ARITHMETICAL COMPLEXITY OF FIRST-ORDER PREDICATE FUZZY LOGICS OVER DISTINGUISHED SEMANTICS

Franco Montagna Carles Noguera
Department of Mathematics and Computer Science
University of Siena
Pian dei Mantellini 44
53100 Siena, Italy
montagna@unisi.it, cnoguera@iiia.csic.es

Abstract

All promiment examples of first-order predicate fuzzy logics are undecidable. This leads to the problem of the arithmetical complexity of their sets of tautologies and satisfiable sentences. This paper is a contribution to the general study of this problem. We propose the classes of first-order core and Δ-core fuzzy logics as a good framework to address these arithmetical complexity issues. We obtain general results providing lower bounds for the complexities associated to arbitrary semantics and we compute upper bounds and exact positions in the arithmetical hierarchy for distinguished semantics: general semantics given by all chains, finite-chain semantics, standard semantics and rational semantics.

Keywords: Arithmetical complexity, Core fuzzy logics, Finite-chain semantics, First-order predicate fuzzy logics, Mathematical Fuzzy Logic, Rational semantics, Standard semantics.

1 Introduction

Since the inception of the theory of fuzzy sets by Zadeh in [45], many logical systems have been proposed to deal with the reasoning with predicates that can be modelled by fuzzy sets. Mathematical Fuzzy Logic is the subdiscipline of Mathematical Logic devoted to the study of these logical systems. The first two examples were taken from the many-valued logic tradition: Łukasiewicz [33] and Gödel-Dummett [19, 10] propositional logics, both complete with respect to a matrix semantics whose truth-values are the real numbers in $[0, 1]$ and where conjunction is truth-functionally interpreted by a continuous t-norm.[1] Later a third system with this feature was introduced: product logic [29], where the corresponding continuous t-norm was the standard product of real numbers. Based on these three main examples, during the last fifteen years the area has experienced a process of increasing generalization giving birth to a multiplicity of fuzzy logics. The first step was taken by Hájek [20] when he proposed the

[1] A t-norm $*$ is a binary operation on the real unit interval that is associative, commutative, non-decreasing and satisfying $x * 1 = x$ for every $x \in [0, 1]$. These functions arose in the theory of probabilistic metric spaces and, after Zadeh's works, were soon proposed as a means to truth-functionally model intersection of fuzzy sets, and hence conjunction in fuzzy logics.

so-called *Basic fuzzy Logic* BL, which turned out to be complete with respect to the semantics of all continuous t-norms. Later on continuity was removed in the logic MTL [12] which is complete with respect to the semantics of all left-continuous t-norms, negation was removed when considering fuzzy logics based on hoops [13], commutativity of t-norms was disregarded in [23], and recently t-norms have been replaced by uninorms in [34]. On the other hand, many interesting axiomatic extensions of MTL have been introduced and proved to be complete with respect to particular kinds of left-continuous t-norms (see e.g. [12, 22, 5, 39, 41]) and, in addition, logics with a higher expressive power have been defined by considering expanded $[0,1]$-valued algebras (with projection Δ, involution $\sim$, truth-constants, etc.). Moreover, for almost all these propositional logics corresponding first-order versions have been introduced in the literature (see e.g. the survey paper [8]).

Nonetheless, Mathematical Fuzzy Logic has been growing not only *in width* but also *in depth* inasmuch, besides introducing new logical systems, scholars have concentrated on studying their properties from several points of view. In the case of first-order predicate fuzzy logics, since most of the prominent examples (namely all first-order versions of consistent axiomatic extensions of MTL) were proved to be undecidable (see [20, 40]), the issue of their arithmetical complexity became an important item in the agenda. An early contribution to such topic was that of Ragaz in [43] where he showed that the tautology problem for the $[0,1]$-valued standard semantics of Łukasiewicz logic was Π_2-complete. Hájek started addressing the problem in a systematic way already in Chapter 6 of his book [20] (which subsumed a couple of previous papers by himself) with several results concerning the position in the arithmetical hierarchy of the sets of tautologies, positive tautologies, satisfiable sentences, and positively satisfiable sentences w.r.t. the standard semantics of the three basic fuzzy logics. Those results were subsequently extended to a rather complete study of arithmetical complexity problems for the standard semantics of the logic BL and its continuous t-norm based axiomatic extensions in a series of papers by Hájek and the first author of the present work [21, 36, 37]. The survey paper [25], besides collecting the mentioned results, provides a new study where the standard semantics is replaced by the so-called *general semantics*, i.e. the one given by models over arbitrary linearly ordered BL-algebras. Finally, the recent works [26, 30, 7, 27] add some new knowledge on the matter by considering respectively fragments of continuous t-norm based logics, semantics based on linearly ordered complete BL-algebras, extensions of Łukasiewicz logic and Gödel logics.

Besides the standard and the general semantics, other kinds of semantics for first-order fuzzy logics, such as the ones given by algebras defined over the rational unit interval or over finite chains, have recently started receiving attention in the literature (see [6, 17]). This points to some new problems that had been neglected so far: What are the arithmetical complexities of the sets of (positive) tautologies and satisfiable sentences w.r.t. the rational and the finite-chain semantics? What are the relations of these sets with those corresponding with the general and the standard semantics?

The present paper intends to expand the landscape of the studies on arithmetical complexity issues for fuzzy logics in two directions: (1) by considering the aforementioned rational and finite-chain semantics and their associated problems, and (2) by widening the framework of first-order fuzzy logics under focus to the classes of core and Δ-core fuzzy logics.[2] These

[2]Roughly speaking, core and Δ-core fuzzy logics are good expansions of, respectively, MTL and MTL_Δ and include in particular their axiomatic extensions and all the logics in enriched languages that have been mentioned before. Those classes have been proved to provide a good level of generality in the works [28, 6].

two layers will amount to a much more general study encompassing (almost all) the previous ones. After a preliminary section that will briefly introduce the needed notions on both propositional and first-order fuzzy logics, Section 3 will present our new approach and new results on arithmetical complexity issues. Namely Section 3.1 will present complexity results for arbitrary semantics (i.e. semantics given by arbitrary classes of linearly ordered algebras) from which we will obtain that all the predicate logics under our scope are undecidable and some consequences for arithmetical complexity of the general semantics; Section 3.2 will be devoted to finite-chain semantics by showing how to obtain uniform upper bounds for its associated complexity problems and obtaining some exact complexity results for prominent logics; finally Section 3.3 will consider the semantics given by standard and rational chains, with a special attention to *canonical rational chains* and the mutual relation between those semantics and that of finite chains.

2 Preliminaries

2.1 Propositional logics

The weakest propositional t-norm based fuzzy logic is the so-called *Monoidal T-norm based Logic* (MTL). It was defined by Esteva and Godo in [12] by means of a Hilbert-style calculus in the language $\mathcal{L} = \{\&, \to, \wedge, \overline{0}\}$ of type $\langle 2, 2, 2, 0\rangle$. $\mathrm{Fm}_{\mathcal{L}}$ will denote the set of all formulae built over a denumerable set of propositional variables using the connectives of $\mathcal{L}$. The only inference rule of the calculus is *modus ponens* and the axiom schemata are the following (taking $\to$ as the least binding connective):

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \& \psi \to \varphi$
(A3)	$\varphi \& \psi \to \psi \& \varphi$
(A4)	$\varphi \wedge \psi \to \varphi$
(A5)	$\varphi \wedge \psi \to \psi \wedge \varphi$
(A6)	$\varphi \& (\varphi \to \psi) \to \varphi \wedge \psi$
(A7a)	$(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$
(A7b)	$(\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$\overline{0} \to \varphi$

The usual defined connectives are introduced as follows:

$\varphi \vee \psi$	as	$((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$	$\neg\varphi$	as	$\varphi \to \overline{0}$
$\varphi \leftrightarrow \psi$	as	$(\varphi \to \psi) \& (\psi \to \varphi)$	$\overline{1}$	as	$\neg\overline{0}$

Also as usual, φ^n will be used as a shorthand for $\varphi \& \overset{n}{\ldots} \& \varphi$, where $\varphi^0 = \overline{1}$. MTL enjoys the following form of local deduction-detachment theorem and substitution rule.

