in \mathcal{CP}(D)$ model of T;
(ii) $T \cup D \vdash_{FCP} P(\varphi \mid \chi) \to \bar{r}$ iff $\mu(\varphi \mid \chi) \leq r$, for each conditional probability $\mu \in \mathcal{CP}(D)$ model of T.

Example 4.2
As examples of interesting deductions with propositional theories, consider the following ones.

– $\psi \to \varphi \vdash_{FCP} P(\varphi \mid \psi)$, for $\psi \in Sat(\mathcal{L})$.

Indeed, by the necessitation rule $\psi \to \varphi \vdash_{FCP} P(\psi \to \varphi \mid \psi)$, and by axiom (FCP1), $\varphi \to \psi \vdash_{FCP} P(\psi \mid \psi) \to_L P(\varphi \mid \psi)$. Finally, using axiom (FCP5) and modus ponens we get $\psi \to \varphi \vdash_{FCP} P(\varphi \mid \psi)$.

– Let T be a probabilistic theory. If $T \vdash P(\varphi \mid \psi \wedge \chi)$ then $T \cup \{\chi\} \vdash_{FCP} P(\varphi \mid \psi)$. In fact, assume $T \vdash_{FCP} P(\varphi \mid \psi \wedge \chi)$, hence $T \cup \{\chi\} \vdash_{FCP} P(\varphi \mid \psi \wedge \chi)$ as well. Clearly, by the necessitation rule $T \cup \{\chi\} \vdash_{FCP} P(\chi \mid \psi)$, and now by axiom (FCP4) $T \cup \{\chi\} \vdash_{FCP} P(\varphi \wedge \chi \mid \psi)$. Since $\varphi \wedge \chi \to \varphi$ is a Boolean tautology, by the necessitation rule we have $\vdash_{FCP} P(\varphi \wedge \chi \to \varphi \mid \psi)$, and by axiom (FCP1) and modus ponens we get $\vdash_{FCP} P(\varphi \wedge \chi \mid \psi) \to_L P(\varphi \mid \chi)$, and finally by modus ponens with $P(\varphi \wedge \chi \mid \psi)$ we get $T \cup \{\chi\} \vdash_{FCP} P(\varphi \mid \psi)$.

5 Consistency, coherent assessments and lower conditional probability

Following de Finetti's research, one of the most important features of the conditional probability approach developed by Coletti and Scozzafava in [10] is based on the possibility of reasoning only from partial conditional probability assessments to an arbitrary family of conditional events (without requiring in principle any specific algebraic structure). However, it must be checked whether such assessments minimally agree with the rules of conditional probability. This consists in requiring that an assessment can be extended at least to a proper conditional probability over $\mathcal{U} \times \mathcal{U}^0$, where $\mathcal{U}$ is the whole Boolean algebra generated by those conditional events, and it is called *coherence*.

First of all, we need to stress out the relationship between Coletti and Scozzafava's framework and our logical framework FCP(ŁΠ). In [10], the authors use a set-theoretical language and speak about *conditional events* as pairs of the form $E|H$, where E and H are basically considered as sets in a (possibly indeterminate) Boolean algebra. Here we model conditional events as pairs $\varphi|\psi$ where φ and ψ are propositions. So, when dealing with sets of conditional events, possibly implicit relationships between (simple) events as sets (e.g. inclusion, incompatibility) need to be explicitly modelled in our framework in a separate way by means of a set of formulas relating the propositions defining the conditional objects. Accordingly, a family of conditional events in [10]'s framework corresponds to a family $\mathcal{C} = \{\varphi_i \mid \chi_i\}_{i \in I}$ of conditional objects together with an associated propositional theory $D_{\mathcal{C}}$ standing for the (possible) logical relationships among the φ_i's and ψ_i's (see Example 5.3). As usual, we also assume that the conditioning propositions χ_i's are not in contradiction with the theory $D_{\mathcal{C}}$, i.e. we assume $\neg\chi_i \notin Con(D_{\mathcal{C}})$ for all $i \in I$. In what follows we will exploit this parallelism to rephrase some results of [10] in our language.

Definition 5.1 (**Coherence** [10])
A probabilistic assessment $\kappa : \mathcal{C} \to [0,1]$ over a family of conditional events $\mathcal{C} = \{\varphi_i \mid \chi_i : i = 1, \ldots, n\}$ is *coherent* if there is a conditional probability μ on $\mathbf{B}([D_{\mathcal{C}}]) \times \mathbf{B}([D_{\mathcal{C}}])^0$, in the sense of Definition 1.1, such that $\kappa(\varphi_i \mid \chi_i) = \mu(\varphi_i \mid \chi_i)$ for all $i = 1, \ldots, n$.

An important result by Coletti and Scozzafava is the characterization of the coherence of an assessment in terms of the existence of a suitable class of simple (non-conditional) coherent probabilities. Indeed, it is shown in [10] that given a set of conditional events, any coherent assessment over such a set can be represented by a family of classical conditional probabilities each of them generated by a class of simple assessments, each one defined over subsets of the atoms of the algebra (see [10, Th. 4]).

In [36] we showed that the notion of coherence of a probabilistic assessment (restricted to rational values) to a set of conditional events is tantamount to the consistency of a suitably defined theory over FCP(ŁΠ)[9].

