Simplified Kripke semantics for K45- and KD45-like Gödel modal logics

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1 Introduction

Possibilistic logic [5, 6]) is a well-known uncertainty logic to reasoning with graded (epistemic) beliefs on classical propositions by means of necessity and possiblity measures. In this setting, epistemic states of an agent are represented by possibility distributions. If W is a set of classical evaluations or possible worlds, for a given propositional language, a normalized possibility distribution on W is a mapping $\pi : W \to [0, 1]$, with $\sup_{w\in W} \pi(w) = 1$. π ranks interpretations according to its plausibility level: $\pi(w) = 0$ means that w is rejected, $\pi(w) = 1$ means that w is fully plausible, while $\pi(w) < \pi(w')$ means that w' is more plausible than w. A possibility distribution π induces a pair of dual possibility and necessity measures on propositions, defined respectively as:

$$\Pi(\varphi) = \sup\{\pi(w) \mid w \in W, w(\varphi) = 1\}$$
$$N(\varphi) = \inf\{1 - \pi(w) \mid w \in W, w(\varphi) = 0\} \ .$$

$N(\varphi)$ measures to what degree φ can be considered certain given the given epistemic, while $\Pi(\varphi)$ measures the degree in which φ is plausible or possible. Both measures are dual in the sense that $\Pi(\varphi) = 1 - N(\neg\varphi)$, so that the degree of possibility of a proposition φ equates the degree in which $\neg\varphi$ is not certain. If the normalized condition over possibility distribution is dropped, then we gain the ability to deal with inconsistency. In [7], a possibility distribution which satisfies $\sup_{w\in W} \pi(w) < 1$ is called sub-normal. In this case, given a set W of classical interpretations, a degree of inconsistency can be defined in the following way:

$$inc(W) = 1 - \sup_{w\in W} \pi(w)$$

When the normalised possibility distribution π is $\{0, 1\}$-valued, i.e. when π is the characteristic function of a subset $\emptyset \neq E \subseteq W$, then the structure (W, π), or better (W, E), can be seen in fact as a KD45 frame. In fact, it is folklore that modal logic KD45, which is sound and complete w.r.t. the class of Kripke frames (W, R) where R is a serial, euclidean and transitive binary relation, also has a simplified semantics given by the subclass of frames (W, E), where E is a non-empty subset of W (understanding E as its corresponding binary relation R_E defined as $R_E(w, w')$ iff $w' \in E$).

When we go beyond the classical framework of Boolean algebras of events to many-valued frameworks, one has to come up with appropriate extensions of the notion of necessity and possibility measures for many-valued events [4]. In the setting of many-valued modal frameworks over Gödel logic, in [1] the authors claim a similar result as above, in the sense of providing a simplified possibilistic semantics for the logic $KD45(\mathbf{G})$ defined by the class of many-valued Kripke models with a many-valued accessibility relation satisfying counterparts of the serial, euclidean and transitive relations. However, it has to be noted that the completeness proof in [1] has some flaws, as reported by Tuyt.[1] In this paper we will report on a correct proof, not only for the completeness of $KD45(\mathbf{G})$ w.r.t. to its corresponding class of possibilistic frames, but also for the weaker logic $K45(\mathbf{G})$ accounting for partially inconsistent possibilistic Kripke frames.

2 The logic $K45(\mathbf{G})$

In their paper [3] Caicedo and Rodríguez consider a modal logic over Gödel logic with two operators $\Box$ and $\Diamond$. The language $\mathcal{L}_{\Box\Diamond}(Var)$ is built from a countable set Var of propositional variables, connectives symbols $\vee, \wedge, \rightarrow, \bot$, and the modal operator symbols $\Box$ and $\Diamond$. We will simply write $\mathcal{L}_{\Box\Diamond}$ assuming Var is known and fixed.

In their work, Caicedo and Rodríguez define the logic $K(\mathbf{G})$ as the smallest set of formulas containing some axiomatic version of Gödel-Dummet propositional calculus; that is, Heyting calculus plus the prelinearity law and the following additional axioms:

$(K_\Box)$	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$	$(K_\Diamond)$	$\Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi)$
$(F_\Box)$	$\Box\top$	(P)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$
$(FS2)$	$(\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi)$	(Nec)	from φ infer $\Box\varphi$

The logic $K45(\mathbf{G})$ is defined by adding to $K(\mathbf{G})$ the following axioms:

$(4_\Box)$	$\Box\varphi \rightarrow \Box\Box\varphi$	$(4_\Diamond)$	$\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$
$(5_\Box)$	$\Diamond\Box\varphi \rightarrow \Box\varphi$	$(5_\Diamond)$	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$

Let $\vdash_G$ denote deduction in Gödel fuzzy logic G. Let $\mathcal{L}(X)$ denote the set of formulas built by means of the connectives $\wedge, \rightarrow$, and $\bot$, from a given subset of variables $X \subseteq Var$. For simplicity, the extension of a valuation $v : X \rightarrow [0,1]$ to $\mathcal{L}(X)$ according to Gödel logic interpretation of the connectives will be denoted v as well. It is well known that G is complete for validity with respect to these valuations. We will need the fact that it is actually sound and complete in the following stronger sense, see [2].

Proposition 2.1. *i) If $T \cup \{\varphi\} \subseteq \mathcal{L}(X)$, then $T \vdash_G \varphi$ implies $\inf v(T) \leq v(\varphi)$ for any valuation $v : X \rightarrow [0,1]$.*

ii) If T is countable, and $T \nvdash_G \varphi_{i_1} \vee .. \vee \varphi_{i_n}$ for each finite subset of a countable family $\{\varphi_i\}_{i\in I}$ there is an evaluation $v : \mathcal{L}(X) \rightarrow [0,1]$ such that $v(\theta) = 1$ for all $\theta \in T$ and $v(\varphi_i) < 1$ for all $i \in I$.

The following are some theorems of $K(G)$, see [3]. The first one is an axiom in Fitting's systems in [8], the next two were introduced in [3], the fourth one will be useful in our completeness proof and is the only one depending on prelinearity. The last is known as the first connecting axiom given by Fischer Servi.

$T1.$	$\neg\Diamond\theta \leftrightarrow \Box\neg\theta$	$T4.$	$(\Box\varphi \rightarrow \Diamond\psi) \vee \Box((\varphi \rightarrow \psi) \rightarrow \psi)$
$T2.$	$\neg\neg\Box\theta \rightarrow \Box\neg\neg\theta$	$T5.$	$\Diamond(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Diamond\psi)$
$T3.$	$\Diamond\neg\neg\varphi \rightarrow \neg\neg\Diamond\varphi$		

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Next we show that in $K45(\mathbf{G})$ some iterated modalities can be simplified. This is in accordance with our intended simplified semantics for $K45(\mathbf{G})$ that will be formally introduced in the next section.

Proposition 2.2. *The logic $K45(\mathbf{G})$ proves the following schemes:*

$(F_{\Diamond\Box})$	$\Diamond\Box\top \leftrightarrow \Diamond\top$	$(F_{\Box\Diamond})$	$\Box\Diamond\top \leftrightarrow \Box\top$
$(U_{\Diamond})$	$\Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi$	$(U_{\Box})$	$\Box\Box\varphi \leftrightarrow \Box\varphi$
$(T4_{\Box})$	$(\Box\varphi \to \Diamond\Box\varphi) \vee \Box\varphi$	$(T4_{\Diamond})$	$(\Box\Diamond\varphi \to \Diamond\varphi) \vee (\Diamond\top \to \Diamond\varphi)$

From now on we will use $ThK45(\mathbf{G})$ to denote the set of theorems of $K45(\mathbf{G})$. We close this section with the following observation: deductions in $K45(\mathbf{G})$ can be reduced to derivations in pure propositional Gödel logic G.

Lemma 2.1. *For any theory T and formula φ in $\mathcal{L}_{\Box\Diamond}$, it holds that $T \vdash_{K45(\mathbf{G})} \varphi$ iff $T \cup ThK45(\mathbf{G}) \vdash_G \varphi$.*

It is worth noticing that for any valuation v such that $v(ThK45(\mathbf{G})) = 1$ there is no formula φ such that $v(\Diamond\top) < v(\nabla\varphi) < 1$ with $\nabla \in \{\Box, \Diamond\}$ because both formulae $(\Box\varphi \to \Diamond\varphi) \vee \Box\varphi$ and $\Diamond\varphi \to \Diamond\top$ are in $ThK45(\mathbf{G})$.

3 Simplified Kripke semantics and completeness

In this section we will show that $K45(\mathbf{G})$ is complete with respect to a class of simplified Kripke Gödel frames.