Proposition 2.1. *For each set of formulae* $\Sigma \cup \{\varphi, \psi, \chi\}$ *it holds:*

$$\Sigma, \varphi \vdash_{\mathrm{MTL}} \psi \text{ iff there is } n \in \mathbb{N} \text{ such that } \Sigma \vdash_{\mathrm{MTL}} \varphi^n \to \psi \qquad (\mathcal{LDT})$$

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{MTL}} \chi(\varphi) \leftrightarrow \chi(\psi). \qquad \text{(Cong)}$$

The algebraic counterpart of MTL logic is the class of MTL-algebras:

Definition 2.2 ([12])**.** *An* MTL*-algebra is an algebra* $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \to^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *of type* $\langle 2, 2, 2, 2, 0, 0 \rangle$ *such that:*

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *is a bounded lattice.*
2. $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *is a commutative monoid with unit* $\overline{1}^{\mathcal{A}}$*.*
3. *The operations* $\&^{\mathcal{A}}$ *and* $\to^{\mathcal{A}}$ *form an adjoint pair:* $a \&^{\mathcal{A}} b \leq c$ *iff* $b \leq a \to^{\mathcal{A}} c$*.*
4. *It satisfies the prelinearity equation:* $(a \to^{\mathcal{A}} b) \vee^{\mathcal{A}} (b \to^{\mathcal{A}} a) = \overline{1}^{\mathcal{A}}$

If the lattice order is total we will say that $\mathcal{A}$ *is a linearly ordered* MTL*-algebra (or just an* MTL*-chain).*

An additional negation operation is defined as $\neg^{\mathcal{A}} a = a \to^{\mathcal{A}} \overline{0}^{\mathcal{A}}$. Similarly, an additional equivalence operation is defined as $a \leftrightarrow^{\mathcal{A}} b = (a \to^{\mathcal{A}} b) \&^{\mathcal{A}} (b \to^{\mathcal{A}} a)$. For the sake of a simpler notation, superscripts in the operations of the algebras will be omitted when they are clear from the context. The class of all MTL-algebras is a variety which will be denoted as $\mathbb{MTL}$. Important examples of MTL-algebras are those defined by left-continuous t-norms, the so-called *standard algebras*. Actually, for every left-continuous t-norm $*$ there is an operation $\Rightarrow$ (the residuum) such that $*$ and $\Rightarrow$ form an adjoint pair, and hence the algebra $[0,1]_* = \langle [0,1], *, \Rightarrow, \min, \max, 0, 1 \rangle$ is an MTL-chain; conversely, any MTL-chain on $[0,1]$ is of this form.

Definition 2.3. *Let* $\mathbb{K}$ *be a class of* MTL*-algebras. We define the consequence relation* $\models_{\mathbb{K}}$ *in the following way:* $\Sigma \models_{\mathbb{K}} \varphi$ *iff for each* $\mathcal{A} \in \mathbb{K}$ *and* $\mathcal{A}$*-evaluation* e*:* $e(\varphi) = \overline{1}^{\mathcal{A}}$ *whenever* $e[\Sigma] \subseteq \{\overline{1}^{\mathcal{A}}\}$*.*

We write $\models_{\mathbb{K}} \varphi$ instead of $\emptyset \models_{\mathbb{K}} \varphi$ and we write $\Sigma \models_{\mathcal{A}} \varphi$ instead of $\Sigma \models_{\{\mathcal{A}\}} \varphi$. The $\mathcal{A}$-evaluation e such that $e[\Sigma] \subseteq \{\overline{1}^{\mathcal{A}}\}$ is called an $\mathcal{A}$-model of Σ. That $\mathbb{MTL}$ is the proper algebraic semantics for MTL is witnessed by the following completeness result.

Theorem 2.4 ([12])**.** *Let* $\Sigma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$*. Then* $\Sigma \vdash_{\mathrm{MTL}} \varphi$ *if and only if* $\Sigma \models_{\mathbb{MTL}} \varphi$*.*

This completeness result can be refined by taking into account the following representation of MTL-algebras.

Proposition 2.5 ([12])**.** *Every* MTL*-algebra is a subdirect product of* MTL*-chains.*

This leads to the completeness of MTL with respect to the class of MTL-chains.

Corollary 2.6. *Let* $\Sigma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$*. Then* $\Sigma \vdash_{\mathrm{MTL}} \varphi$ *if and only if* $\Sigma \models_{\{\mathrm{MTL}\text{-}\mathit{chains}\}} \varphi$*.*

The following result is a further refinement which justifies why MTL is the weakest t-norm based fuzzy logic, the *standard completeness* theorem:

Theorem 2.7 ([32])**.** *Let* $\Sigma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$*. Then* $\Sigma \vdash_{\mathrm{MTL}} \varphi$ *if and only if*
$\Sigma \models_{\{[0,1]_* \mid * \text{ left-continuous t-norm}\}} \varphi$*.*

Axiom schema	Name
$\neg\neg\varphi \to \varphi$	Involution (Inv)
$\neg\varphi \vee ((\varphi \to \varphi\&\psi) \to \psi)$	Cancellation (Can)
$\neg(\varphi\&\psi) \vee ((\psi \to \varphi\&\psi) \to \varphi)$	Weak Cancellation (WCan)
$\varphi \to \varphi\&\varphi$	Contraction (C)
$\varphi^{n-1} \to \varphi^n$	n-Contraction (C_n)
$\varphi \wedge \neg\varphi \to \overline{0}$	Pseudocomplementation (PC)
$\varphi \wedge \psi \to \varphi\&(\varphi \to \psi)$	Divisibility (Div)
$(\varphi\&\psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi\&\psi)$	Weak Nilpotent Minimum (WNM)
$\varphi \vee \neg\varphi$	Excluded Middle (EM)

Table 1: Some usual axiom schemata in fuzzy logics.

Most of the well-known fuzzy logics can be presented as axiomatic extensions of MTL. Tables 1 and 2 collect some axiom schemata and the axiomatic extensions of MTL that they define.

MTL is actually an algebraizable logic in the sense of Blok and Pigozzi (see [3]) and $\mathbb{MTL}$ is its equivalent algebraic semantics. This implies that all axiomatic extensions of MTL are also algebraizable and their equivalent algebraic semantics are the subvarieties of $\mathbb{MTL}$ defined by the translations of the axioms into equations. In particular, there is an order-reversing isomorphism between axiomatic extensions of MTL and subvarieties of $\mathbb{MTL}$:

1. If $\Sigma \subseteq \mathrm{Fm}_{\mathcal{L}}$ and L is the extension of MTL obtained by adding the formulae of Σ as axiom schemata, then the equivalent algebraic semantics of L is the subvariety of $\mathbb{MTL}$ axiomatized by the equations $\{\varphi \approx \overline{1} \mid \varphi \in \Sigma\}$. We denote this variety by $\mathbb{L}$ and we call its members L-*algebras*. There are two exceptions to that rule: the algebras associated to Ł are called MV-*algebras* following the terminology of Chang in [4] and the algebras associated to the Classical Propositional Calculus (CPC for short) are called, of course, *Boolean algebras*. Moreover, since each L-algebra is representable as a subdirect product of L-chains, the completeness of MTL with respect to chains is inherited by L.

2. Let $\mathbb{L} \subseteq \mathbb{MTL}$ be the subvariety axiomatized by a set of equations Λ. Then the logic associated to $\mathbb{L}$ is the axiomatic extension L of MTL given by the axiom schemata $\{\varphi \leftrightarrow \psi \mid \varphi \approx \psi \in \Lambda\}$.

In particular, if $*$ is a left-continuous t-norm, we denote by L_* the axiomatic extension of MTL corresponding to the variety generated by $[0,1]_*$.

Important examples of fuzzy logics studied in the literature transcend the framework of axiomatic extensions of MTL we have presented so far because they are expansions in richer languages. Well-known examples are the expansions with Baaz's Delta projection connective Δ [1], expansions with an involutive negation $\sim$ [14, 9, 18], expansions with other conjunction or implication connectives [15, 31, 38], or expansions with intermediate truth-constants [16, 11, 42, 44].

Let us consider MTL_Δ as the basic logic with Baaz's Delta connective. It is obtained by enriching the language with the unary connective Δ and adding to the Hilbert-style system of MTL the deduction rule of necessitation (from φ infer $\Delta\varphi$) and the following axiom schemata:

Logic	Additional axiom schemata	References
SMTL	(PC)	[22]
ΠMTL	(Can)	[22]
WCMTL	(WCan)	[39]
IMTL	(Inv)	[12]
WNM	(WNM)	[12]
NM	(Inv) and (WNM)	[12]
C_nMTL	(C_n)	[5]
C_nIMTL	(Inv) and (C_n)	[5]
BL	(Div)	[20]
SBL	(Div) and (PC)	[14]
Ł	(Div) and (Inv)	[20, 33]
Π	(Div) and (Can)	[29]
G	(C)	[20, 10, 19]
CPC	(EM)	

Table 2: Some axiomatic extensions of MTL obtained by adding the corresponding additional axiom schemata.