THEOREM 5.2 ([36])
Let κ be a rational assessment to a family of conditional events $\mathcal{C} = \{\varphi_i \mid \chi_i : i = 1, \ldots, n\}$. Let $\alpha_i = \kappa(\varphi_i \mid \chi_i)$, for $i = 1, \ldots, n$. Then κ is coherent iff the theory $T_\kappa = \{P(\varphi_i \mid \chi_i) \equiv \overline{\alpha_i} : i = 1, \ldots, n\} \cup D_{\mathcal{C}}$ is consistent in FCP(ŁΠ), i.e. $T_\kappa \cup D_{\mathcal{C}} \not\vdash_{FCP} \bar{0}$.

EXAMPLE 5.3
Suppose you are calling your friend Sally at her cell phone and consider the following events:

φ: Sally hears the phone ringing
ψ Sally is out
χ: The cell phone is at home
ρ: Sally answers the call

You know Sally's apartment is small so it is easy for her to hear the phone when ringing, and it is usually the case that if she does not answer it is because she has forgotten to take the cell phone when she is going out. This allows you, considering the set of conditional events $\mathcal{C} = \{\varphi|\neg\psi, \psi \wedge \chi|\neg\rho\}$, to make the following conditional probability assessment $\kappa : \mathcal{C} \to [0, 1]$, with $\kappa(\varphi \mid \neg\psi) = 0.9, \kappa(\psi \wedge \chi \mid \neg\rho) = 0.7$. On the other hand, it is clear that if Sally is out, and she has left her cell phone at home, she cannot hear the phone ringing, and hence she cannot answer the call either. In other words, this means that if ψ and χ are true logically implies that φ cannot be true, and this in turn implies that ρ that cannot be true. All this information is modelled in our logic FCP(ŁΠ) by both the modal theory

$$T_\kappa = \{P(\varphi \mid \neg\psi) \equiv \overline{0.9}, P(\psi \wedge \chi \mid \neg\rho) \equiv \overline{0.7}\}$$

and the propositional theory

$$D_{\mathcal{C}} = \{\psi \wedge \chi \to \neg\varphi, \rho \to \varphi\},$$

the latter making explicit the implicit relationships among the events (assuming these are all we know). The coherence of the assessment κ is then equivalent to the consistency of the theory $T_\kappa \cup D_{\mathcal{C}}$ in the logic FCP(ŁΠ), i.e. to the fact that $T_\kappa \cup D_{\mathcal{C}} \not\vdash_{FCP} \overline{0}$. Notice that the consideration of the non-modal theory $D_{\mathcal{C}}$ together with the probabilistic (modal) theory T_κ is very important, for instance from $T_\kappa \cup D_{\mathcal{C}}$, we can derive in FCP(ŁΠ) formulas like $P(\varphi \mid \neg\rho) \to_L \overline{0.3}$ or $P(\rho \mid \neg\psi) \to_L \overline{0.9}$, which we cannot derive from T_κ alone.

Sometimes, we might not be able to assess precise conditional probability values for a family of conditional events, but we can rather provide just a vector of lower (or upper) bounds for those values. In such situations where we have to deal with *imprecise*

[9]A very similar result was provided by Flaminio and Montagna in [15] in the framework of their logic FP(SŁΠ).

assessments of probabilities, the notion of coherence has been naturally generalized by Biazzo and Gilio[10].

Definition 5.4 (**Generalized Coherence** [4, 19])
Let $\mathcal{C} = \{\varphi_1|\chi_1, \ldots, \varphi_n|\chi_n\}$ be a family of conditional events. A probabilistic assessment of lower values $\lambda : \mathcal{C} \to [0,1]$ on $\mathcal{C}$ is said to be *g-coherent* iff there exists a (precise) coherent assessment $\kappa : \mathcal{C} \to [0,1]$ which is consistent with λ, that is, such that $\kappa(\varphi_i|\chi_i) \geq \lambda(\varphi_i|\chi_i)$ for each i.

The above definition also works when dealing with upper bounds and with intervals. Indeed, for any conditional event $\varphi|\chi$, such that $\mu(\varphi|\chi) \leq \beta$, we have the inequality $\mu(\neg\varphi|\chi) \geq 1 - \beta$. Therefore, we can determine *g*-coherence in the presence of interval-valued assessments $\kappa(\varphi_i|\chi_i) = [\alpha_i, \beta_i]$ standing for constraints of the type

$$\alpha_i \leq Pr(\varphi_i|\chi_i) \leq \beta_i.$$

In the following theorem we prove that the generalized coherence of any imprecise rational assessment of conditional probabilities coincides with the consistency of a suitably defined theory over FCP(ŁΠ). In the following $\mathcal{I}[0,1]$ denotes the set of closed intervals in $[0,1]$.

Theorem 5.5
Let $\kappa^g : \mathcal{C} \to \mathcal{I}[0,1]$, be a rational generalized probabilistic assessment on a family $\mathcal{C} = \{\varphi_1|\chi_1, \ldots, \varphi_n|\chi_n\}$ of conditional events. Let $\kappa^g(\varphi_i \mid \chi_i) = [\alpha_i, \beta_i]$. Then κ^g is *g*-coherent iff the theory $T_{\kappa^g} = \{(\overline{\alpha_i} \to_L P(\varphi_i \mid \chi_i))\&(P(\varphi_i \mid \chi_i) \to_L \overline{\beta_i}) \mid i = 1, \ldots, n\} \cup D_{\mathcal{C}}$ is consistent in FCP(ŁΠ), i.e. iff $T_{\kappa^g} \not\vdash_{FCP} \overline{0}$.