Definition 3.1. *A* (normalised) possibilistic Kripke frame*, or* Π*-frame, is a structure* $\langle W, \pi\rangle$ *where* W is a non-empty set of worlds, and $\pi : W \to [0,1]$ *is a (resp. normalised) possibility distribution over* W*.*

A (resp. normalised) possibilistic Gödel Kripke model *is a triple* $\langle W, \pi, e\rangle$ *where* $\langle W, \pi\rangle$ *is a* Π*-frame frame and* $e : W \times Var \to [0,1]$ *provides a Gödel evaluation of variables in each world. For each* $w \in W$*,* $e(w, -)$ *extends to arbitrary formulas in the usual way for the propositional connectives and for modal operators in the following way:*

$$e(w, \Box\varphi) := \inf_{w'\in W}\{\pi(w') \Rightarrow e(w', \varphi)\}$$
$$e(w, \Diamond\varphi) := \sup_{w'\in W}\{\min(\pi(w'), e(w', \varphi))\}.$$

Observe that the evaluation of formulas beginning with a modal operator is in fact independent from the current world. Also note that the $e(-, \Box\varphi)$ and $e(-, \Diamond\varphi)$ are in fact generalisations for Gödel logic propositions of the necessity and possibility degrees of φ introduced in Section 1 for classical propositions, although now they are not dual (with respect to Gödel negation) any longer.

In the rest of this abstract we briefly sketch a weak completeness proof of the logic $K45(\mathbf{G})$ (resp. $KD45(\mathbf{G})$) with respect to the class $\Pi\mathcal{G}$ (resp. $\Pi^*\mathcal{G}$) of (resp. normalised) possibilistic Gödel Kripke models. In fact one can prove a little more, namely completeness for deductions from finite theories.

In what follows, for any formula φ we denote by $Sub(\varphi) \subseteq \mathcal{L}_{\Box\Diamond}$ the set of subformulas of φ and containing the formula $\bot$. Moreover, let $X := \{\Box\theta, \Diamond\theta : \theta \in \mathcal{L}_{\Box\Diamond}\}$ be the set of formulas in $\mathcal{L}_{\Box\Diamond}$ beginning with a modal operator; then $\mathcal{L}_{\Box\Diamond}(Var) = \mathcal{L}(Var \cup X)$. That is, any formula in $\mathcal{L}_{\Box\Diamond}(Var)$ may be seen as a propositional Gödel formula built from the extended set of propositional variables $Var \cup X$. In addition, for a given formula φ, let $\sim_\varphi$ be equivalence relation in $[0,1]^{Var\cup X} \times [0,1]^{Var\cup X}$ defined as follows:

$$u \sim_\varphi w \text{ iff } \forall \psi \in Sub(\varphi) : u(\Box\psi) = w(\Box\psi) \text{ and } u(\Diamond\psi) = w(\Diamond\psi).$$

Now, assume that a formula φ is not a theorem of $K45(\mathbf{G})$. Hence by completeness of Gödel calculus and Lemma 2.1, there exists a Gödel valuation v such that $v(ThK45(\mathbf{G})) = 1$ and $v(\varphi) < 1$. Following the usual canonical model construction, once fixed the valuation v, we define next a canonical $\Pi\mathcal{G}$-model M^v_φ in which we will show φ is not valid.

The *canonical model* $M^v_\varphi = (W^v, \pi^\varphi, e^\varphi)$ is defined as follows:

- W^v is the set $\{u \in [0,1]^{Var\cup X} \mid u \sim_\varphi v \text{ and } u(ThK45(\mathbf{G})) = 1\}$.
- $\pi^\varphi(u) = \inf_{\psi \in Sub(\varphi)}\{\min(v(\Box\psi) \to u(\psi), u(\psi) \to v(\Diamond\psi))\}$.
- $e^\varphi(u, p) = u(p)$ for any $p \in Var$.

Completeness will follow from the next truth-lemma, whose proof is rather involved.

Lemma 3.1 (Truth-lemma)**.** *$e^\varphi(u,\psi) = u(\psi)$ for any $\psi \in Sub(\varphi)$ and any $u \in W^v$.*

Actually, the same proof for weak completeness easily generalizes to get completeness for deductions from finite theories.

Theorem 3.1 (Finite strong completeness)**.** *For any finite theory T and formula φ in $\mathcal{L}_{\Box\Diamond}$, we have:*

- $T \models_{\Pi\mathcal{G}} \varphi$ *implies* $T \vdash_{K45(\mathbf{G})} \varphi$
- $T \models_{\Pi^*\mathcal{G}} \varphi$ *implies* $T \vdash_{KD45(\mathbf{G})} \varphi$

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