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$

(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$

(Δ3) $\Delta\varphi \to \varphi$

(Δ4) $\Delta\varphi \to \Delta\Delta\varphi$

(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

It is easily provable that MTL_Δ enjoys (Cong) but not ($\mathcal{LDT}$) (cf. Proposition 2.1). However it enjoys another form of deduction theorem (in fact it is a global deduction-detachment theorem).

Proposition 2.8. *For each set of formulae $\Sigma \cup \{\varphi, \psi\}$ holds:*

$$\Sigma, \varphi \vdash_{\mathrm{MTL}_\Delta} \psi \textit{ iff } \Sigma \vdash_{\mathrm{MTL}_\Delta} \Delta\varphi \to \psi \qquad (\mathcal{DT}_\Delta)$$

Analogously to the case of MTL, it can be seen that MTL_Δ is also an algebraizable logic whose equivalent algebraic semantics is given by the so-called MTL_Δ-*algebras* (introduced in [12]). They are the expansions of MTL-algebras with an extra operation that interprets Δ.

Two very general classes of logics have been considered in the literature (see [28, 8, 6]) to encompass t-norm based fuzzy logics expanding MTL or MTL_Δ:

Definition 2.9. *We say that a finitary logic* L *in a countable language is a* core fuzzy logic *if*

- L *expands* MTL*,*
- L *satisfies* (Cong)*,*

- L *satisfies* ($\mathcal{LDT}$).

Definition 2.10. *We say that a finitary logic* L *in a countable language is a* Δ-core fuzzy logic *if*

- L *expands* MTL_Δ,
- L *satisfies* (Cong),
- L *satisfies* ($\mathcal{DT}_\Delta$).

2.2 First-order predicate logics

We introduce first-order predicate fuzzy logics in the standard way (see e.g. [20, 8]). Let L be a (Δ)-core fuzzy logic and Γ a predicate language. We denote by Sent_Γ the set of sentences (closed formulae) over Γ. For each L-chain $\mathcal{A}$, an $\mathcal{A}$-structure is $\mathbf{M} = \langle M, \langle P_\mathbf{M}\rangle_{P\in\Gamma}, \langle f_\mathbf{M}\rangle_{f\in\Gamma}\rangle$ where $M \neq \emptyset$, for each predicate symbol P of arity n, $P_\mathbf{M}$ is an n-ary $\mathcal{A}$-fuzzy relation on M (a mapping $M^n \to A$), and for each n-ary function symbol f, $f_\mathbf{M}$ is a mapping $M^n \to M$ (0-ary functions are interpreted as a constant value in M). Having this, one defines for each formula φ (of the given language), the *truth value* $\|\varphi\|^\mathcal{A}_{\mathbf{M},v}$ of φ in $\mathbf{M}$ determined by the L-chain $\mathcal{A}$ and evaluation v of free variables of φ in M in the usual Tarskian way. In particular, the truth value of a universally quantified formula is the infimum of truth values of all its instances, similarly for $\exists$ and supremum. A structure $\mathbf{M}$ is *safe* if the truth value is defined for each φ and v.

By $\langle \mathbf{M}, \mathcal{A}\rangle \models \varphi$ we denote that $\|\varphi\|^\mathcal{A}_{\mathbf{M},v} = \overline{1}^\mathcal{A}$ for each $\mathbf{M}$-evaluation v. When $\mathcal{A}$ is known from the context we write $\mathbf{M} \models \varphi$. We say that $\langle \mathbf{M}, \mathcal{A}\rangle$ is a model of a theory (i.e. a set of *sentences*) T to mean that $\mathcal{A}$ is an L-chain, $\mathbf{M}$ is a safe $\mathcal{A}$-structure and $\langle \mathbf{M}, \mathcal{A}\rangle \models \alpha$ for each $\alpha \in T$. To simplify matters, we use the expression "$\langle \mathbf{M}, \mathcal{A}\rangle$ is a model" meaning that $\langle \mathbf{M}, \mathcal{A}\rangle$ is a model of the empty theory. If we say "for each model $\langle \mathbf{M}, \mathcal{A}\rangle$" we mean "for each L-chain $\mathcal{A}$ and each safe $\mathcal{A}$-structure $\mathbf{M}$". Finally, by $\|\varphi(a_1, \ldots, a_n)\|^{\langle \mathbf{M},\mathcal{A}\rangle}$ we mean $\|\varphi(x_1, \ldots, x_n)\|^\mathcal{A}_{\mathbf{M},v}$ for $v(x_i) = a_i$.

Definition 2.11. *Let* L *be a (Δ-)core fuzzy logic and* Γ *a predicate language. The logic* L$\forall$ *has the deduction rules of* L *and generalization (from* φ *infer* $(\forall x)\varphi$*) and its axioms are:*

(P)	*the axioms resulting from the axioms of* L *by the substitution of propositional variables with formulae of* Γ,
($\forall$1)	$(\forall x)\varphi(x) \to \varphi(t)$, *where t is substitutable for x in φ,*
($\exists$1)	$\varphi(t) \to (\exists x)\varphi(x)$, *where t is substitutable for x in φ,*
($\forall$2)	$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, *where x is not free in χ,*
($\exists$2)	$(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, *where x is not free in χ,*
($\forall$3)	$(\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi)$, *where x is not free in χ,*
($\approx$1)	$(\forall x)(\forall y)((x \approx y) \vee \neg(x \approx y))$,
($\approx$2)	$(\forall x) x \approx x$,
($\approx$3)	$(\forall x)(\forall y)(\forall \vec{z})(x \approx y \to (\varphi(x, \vec{z}) \leftrightarrow \varphi(y, \vec{z})))$, *where y is substitutable for x in φ.*

The completeness theorem for first-order BL was proven in [20] and the completeness theorems of other predicate fuzzy logics defined in the literature were proven in the corresponding papers. They can be formulated in the following general way:

Theorem 2.12. *Let* L *be a* (Δ-)*core fuzzy logic,* T *a theory, and* φ *a formula. Then the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $\langle \mathbf{M}, \mathcal{A} \rangle \models \varphi$ *for each model* $\langle \mathbf{M}, \mathcal{A} \rangle$ *of the theory* T.

Sometimes this result can be refined by considering models over a proper subclass of L-chains. Such completeness properties with respect to several distinguished semantics have been systematically studied in [6] in several different forms. For the purposes of this paper we only need the following kinds of completeness:

Definition 2.13. *Let* L *be a* (Δ-)*core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. We say that* L$\forall$ *has the* Finite Strong $\mathbb{K}$-Completeness property *(*FS$\mathbb{K}$C *for short) if for every finite theory* T *and formula* φ *the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $\langle \mathbf{M}, \mathcal{A} \rangle \models \varphi$ *for each model* $\langle \mathbf{M}, \mathcal{A} \rangle$ *of the theory* T *such that* $\mathcal{A} \in \mathbb{K}$.

We say that L$\forall$ *has the* $\mathbb{K}$-Completeness property *(*$\mathbb{K}$C *for short) if the equivalence holds for the empty theory.*

We will refer to the particular cases when $\mathbb{K}$ is the class $\mathcal{R}$ of L-chains over the real unit interval $[0, 1]$ (standard chains), the class $\mathcal{Q}$ of L-chains over the rational unit interval $[0, 1]^{\mathbb{Q}}$, or the class $\mathcal{F}$ of all non-trivial finite L-chains.