Proof. Suppose T_{κ^g} is consistent. Then, by completeness of FCP(ŁΠ), the class of models of T_{κ^g} is non-empty. Then if $M_j = \langle \Omega_{D_{\mathcal{C}}}, \mathbf{B}(\Omega_{D_{\mathcal{C}}}), e_\mu, \mu\rangle$ is one of such models, then $\alpha_i \leq \mu(\varphi_i \mid \chi_i) \leq \beta_i$. Hence the assessment $\kappa(\varphi_i \mid \chi_i) = \mu(\varphi_i \mid \chi_i)$ is coherent, hence κ^g is *g*-coherent.

Conversely, suppose that κ^g is *g*-coherent. Then, there exists a conditional probability μ on $\mathbf{B}(\Omega_{D_{\mathcal{C}}}) \times \mathbf{B}(\Omega_{D_{\mathcal{C}}})^0$ such that $\mu(\varphi_i|\chi_i) \in [\alpha_i, \beta_i]$. This probability μ induces a probabilistic Kripke structure $\langle \Omega_{D_{\mathcal{C}}}, \mathbf{B}(\Omega_{D_{\mathcal{C}}}), e_\mu, \mu\rangle$ that is a model of T_{κ^g}. ∎

Finally, let us remark that in [10] the authors also deal with the notion of *coherent lower* and *upper conditional probability*. Given an arbitrary set $\mathcal{C}$ of conditional events, a coherent lower (upper) conditional probability on $\mathcal{C}$ is an assessment $\underline{\mu} : \mathcal{C} \to [0,1]$ (resp. an assessment $\overline{\mu} : \mathcal{C} \to [0,1]$) such that there exists a non-empty *dominating family* $\mathcal{P} = \{\kappa(\cdot|\cdot)\}$ of *coherent* conditional probabilities on $\mathcal{C}$ whose lower (resp. upper) envelope is $\underline{\mu}$ (resp. $\overline{\mu}$), that is, for every $\varphi|\psi \in \mathcal{C}$,

$$\underline{\mu}(\varphi|\psi) = \inf_{\kappa\in\mathcal{P}} \kappa(\varphi|\psi) \ \ (\mathit{resp.}\ \overline{\mu}\,(\varphi|\psi) = \sup_{\kappa\in\mathcal{P}} \kappa(\varphi|\psi)).$$

Moreover, they show that if $\mathcal{C}$ is finite, there exists a dominating family $\mathcal{P}' \supseteq \mathcal{P}$ such that $\underline{\mu}(\varphi|\psi) = \min_{\kappa\in\mathcal{P}'} \kappa(\varphi|\psi)$ (resp. $\overline{\mu}(\varphi|\psi) = \max_{\kappa\in\mathcal{P}'} \kappa(\varphi|\psi)$). These notions also have a

[10]Notice that the notion of coherence used in [4, 19] is actually given in terms of random gains using de Finetti's betting scheme, but shown to be equivalent to the ones given in Definitions 5.1 and 5.4.

corresponding representation in FCP(ŁΠ). Indeed, given a finite modal theory T and a propositional theory D, one can compute in FCP(ŁΠ), for each conditional object $\varphi|\psi$, the greatest lower bound and the lowest upper bound for the coherent probability values induced by $T \cup D$. Indeed, let

$$\underline{\kappa}(\varphi|\psi) = \sup\{r \mid T \cup D \vdash_{FCP} \overline{r} \rightarrow_L P(\varphi \mid \psi)\},$$
$$\overline{\kappa}(\varphi|\psi) = \inf\{s \mid T \cup D \vdash_{FCP} P(\varphi \mid \psi) \rightarrow_L \overline{s}\}.$$

It is not difficult to check that $\underline{\kappa}$ and $\overline{\kappa}$ are coherent lower and upper probabilities in the above sense. By the finite strong completeness of $Ł\Pi\frac{1}{2}$, when the above infimum and supremum are rational numbers, they actually become a minimum and a maximum respectively. In such a case it then holds

$$T \cup D \vdash_{FCP} \overline{\alpha} \rightarrow_L P(\varphi \mid \psi) \ \text{ and } \ T \cup D \vdash_{FCP} P(\varphi \mid \psi) \rightarrow_L \overline{\beta},$$

where $\alpha = \underline{\kappa}(\varphi|\psi)$ and $\beta = \overline{\kappa}(\varphi|\psi)$.

6 Compactness of coherent assessments

Very recently, Flaminio has shown [16] the compactness of coherent probabilistic assessments to conditional events, both under Flaminio and Montagna's probabilistic logic FP(SŁΠ) and under our logic FCP(ŁΠ). In particular, for FCP(ŁΠ), he provides the following theorem.

THEOREM 6.1 (**Compactness**)
Let us consider a modal theory $T = \{P(\varphi_i \mid \psi_i) \equiv \overline{\alpha_i}\}_{i \in I}$ over FCP(ŁΠ). Then T is satisfiable iff every finite subtheory of T is satisfiable.

The proof is based on the well-known theorems of Łos on the ultraproduct model and on the related theorem of compactness. Roughly speaking, what Łos shows is how to define a model for an arbitrary set of formulas out of models of their finite subsets. Particularized to our framework, Łos' theorem reads as follows.

THEOREM 6.2 (cf. [7])
Let Γ be an arbitrary theory over FCP(ŁΠ). Let $S_\omega(\Gamma) = \{T_i\}_{i \in I}$ be the (countable) set of all finite subtheories of Γ, and for every $i \in I$, let M_i be a conditional probabilistic Kripke structure model of T_i. Then, there exists an ultrafilter[11] $\mathcal{F}$ over I such that the ultraproduct structure[12] $(\prod_{i \in I} M_i)/\mathcal{F}$ is a model of Γ.

