

DE MORGAN TRIPLES REVISITED

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Abstract

In this paper we overview basic known results about the varieties generated by De Morgan triples and about the problem to find equations defining the variety generated by a concrete De Morgan triple. We also provide some alternative proofs and some new results, specially for the case of Łukasiewicz De Morgan triples.

1 Introduction

De Morgan triples are algebraic structures over the real unit interval, defined by the minimum, the maximum, a t-norm, an involutive negation and the t-conorm obtained by duality. Namely $([0, 1], \leq, *, n, \oplus)$ is a De Morgan triple if $*$ is a t-norm, n is an involutive (strong) negation and \oplus is the t-conorm defined by duality from $*$ and n (i.e. $x \oplus y = n(n(x) * n(y))$). Actually, any De Morgan triple is characterized by a t-norm $*$ and an involutive negation n , therefore for the sake of a simpler notation, we will denote De Morgan triples as pairs $(*, n)$. Throughout this paper, we will denote the *standard negation* by n_s (i.e. $n_s(x) = 1 - x$) and the De Morgan triples defined by the minimum ($*_G$), product ($*_{\Pi}$) and Łukasiewicz ($*_L$) t-norms and the standard negation n_s will be called *standard* (minimum, product or Łukasiewicz) *De Morgan triples*.

The literature about these structures is quite large and also the use of them in different applications (see for example [20, 21, 1, 2, 17, 18]). De Morgan triples and their isomorphisms were firstly studied by Garcia and Valverde in [11] and posteriorly by Gehrke et al. in [12]. From the logical point of

view they have been first studied by Gerkhe et al. in [13]. In that paper the authors consider logics associated to De Morgan triples only having an involutive negation and two conjunctions and disjunctions (corresponding to min, max and the connectives associated to a t-norm and its dual t-conorm) and the constants 0 and 1 (thus implication is not involved). In the setting of the so-called Mathematical Fuzzy Logic (considering also the residuated implication as connective) the most direct precedent is the paper by Monteiro [15] and Sankaranavar [16] about Heyting algebras with an involutive negation, later generalized to residuated lattices with an involutive negation in [4]. But the most relevant papers in mathematical fuzzy logic strictus sensus are [8, 10, 6, 7]. The first paper [8] studies the logic obtained by adding an involutive negation to a strict BL logic, and in particular to Product and Gödel logics. The papers [10, 6, 7] make further studies in the same topic in a more general setting.

This paper is a summary of basic results (with some new results specially for Łukasiewicz triples) about the varieties generated by De Morgan triples and the problem to find equations defining the variety generated by a concrete De Morgan triple. The generated varieties are understood in this paper as subvarieties of the variety of enriched De Morgan algebras of the kind $(A, \wedge, \vee, \odot, \neg, 0, 1)$, where $(A, \wedge, \vee, \neg, 0, 1)$ is a De Morgan algebra¹ and $(A, \leq, \odot, 1)$ is a commutative ordered monoid² such that $x \odot 0 = 0$ for all $x \in A$. Therefore, when talking in this paper about axiomatizations of varieties generated by De Morgan triples, they will

¹That is, $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\neg : A \rightarrow A$ is an involutive negation on A .

²Here \leq denotes the order defined by the lattice operations \wedge and \vee .

already assume the following set of axioms:

(EDM1) axioms of bounded distributive lattices for $\wedge, \vee, 0, 1$

(EDM2) $\neg(\neg x) = x$

(EDM3) $\neg(x \wedge y) = \neg x \vee \neg y$

(EDM4) axioms of commutative ordered monoids for $\odot, 1$, plus $x \odot 0 = 0$.

