An extension of Gödel logic for reasoning under both vagueness and possibilistic uncertainty

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Abstract. In this paper we introduce a logic called $\text{FNG}_{\sim}(\mathbb{Q})$ that combines the well-known Gödel logic with a strong negation, rational truth-constants and Possibilistic logic. In this way, we can formalize reasoning involving both vagueness and (possibilistic) uncertainty. We show that the defined logical system is useful to capture the kind of reasoning at work in the medical diagnosis system CADIAG-2, and we finish by pointing out some of its potential advantages to be developed in future work.

1 Introduction

In the field of uncertain reasoning, many formalisms (e.g., [6], [16], [18]) have been developed to deal with different measures of uncertainty. The most general notion of uncertainty is captured by monotone set functions with two natural boundary conditions. In the literature, these functions have received several names, like *Sugeno measures* [17], *plausibility measures* [12] or *capacities* [1]. In its simplest form, given a *Boolean algebra* of events $\wp = (U, \wedge, \vee, ', \bot, \top)$, a Sugeno measure is a mapping $\mu : U \to [0, 1]$ satisfying $\mu(\top) = 1$ and $\mu(\bot) = 0$, and the monotonicity condition $\mu(x) \leq \mu(y)$ whenever $x \leq^{\wp} y$, where $\leq^{\wp}$ is the lattice order in $\wp$. Many popular uncertainty measures, like probabilities [18], Dempster-Shafer plausibility and belief functions [16], or possibility and necessity measures [6], can be therefore seen as particular classes of Sugeno measures. In this paper, we focus on possibilistic models of uncertainty.

Recall that a *possibility measure* on a (finite) Boolean algebra of events $\wp = (U, \wedge, \vee, ', \bot, \top)$ is a Sugeno measure μ^* satisfiying the following $\vee$-decomposition property

$$\mu^*(u \vee v) = \max(\mu^*(u), \mu^*(v)),$$

while a *necessity measure* is a Sugeno measure μ_* satisfying the $\wedge$-decomposition property

$$\mu_*(u \wedge v) = \min(\mu_*(u), \mu_*(v)).$$

Actually, in presence of these decomposition properties, there is no need for the monotonicity condition since it easily follows from each one of them. Possibility and necessity measures are *dual* in the sense that if μ^* is a possibility measure, then the mapping $\mu_*(u) = 1 - \mu^*(u')$ is a necessity measure, and vice versa. If U is the power set of a (finite) set X, then any dual pair of measures (μ^*, μ_*) on U is induced by a *normalized possibility distribution*, namely a mapping $\pi : X \to [0,1]$ such that $\sup_{x \in X} \pi(x) = 1$, and, for any $A \subseteq X$,

$$\mu^*(A) = \sup\{\pi(x) | x \in A\} \text{ and } \mu_*(A) = \inf\{1 - \pi(x) | x \notin A\}.$$

On the other hand, formal computational models of vague statements usually resort to some sort of fuzzy logic. Fuzzy logics rely on the idea that truth comes in degrees. The inherent vagueness in many real-life declarative statements makes it impossible to always claim either their full truth or full falsity. For this reason, propositions are taken as statements that can be potentially evaluated as being partially true.

Probably the most studied and developed many-valued systems related to fuzzy logic are those corresponding to logical calculi with the real interval $[0,1]$ as set of truth-values and built up from a conjunction & and an implication $\to$, interpreted respectively by a continuous t-norm $*$ and its residuum $\Rightarrow$, and where the negation is defined as $\neg\varphi = \varphi \to \overline{0}$, with $\overline{0}$ being the truth-constant for falsity. In the framework of these logics, called *t-norm based fuzzy logics*, each continuous t-norm $*$ uniquely determines a semantical (propositional) calculus $PC(*)$ over formulas defined in the usual way from a countable set of propositional variables, connectives $\wedge$, & and $\to$ and truth-constant $\overline{0}$ [13]. Evaluations of propositional variables are mappings e assigning each propositional variable p a truth-value $e(p) \in [0,1]$, which extend truth-functionally and univocally to compound formulas as follows:

$$\begin{aligned} e(\overline{0}) &= 0 \\ e(\varphi \& \psi) &= e(\varphi) * e(\psi) \\ e(\varphi \to \psi) &= e(\varphi) \Rightarrow e(\psi) \end{aligned}$$