This result would directly lead to the compactness of consistency in FCP(ŁΠ) if the ultraproduct model $(\prod_{i \in I} M_i)/\mathcal{F}$ was a conditional probabilistic Kripke structure of the class. Unfortunately, the class of conditional probabilistic Kripke structures is not

[11] Recall that, given a non-empty set S, an *ultrafilter* $\mathcal{F}$ over S is a collection of subsets of S such that $\emptyset \in \mathcal{F}$; if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$; if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$; and for each $A \subseteq S$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

[12] See next paragraph for the definition of this structure.

closed under ultraproduct. Indeed, if $M_i = (W_i, \mathcal{U}_i, \mu_i, e_i)$ then the ultraproduct structure $(\prod_{i \in I} M_i)/\mathcal{F}$ is a structure $(W, \mathcal{U}, \mu^*, e)$, where:

(i) $W = \prod_{i \in I} W_i/\mathcal{F}$ is the direct product of the W_i's modulo $\mathcal{F}$ [13];
(ii) $\mathcal{U} = \prod_{i \in I} \mathcal{U}_i/\mathcal{F}$ is again the direct product of the $\mathcal{U}_i$'s modulo $\mathcal{F}$;
(iii) $\mu^* : \mathcal{U} \times \mathcal{U}^0 \to [0,1]^*$ is a non-standard conditional probability, where $[0,1]^*$ denotes the ultrapower of $[0,1]$ modulo $\mathcal{F}$, and $\mu^*(A \mid B)$ is defined as the $\mathcal{F}$-equivalence class of $(\mu_1(A_1 \mid B_1), \mu_2(A_2 \mid B_2), \ldots)$ where the A_i's and B_i's are the i-th projections of A and B respectively;
(iv) $e : \mathcal{L} \times W \to \{0,1\}$ is defined as $e(\varphi, \overline{w}) = 1$ if $\{i \in I \mid e_i(\varphi, w_i) = 1\} \in \mathcal{F}$, $e(\varphi, \overline{w}) = 0$, otherwise.

However, as Flaminio shows, by letting μ be the standard part of the non-standard conditional probability μ^*, the structure $(W, \mathcal{U}, \mu, e)$ becomes a (standard) conditional probabilistic Kripke structure which is still model of Γ.

Now, it is not difficult to check that Flaminio's proof also works for more general kinds of theories involving $Ł\Pi\frac{1}{2}$ connectives with continuous truth-functions, i.e., Łukasiewicz connectives and Product conjunction.

Theorem 6.3 (**General Compactness**)
Let T be a modal theory over FCP(ŁΠ) whose formulas only involve (at most) truth-constants and the $\&$, $\to_L$, $\odot$ connectives. Then T is satisfiable iff every finite subtheory of T is satisfiable.

Proof. Left-to-right direction is easy. To prove the converse, suppose that every finite subtheory T_i of T is satisfiable. Then, for any T_i there exists some model M_i such that $M_i \models T_i$. Then, by the above Łos theorem 6.2 there exists an ultrafilter $\mathcal{F}$ such that $\prod_{i \in I} M_i/\mathcal{F} \models T$. However, as mentioned, the model obtained is based on a non-standard probability μ^*, so is the evaluation e_{μ^*}. Still, like in [16] we can get a standard model by recovering the standard part, so if St denotes the standard part, we define $e(\Phi) = St(e_{\mu^*}(\Phi))$. Then $e(\Phi \circ \Psi) = St(e_{\mu^*}(\Phi \circ \Psi)) = St(e_{\mu^*}(\Phi)) \circ St(e_{\mu^*}(\Psi))$, for $\circ \in \{\&, \to_L, \odot\}$. The respect of the behavior of the connectives is guaranteed by the continuity of their related truth-functions. Then the "standardized" structure conserves the necessary requirements for being a model of T. ∎

A parallel result for Flaminio and Montagna's logic FP(SŁΠ) also holds.

Following the above strategy we can also prove compactness for *generalized coherence* [4, 19]. This corresponds to state compactness of coherence for interval-valued conditional probability assessments of the kind

$$\kappa = \{\beta_i \le Pr(\varphi_i \mid \psi_i) \le \alpha_i\}_{i \in I},$$

which amounts in turn to compactness for the consistency of the theory

$$T = \{\overline{\alpha_i} \to_L P(\varphi_i \mid \psi_i), P(\varphi_i \mid \psi_i) \to_L \overline{\beta_i}\}_{i \in I}.$$

Now, given Theorem 6.3, we directly have as a consequence the following corollary.

[13] $(w_1, w_2, \ldots), (w'_1, w'_2, \ldots) \in \prod_{i \in I} W_i$ are equivalent modulo $\mathcal{F}$ iff $\{i \in I \mid w_i = w'_i\} \in \mathcal{F}$.

Corollary 6.4
Let $T = \{\overline{\alpha_i} \rightarrow_L P(\varphi_i \mid \psi_i), P(\varphi_i \mid \psi_i) \rightarrow_L \overline{\beta_i}\}_{i \in I}$ be a modal theory in FCP(ŁΠ). Then T is consistent iff every finite subtheory of T is consistent.

This corollary clearly yields as a direct consequence the following result of compactness for generalized coherent probabilistic assessments to conditional events (in e.g. [10] such a result is mentioned for single-valued coherent assessments).