2 On De Morgan triples defined over the three basic t-norms

In this section, we recall basic results on De Morgan triples defined by Łukasiewicz, Product and Minimum t-norms together with an involutive negation. Basic results on these three cases are:

- The minimum t-norm case is the most easy one since any pair of De Morgan triples defined by the minimum t-norm and an arbitrary involutive negation are trivially isomorphic (see [11]). Thus the set of valid equations satisfied by them are the same, independently of the particular De Morgan triple. Of course these De Morgan triples are finitely axiomatizable by just adding to (EDM1) – (EDM4) the following single additional axiom:

(MIN) $x \odot y = x \wedge y$

- The Łukasiewicz case is well known for the standard De Morgan triple $(*_L, n_s)$. The variety generated by this triple is the variety of MV-algebras, finitely axiomatizable by the following equations in a language using a negation operation \neg , the disjunction operation \oplus that is dual to \odot wrt \neg , and the constant 0 (see for example [3]):

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

(MV2) $x \oplus y = y \oplus x$

(MV3) $x \oplus 0 = x$

(MV4) $\neg(\neg x) = x$

(MV5) $x \oplus \neg 0 = \neg 0$

(MV6) $(\neg(\neg x \oplus y)) \oplus y = (\neg(\neg y \oplus x)) \oplus x$

But if we take the Łukasiewicz t-norm and an involutive negation different from the standard, then equation (MV6) is not valid anymore. Take, for example the involutive negation defined by:

$$n(x) = \begin{cases} 1 - 3x, & \text{if } x \leq \frac{1}{4} \\ \frac{1}{3} - \frac{1}{3}x, & \text{otherwise} \end{cases}$$

Taking $x = \frac{1}{4}, y = \frac{1}{6}$, the left hand of (MV6) is $\frac{1}{6}$ while the right hand of (MV6) is $\frac{1}{4}$.

We devote Section 3 below to the study of the varieties generated by Łukasiewicz De Morgan triples.

- The case of product t-norm has been largely studied. Observe first that in [8], in order to show that Product logic with an involutive negation is not standard complete, it is proved that the equation $(\neg x \odot x) \leq (\neg(\neg x \odot x))^3$ is valid on the standard product De Morgan triple $(*_\Pi, n_s)$ but not in a product De Morgan triple $(*_\Pi, n)$ with $n \neq n_s$. Section 4 is devoted to discuss the varieties generated by the De Morgan triples defined by product t-norm and an involutive negation.

3 On the lattice of subvarieties generated by Łukasiewicz De Morgan triples

Observe first that the identity is the unique automorphism of the algebraic structure defined over $[0, 1]$ by Łukasiewicz t-norm $*_L$. This easily follows from the fact that any such an automorphism has to be the identity over the rationals and, since $*_L$ is continuous, this implies that it must be the identity over the whole real interval $[0, 1]$. As a consequence we obtain the following result.

Proposition 3.1. *There are as many non-isomorphic Łukasiewicz De Morgan triples as different involutions can be defined on the real unit interval.*

And for the generated varieties we can prove the following result.

Theorem 3.2. *The lattice of subvarieties generated by Łukasiewicz De Morgan triples has infinite height and infinite width.*

Proof. Take a sequence of involutive negations $\{n_k \mid k \in \mathbb{N}, k \geq 2\}$ such that the fix point of n_k is $\frac{k-1}{k}$ and $n_k(\frac{k-2}{k}) = \frac{(k-k)-1}{k-k}$ and consider the corresponding De Morgan triples $T_k = (*_L, n_k)$ for each $k \in \mathbb{N}$. Notice that $(\frac{k-1}{k})^r = 0$ (where $x^k = x *_L \dots *_L x$) if and only if $r \geq k$, $x \wedge n_k(x) \leq \frac{k-1}{k}$, and $(n_k((x \vee n_k(x))^2))^r \leq \frac{k-1}{k}$ if and only if $r \geq k-1$. From these results it is obvious that :

- For $m \in \mathbb{N}$, the equation $(x \wedge \neg x)^m = 0$ (A_m)

is satisfied in any Łukasiewicz De Morgan triple T_k such that $m \geq k$ and it is not satisfied in each T_k such that $m < k$.