A formula φ is said to be a 1-tautology of $PC(*)$ if $e(\varphi) = 1$ for each evaluation e. The set of all 1-tautologies of $PC(*)$ will be denoted as $TAUT(*)$. For instance, the well-known *Gödel logic* G is one of the three outstanding t-norm based fuzzy logic calculi corresponding to the choice $* = *_G$, where

$$\begin{aligned} x *_G y &= \min(x, y) \\ x \Rightarrow_G y &= \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise.} \end{cases} \end{aligned}$$

The set $TAUT(*_G)$ is finitely axiomatizable, for instance, as the schematic extension of Hájek's BL with the idempotency axiom $\varphi \to \varphi \& \varphi$ [13].

In some situations, like in the medical diagnosis system CADIAG-2, one has to deal with statements referring to both uncertainty and vagueness (in the sense of gradual properties). CADIAG-2 consists of a knowledge base in the form of a set of if-then rules that relate medical entities (i.e., symptoms on the one hand and diagnoses on the other hand). The rules are defined along with a certain degree of confirmation which intuitively expresses the degree to which the antecedent confirms the consequent. CADIAG-2 considers statements about symptoms as being gradual, where grades refer to the intensity with which symptoms are observed. The second class of propositions in CADIAG-2 refers to diagnoses. It is often the case not, or not yet, possible to confirm or to exclude a diagnosis with certainty. Thus, to each diagnosis, a degree of certainty is associated.

CADIAG-2 also allows facts be qualified by degrees, providing a measure of certainty of crisp statements or a degree of presence of vague statements. Hence, CADIAG-2 shows a challenging feature, i.e., it combines notions of linguistic vagueness and uncertainty. The way those (certainty) values were interpreted and the compositional way they were handled by the inference procedures varied from one approach to another. Several approaches have been developed to provide a clear basis for CADIAG-2. However, in all of these approaches, there was a mismatch between the intended semantics of the (certainty) degrees and the way they were used. Indeed, in some approaches the certainty values were interpreted probabilistically (in some form or another), like in [14], but the propagation rules were either not sound or they were making too strong conditional independence assumptions. On the other hand, other approaches interpreted certainty degrees as truth degrees in a truth-functional many-valued of fuzzy logic setting (see, e.g. [3]). Here the problem was the misuse of partial degrees of truth as belief degrees. This kind of confusion was quite common, but Hájek (in his monograph [13]) had already clear this distinction in mind. He argues a very important issue to distinguish uncertainty measures and truth values in a logical setting, many-valued logics are truth functional but uncertainty measures are not.

Clearly, since uncertainty and vagueness are semantically quite different, it is important to have a unifying formalism for the medical expert system CADIAG-2, which allows us to deal with both uncertainty and vagueness. Any formalism disallowing a unified platform for handling both uncertainty and vagueness is therefore inapt to capture this knowledge and entails the danger of fallacies due to misplaced precision.

We propose an alternative framework for the medical expert system CADIAG-2. Our approach is guided by the idea to use a logical calculus that can deal with both uncertainty and vagueness, and to find an interpretation for CADIAG-2's rules within this unified formalization. In our approach, we make a strong distinction between degrees of uncertainty due to a state of incomplete knowledge and intermediary degrees of truth due to the presence of vague propositions. Our framework to deal with uncertainty is possibilistic logic. The reader is referred to [7] for an extensive overview on possibilistic logic. The proposed logic provides a satisfying conceptual framework for CADIAG-2.

2 The logic $\mathbf{FNG}_{\sim}(\mathbb{Q})$ and its possibilistic semantics

Not surprisingly, the logic which we have in mind is an extension of the logic $G_{\sim}(\mathbb{Q})$, that is Gödel logic extended by the standard negation $\sim$ as well as with rational truth constants[3]. More precisely we will define a logic, called $\mathrm{FNG}_{\sim}(\mathbb{Q})$, F for fuzzy and N for necessity, which will include an extension of possibilistic logic embedded inside the fuzzy logic $\mathrm{G}_{\sim}(\mathbb{Q})$, where it is possible to express e.g. statements very close to the so-called *certainty fuzzy if-then rules* [4] of the form "the more φ is true, the more certain ψ holds "[4], where φ is a fuzzy proposition and ψ is Boolean proposition, as in "the younger a man, the more certainly he is single", or "the higher the fever, the more likely there is an infection".