Theorem 6.5 (**Compactness of Imprecise Coherent Assessments**)
Let $\kappa^g : \mathcal{C} \to \mathcal{I}[0,1]$ be an imprecise assessment of conditional probability over a class of conditional events $\mathcal{C} = \{\varphi_i \mid \psi_i\}_{i \in I}$ with rational bounds, i.e. for each $i \in I$, if $\kappa^g(\varphi_i \mid \psi_i) = [\alpha_i, \beta_i]$ then $\alpha_i, \beta_i \in \mathbb{Q}$. For every finite subset $\mathcal{J}$ of $\mathcal{C}$, let $\kappa^g_{\upharpoonright \mathcal{J}}$ denote the restriction of κ^g to $\mathcal{J}$. Then

$$\kappa^g \text{ is } g-\text{coherent iff for every finite } \mathcal{J} \subseteq C, \kappa^g_{\upharpoonright \mathcal{J}} \text{ is } g-\text{coherent.}$$

These compactness results directly refer to the probabilistic logics FCP(ŁΠ) and FP(SŁΠ) without mentioning a possible similar result for the base logic ŁΠ$\frac{1}{2}$. A study of compactness of many fuzzy logics was presented by Cintula and Navara in [9]. The notion of *satisfiability* proposed there generalizes the classical one, since it admits various degrees of simultaneous satisfiability.

Definition 6.6 ([9])
For a set Γ of formulas in a fuzzy logic and $K \subseteq [0,1]$, we say that Γ is *K-satisfiable* is there exists an evaluation e such that $e(\varphi) \in K$ for all $\varphi \in \Gamma$. The set Γ is said to be *finitely K-satisfiable* if each finite subset of Γ is K-satisfiable. A logic is said to be *K-compact* if K-satisfiability is equivalent to finite K-satisfiability. A logic satisfies the *compactness property* if it is K-compact for each closed subset K of [0, 1].

In particular Cintula and Navara comment that the same proof they provide for the compactness of Łukasiewicz logic (originally proved by Butnariu, Klement and Zafrany [6]) also works for other fuzzy logics with connectives interpreted by continuous functions.

Theorem 6.7 ([9, 6])
Let L be any fuzzy logic whose connectives only have continuous truth-functions. Then L has the compactness property.

The proof in [9] runs as follows. Assume the language of L is built from a countable set Var of propositional variables. Let Γ a theory over L such that every finite subset Γ' is K-satisfiable. For each $\varphi \in \Gamma$, define $H_\varphi : [0,1]^{Var} \to [0,1]$ by $H_\varphi(e) = e(\varphi)$, which is continuous by hypothesis. Then $H_\varphi^{-1}(K)$ is a closed subset of $[0,1]^{Var}$, which is compact in the product topology. Since Γ' is K-satisfiable, the intersection $\cap_{\varphi \in \Gamma'} H_\varphi^{-1}(K)$ is non-empty for every finite $\Gamma' \subset \Gamma$, hence by compactness of $[0,1]^{Var}$, $\cap_{\varphi \in \Gamma} H_\varphi^{-1}(K)$ is non-empty as well. Then any evaluation e in this intersection is such that $e(\varphi) \in K$ for all $\varphi \in \Gamma$. Since the important point in the proof is that the functions H_φ are continuous, the following corollary is a direct consequence of the above theorem.

Corollary 6.8
Let T be a theory in a given fuzzy logic L whose formulas only involve connectives having continuous truth-functions. Then the compactness property holds w.r.t. T.

Such a corollary can be clearly applied to the *continuous* fragment of $\text{Ł}\Pi\frac{1}{2}$. Now, considering that the completeness proof of FCP(ŁΠ) shows that one can translate a modal theory over FCP(ŁΠ) into a theory over $\text{Ł}\Pi\frac{1}{2}$, we also obtain compactness results for modal (probabilistic) theories over FCP(ŁΠ) which do not involve the product implication connective $\rightarrow_\Pi$. Therefore, all the above results concerning compactness can be proved also by relying on the fact that in such theories only continuous truth-functions are involved. Indeed, if T is any theory in FCP(ŁΠ) or in FP(SŁΠ) such that product implication does not appear in T and each of its finite subtheories is satisfiable, the by the above corollary T itself is satisfiable. Such compactness is consequently transmitted to the respective probabilistic assessments, easily yielding then the above compactness results for simple and generalized conditional assessments.

7 Applications to knowledge representation

It is worth pointing out that the logic FCP(ŁΠ) is actually very powerful from a knowledge representation point of view. Indeed, it allows to express several kinds of statements about conditional probability, from purely comparative staments like "the conditional event $\varphi \mid \chi$ is at least as probable as the conditional event $\psi \mid \delta$" as

$$P(\psi \mid \delta) \rightarrow_L P(\varphi \mid \chi),$$

or numerical probability statements like

- "the probability of $\varphi \mid \chi$ is 0.8" as $P(\varphi \mid \chi) \equiv \overline{0.8}$,
- "the probability of $\varphi \mid \chi$ is at least 0.8" as $\overline{0.8} \rightarrow_L P(\varphi \mid \chi)$,
- "the probability of $\varphi \mid \chi$ is at most 0.8" as $P(\varphi \mid \chi) \rightarrow_L \overline{0.8}$,
- "$\varphi \mid \chi$ has positive probability" as $\neg_\Pi \neg_\Pi P(\varphi \mid \chi)$,

or even staments about *independence*, like "φ and ψ are independent given χ" as

$$P(\varphi \mid \chi \wedge \psi) \equiv P(\varphi \mid \chi).$$

Another interesting issue is the possibility of modelling default reasoning by means of conditional events and probabilities. This has been largely explored in the literature. Actually, from a semantical point of view, the logical framework that FCP(ŁΠ) offers is very close to the so-called *model-theoretic probabilistic logic* in the sense of Biazzo et al's approach [5], and the links established there to probabilistic reasoning under coherence and default reasoning[14]. Actually, FCP(ŁΠ) can provide a (syntactical) deductive system for such a rich framework.