- For $r \in \mathbb{N}$, the equation

$$(\neg((x \vee \neg x)^2))^r \leq y \vee \neg y \quad (B_r)$$

is satisfied in any Łukasiewicz De Morgan triple T_k such that $r \geq k - 1$ and it is not satisfied in each T_k such that $1 < r < k - 1$.

This proves the infinite width of the lattice of subvarieties. In order to prove the infinite height, consider the sets (parametrized by $m \in \mathbb{N}$) of Łukasiewicz De Morgan triples $\mathcal{T}_m = \{T_k \mid 2 \leq k - 1 \leq m\}$. Obviously $\mathcal{T}_m \subset \mathcal{T}_{m+1}$ and thus the subset relation is also true for their generated varieties. Moreover it is also obvious that equation (A_m) is satisfied by all De Morgan triples of \mathcal{T}_m but not for all De Morgan triples of \mathcal{T}_{m+1} and so the subset relation between the generated varieties is strict. \square

Some interesting open problems are:

- 1) whether the infinite height and width in Theorem 3.2 is countable or not;
- 2) to study under which conditions two different Łukasiewicz De Morgan triples generate the same variety;
- 3) the (finite or not) axiomatization of the different varieties generated by Łukasiewicz De Morgan triples, or equivalently, to find a (finite or not) family of equations characterizing each Łukasiewicz De Morgan triple.

4 On the lattice of subvarieties generated by product De Morgan triples

We consider here product De Morgan triples, i.e. triples $(*, n)$ such that $*$ is isomorphic to the standard product t-norm $*_{\Pi}$. A first result shows the important role played by the set \mathcal{M}_{Π} of the product De Morgan triples $(*_{\Pi}, n)$ defined by the standard product t-norm $*_{\Pi}$ and any involutive negation n with $\frac{1}{2}$ as its fix point.

Proposition 4.1. *Any product De Morgan triple $(*, n)$ is isomorphic to a product De Morgan triple from \mathcal{M}_{Π} .*

Proof. Let $*$ be a t-norm isomorphic to the product t-norm $*_{\Pi}$, and let $f : [0, 1]_* \rightarrow [0, 1]_{*_{\Pi}}$ be such isomorphism. Then $(*, n)$ is isomorphic to the product De Morgan triple $(*_{\Pi}, \bar{n})$ where the involutive

negation \bar{n} is defined as $\bar{n}(x) = f(n(f^{-1}(x)))$. Denote by s the fix point of \bar{n} . On the other hand, it is well known that any automorphism of the algebraic structure defined by the product t-norm is of the form $x \rightarrow x^a$ for a fixed positive real a , where x^a denotes usual exponential. The automorphism g defined by taking a such that $s^a = \frac{1}{2}$ does the job, indeed $g \circ f$ gives the desired isomorphism since g transforms the involutive negation \bar{n} into a new involutive negation with fix point $\frac{1}{2}$. \square

Thus, in order to study the subvarieties generated by product De Morgan triples one only needs to consider as generators the chains belonging to \mathcal{M}_{Π} . The main result in this section is stated in the following theorem.

Theorem 4.2. *The lattice of subvarieties generated by product De Morgan triples has infinite height and infinite (uncountable) width.*

The result is really surprising if we take into account that the lattice of subvarieties of product-algebras contains Boolean algebras as the only proper subvariety and thus the addition of an involutive negation gives rise to a continuum of subvarieties.

The proof of the next theorem is based on results in the paper [13]. Actually, in that paper only the infinite (uncountable) width result is proved, while the infinite height result is proved in [6]. Next we follow [13] for the proof of the uncountable width result, and from there we provide a new proof of the infinite height result.

Theorem 4.3. *Let $(*, n)$ and $(*, \eta)$ be two product De Morgan triples. The variety generated by $(*, n)$ is comparable with the variety generated by $(*, \eta)$ if and only if the triples $(*, n)$ and $(*, \eta)$ are isomorphic.*

By the previous result, we can restrict ourselves to chains of \mathcal{M}_{Π} . Obviously two chains of \mathcal{M}_{Π} are isomorphic if and only if they are the same chain and what this theorem says is that two different chains from \mathcal{M}_{Π} generate incomparable subvarieties. To prove this statement we need several lemmas.