On the other hand, Montagna and Ono proved the following undecidability result:

Theorem 2.14 ([40])**.** *For every consistent axiomatic extension* L *of* MTL*, the first-order logic* L$\forall$ *is undecidable.*

The non-recursiveness of the sets of tautologies of these logics poses the problem of determining their position in the arithmetical hierarchy, and the same for the sets of satisfiable sentences. Let us briefly recall the main notions that we will need from recursion theory. We denote the set of natural numbers by $\mathbb{N}$ and consider the complexity of sets of objects that can be encoded as subsets of $\mathbb{N}$ (or, in general, of $\mathbb{N}^n$ for some finite n). Decidable or computable sets (or relations, or functions) are called *recursive*. $X \subseteq \mathbb{N}$ is Σ_1 (*recursively enumerable*) iff $X = \{n \mid \exists m R(n, m)\}$ for some recursive relation $R \subseteq \mathbb{N}^2$. $X \subseteq \mathbb{N}$ is Π_1 iff $X = \{n \mid \forall m R(n, m)\}$ for some recursive relation $R \subseteq \mathbb{N}^2$. $X \subseteq \mathbb{N}$ is Σ_2 iff $X = \{n \mid \exists k \forall m R(n, k, m)\}$ for some recursive relation $R \subseteq \mathbb{N}^3$. In general: $X \subseteq \mathbb{N}$ is Π_n iff it is defined by adding a sequence of n alternating quantifiers starting with universal in front of a recursive relation, and $X \subseteq \mathbb{N}$ is Σ_n iff it is defined by adding a sequence of n alternating quantifiers starting with existential in front of a recursive relation. Let Λ be Σ_n or Π_n; $X \subseteq \mathbb{N}$ is Λ-*hard* iff for every Λ-set $Y \subseteq \mathbb{N}$ there is a recursive function f such that $Y = \{n \mid f(n) \in X\}$. $X \subseteq \mathbb{N}$ is Λ-*complete* iff it is Λ and Λ-hard. The set of tautologies of classical logic, $\mathrm{TAUT}(\mathcal{B}_2)$, is Σ_1-complete. The set of its satisfiable sentences, $\mathrm{SAT}(\mathcal{B}_2)$, is Π_1-complete. $X \subseteq \mathbb{N}$ is *arithmetical* iff it is Σ_n or Π_n for some n. Let PA be the axiomatic system of Peano Arithmetic and let $\mathbf{N}$ be its standard model. The set of sentences true in that model, $Th(\mathbf{N})$, is non-arithmetical.

3 Arithmetical complexity results

3.1 General results and general semantics

We will work with arbitrary classes of linearly ordered MTL-algebras or its expansions corresponding to (Δ-)core fuzzy logics in richer languages. These classes will be usually denoted by $\mathbb{K}$, and we will always assume (to avoid dealing with non-interesting trivial cases) that they are not empty and do not contain the trivial algebra. When $\mathbb{K}$ is a class of (expansions of) MTL-chains and no further condition is assumed, we just say for simplicity that it is *a class of chains.*

Definition 3.1. *Given a class $\mathbb{K}$ of chains we define the following sets of sentences:*

1. $\mathrm{TAUT}_1(\mathbb{K}) = \{\varphi \in \mathrm{Sent}_\Gamma \mid$ *for every $\mathcal{A} \in \mathbb{K}$ and every $\mathcal{A}$-structure* $\mathbf{M}$, $\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = \overline{1}^{\mathcal{A}}\}$.
2. $\mathrm{TAUT}_{pos}(\mathbb{K}) = \{\varphi \in \mathrm{Sent}_\Gamma \mid$ *for every $\mathcal{A} \in \mathbb{K}$ and every $\mathcal{A}$-structure* $\mathbf{M}$, $\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} > \overline{0}^{\mathcal{A}}\}$.
3. $\mathrm{SAT}_1(\mathbb{K}) = \{\varphi \in \mathrm{Sent}_\Gamma \mid$ *there exist $\mathcal{A} \in \mathbb{K}$ and an $\mathcal{A}$-structure* $\mathbf{M}$ *such that* $\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = \overline{1}^{\mathcal{A}}\}$.
4. $\mathrm{SAT}_{pos}(\mathbb{K}) = \{\varphi \in \mathrm{Sent}_\Gamma \mid$ *there exist $\mathcal{A} \in \mathbb{K}$ and an $\mathcal{A}$-structure* $\mathbf{M}$ *such that* $\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} > \overline{0}^{\mathcal{A}}\}$.

Definition 3.2. *Let* L$\forall$ *be a first-order (Δ-)core fuzzy logic. Then:*

- *If $\mathbb{K}$ is the class of all* L*-chains (the general semantics), we write* $\mathrm{genTAUT}_1(\mathrm{L}\forall)$ *instead of* $\mathrm{TAUT}_1(\mathbb{K})$.
- *If $\mathbb{K}$ is the class of all* L*-chains whose lattice reduct is the real unit interval* $[0,1]$*, we write* $\mathrm{stTAUT}_1(\mathrm{L}\forall)$ *instead of* $\mathrm{TAUT}_1(\mathbb{K})$.
- *If $\mathbb{K}$ is the class of all* L*-chains whose lattice reduct is the rational unit interval* $[0,1]^{\mathbb{Q}}$*, we write* $\mathrm{ratTAUT}_1(\mathrm{L}\forall)$ *instead of* $\mathrm{TAUT}_1(\mathbb{K})$*. If $\mathbb{K}$ consists of a single* L*-chain over* $[0,1]^{\mathbb{Q}}$ *which, for some reason, is considered the* canonical *rational* L*-chain, then we write* $\mathrm{canratTAUT}_1(\mathrm{L}\forall)$ *instead of* $\mathrm{TAUT}_1(\mathbb{K})$.
- *If $\mathbb{K}$ is the class of all finite* L*-chains, we write* $\mathrm{finTAUT}_1(\mathrm{L}\forall)$ *instead of* $\mathrm{TAUT}_1(\mathbb{K})$.

We define analogous notations for the sets $\mathrm{TAUT}_{pos}(\mathbb{K})$, $\mathrm{SAT}_1(\mathbb{K})$ *and* $\mathrm{SAT}_{pos}(\mathbb{K})$ *in all the cases.*

Given a first-order (Δ-)core fuzzy logic L$\forall$, we write $\Sigma \models_{\mathrm{st(L\forall)}} \varphi$ ($\Sigma \models_{\mathrm{rat(L\forall)}} \varphi$ or $\Sigma \models_{\mathrm{fin(L\forall)}} \varphi$, respectively) meaning that $\Sigma \models_{\mathbb{K}} \varphi$ when $\mathbb{K}$ is the class consisting of all standard (rational or finite, respectively) L-chains. Moreover $\mathrm{stCons}(\mathrm{L}\forall, \Sigma)$, $\mathrm{ratCons}(\mathrm{L}\forall, \Sigma)$ and $\mathrm{finCons}(\mathrm{L}\forall, \Sigma)$ denote the sets $\{\varphi \in \mathrm{Sent}_\Gamma \mid \Sigma \models_{\mathrm{st(L\forall)}} \varphi\}$, $\{\varphi \in \mathrm{Sent}_\Gamma \mid \Sigma \models_{\mathrm{rat(L\forall)}} \varphi\}$ and $\{\varphi \in \mathrm{Sent}_\Gamma \mid \Sigma \models_{\mathrm{fin(L\forall)}} \varphi\}$, respectively.

From the results in [6] it is easy to obtain:

Lemma 3.3. *Let* L$\forall$ *be a first-order (Δ-)core fuzzy logic. If $\mathcal{A}$ and $\mathcal{B}$ are* L*-chains such that there is a σ-embedding (i.e. an embedding preserving all existing infima and suprema) from $\mathcal{A}$ into $\mathcal{B}$, then:*

- $\mathrm{TAUT}_1(\mathcal{B}) \subseteq \mathrm{TAUT}_1(\mathcal{A})$,
- $\mathrm{TAUT}_{pos}(\mathcal{B}) \subseteq \mathrm{TAUT}_{pos}(\mathcal{A})$,
- $\mathrm{SAT}_1(\mathcal{A}) \subseteq \mathrm{SAT}_1(\mathcal{B})$, *and*
- $\mathrm{SAT}_{pos}(\mathcal{A}) \subseteq \mathrm{SAT}_{pos}(\mathcal{B})$.

