The preserving non-falsity companion of the Nilpotent Minimum Logic

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We recall that a logic L is said to be *paraconsistent* with respect to a negation connective $\neg$ when it contains a $\neg$-contradictory but not trivial theory. Assuming that L is (at least) Tarskian, this is equivalent to say that the $\neg$-explosion rule $\frac{\varphi \quad \neg\varphi}{\psi}$ is not valid in L.

The 3-valued logic J_3 introduced by D'Ottaviano and da Costa in [2] is one of the well known paraconsistent logics and it can be defined (up to language) as the logic given by the matrix $\langle \mathbf{MV_3}, \{\frac{1}{2}, 1\}\rangle$ where $\mathbf{MV_3}$ is the 3 element MV-chain. Notice that J_3 is strongly related with the 3-valued Łukasiewicz logic Ł$_3$ as $\langle \mathbf{MV_3}, \{1\}\rangle$ is a matrix semantics for Ł$_3$. Moreover, these two logics are equivalent deductive systems in the Blok-Pigozzy sense [1]. Notice that, while Ł$_3$ is explosive and truth-preserving (1 being full truth), J_3 is paraconsistent and non-falsity-preserving, because it preserves every element different from 0 (0 being false). We call J_3 the *non-falsity companion of* Ł$_3$.

The *nilpotent minimum logic*, NML for short, was firstly introduced by Esteva and Godo in [3] in order to formalize the logic of the nilpotent minimum t-norm, that was defined by Fodor in [4] as an example of an involutive left continuous t-norm which is not continuous. NML is obtained from the monoidal t-norm logic MTL defined in [3], by adding the involutive condition axiom (INV) $\neg\neg\varphi \to \varphi$ and the (weak) nilpotent minimum condition axiom (WNM) $(\psi * \varphi \to \bot) \vee (\psi \wedge \varphi \to \psi * \varphi)$. It is well known that NML is algebraizable and the class $\mathbb{NM}$ of all nilpotent minimum algebras is its equivalent algebraic quasivariety semantics [3]. Moreover, NML is sound and strong complete with respect the standard NM-algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ [7]. That is, NML is the logic defined by the matrix $\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, \{1\}\rangle$. The aim of this talk is to axiomatize and characterize the non-falsity companions of NML and its axiomatic extensions.

Let $\mathbf{A}$ be a subalgebra of $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$, then the finitary logic L defined by $\langle \mathbf{A}, \{1\}\rangle$ is an axiomatic extension (not necessarily proper) of NML. We call nf-L the non-falsity companion of L. That is, nf-L is the finitary logic defined by the matrix $\langle \mathbf{A}, (0,1] \cap A\rangle$. Consider now the following *restricted* inference rule, which is intended for axiomatising nf-L::

- Restricted Square Modus Ponens for L (r-MP^2 for L):

  From φ and $\varphi \to \neg(\neg\psi)^2$ derive ψ, whenever $\vdash_L \varphi \to \neg(\neg\psi)^2$.

It is not hard to see that from (r-MP^2 for L) we can derive the following restricted version of Modus Ponens:

- Restricted Modus Ponens for L (r-MP for L):

  From φ and $\varphi \to \psi$ derive ψ, whenever $\vdash_L \varphi \to \psi$

Note that both inference rules involve conditions on the derivability of formulas in the logic L. Since any axiomatic extension of NML is complete w.r.t at most two subalgebras of $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ [5] we obtain the following result.

Theorem 1. *Let L be an axiomatic extension of NML. The following axiomatization*

- *Axioms: those of L*
- *Rules: Adjunction* $\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}$ *and* *(r-MP²) for L*

is a sound and complete axiomatisation of nf-*L.*

For the case of finite-valued axiomatic extensions NM_n, unlike the Lukasiewicz case [1, Th.5.2], we prove that nf-NM_n is not equivalent to NM_n. With an abuse of language, $\mathcal{N}_k$ denotes the matrix $\langle \mathbf{NM_k}, \{1\}\rangle$ and $\mathcal{J}_k$ will denote the matrix $\langle \mathbf{NM_k}, \{\frac{1}{k-1}, \frac{2}{k-1}, \ldots, 1\}\rangle$ where $\mathbf{NM_k}$ is the k-element NM-chain. It is shown in [6] that any finitary extension of NM_n is complete w.r.t. following set of matrices $\{\mathcal{N}_{2k}, \mathcal{N}_{2m+1}, \mathcal{N}_2 \times \mathcal{N}_{2r+1}\}$ for some $0 \leqslant m \leqslant r \leqslant k \leqslant n$, For the case of nf-$\mathrm{NM}_n$ we cannot accomplish this reduction, but the following one that is restricted to finitary extensions defined by finite products of $\mathcal{J}_k$'s.

Theorem 2. *Let L be a finitary extension of* nf-*NML defined by* $\mathcal{J}_{k_1} \times \cdots \times \mathcal{J}_{k_s}$. *Then L is complete w.r.t a finite set of the following matrices:*

(i) $\mathcal{J}_n$ *for some positive integer* $n > 1$.

(ii) $\mathcal{J}_n \times \mathcal{J}_k$ *for some positive integers* $n \neq k$.

(iii) $\mathcal{J}_{2n} \times \mathcal{J}_{2k} \times \mathcal{J}_{2l+1}$ *for some positive integers* $l < n < k$.

(iv) $\mathcal{J}_{2n} \times \mathcal{J}_{2m+1} \times \mathcal{J}_{2l+1}$ *for some positive integers* $m < n$ *and* $m < l$.

Moreover every different matrix of these four types defines a different logic

Finally, next result charcaterizes all finite maximal paraconsistent extensions nf-NML

Theorem 3. *The only finite matrices defining maximal paraconsitent extesnions of* nf-*NML are* $\mathcal{J}_3$, $\mathcal{J}_4$ *and* $\mathcal{J}_3 \times \mathcal{J}_4$.

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