Starting from the basic ideas exposed by Hájek in [13], various kinds of uncertainties can be studied by using various kinds of modal-fuzzy logics (see, e.g., [10, 11]). The very basic idea allowing a treatment of the certainty of classical (crisp) events inside a fuzzy-logical setting consists of interpreting the certainty degree of a (classical) proposition ϕ as the truth value of a modal proposition $\Box\phi$ which reads "ϕ is certain".

Following the same approach, we define below the logic $\mathrm{FNG}_{\sim}(\mathbb{Q})$ which combines in the same logic a formal treatment of fuzziness and uncertainty aspects, the latter under the possibilistic semantics of necessity measures.

Language. We start from a countable set of Boolean propositional variables $V_B = \{\delta_1, \delta_2, ...\}$ and a countable set $V_{MV} = \{\sigma_1, \sigma_2, ...\}$ of *many-valued propositional variables*. We assume that the two sets V_B and V_{MV} are disjoint. Then formulas of $\mathrm{FNG}_{\sim}(\mathbb{Q})$ are defined as follows:

- The set BFm consists of *Boolean formulas* built from the countable set of propositional variables V_B using the classical logic connectives. $\top$ and $\bot$ will continue denoting the truth constants *true* and *false* respectively. Boolean formulas will be denoted by lower case greek letters $\varphi, \psi, \ldots$
- *Box-formulas* are formulas of the kind $\Box\varphi$, where $\varphi \in BFm$.
- General *FNG-formulas* are then built up from the countable set of many-valued propositional variables V_{MV} together with $\Box$-formulas (taken as new many-valued variables) using $\mathrm{G}_{\sim}(\mathbb{Q})$ connectives $(\wedge, \to, \sim)$ and rational truth constants $\overline{r}$ for every rational $r \in [0, 1]$. We shall denote them by lower case greek letters ζ, ξ, τ, η.

For instance, if ζ is a FNG-formula and $\varphi \in BFm$ is a Boolean formula, then $\overline{0.5} \to (\zeta \to \Box\varphi)$ is a FNG-formula, while $\overline{0.5} \to (\zeta \to \varphi)$ is not.

Note that, as in $\mathrm{G}_{\sim}(\mathbb{Q})$, other connectives are definable, notably $\neg\zeta$ is $\zeta \to \overline{0}$ (Gödel negation), $\Delta\zeta$ is $\neg\sim\zeta$ (Monteiro-Baaz connective), $\zeta \vee \xi$ is $((\zeta \to \xi) \to \xi) \wedge ((\xi \to \zeta) \to \zeta)$ (max disjunction), and $\zeta \leftrightarrow \xi$ is $\zeta \to \xi) \wedge (\xi \to \zeta)$ (equivalence).

[3] Due to lack of space, we cannot include preliminaries on basic notions regarding Gödel logic and its expansions with truth-constants, with Monteiro-Baaz's operator Δ and with an involutive negation, that will be used throughout this section. Instead, the reader is referred to [13, 8, 9] for the necessary background.

[4] Informally interpreted as *truth-degree*$(\varphi) \leq$ *certainty-degree*(ψ).

Semantics. The intended possibilistic semantics is given by what we call *mixed possibilistic models*. A mixed possibilistic model is a pair

$$\mathbf{M} = (v, \langle W, e, \pi \rangle)$$

where $v : V_{MV} \to [0,1]$ is a $[0,1]$-valued interpretation of the many-valued propositional variables, and $\mathbf{M} = \langle W, e, \pi \rangle$ is a possibilistic Kripke model, where W is a non-empty set whose elements are called nodes (or states or possible worlds), $e : W \times V_B \to 0, 1$ provides for each world w a $\{0,1\}$-evaluation of Boolean propositional variables, and $\pi : W \to [0,1]$ is a normalized possibility distribution on W, i.e π: is such that $\max_{w \in W} \pi(w) = 1$. In other words, π models a consistent belief state, in the sense that at least one possible world has to be fully plausible. An evaluation of Boolean propositional variables extends to an evaluation of Boolean formulas of BFm in the usual way. For each $\varphi \in BFm$, we shall write $[\varphi]$ to denote the set of worlds which are a model of φ, i.e. $[\varphi] = \{w \in W : e(w, \varphi) = 1\}$.