Lemma 3.4. *Let $\mathbb{K}$ be a class of chains and let $\sim$ be a negation operation present in all members of $\mathbb{K}$.*[3] *Then for every $\varphi \in \mathrm{Sent}_\Gamma$:*

1. $\varphi \in \mathrm{TAUT}_{pos}(\mathbb{K})$ *iff* $\sim \varphi \notin \mathrm{SAT}_1(\mathbb{K})$
2. $\varphi \in \mathrm{SAT}_{pos}(\mathbb{K})$ *iff* $\sim \varphi \notin \mathrm{TAUT}_1(\mathbb{K})$

When the negation is involutive (i.e. $\sim\sim a = a$ for every a) or strict (i.e. $\sim a = \overline{0}$ for every $a \neq \overline{0}$), we have some additional relations:

Lemma 3.5. *Let $\mathbb{K}$ be a class of chains with an involutive negation $\sim$ (in particular, $\mathbb{K}$ can be a class of expansions of* IMTL*-chains). Then for every $\varphi \in \mathrm{Sent}_\Gamma$:*

1. $\varphi \in \mathrm{SAT}_1(\mathbb{K})$ *iff* $\sim \varphi \notin \mathrm{TAUT}_{pos}(\mathbb{K})$
2. $\varphi \in \mathrm{TAUT}_1(\mathbb{K})$ *iff* $\sim \varphi \notin \mathrm{SAT}_{pos}(\mathbb{K})$

Lemma 3.6. *Let $\mathbb{K}$ be a class of chains with a strict negation $\sim$ (in particular, $\mathbb{K}$ can be a class of expansions of* SMTL*-chains). Then for every $\varphi \in \mathrm{Sent}_\Gamma$:*

1. $\varphi \in \mathrm{TAUT}_{pos}(\mathbb{K})$ *iff* $\sim\sim \varphi \in \mathrm{TAUT}_1(\mathbb{K})$
2. $\varphi \in \mathrm{SAT}_{pos}(\mathbb{K})$ *iff* $\sim\sim \varphi \in \mathrm{SAT}_1(\mathbb{K})$

For chains with Δ we have the following:

Lemma 3.7. *Let $\mathbb{K}$ be a class of chains with the Δ operation. Then for every $\varphi \in \mathrm{Sent}_\Gamma$:*

1. $\varphi \in \mathrm{SAT}_1(\mathbb{K})$ *iff* $\neg\Delta\varphi \notin \mathrm{TAUT}_1(\mathbb{K})$
2. $\varphi \in \mathrm{TAUT}_1(\mathbb{K})$ *iff* $\neg\Delta\varphi \notin \mathrm{SAT}_1(\mathbb{K})$

We can obtain some lower bounds for the complexity of some of these problems. First we consider the SAT problems.

Proposition 3.8. *For every class $\mathbb{K}$ of chains, $\mathrm{SAT}_1(\mathbb{K})$ is Π_1-hard.*

Proof. If $\varphi \in \mathrm{Sent}_\Gamma$ is a sentence and $\{P_i \mid 1 \leq i \leq n\}$ are the predicate symbols from Γ appearing in φ, we define the sentence $Crisp(\varphi) = \bigwedge_{1\leq i\leq n} \forall \overrightarrow{x}(P_i(\overrightarrow{x}) \vee \neg P_i(\overrightarrow{x}))$. Now just observe that for every $\varphi \in \mathrm{Sent}_\Gamma$, $\varphi \in \mathrm{SAT}(\mathcal{B}_2)$ iff $Crisp(\varphi)\&\varphi \in \mathrm{SAT}_1(\mathbb{K})$, and since the satisfiability problem in classical logic is Π_1-hard so it must be $\mathrm{SAT}_1(\mathbb{K})$. $\square$

[3] Of course, in any class $\mathbb{K}$ there is always a negation operation present in all its members: $\neg a = a \to \overline{0}$; however we prefer this general formulation to cope with other possible negations in logics expanded with extra connectives.

Proposition 3.9. *For every class $\mathbb{K}$ of chains with an involutive negation $\sim$, $\mathrm{SAT}_{pos}(\mathbb{K})$ is Π_1-hard.*

Proof. First observe that the set $\{\sim \varphi \in \mathrm{Sent}_\Gamma \mid \sim \varphi \in \mathrm{TAUT}_1(\mathbb{K})\}$ is Σ_1-hard (it follows from the fact, proved in Theorem 3.15, that $\mathrm{TAUT}_1(\mathbb{K})$ is Σ_1-hard and for every φ, $\varphi \in \mathrm{TAUT}_1(\mathbb{K})$ iff $\sim (\sim \varphi) \in \mathrm{TAUT}_1(\mathbb{K})$). Therefore, the complement of this set is Π_1-hard. On the other hand, by Lemma 3.4 for every $\varphi \in \mathrm{Sent}_\Gamma$ we have: $\sim \varphi \notin \mathrm{TAUT}_1(\mathbb{K})$ iff $\varphi \in \mathrm{SAT}_{pos}(\mathbb{K})$, and hence $\mathrm{SAT}_{pos}(\mathbb{K})$ is Π_1-hard. $\square$

Proposition 3.10. *For every class $\mathbb{K}$ of chains with a strict negation $\sim$, $\mathrm{SAT}_{pos}(\mathbb{K})$ is Π_1-hard.*

Proof. For every $\varphi \in \mathrm{Sent}_\Gamma$ we denote as $\varphi^{\sim\sim}$ the sentence resulting from φ by adding double negation $\sim\sim$ to all atoms. Then we claim that for every $\varphi \in \mathrm{Sent}_\Gamma$, $\varphi \in \mathrm{SAT}(\mathcal{B}_2)$ iff $\varphi^{\sim\sim} \in \mathrm{SAT}_{pos}(\mathbb{K})$. Indeed, one direction is obvious and for the other one assume that $\varphi^{\sim\sim} \in \mathrm{SAT}_{pos}(\mathbb{K})$, i.e. there is $\mathcal{A} \in \mathbb{K}$ and an $\mathcal{A}$-model $\mathbf{M}$ such that $\|\varphi^{\sim\sim}\|^{\mathcal{A}}_{\mathbf{M}} > \overline{0}^{\mathcal{A}}$; it implies that $\|\varphi^{\sim\sim}\|^{\mathcal{A}}_{\mathbf{M}} = \overline{1}^{\mathcal{A}}$ (because $\varphi^{\sim\sim}$ is crisp). We define a model $\mathbf{M}'$ over $\mathcal{B}_2$ with the domain of $\mathbf{M}$ in the following way: for every n-ary predicate P and elements $a_1, \dots, a_n$ in the domain, define $P_{\mathbf{M}'}(a_1, \dots, a_n) = \overline{1}^{\mathcal{A}}$ if $P_{\mathbf{M}}(a_1, \dots, a_n) \neq \overline{0}^{\mathcal{A}}$ and $P_{\mathbf{M}'}(a_1, \dots, a_n) = \overline{0}^{\mathcal{A}}$ otherwise. Then, by induction on the subformulae of φ, we obtain $\|\varphi\|^{\mathcal{B}_2}_{\mathbf{M}'} = \|\varphi^{\sim\sim}\|^{\mathcal{A}}_{\mathbf{M}}$. $\square$

Open problem: Show that for every class $\mathbb{K}$ of chains, the set $\mathrm{SAT}_{pos}(\mathbb{K})$ is Π_1-hard.

Now we consider the TAUT problems.[4] In the sequel, for every sentence φ, 2φ denotes the sentence $\neg(\neg\varphi)^2$.

Lemma 3.11. *Let* L *be any (Δ-)core fuzzy logic. For every sentence φ, $2\varphi \vee 2(\neg\varphi) \in$ $\mathrm{genTAUT}_1(\mathrm{L}\forall)$.*

Proof. Let $\mathcal{A}$ be an L-chain and $\mathbf{M}$ an $\mathcal{A}$-model. If $\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} \leq \|\neg\varphi\|^{\mathcal{A}}_{\mathbf{M}}$, then $\|(\neg\neg\varphi)^2\|^{\mathcal{A}}_{\mathbf{M}} = \overline{0}^{\mathcal{A}}$. If $\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} > \|\neg\varphi\|^{\mathcal{A}}_{\mathbf{M}}$, then $\|(\neg\varphi)^2\|^{\mathcal{A}}_{\mathbf{M}} = \overline{0}^{\mathcal{A}}$. In either case we have $\|\neg(\neg\varphi)^2 \vee \neg(\neg\neg\varphi)^2\|^{\mathcal{A}}_{\mathbf{M}} = \overline{1}^{\mathcal{A}}$. $\square$

Definition 3.12. *Let φ be a sentence. Consider its prenex normal form in classical logic, $Q_1x_1 \dots Q_nx_n\ \psi(x_1, \dots, x_n)$, where ψ is a lattice combination of literals. We define a formula $\varphi^\star$ by induction as follows: if φ is a literal, then $\varphi^\star = 2\varphi$; * commutes with quantifiers, $\wedge$ and $\vee$.*

Lemma 3.13. *Let φ be a lattice combination of literals,* L *be a (Δ-)core fuzzy logic and $\mathbb{K}$ a class of* L*-chains. The following are equivalent:*

(1) φ is a classical propositional tautology.