Here, following the work on default reasoning proposed in [10] in the framework of coherent conditional probability, we show how to define over FCP(ŁΠ) a notion of default rule and default entailment using the deductive machinery of FCP(ŁΠ). First we introduce the basic notions for treating defaults w.r.t. coherent conditional probabilities. Then, we develop the related logical treatment, exploiting the tools provided by FCP(ŁΠ). These two approaches will be shown to be equivalent when we take into account only rational assessments.

[14] See also [39] for another recent probabilistic logic approach to model defaults.

As stressed out above, in [10] the authors use a set-theoretical language and treat conditional events as pairs $E|H$, where E and H are basically considered as sets in a (possibly indeterminate) Boolean algebra. Here we model conditional events as pairs $\varphi|\psi$ where φ and ψ are propositions. So, we model in our framework the implicit relationships between simple events by framing them into a set of formulas.

Now, let κ be a coherent assessment of conditional probability on $\mathcal{C}$. According to the above section this means that there exists at least one conditional probability μ on $\mathbf{B}(\Omega') \times \mathbf{B}(\Omega')^0$, where $\Omega' = \Omega_{D_C}$, extending κ, that is, $\kappa(\varphi_i \mid \chi_i) = \mu(\varphi_i \mid \chi_i)$ for each $i \in I$. Let $\mathcal{E}(\kappa)$ be the set of coherent conditional probabilities extending κ. In this context, a conditional object $\varphi|\chi$ is called a *default* when the coherent assessment κ univocally determines that its probability is 1.

Definition 7.1
Given a coherent assessment κ over a class of conditional events $\mathcal{C}$, a conditional object $\varphi|\chi$ is a default with respect to κ, written $\chi \rightsquigarrow_\kappa \varphi$, if for any $\mu \in \mathcal{E}(\kappa)$ we have $\mu(\varphi \mid \chi) = 1$.

Actually, $\rightsquigarrow_\kappa$ defines a consequence relation among events (propositional formulas) which, due to the possibility of coherent conditional probabilities of assigning zero probabilities to the conditioning events, enjoys the core properties of nonmonotonic reasoning characterizing the system $\mathbf{P}$ [10] of preferential entailment:

(1) *Reflexivity:* $\varphi \rightsquigarrow_\kappa \varphi$.
(2) *Left logical equivalence:* if $\varphi \leftrightarrow \psi \in Taut(\mathcal{L})$ and $\varphi \rightsquigarrow_\kappa \chi$ then $\psi \rightsquigarrow_\kappa \chi$.
(3) *Right weakening:* if $\varphi \to \psi \in Taut(\mathcal{L})$ and $\chi \rightsquigarrow_\kappa \varphi$ then $\chi \rightsquigarrow_\kappa \psi$.
(4) *And:* if $\varphi \rightsquigarrow_\kappa \psi$ and $\varphi \rightsquigarrow_\kappa \chi$ then $\varphi \rightsquigarrow_\kappa \psi \wedge \chi$.
(5) *Cautious Monotonicity:* if $\varphi \rightsquigarrow_\kappa \psi$ and $\varphi \rightsquigarrow_\kappa \chi$ then $\varphi \wedge \psi \rightsquigarrow_\kappa \chi$.
(6) *Or:* if $\varphi \rightsquigarrow_\kappa \chi$ and $\psi \rightsquigarrow_\kappa \chi$ then $\varphi \vee \psi \rightsquigarrow_\kappa \chi$.

We shift now to FCP(ŁΠ). Here, given a modal theory T and a propositional theory D, we define a default w.r.t. the pair (T, D), as any modal formula $P(\varphi \mid \chi)$ which follows from $T \cup D$.

Definition 7.2
Given a modal theory T and a propositional theory D over FCP(ŁΠ) a modal formula $P(\varphi \mid \chi)$ is a default with respect to $T \cup D$, written $\chi \rightsquigarrow_{T,D} \varphi$, iff $T \cup D \vdash_{FCP} P(\varphi \mid \chi)$.

It is now easy to show that $\rightsquigarrow_{T,D}$ is a preferential consequence relation (see e.g. [33]).

Theorem 7.3
$\rightsquigarrow_{T,D}$ is a preferential consequence relation, i.e. it satisfies the above six properties characterizing system $\mathbf{P}$.

Proof. For proving the corresponding properties it suffices to check that the following deductions hold in FCP(ŁΠ)

(1) *Reflexivity:* $\vdash_{FCP} P(\varphi \mid \varphi)$;
(2) *Left logical equivalence:* $\{(\varphi \leftrightarrow \psi), P(\chi \mid \varphi)\} \vdash_{FCP} P(\chi \mid \psi)$;
(3) *Right weakening:* $\{(\varphi \to \psi), P(\varphi \mid \chi)\} \vdash_{FCP} P(\psi \mid \chi)$;
(4) *And:* $\{P(\psi \mid \varphi), P(\chi \mid \varphi)\} \vdash_{FCP} P(\psi \wedge \chi \mid \varphi)$;
(5) *Cautious Monotonicity:* $\{P(\psi \mid \varphi), P(\chi \mid \varphi)\} \vdash_{FCP} P(\chi \mid \varphi \wedge \psi)$;
(6) *Or:* $\{P(\chi \mid \varphi), P(\chi \mid \psi)\} \vdash_{FCP} P(\chi \mid \varphi \vee \psi)$.