Given a mixed possibilistic model $\mathbf{M}$ for $\text{FNG}_{\sim}(\mathbb{Q})$ and a FNG-formula ζ, the truth value of ζ in $\mathbf{M}$, denoted by $\|\zeta\|_{\mathbf{M}}$, is inductively defined as follows:

- if ζ is a propositional variable from the set V_{MV}, then $\|\zeta\|_{\mathbf{M}} = v(\zeta)$,
- if ζ is a Box-formula $\Box\varphi$, where $\varphi \in BFm$, then $\|\zeta\|_{\mathbf{M}} = N([\varphi]|\pi)$, where $N(.|\pi)$ is the necessity measure induced by π on the power set of W, defined as $N([\varphi]|\pi) = \inf_{w \notin [\varphi]}(1 - \pi(w))$.
- if ζ is a compound FNG-formula, then its truth-value $\|\zeta\|_{\mathbf{M}}$ is computed using the truth functions of $\text{G}_{\sim}(\mathbb{Q})$.

The notions of *satisfaction* and *validity* of a formula are defined as usual, as well as the notion of *logical consequence* of a formula from a set of formulas, that we will denote by $\models_{FNG}$. For instance, it is very easy to check that $\overline{0.6} \to \Box\varphi$, saying that φ is certain at least to the degree 0.6, is a logical consequence of the set of FNG-formulas $\{\overline{0.8} \to \sigma_1, \overline{0.6} \leftrightarrow \sigma_2, \overline{0.7} \to (\sigma_1 \wedge \sigma_2 \to \Box\varphi)\}$, saying that σ_1 is true at least to the degree 0.8, σ_2 is true to the degree 0.6 and that, for every $r \in [0,1]$, if $\sigma_1 \wedge \sigma_2$ is true at least to the degree r, then φ is certain at least to the degree $\min(r, 0.7)$.

Axioms and rules. $\text{FNG}_{\sim}(\mathbb{Q})$ has the following axioms and rules:

- Axioms of classical propositional logic for BFm-formulas.
- Axioms and rules of $\text{G}_{\sim}(\mathbb{Q})$ for FNG-formulas.
- The following axiom schemata for the modality $\Box$:
  (N1) $\sim\Box\bot$
  (N2) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
  (N3) $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$
- The *modus ponens* rule of (for BFm and FNG-formulas).
- The *necessitation* rule for $\Box$: from φ derive $\Box\varphi$, for any Boolean formula φ.

A theory Γ over $\text{FNG}_{\sim}(\mathbb{Q})$ is a set of FNG-formulas. We define a notion of proof (denoted by $\vdash_{FNG}$) from a theory in the usual way from the above axioms and rules.

Using standard techniques, it is not hard to prove that $\text{FNG}_{\sim}(\mathbb{Q})$ is indeed *sound and strongly complete* for deductions from finite theories with respect to the class of mixed possibilsitic models, i.e., for every finite theory T and formula ζ, it holds that $T \models_{FNG} \zeta$ iff $T \vdash_{FNG} \zeta$. Details of the completeness proof are to be found in a longer version of this manuscript in preparation.

3 CADIAG-2 and the logic $\text{FNG}_{\sim}(\mathbb{Q})$

As we have already mentioned, CADIAG-2 is a knowledge-based system for medical diagnosis, whose (weighted) if-then rules may combine vague with uncertain knowledge. In this section we explain how the logic $\text{FNG}_{\sim}(\mathbb{Q})$ can represent CADIAG-2 if-then rules as well as the data associated to a patient. The latter is the data describing the state of the patient and it is the input of a particular run of CADIAG-2. We will specify how the input data and the rules translate to formulas of $\text{FNG}_{\sim}(\mathbb{Q})$. We will also compare the CADIAG-2 inference mechanism with proofs in the logic $\text{FNG}_{\sim}(\mathbb{Q})$. We will examine the situation from the theoretical side. Actually, instead of referring to CADIAG-2 system itself, we will usually refer to the formal calculus CadL defined in [3] (see also [2]) capturing its operational semantics in a logical framework.