Lemma 4.4. *If $(*_{\Pi}, n)$ belongs to the variety generated by $(*_{\Pi}, \eta)$, then $n(\frac{1}{2^k}) = \eta(\frac{1}{2^k})$ for all $k \in \mathbb{N}$.*

Proof. Consider for any $k, l, m \in \mathbb{N}$, the equations³

³Remember that $a \leq b$ is equivalent to $a \wedge b = a$.

$$(n((x \vee n(x))^k))^l \leq (y \vee n(y))^m \quad (1)$$

and

$$(n((x \wedge n(x))^k))^l \geq (y \wedge n(y))^m. \quad (2)$$

Equation (1) is valid over $(*_\Pi, n)$ if the inequality holds for any $a, b \in [0, 1]$, which is equivalent to

$$\max_{a \in [0,1]} (n(a \vee n(a))^k)^l \leq \min_{b \in [0,1]} (b \vee n(b))^m.$$

It is obvious that these maxima and minima are attained at $\frac{1}{2}$, the fix point of the negations. Thus (1) holds in $(*_\Pi, n)$ if and only if

$$n\left(\frac{1}{2^k}\right) \leq \left(\frac{1}{2}\right)^{\frac{m}{l}}.$$

An analogous reasoning proves that (2) holds in $(*_\Pi, n)$ if and only if

$$n\left(\frac{1}{2^k}\right) \geq \left(\frac{1}{2}\right)^{\frac{m}{l}}.$$

If $(*_\Pi, n)$ belongs to the variety generated by $(*_\Pi, \eta)$, the inequalities that hold for $(*_\Pi, n)$ must hold for $(*_\Pi, \eta)$ as well, and being the set $\{(\frac{1}{2})^{\frac{m}{l}} \mid l, m \in \mathbb{N}\}$ dense in the real unit interval, we conclude that for each $k \in \mathbb{N}$, $n(\frac{1}{2^k}) = \eta(\frac{1}{2^k})$. \square

Now for any involutive negation n with fix point $\frac{1}{2}$, define the set

$$M(n) = \{(n(\frac{1}{2^k}))^l \mid k, l \in \mathbb{N}\}.$$

Lemma 4.5. *For any involutive negation n with fix point $\frac{1}{2}$, $M(n)$ is dense in the real unit interval.*

Proof. The sequence $\{n(\frac{1}{2^k}) \mid k \in \mathbb{N}\}$ is an increasing sequence with limit 1, and thus for any $\varepsilon > 0$ there is k_0 such that $1 - n(\frac{1}{2^{k_0}}) < \varepsilon$. But $1 - b < \varepsilon$ implies $b^m - b^{m+1} = b^m(1 - b) < (1 - b) < \varepsilon$. Thus it follows that for each real in $[0, 1]$ there is an element in the sequence $\{(n(\frac{1}{2^k}))^l \mid k, l \in \mathbb{N}\}$ whose difference from it is at most ε . \square

Lemma 4.6. *If $(*_\Pi, n)$ belongs to the variety generated by $(*_\Pi, \eta)$, then n and η coincide on $M(n)$.*

Proof. If $(*_\Pi, n)$ belongs to the variety generated by $(*_\Pi, \eta)$, we know from Lemma 4.4, that for all $k, l \in \mathbb{N}$, $(n(\frac{1}{2^k}))^l = (\eta(\frac{1}{2^k}))^l$ and thus the sets $M(n)$ and $M(\eta)$ coincide. Now consider the inequalities:

$$(n((n((x \wedge n(x))^k))^l))^r \leq (y \vee n(y))^m \quad (3)$$

and

$$(n((n((x \vee n(x))^k))^l))^r \geq (y \wedge n(y))^m \quad (4)$$

By an argument similar to that in Lemma 4.4 we obtain that (3) holds in $(*_\Pi, n)$ if, and only if,

$$n\left(\left(n\left(\frac{1}{2^k}\right)\right)^l\right) \leq \left(\frac{1}{2}\right)^{\frac{m}{r}},$$

and that (4) holds in $(*_\Pi, n)$ if, and only if,

$$n\left(\left(n\left(\frac{1}{2^k}\right)\right)^l\right) \geq \left(\frac{1}{2}\right)^{\frac{m}{r}}.$$