It is easy to check that (1) holds by FCP5 and (2) by the substitution of equivalents rule. As for (3), by necessitation for P we have $P(\varphi \to \psi \mid \chi)$ which, along with FCP1 and $P(\varphi \mid \chi)$, by applying modus ponens, implies $P(\psi \mid \chi)$.

As for (4), notice that $\psi \to (\chi \to (\psi \wedge \chi)) \in Taut(\mathcal{L})$. By necessitation for P we obtain $\vdash_{FCP} P(\psi \to (\chi \to (\psi \wedge \chi)) \mid \varphi)$. Now, by FCP1 and modus ponens (twice) we get $\vdash_{FCP} P(\psi \mid \varphi) \to_L P((\chi \to \psi \wedge \chi)|\varphi)$ and $P(\chi \mid \varphi) \to_L P(\psi \wedge \chi \mid \varphi)$, hence $\{P(\psi \mid \varphi), P(\chi \mid \varphi)\} \vdash_{FCP} P(\psi \wedge \chi \mid \varphi)$.

As for (5), by FCP4 we have $\vdash_{FCP} P(\chi \wedge \psi \mid \varphi) \equiv P(\chi \mid \varphi \wedge \psi) \odot P(\psi \mid \varphi)$, hence $\vdash_{FCP} P(\chi \wedge \psi \mid \varphi) \to_L P(\chi \mid \varphi \wedge \psi) \odot P(\psi \mid \varphi)$ as well. By (4), $\{P(\psi \mid \varphi), P(\chi \mid \varphi)\} \vdash_{FCP} P(\psi \wedge \chi \mid \varphi)$, then by modus ponens, $\{P(\psi \mid \varphi), P(\chi \mid \varphi)\} \vdash_{FCP} P(\chi \mid \varphi \wedge \psi) \odot P(\psi \mid \varphi)$, hence $\{P(\psi \mid \varphi), P(\chi \mid \varphi)\} \vdash_{FCP} P(\chi \mid \varphi \wedge \psi)$ as well, since $\Phi \odot \Psi \to_L \Phi$ is a theorem of $Ł\Pi\frac{1}{2}$.

Finally, let us consider (6). Notice that the following equivalences $P(\chi \mid \varphi \vee \psi) \equiv P(\chi \wedge (\varphi \vee \psi) \mid \varphi \vee \psi) \equiv P((\chi \wedge \varphi) \vee (\chi \wedge \psi) \mid \varphi \vee \psi)$ are provable in FCP(ŁΠ). By FCP3, $\vdash_{FCP} P((\chi \wedge \varphi) \vee (\chi \wedge \psi) \mid \varphi \vee \psi) \equiv (P(\chi \wedge \varphi \mid \varphi \vee \psi) \to_L P(\varphi \wedge \psi \wedge \chi \mid \varphi \vee \psi)) \to_L P(\chi \wedge \psi \mid \varphi \vee \psi)$. By FCP4, we have $\vdash_{FCP} P(\chi \wedge \varphi \mid \varphi \vee \psi) \equiv P(\chi \mid \varphi) \odot P(\varphi \mid \varphi \vee \psi)$, $\vdash_{FCP} P(\chi \wedge \psi \mid \varphi \vee \psi) \equiv P(\chi \mid \psi) \odot P(\psi \mid \varphi \vee \psi)$, and $\vdash_{FCP} P(\varphi \wedge \psi \wedge \chi \mid \varphi \vee \psi) \equiv P(\chi \mid \varphi \wedge \psi) \odot P(\varphi \wedge \psi \mid \varphi \vee \psi)$. Now, by the premises and the foregoing, it is easy to check that

$$\{P(\chi \mid \varphi), P(\chi \mid \psi)\} \vdash_{FCP} \tag{*}$$

$$P(\chi \mid \varphi \vee \psi) \equiv (P(\varphi \mid \varphi \vee \psi) \to_L P(\chi \mid \varphi \wedge \psi) \odot P(\varphi \wedge \psi \mid \varphi \vee \psi)) \to_L P(\psi \mid \varphi \vee \psi).$$

On the other hand, since $\vdash_{FCP} P(\varphi \vee \psi \mid \varphi \vee \psi)$ and, by FCP3, $\vdash_{FCP} P(\varphi \vee \psi \mid \varphi \vee \psi) \equiv (P(\varphi \mid \varphi \vee \psi) \to_L P(\varphi \wedge \psi \mid \varphi \vee \psi)) \to_L P(\psi \mid \varphi \vee \psi)$, we also have $\vdash_{FCP} (P(\varphi \mid \varphi \vee \psi) \to_L P(\varphi \wedge \psi \mid \varphi \vee \psi)) \to_L P(\psi \mid \varphi \vee \psi)$. Since $(\Phi \to_L \Psi \odot \Gamma) \to_L (\Phi \to_L \Gamma)$ is a theorem of $Ł\Pi\frac{1}{2}$ we also have $\vdash_{FCP} (P(\varphi \mid \varphi \vee \psi) \to_L P(\chi \mid \varphi \wedge \psi) \odot P(\varphi \wedge \psi \mid \varphi \vee \psi)) \to_L P(\psi \mid \varphi \vee \psi)$. Finally, from this and (*) one can conclude that $\{P(\chi \mid \varphi), P(\chi \mid \psi)\} \vdash_{FCP} P(\chi \mid \varphi \vee \psi)$, which proves (6). ■

We can define now a natural notion of *default entailment*. Let $K = \{\chi_i \rightsquigarrow \varphi_i\}_{i \in I}$ be a conditional knowledge-base. We define a corresponding theory in FCP(ŁΠ) by putting $T^K = \{P(\varphi_i \mid \chi_i)\}_{i \in I}$.