A knowledge-base of CADIAG-2 (or a theory in CadL) basically deals with two kinds of basic information variables, corresponding to symptoms (including signs), and to diagnoses (diseases and therapies). Given a finite set $S = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ of symptoms and a finite set $D = \{\delta_1, \delta_2, \ldots, \delta_n\}$ of diagnoses, propositions of CadL are built from the variables by means of three connectives "and" ($\wedge$), "or" ($\vee$) and "not" ($\neg$), and graded propositions are of the form (ζ, r), where ζ is a proposition and $r \in [0, 1]$. A graded if-then rule is represented in CadL as a pair $(\zeta \to \varphi, r)$ where ζ is a proposition, φ is a literal (a variable from $S \cup D$ or its negation), and $r \in [0, 1]$. Finally, an input for a run of the system is a set of graded atomic propositions $(\sigma_1, r_1), (\sigma_2, r_2) \ldots$ and $(\delta_1, d_1), (\delta_2, d_2) \ldots$ referring to the available data about the presence of a set of symptoms and diagnoses of a given patient. [5]

In order to define a translation of CadL formulas into $\text{FNG}_{\sim}(\mathbb{Q})$ formulas, first of all we make a formal distinction between symptoms and diagnoses because we formally distinguish between degrees of presence (for symptoms) and degrees of certainty (for diagnoses). Thus, we shall identify each symptom and diagnosis appearing in a CadL theory with a unique atomic proposition from the sets V_{MV} and V_B, respectively, of the language of $\text{FNG}_{\sim}(\mathbb{Q})$. In other words, we assume $S = \{\sigma_1, \sigma_2, \ldots, \sigma_m\} \subseteq V_{MV}$ and $D = \{\delta_1, \delta_2, \ldots, \delta_n\} \subseteq V_B$. Furthermore, let $Lit(D) = \{\delta, \neg\delta \mid \delta \in D\}$ where $\neg\delta$ is the negation of δ, and let $D^{\Box} = \{\Box\varphi : \varphi \in Lit(D)\}$. Recall that $\Box\delta$ means we are certain that the disease δ is present, while

[5] For convenience, from now on we consider all weights being actually rational numbers from $[0, 1]$.

$\Box\neg\delta$ means we are certain that the disease δ is not present (or equivalently, the disease δ is impossible).

The translation of CadL propositions is done as follows. Consider the set M of FNG-formulas built from the set of many-valued propositional variables S and from Box-formulas $\Box\varphi$ for each $\varphi \in Lit(D)$, using the connectives $\wedge, \vee, \sim$ of $\mathrm{FNG}_{\sim}(\mathbb{Q})$. Then, we shall also identify each compound entity in CadL with the respective many-valued formulas from the set M. For instance, assume that we are given the following compound proposition: $(\sigma_1 \vee \neg\delta_1) \wedge (\delta_2 \wedge \neg\sigma_2)$ where σ_1, σ_2 are symptoms and δ_1, δ_2 are diagnoses, then we translate it to the respective FNG formula $(\sigma_1 \vee \Box\neg\delta_1) \wedge \Box\delta_2 \ \wedge \sim\sigma_2$. We shall denote formulas from M by lower case greek letters ζ, ξ, τ, η.

Let us now consider the input of a specific run of CADIAG-2. It consists typically of weighted symptoms, but is also allowed to contain information about confirmed or excluded diagnoses. The two cases are to be distinguished.

- Assume that the graded proposition (σ_i, r) is provided, where $\sigma_i \in S$ is a symptom and r is its degree of presence. Hence, we translate it to the following FNG-formula $\overline{t} \leftrightarrow \sigma_i$. Note that, as particular cases, when $r = 1$ and $r = 0$ the FNG-formula $t \leftrightarrow \sigma_i$ is equivalent to σ_i and $\sim\sigma_i$, respectively.
- Assume that the graded proposition (δ_i, r) is provided, where $\delta_i \in D$ is a diagnosis and $r > 0$. In this case r expresses a degree of uncertainty; we translate it to the following FNG-formula $\overline{r} \to \Box\delta_i$, meaning that we are certain to a degree at least t that the disease δ_i is present. Note that the degree assigned to δ_i may be increased at a later point of a CadL run[6]. As a particular case, when $t = 1$, the FNG-formula $\overline{r} \to \Box\delta_i$ is equivalent to $\Box\delta_i$. However, in case that $t = 0$, the graded proposition $(\delta_i, 0)$ means in CadL that the diagnosis δ_i is excluded, and hence it will be translated to the following FNG-formula $\Box\neg\delta_i$.

We next turn to the translations of rules in the language of CadL. Rules contained in a knowledge base are classified as being of three types: (C), (me), and (ao). We consider first the rules of type (C). It is necessary to differentiate its contents. Namely, there are three kinds of rules that all belong to the type (C).