(2) $\varphi^\star$ is an L*-tautology.*

(3) $\varphi^\star$ is a tautology for every chain in $\mathbb{K}$.

[4] Some of the techiques we will use in the following results to deal with these problems are quite similar to those used in [35, Chapters 7 and 8] and in [2] for some computational complexity and decidability issues.

(4) $\varphi^\star$ *is a positive tautology for every chain in* $\mathbb{K}$.

Proof. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious. We prove (1) $\Rightarrow$ (2). By distributivity, φ can be equivalently written as $\bigwedge_{i=1}^{n}\bigvee_{j=1}^{n_i}\alpha_{i,j}$, where $\alpha_{i,j}$ are literals. Thus, φ is a classical tautology iff for every $i\in\{1,\dots,n\}$, $\bigvee_{j=1}^{n_i}\alpha_{i,j}$ is a classical tautology. This is the case if for every $i\in\{1,\dots,n\}$ there are $j_1,j_2\in\{1,\dots,n_i\}$ such that $\alpha_{i,j_1}=\neg\alpha_{i,j_2}$. Hence, $2\alpha_{i,j_1}\vee 2\alpha_{i,j_2}$ is an L-tautology by previous lemma and, since this formula implies $\bigvee_{j=1}^{n_i}2\alpha_{i,j}$, we have that $\varphi^\star$ is an L-tautology. We finally prove (4) $\Rightarrow$ (1) by contraposition. If φ is not a classical propositional tautology, then there is an evaluation e on $\mathcal{B}_2$ such that $e(\varphi)=0$. Since $\varphi^\star$ and φ are equivalent in classical logic, we also have $e(\varphi^\star)=0$. Now, given any $\mathcal{A}\in\mathbb{K}$, it is clear that e can also be seen as an evaluation on $\mathcal{A}$ and $e(\varphi^\star)=\overline{0}^{\mathcal{A}}$. □

Lemma 3.14. *Let* $\varphi=\exists x_1\dots\exists x_n\ \psi(x_1,\dots,x_n)$*, where* ψ *is a lattice combination of literals, be a purely existential formula,* L *be a* (Δ-)*core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. The following are equivalent:*

(1) $\varphi\in\mathrm{TAUT}(\mathcal{B}_2)$.

(2) $\varphi^\star\in\mathrm{genTAUT}_1(\mathrm{L}\forall)$.

(3) $\varphi^\star\in\mathrm{TAUT}_1(\mathbb{K})$.

(4) $\varphi^\star\in\mathrm{TAUT}_{pos}(\mathbb{K})$.

Proof. Again, (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious. (4) $\Rightarrow$ (1) is proved as in the previous lemma. We prove (1) $\Rightarrow$ (2). Suppose that φ is a classical tautology. By Herbrand's Theorem, there is a classical propositional tautology of the form $\bigvee_{i=1}^{m}\psi(t_1^i,\dots,t_n^i)$, where the t_j^i's are closed terms. By the previous lemma, recalling that * commutes with $\vee$, we have that $\bigvee_{i=1}^{m}\psi^\star(t_1^i,\dots,t_n^i)\in\mathrm{genTAUT}_1(\mathrm{L}\forall)$. By an easy proof in L$\forall$, we can derive $\varphi^\star=\exists x_1\dots\exists x_n\ \psi^\star(x_1,\dots,x_n)$, and hence we have proved (2). □

Theorem 3.15. *For every class* $\mathbb{K}$ *of chains, the sets* $\mathrm{TAUT}_1(\mathbb{K})$ *and* $\mathrm{TAUT}_{pos}(\mathbb{K})$ *are* Σ_1*-hard.*

Proof. The set of provable existential formulae of first-order classical logic is Σ_1-hard. Indeed, the Herbrand form φ^H of any sentence φ is purely existential, and φ is provable iff φ^H is provable. The claim now follows from the previous lemma. □

This theorem, in particular, solves a couple of open problems recently proposed by Hájek in [27]; namely given a set $\mathbb{K}$ of standard BL-chains such that its corresponding logic $\mathrm{L}_{\mathbb{K}}\forall$ is recursively axiomatizable, show that $\mathrm{genTAUT}_1(\mathrm{L}_{\mathbb{K}}\forall)$ and $\mathrm{genTAUT}_{pos}(\mathrm{L}_{\mathbb{K}}\forall)$ are Σ_1-hard.

On the other hand, completeness with respect to a Hilbert-style calculus gives upper bounds for the complexity:

Proposition 3.16. *Let* L *be a recursively axiomatizable* (Δ-)*core fuzzy logic and* $\mathbb{K}$ *be a class of* L*-chains. If* L$\forall$ *has the* FS$\mathbb{K}$C*, then* $\mathrm{TAUT}_1(\mathbb{K})$ *and* $\mathrm{TAUT}_{pos}(\mathbb{K})$ *are* Σ_1*, while* $\mathrm{SAT}_1(\mathbb{K})$ *and* $\mathrm{SAT}_{pos}(\mathbb{K})$ *are* Π_1.

Proof. $\mathrm{TAUT}_1(\mathbb{K})$ is Σ_1 because it is the set of theorems of a recursively axiomatizable logic. Using Lemma 3.4 ($\varphi \in \mathrm{SAT}_{pos}(\mathbb{K})$ iff $\sim \varphi \notin \mathrm{TAUT}_1(\mathbb{K})$) we obtain that $\mathrm{SAT}_{pos}(\mathbb{K})$ is Π_1. As regards to $\mathrm{SAT}_1(\mathbb{K})$, notice that for every $\varphi \in \mathrm{Sent}_\Gamma$, $\varphi \in \mathrm{SAT}_1(\mathbb{K})$ iff $\varphi \not\models_{\mathbb{K}} \overline{0}$ iff $\varphi \nvdash_{\mathrm{L}\forall} \overline{0}$. Using again Lemma 3.4 (now $\varphi \in \mathrm{TAUT}_{pos}(\mathbb{K})$ iff $\sim \varphi \notin \mathrm{SAT}_1(\mathbb{K})$) we obtain that $\mathrm{TAUT}_{pos}(\mathbb{K})$ is Σ_1. □

In particular, since a first-order logic is always complete with respect to the semantics of all chains, we obtain:

Corollary 3.17. *For every recursively axiomatizable first-order (Δ-)core fuzzy logic* L$\forall$, $\mathrm{genTAUT}_1(\mathrm{L}\forall)$ *and* $\mathrm{genTAUT}_{pos}(\mathrm{L}\forall)$ *are* Σ_1*-complete,* $\mathrm{genSAT}_1(\mathrm{L}\forall)$ *is* Π_1*-complete and* $\mathrm{genSAT}_{pos}(\mathrm{L}\forall)$ *is* Π_1*. Moreover, if* L *has an involutive or strict negation, then* $\mathrm{genSAT}_{pos}(\mathrm{L}\forall)$ *is* Π_1*-complete.*

Moreover, it yields the following generalization of the undecidability result in [40]:

Corollary 3.18. *For every (Δ)-core fuzzy logic, the first-order logic* L$\forall$ *is undecidable.*

See all the results for the general semantics in Table 3.

	All logics	Logics with involutive or strict negation
$\mathrm{genTAUT}_1(\mathrm{L}\forall)$	Σ_1-complete	Σ_1-complete
$\mathrm{genSAT}_1(\mathrm{L}\forall)$	Π_1-complete	Π_1-complete
$\mathrm{genTAUT}_{pos}(\mathrm{L}\forall)$	Σ_1-complete	Σ_1-complete
$\mathrm{genSAT}_{pos}(\mathrm{L}\forall)$	Π_1	Π_1-complete

Table 3: Complexity results for the general semantics.

In this way we have generalized to the framework of first-order (Δ)-core fuzzy logics the only complexity results that were known so far with respect to the general semantics: those for logics L_* based on a continuous t-norm $*$ (completely solved by Hájek in [21, 25]; see the results in Table 4).

	Complexity
$\mathrm{genTAUT}_1(\mathrm{L}_*\forall)$	Σ_1-complete
$\mathrm{genSAT}_1(\mathrm{L}_*\forall)$	Π_1-complete
$\mathrm{genTAUT}_{pos}(\mathrm{L}_*\forall)$	Σ_1-complete
$\mathrm{genSAT}_{pos}(\mathrm{L}_*\forall)$	Π_1-complete

Table 4: Complexity results for the general semantics when $*$ is continuous t-norm.