Definition 7.4
A default $\delta \rightsquigarrow \psi$ follows from K, written $K \vdash^*_{FCP} \delta \rightsquigarrow \psi$, iff $T^K \vdash_{FCP} P(\psi \mid \delta)$.

As a direct consequence we have the following corollary.

Corollary 7.5
The inference rules of system **P** are sound w.r.t. $\vdash^*_{FCP}$.

Now, due to FCP(ŁΠ)-probabilistic completeness, the notions of default for coherent assessments and defaults over FCP(ŁΠ) clearly are strictly related. Indeed they can be shown to be equivalent. However, once again, we must take into account only rational assessments, since we cannot represent reals in FCP(ŁΠ)-theories. The following results strengthen the idea that FCP(ŁΠ) strongly captures the concept of coherent conditional probability.

THEOREM 7.6
Let $\mathcal{C}=\{\varphi_i \mid \chi_i\}_{i\in I}$ be a finite family of conditional events, let κ be a rational coherent probability assessment on $\mathcal{C}$. Define the following theory $T=\{P(\varphi_i \mid \chi_i)\equiv \overline{\alpha_i} \mid i\in I, \alpha_i=\kappa(\varphi_i \mid \psi_i)\}$ over FCP(ŁΠ). Then the following condition holds:

$$\chi \leadsto_{\kappa} \varphi \textit{ iff } \chi \leadsto_{T, D_{\mathcal{C}}} \varphi.$$

PROOF. Suppose that $\chi \leadsto_{\kappa}\varphi$. This means that for any conditional probability $\mu\in\mathcal{E}(\kappa)$, $\mu(\varphi \mid \chi)=1$. But, by definition, $\chi\leadsto_{\kappa}\varphi$ iff $\mathcal{E}(\kappa)$ is the set of probability models of T defined on $\Omega_{D_{\mathcal{C}}}$, hence $T\cup D_{\mathcal{C}}\models_{FCP} P(\varphi \mid \chi)$, and by completeness, iff $T\cup D_{\mathcal{C}}\vdash_{FCP} P(\varphi \mid \chi)$, hence iff $\chi\leadsto_{T, D_{\mathcal{C}}} \varphi$. ■

Following Lehmann and Magidor's ideas [33], we can also define *rational* consequence relations with coherent probabilistic semantics (actually, in [33] they use non-standard probabilistic models). So as to do it, we need to fix a single probabilistic Kripke structure $M=\langle W,\mathcal{U},\mu,e\rangle$ and then define the following consequence relation $\leadsto_M$ on propositional (non-modal) formulas:

$$\varphi\leadsto_M\psi \text{ iff } M\models P(\psi \mid \varphi),$$

or equivalently, iff $\mu([\psi]_W \mid [\varphi]_W)=1$ (assuming $[\varphi]_W\neq\emptyset$). This consequence relation can be easily shown to be also a preferential relation, but moreover it can be shown to satisfy the further rational property:

7. Rational Monotonicity[15]: if $\varphi\leadsto_M\psi$ and $\varphi\not\leadsto_M\neg\chi$ then $\varphi\wedge\chi\leadsto_M\psi$,

where the notation $\varphi\not\leadsto_M\psi$ means that the pair (φ,ψ) is not in the consequence relation $\leadsto_M$, i.e. that $M\not\models P(\psi \mid \varphi)$, i.e. that $\mu(\psi \mid \varphi)<1$. This is a consequence of the validity of the following derivation in FCP(ŁΠ): $\{P(\psi \mid \varphi), \neg_L\Delta P(\neg\chi \mid \varphi)\}\vdash_{FCP} P(\psi \mid \varphi\wedge\chi)$. Notice that Rational Monotonicity does not hold in general for the notion of default introduced in Definition 7.2.

8 Conclusions

In this paper we have investigated several aspects of the fuzzy modal logic FCP(ŁΠ) which allows reasoning about coherent conditional probability in the sense of de Finetti. To conclude, we would like to point out some open problems which deserve further investigations.

First, it remains to be studied whether we could use logics weaker than $\text{Ł}\Pi\frac{1}{2}$. Indeed, we could define, as above, a conditional probability logic over ŁΠ (i.e. without rational truth-constants in the language), yielding a kind of *qualitative* probability logic where we could reason for instance about comparative and conditional probability independence statements. Notice that a notion of φ being probable when φ is more probable than $\neg\varphi$, as considered in [29], could also still be defined in such a logic by the formula $\nabla(P(\varphi \mid \top)\ominus P(\neg\varphi \mid \top))$.

[15]The same property has been presented in the literature under different, but equivalent, formulations. After Adams (see e.g. [1, 19]), it is also known as *Disjunctive Weak Rational Monotony*.

Second, we plan to study in more detail the links among FCP(ŁΠ) and different kinds of probabilistic nonmonotonic consequence relations as those defined by Łukasiewicz in [34, 35]. In fact, in that framework, a (strict or defeasible) conditional constraint $(\psi|\varphi)[l, u]$ syntactically corresponds to the FCP(ŁΠ)-formula $(\bar{l} \to_L P(\psi \mid \varphi))\&(P(\psi \mid \varphi) \to_L \overline{u})$. We think FCP(ŁΠ) may provide a suitable framework where to define and compare the different notions of probabilistic default reasoning introduced in [35]. Finally, possible links to the very recent work by Arló-Costa and Parikh [2] on conditional probability and deafeasible reasoning deserve attention.

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