- First, there are the symptom-symptom rules, which are of the form $(\sigma_i \to \sigma_j, 1)$, where $\sigma_i, \sigma_j \in S$ denote symptoms. According to the manipulation rule (c) of CadL (see [3, 2]), the above rule is to be interpreted as specifying that the degree of σ_j is at least as high as the one of σ_i. Hence, we translate that rule to the following FNG-formula $\sigma_i \to \sigma_j$.
- The second group of rules of type (C) are the diagnosis-diagnosis rules, which are of the form $(\delta_i \to \delta_j, 1)$, where $\delta_i, \delta_j \in D$ are diagnoses. These rules resemble the symptom-symptom rules and the translation of these rules is straightforward: $\Box\delta_i \to \Box\delta_j$.

[6] This is why we interpret r as a lower bound for the certainty degree on δ_i rather than an equality.

– The last group of rules of type (C) are the symptom-diagnosis rules, of the form $(\tau \to \delta, d)$, where τ is a (compound) proposition and $\delta \in D$ refers to a diagnosis. These rules could be considered as the kernel of the inference of CadL and they express that δ is certain to the degree d given the proviso that τ is true. Again, according to the CadL manipulation rule (c) such a rule is to be interpreted as "we are certain that δ is true with a necessity degree at least equal to $\min\{\|\tau\|, d\}$, where $\|\tau\|$ is the truth value of τ. Therefore, the translation into a FNG formula is again straightforward:

$$\overline{d} \wedge \tau^* \to \Box\delta$$

where τ^* is the translation into its corresponding FNG formula of the CadL proposition τ. Due to the residuation property, equivalent translations would also be $\overline{d} \to (\tau^* \to \Box\delta)$ and $\tau^* \to (\overline{d} \to \Box\delta)$.

We finally turn to the rules of type (me) and (ao). They are of the form $(\tau \to \neg\delta, 1)$ and $(\neg\tau \to \neg\delta, 1)$, respectively, where τ is a possibly compound CadL proposition and $\delta \in D$ refers to a diagnosis. According to the CadL manipulation rules (me) and (ao), we translate these rules into the following FNG formulas

$$\triangle\tau^* \to \Box\neg\delta \qquad\qquad \triangle(\sim\tau^*) \to \Box\neg\delta$$

respectively, where again τ^* is the translation into its corresponding FNG formula of the CadL proposition τ. The role of Monteiro-Baaz's operator $\triangle$[7] in the translation of the rules of type (me) and (ao) is obvious, we just have to recall that a CadL rule (me) is applicable if the respective proposition τ is fully present (i.e., its truth value is 1), but not if it is present to any degree strictly smaller than 1. The case of rules (ao) is similar.

Rules of type (me) or (ao) are also used to express relationships between other kinds of entities, namely between two symptoms, or between two diagnoses. The translation of the rules is analogous, e.g., in the case of two symptoms, the translation of such rules is respectively:

$$\triangle\sigma_1 \to \sim\sigma_2, \qquad\qquad \triangle(\sim\sigma_1) \to \sim\sigma_2.$$

From the above, we conclude that the data processed by CadL is translated into a theory of $\mathrm{FNG}_\sim(\mathbb{Q})$ in a way that preserves the contents. However, we have not yet addressed the question what CADIAG-2 (or CadL) on the one hand and $\mathrm{FNG}_\sim(\mathbb{Q})$ on the other hand do with this information. In the following we will show that $\mathrm{FNG}_\sim(\mathbb{Q})$ allows us to draw all possible conclusions in CadL. For this purpose, let us consider the inference rules of CadL (the reader is referred to [3, 2] for details) and check that for every inference rule of CadL, we can specify a corresponding valid rule in the logic $\mathrm{FNG}_\sim(\mathbb{Q})$. Indeed, for all FNG-formulas $\zeta, \tau \in M$, for all Boolean formulas $\delta \in D$ and all rational numbers $r, s, d \in (0, 1] \cap \mathbb{Q}$, we have that the following inferences are valid (and hence provable) in $\mathrm{FNG}_\sim(\mathbb{Q})$:

[7] Recall that this operator is interpreted in a model $\mathbf{M}$ as follows : $\|\Delta\tau\|_{\mathbf{M}} = 1$ if $\|\tau\|_{\mathbf{M}} = 1$, $\|\Delta\tau\|_M = 0$ otherwise.