But the same conditions are valid for η and thus, reasoning again as in Lemma 4.4, we obtain $n(a) = \eta(a)$, for all $a \in M(n) = M(\eta)$. \square

We have thus shown that if $(*_\Pi, n)$ belongs to the variety generated by $(*_\Pi, \eta)$, then n and η agree on a dense set and since involutive negations are continuous functions, they coincide in the full real unit interval. This ends the proof of Theorem 4.3.

The two families of equations used in the proofs above have been considered separately. However the first family is a special case of the second. Namely, taking $k = 1$ equation (3) becomes (1), and (4) becomes (2) as an easy computation shows. Thus, in fact, we only have one family of equations used to separate the subvarieties. From Theorem 4.3 it follows that there are as many incomparable subvarieties as involutive negations having $\frac{1}{2}$ as fixed point. Therefore there are uncountable many of the latter. Summarizing, we have the following result.

Corollary 4.7. *The set of subvarieties generated by single product De Morgan triples contains an uncountable number of pair-wise incomparable subvarieties. Furthermore these subvarieties are separable by the following family of equations:*

$$(n((n((x \wedge n(x))^k))^l))^r \leq (y \vee n(y))^m.$$

Next we will prove the infinite height part of Theorem 4.2. Fix $k_0, m_0 \in \mathbb{N}$ and an strictly increasing sequence of naturals $\{l_i\}_{i \in \mathbb{N}}$. Then define the sequence $\{T_i\}_{i \in \mathbb{N}}$ of subsets of \mathcal{M}_Π as

$$T_i = \{(*_\Pi, n) \in \mathcal{M}_\Pi \mid n\left(\frac{1}{2^{k_0}}\right) \leq \left(\frac{1}{2}\right)^{\frac{m_0}{l_i}}\}.$$

Since $\{\frac{m_0}{l_i}\}_{i \in \mathbb{N}}$ is a decreasing sequence with limit 0, for all $i \in \mathbb{N}$, $T_i \subset T_{i+1}$ and the same inclusions hold true for the varieties generated by these families. Finally an easy observation shows that these inclusions are proper, since the equation (1) for

k_0, m_0 and l_i is valid for T_i but not for T_{i+1} . Thus we have an infinite sequence of strict inclusions of subvarieties, and thus the height of the lattice of subvarieties is, at least, countably infinite.

Remark In [6] and in [14], further insights into the subvarieties of product De Morgan triples and strict De Morgan triples⁴ can be found. In order to separate the subvarieties, in [6] Cintula et al. use a different family of equations. For each natural n , they define the equation⁵

$$n((n(x^n))^n) = x \quad (D_n)$$

and prove, using these equations, that the lattice of subvarieties of product De Morgan triples and strict De Morgan triples contain a sublattice isomorphic to the lattice of natural numbers (\mathbb{N}, \preceq) with the order \preceq defined by: $1 \preceq n$ for all $n \in \mathbb{N}$ and $n \preceq m$ if there is a natural k such that $n^k = m$. So defined, it is clear that (\mathbb{N}, \preceq) has infinite width and infinite height.

One interesting question is to know whether or not there are finitely-many characterizing equations for each product De Morgan triple of \mathcal{M}_Π , or equivalently, whether there is a finite equational basis for the subvariety generated by each product De Morgan triple of \mathcal{M}_Π . The question makes sense because in both papers [13, 7] the authors give different sets of separating equations, i.e. defining different subvarieties, but in each case they need an infinite number of equations to axiomatize the subvariety generated by each one of the product De Morgan triple.