3.2 Complexity of finite-chain semantics

Let $\mathcal{A}$ be any finite chain and let $\overline{0} = a_1 < \ldots < a_n = \overline{1}$ be the elements of A in increasing order. $\mathrm{L}_\mathcal{A}$, the first-order many-valued logic based on $\mathcal{A}$, is defined semantically as follows: $\mathrm{L}_\mathcal{A}$ has a language $\Gamma_\mathcal{A}$ containing, besides parentheses (and), variables, predicate symbols, function symbols, a k-ary connective F for each k-ary operation $F^\mathcal{A}$ on $\mathcal{A}$ (different symbols for different operations), plus the quantifiers $\exists$ and $\forall$. For each connective F introduced in this way, we refer to $F^{\mathcal{A}}$ as *the realization of* F *in* $\mathcal{A}$. Since $\mathcal{A}$ is finite, each $\mathcal{A}$-structure $\mathbf{M} = \langle M, \langle P_{\mathbf{M}}\rangle_{P\in\Gamma_{\mathcal{A}}}, \langle f_{\mathbf{M}}\rangle_{f\in\Gamma_{\mathcal{A}}}\rangle$ is safe, because universal quantifiers are interpreted by taking the minimum value of instances, and existential quantifiers by taking the maximum value of instances.

For every set $T \cup \{\phi\}$ of sentences of $\mathrm{L}_{\mathcal{A}}$, the consequence relation $\models_{\mathrm{L}_{\mathcal{A}}}$ in $\mathrm{L}_{\mathcal{A}}$ is defined as follows: $T \models_{\mathrm{L}_{\mathcal{A}}} \phi$ iff for every $\mathcal{A}$-structure $\mathbf{M}$, one has that if $\langle \mathbf{M}, \mathcal{A}\rangle \models \psi$ for all $\psi \in T$, then $\langle \mathbf{M}, \mathcal{A}\rangle \models \phi$. If $\emptyset \models_{\mathrm{L}_{\mathcal{A}}} \phi$, then we say that ϕ is an *$\mathcal{A}$-1-tautology* and we write $\models_{\mathrm{L}_{\mathcal{A}}} \phi$.

Theorem 3.19. *For every finite chain $\mathcal{A}$, the set of $\mathcal{A}$-1-tautologies is Σ_1.*

Proof. We will associate to $\mathcal{A}$ a recursively axiomatized classical first-order theory $T_{\mathcal{A}}$ and to every sentence ϕ of $\mathrm{L}_{\mathcal{A}}$ a formula ϕ^T in the language of $T_{\mathcal{A}}$ such that the map $\phi \mapsto \phi^T$ is computable and ϕ is an $\mathcal{A}$-1-tautology iff ϕ^T is a theorem of $T_{\mathcal{A}}$. This will clearly suffice to prove the theorem. First of all, the theory $T_{\mathcal{A}}$ has all function symbols in $\mathrm{L}_{\mathcal{A}}$. Moreover $T_{\mathcal{A}}$ has a constant symbol c^T for each element c of A, plus an additional constant u (for undefined) and an additional k-ary functional symbol f_ϕ for each formula ϕ with k free variables (the intended meaning is that $f_\phi(d_1,\ldots,d_k) = \|\phi(d_1,\ldots,d_k)\|^{\mathcal{A}}_{\mathbf{M}}$; in particular, if ϕ is a sentence, then f_ϕ is a constant). $T_{\mathcal{A}}$ has two binary predicate symbols $=$ and $<$: the intended meaning of $x = y$ is that x is equal to y, and the intended meaning of $x < y$ is that $x, y \in A$ and x is less than y in the order of $\mathcal{A}$. Finally $T_{\mathcal{A}}$ has two unary predicate symbols M and A. The intended meanings of $M(v)$ and of $A(v)$ are: v is in the domain M of individuals of the first-order structure we are referring to, and v is an element of the algebra $\mathcal{A}$, respectively.

It is a little bit boring to write all the axioms of $T_{\mathcal{A}}$, therefore we only describe them informally and we leave the obvious formal translation to the reader. Roughly speaking, we have:

(0) Identity axioms for $=$.

(1) A group of axioms which say that the domain M of individuals is disjoint from A and u is neither in M nor in A.

(2) An axiom saying that every element is either in A or in M or u.

(3) Axioms describing the structure of $\mathcal{A}$, that is: (3a) $c_i^T < c_j^T$ for each $1 \le i < j \le n$, and $\neg(c_i^T < c_j^T)$ for each $j \le i$; (3b) axioms of the form $\neg(c_i^T = c_j^T)$ for each $i \neq j$; (3c) axioms of the form $F(e_1^T,\ldots,e_k^T) = e^T$ for each k-ary connective F and for all $e_1,\ldots,e_k \in A$ such that $F^{\mathcal{A}}(e_1,\ldots,e_k) = e$; (3d) the axiom $\forall v(A(v) \leftrightarrow (v = c_1^T \vee \ldots \vee v = c_n^T)$ saying that $A = \{c_1,\ldots,c_n\}$; (3e) axioms saying that for every connective F corresponding to an operation $F^{\mathcal{A}}$, $F(x_1,\ldots,x_k)$ is undefined (i.e. it is equal to u) if some of the x_i is not in A; (3f) an axiom saying that if $x < y$ then $x, y \in A$.

(4) Axioms describing the structure $\mathbf{M}$, that is: (4a) for every constant symbol d of $\mathrm{L}_{\mathcal{A}}$, an axiom saying that $d \in M$; (4b) for every k-ary function symbol g of $\mathrm{L}_{\mathcal{A}}$, an axiom saying that for all $x_1,\ldots,x_k$, $g(x_1,\ldots,x_k) \in M$ if $x_1,\ldots,x_k \in M$ and $g(x_1,\ldots,x_k) = u$ otherwise.

(5) Axioms describing the behavior of $\|\phi(v_1,\ldots,v_k)\|^{\mathcal{A}}_{\mathbf{M}}$, that is: (5a) if $v_1,\ldots,v_k$ are all in M, then $f_\phi(v_1,\ldots,v_k)$ is in M, otherwise $f_\phi(v_1,\ldots,v_k) = u$; (5b) for every k-ary connective F of $\mathrm{L}_{\mathcal{A}}$, $f_{F(\phi_1,\ldots,\phi_k)}(v_1,\ldots,v_l) = F(f_{\phi_1}(v_1,\ldots,v_l),\ldots,f_{\phi_k}(v_1,\ldots,v_l)$ (thus $f_{F(\phi_1,\ldots,\phi_k)}(v_1,\ldots,v_l) = u$ if for some i, $f_{\phi_i}(v_1,\ldots,v_l) = u$, otherwise $f_{F(\phi_1,\ldots,\phi_k)}(v_1,\ldots,v_l) \in A$); (5c) an axiom saying that for $j = 1,\ldots,n$, $f_{\forall v\phi}(v_1,\ldots,v_k) = c_j$ iff (i) $v_1,\ldots,v_k \in M$, (ii) for all $v \in M$, $f_\phi(v,v_1,\ldots,v_k) \ge c_j$ and (iii) for some $v \in M$, $f_\phi(v,v_1,\ldots,v_k) = c_j$; (5d) an axiom saying that for $j = 1,\ldots,n$, $f_{\exists v\phi}(v_1,\ldots,v_k) = c_j$ iff (i) $v_1,\ldots,v_k \in M$, (ii) for all $v \in M$, $f_\phi(v,v_1,\ldots,v_k) \le c_j$ and (iii) for some $v \in M$, $f_\phi(v,v_1,\ldots,v_k) = c_j$.