- $\Box\neg\delta \vdash {\sim}\Box\delta$
- $\Box\neg\delta \vdash {\sim}(\zeta \wedge \Box\delta)$
- ${\sim}\zeta \vdash {\sim}(\tau \wedge \zeta)$
- $\overline{r} \leftrightarrow \zeta,\ \overline{s} \leftrightarrow \tau \vdash \overline{\min(r,s)} \leftrightarrow (\zeta \wedge \tau)$
- $\overline{r} \leftrightarrow \zeta,\ \overline{s} \leftrightarrow \tau \vdash \overline{\max(r,s)} \leftrightarrow (\zeta \vee \tau)$
- $\overline{r} \to \zeta,\ \overline{s} \to \tau \vdash \overline{\min(r,s)} \to (\zeta \wedge \tau)$
- $\overline{r} \to \zeta,\ \overline{s} \to \tau \vdash \overline{\max(r,s)} \to (\zeta \vee \tau)$
- $\overline{r} \leftrightarrow \zeta \vdash \overline{1-r} \leftrightarrow {\sim}\zeta$
- $\overline{r} \leftrightarrow \zeta \vdash \overline{r} \to (\zeta \vee \tau)$
- $\overline{r} \to \zeta \vdash \overline{r} \to (\zeta \vee \tau)$
- $\overline{r} \to \zeta,\ \overline{d} \wedge \zeta \to \tau \vdash \overline{\min(d,r)} \to \tau$
- $\overline{r} \to \zeta,\ \overline{d} \wedge \zeta \to \Box\delta \vdash \overline{\min(d,r)} \to \Box\delta$
- $\triangle\tau \to \Box\neg\delta,\ \tau \vdash \Box\neg\delta$
- $\triangle{\sim}\tau \to \Box\neg\delta,\ {\sim}\tau \vdash \Box\neg\delta$
- $\overline{r} \to \zeta,\ \overline{s} \to \zeta \vdash \overline{\max(r,s)} \to \zeta$

Therefore we conclude that the logic $\text{FNG}_{\sim}(\mathbb{Q})$ can reproduce the inference of the CadL system. One difference is certainly present, $\text{FNG}_{\sim}(\mathbb{Q})$ is stricly stronger than CadL. In other words, $\text{FNG}_{\sim}(\mathbb{Q})$ can produce inferences that CadL cannot, for instance, the following rule concerning $\sim$ is also valid in $\text{FNG}_{\sim}(\mathbb{Q})$: $\overline{r} \to \zeta \vdash \zeta \to \overline{1-r}$, since if r is a lower bound for the truth value of ζ, then $1-r$ is an upper bound of the truth value of ζ.

4 Final remarks and future work

We end by addressing the special feature in CADIAG-2's inference engine, related to the particular role played by the truth value 0. It is a special feature of CADIAG-2 that sharp values dominate over intermediate ones. For example, a medical entity δ, where δ is a diagnosis, may be assigned the certainty value $r \in (0,1) \cap \mathbb{Q}$ at some step in the inference process and it may be the case that 0 is also assigned to it (that is to say, it is considered false with certainty or impossible) in a later step. Hence, according to CADIAG-2, the former value r of δ becomes obsolete once δ is assigned 0.

This means that in such a situation, both the FNG-formulas $\Box\neg\delta$ and $\overline{r} \to \Box\delta$, with $r > 0$, could be provable in the logic $\text{FNG}_{\sim}(\mathbb{Q})$ from a theory obtained by the translation of a CADIAG-2 knowledge base, expressing the constraints $N(\neg\delta) = 1$ and $0 < r \leq N(\delta)$. But having $\min(N(\delta), N(\neg\delta)) = r > 0$ is in conflict with the postulates of possibility theory, that stipulates $\min(N(\delta), N(\neg\delta)) = 0$ for any δ. Therefore, the particular role assigned to 0 in CADIAG-2 may lead to deal with theories with (partial) possibilistic inconsistencies, and thus being $\text{FNG}_{\sim}(\mathbb{Q})$-inconsistent as well.

Part of our future work will address non-monotonicity aspects arising when dealing with (partial) inconsistency in $\text{FNG}_{\sim}(\mathbb{Q})$ theories, either by adapting well-known techniques already used in Possibilistic logic [5], or by designing particular revision or inconsistency repairing mechanisms specially suited for the particular case of CadL in the line of [15] for the case of dealing with a probabilistic semantics.

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