5 Final remarks

In the setting of Mathematical Fuzzy logic, the expansions of t-norm based logics with an involutive negation was firstly studied by Esteva et al. in [8] and posteriorly by Flaminio and Marchioni in [10], Cintula et al. in [6] and finally by Haniková and Savický in [14]. The basic difference with the work on De Morgan triples is the use of one further operation, the residuated implication \Rightarrow , and of its associated negation defined as $x \Rightarrow 0$. The first paper mostly deals with completeness results for logics of continuous t-norms expanded with an involutive negation, the second one generalizes it to the more general setting of logics of left-continuous

⁴That is, those triples defined by continuous t-norms * satisfying $x * x > 0$ for each $x > 0$.

⁵We have translated the equation to our notation.

t-norms, and the third one deals with some simplifications on the axioms and with some subvarieties defined by additional axioms (among them the ones we have commented in the remark of the preceding section). In the last paper, Haniková and Savický give a set of equations for product logic with an involutive negation that separate the varieties generated by real product logic chains (that cannot be used in our setting because they use the implication and its associated negation that we do not have in the De Morgan triples). On the other hand they characterize the the strict continuous t-norms defining De Morgan triples for which the result of Theorem 4.3 still remains valid.

Still in the setting of residuated t-nom based logics, the problem of whether the logic corresponding to the expansion of the standard product chain with an involutive is finitely axiomatizable has been (partially) adressed in the literature. As far as we know, only the case of the logic of the standard product chain and the standard negation has been proved to be finitely axiomatizable (when using the product implication and thus in a more general setting than the De Morgan triples). The proof of this result is not trivial and is based on the study of the logic $\mathbb{L}\Pi$ combining both product and Łukasiewicz logics [9]. The original definition of $\mathbb{L}\Pi$ is done in a language with four basic connectives (both Łukasiewicz and product conjunctions and implications), and is complete with respect to the standard $\mathbb{L}\Pi$ -chain, the chain defined over $[0, 1]$ by Łukasiewicz and product t-norms and their associated residuated implications. However, a very nice result due to Cintula (see [5, 6, 19] for further details and references) proves that $\mathbb{L}\Pi$ is also complete with respect to the expansion of the standard product chain with the standard negation. Indeed, he observes that, over this structure, the standard Łukasiewicz conjunction and implication operations are definable as follows:

$$(C) \quad x *_L y = x *_\Pi n_s(x \Rightarrow_\Pi n_s(y))$$

$$(I) \quad x \Rightarrow_L y = n_s(x *_\Pi n_s(x \Rightarrow_\Pi y))$$

and, following this idea, it is proved in [5, 19] that $\mathbb{L}\Pi$ can be defined as an axiomatic extension of the product logic with an involutive negation by either defining $\&_L$ as in (C), i.e., as $\varphi \& \sim (\varphi \rightarrow \sim \psi)$ and adding the axiom:

$$\bullet \quad \varphi \&_L \psi \rightarrow_L \psi \&_L \varphi$$

or defining \rightarrow_L as in (I), i.e. as $\sim (\varphi \& \sim (\varphi \rightarrow \psi))$

and adding the axiom:

- $(\varphi \rightarrow_{\mathbb{L}} \psi) \rightarrow_{\mathbb{L}} ((\psi \rightarrow_{\mathbb{L}} \chi) \rightarrow_{\mathbb{L}} (\varphi \rightarrow_{\mathbb{L}} \chi))$

This implies that if one takes an involutive negation different from n_s , the resulting operation in (C) is not commutative any longer and, analogously, the resulting function in (I) is not transitive any longer (thus they do not coincide with Łukasiewicz t-norm and its residuum). As a consequence, the logic that is complete with respect to the standard product chain with the standard involutive negation is in fact $\mathbb{L}\Pi$, hence it is finitely axiomatizable.⁶

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⁶This does not imply that the corresponding De Morgan triple be finitely axiomatizable since here we are using the product implication.