Lemma 3.20. *(a) Let* $\mathbf{M}$ *be an* $\mathcal{A}$*-structure for* $\mathrm{L}_{\mathcal{A}}$*. Then there is a model* $\mathbf{M}^*$ *of* $T_{\mathcal{A}}$ *(in the sense of classical logic) such that for every sentence* ϕ *of* $\mathrm{L}_{\mathcal{A}}$ *and for every* $c_i \in A$ *one has:* $\mathbf{M}^* \models f_\phi = c_i^T$ *iff* $\|\phi\|_{\mathbf{M}}^{\mathcal{A}} = c_i$*.*
(b) Let $\mathbf{N}$ *be a model of* $T_{\mathcal{A}}$ *(again, in the sense of classical logic). Then there is a* $\mathcal{A}$*-structure* $\mathbf{N}^+$ *for* $\mathrm{L}_{\mathcal{A}}$ *such that for every sentence* ϕ *of* $\mathrm{L}_{\mathcal{A}}$ *and for every* $c_i \in A$ *one has:* $\mathbf{N} \models f_\phi = c_i^T$ *iff* $\|\phi\|_{\mathbf{N}^+}^{\mathcal{A}} = c_i$*.*

Proof. (a) Given $\mathbf{M}$, we can assume without loss of generality that $M \cap A = \emptyset$. Let $u^* \notin M \cup A$, and consider the model $\mathbf{M}^*$ whose universe is $M^* = M \cup A \cup \{u^*\}$ and whose constants, operations and predicates are as follows:

(i) If c_i^T is a constant for an element of A, then $(c_i^T)^{\mathbf{M}^*} = c_i$; if c is a constant of $\mathrm{L}_{\mathcal{A}}$, then $c^{\mathbf{M}^*} = c^{\mathbf{M}}$; if c is a constant of the form f_ϕ, ϕ a sentence of $\mathrm{L}_{\mathcal{A}}$, then $c^{\mathbf{M}^*} = \|\phi\|_{\mathbf{M}}^{\mathcal{A}}$. Finally, u is interpreted as u^*.

(ii) If f is a k-ary function symbol in $\mathrm{L}_{\mathcal{A}}$, then $f^{\mathbf{M}^*}$ is defined by $f^{\mathbf{M}^*}(d_1, \ldots, d_k) = f^{\mathbf{M}}(d_1, \ldots, d_k)$ if $d_{1,\ldots,}d_k \in M$, and $f^{\mathbf{M}^*}(d_1, \ldots, d_k) = u^*$ otherwise; if F is a k-ary connective of $\mathrm{L}_{\mathcal{A}}$, then $F^{\mathbf{M}^*}(d_1, \ldots, d_k) = F^{\mathcal{A}}(d_1, \ldots, d_k)$ if $d_1, \ldots, d_k \in A$, and $F^{\mathbf{M}^*}(d_1, \ldots, d_k) = u^*$ otherwise; if $\phi(v_1, \ldots, v_k)$ is a formula of $\mathrm{L}_{\mathcal{A}}$ with free variables $v_1, \ldots, v_k$, then $f_\phi^{\mathbf{M}^*}$ is defined by $f_\phi^{\mathbf{M}^*}(d_1, \ldots, d_k) = \|\phi(d_1, \ldots, d_k)\|_{\mathbf{M}}^{\mathcal{A}}$ if $d_1, \ldots, d_k \in M$, and $f_\phi^{\mathbf{M}^*}(d_1, \ldots, d_k) = u^*$ otherwise.

(iii) $\mathbf{M}^* \models d = e$ iff d is equal to e; $\mathbf{M}^* \models d < e$ iff $d, e \in A$ and $d < e$ in the order of $\mathcal{A}$; $\mathbf{M}^* \models M(d)$ iff $d \in M$ and $\mathbf{M}^* \models A(d)$ iff $d \in A$.

It is clear that for every formula $\phi(v_1, \ldots, v_k)$, for every $c_i \in A$ and for every $d_1, \ldots, d_k \in M$: $\mathbf{M}^* \models f_\phi(d_1, \ldots, d_k) = c_i^T$ iff $\|\phi(d_1, \ldots, d_k)\|_{\mathbf{M}}^{\mathcal{A}} = c_i$, and (a) follows.

(b) Let $\mathbf{N}$ be a model of $T_{\mathcal{A}}$; we define an algebra $\mathcal{A}^+$ and an $\mathcal{A}^+$-structure $\mathbf{N}^+$ based on $\mathcal{A}^+$ as follows:

(i) The domain A^+ of $\mathcal{A}^+$ is the set $\{d \in N \mid \mathbf{N} \models A(d)\}$ and the operations of $\mathcal{A}^+$ are the restrictions to A^+ of the operations $F^{\mathbf{N}}$ of $\mathbf{N}$ such that F is a connective of $\mathrm{L}_{\mathcal{A}}$. Trivially, $\mathcal{A}^+$ is isomorphic to $\mathcal{A}$ (here we use in a crucial way the fact that $\mathcal{A}$ is finite).

(ii) $N^+ = \{d \in N \mid \mathbf{N} \models M(d)\}$; for every constant c of $\mathrm{L}_{\mathcal{A}}$, $c^{\mathbf{N}^+} = c^{\mathbf{N}}$; for every k-ary function symbol g of $\mathrm{L}_{\mathcal{A}}$, $g^{\mathbf{N}^+}$ is the function from $(N^+)^k$ into N^+ defined for all $d_1, \ldots, d_k \in N^+$, by $g^{\mathbf{N}^+}(d_1, \ldots, d_k) = g^{\mathbf{N}}(d_1, \ldots, d_k)$ (i.e. $g^{\mathbf{N}^+}$ is the restriction of $g^{\mathbf{N}}$ to $(N^+)^k$).

(iii) For every k-ary predicate P and for every $d_1, \ldots, d_k \in N^+$, $\|P(d_1, \ldots, d_k)\|_{\mathbf{N}^+}^{\mathcal{A}} = f_P^{\mathbf{N}}(d_1, \ldots, d_k)$.

Then $\|.\|_{\mathbf{N}^+}^{\mathcal{A}}$ uniquely extends to all formulae in such a way that for every formula $\phi(v_1, \ldots, v_k)$, for every $c_i \in A$ and for every $d_1, \ldots, d_k \in M$: $\mathbf{N} \models f_\phi(d_1, \ldots, d_k) = c_i^T$ iff $\|\phi(d_1, \ldots, d_k)\|_{\mathbf{N}^+}^{\mathcal{A}} = c_i$, and (b) follows. □

Continuing with the proof of Theorem 3.19, it suffices to associate to every sentence ϕ of $\mathrm{L}_{\mathcal{A}}$ the formula $f_\phi = \overline{1}^T$ (remind that $\overline{1}$ is the top element of $\mathcal{A}$). Then by Lemma 3.20 we have that the following are equivalent:

(i) There is an $\mathcal{A}$-structure $\mathbf{M}$ such that $\|\phi\|_{\mathbf{M}}^{\mathcal{A}} \neq \overline{1}$, and

(ii) There is a model $\mathbf{N}$ of $T_{\mathcal{A}}$ such that $f_\phi = \overline{1}^T$ is not valid in $\mathbf{N}$.

Thus we conclude that ϕ is an $\mathcal{A}$-1-tautology iff $T_{\mathcal{A}} \vdash f_\phi = \overline{1}^T$, and the set of $\mathcal{A}$-1-tautologies is Σ_1. This ends the proof. □

We have seen that $\mathrm{TAUT}_1(\mathcal{A})$ is Σ_1. Similar arguments show that for every sentence ϕ of $\mathrm{L}_{\mathcal{A}}$ we have:

- $\phi \in \mathrm{TAUT}_{pos}(\mathcal{A})$ iff $T_{\mathcal{A}} \vdash \overline{0}^T < f_\phi$,
- $\phi \in \mathrm{SAT}_1(\mathcal{A})$ iff $T_{\mathcal{A}}$ plus $f_\phi = \overline{1}^T$ is consistent,
- $\phi \in \mathrm{SAT}_{pos}(\mathcal{A})$ iff $T_{\mathcal{A}}$ plus $f_\phi > \overline{0}^T$ is consistent.

Theorem 3.21. *Let $\mathcal{A}$ be a finite chain.* $\mathrm{TAUT}_1(\mathcal{A})$ *and* $\mathrm{TAUT}_{pos}(\mathcal{A})$ *are in Σ_1. Moreover,* $\mathrm{SAT}_1(\mathcal{A})$ *and* $\mathrm{SAT}_{pos}(\mathcal{A})$ *are in Π_1.*

Observe that the proof of this theorem would be completely analogous if instead of a linearly ordered algebra $\mathcal{A}$ would be an arbitrary finite algebra (in a finite language), as this was the essential requirement to build the classical first-order theory $T_{\mathcal{A}}$.

By the general hardness results from Section 3.1 we obtain:

Corollary 3.22. *For every finite chain $\mathcal{A}$,* $\mathrm{SAT}_1(\mathcal{A})$ *is Π_1-complete. Moreover:*

1. $\mathrm{TAUT}_1(\mathcal{A})$ *and* $\mathrm{TAUT}_{pos}(\mathcal{A})$ *are Σ_1-complete.*
2. $\mathrm{SAT}_1(\mathcal{A})$ *is Π_1-complete.*
3. *If $\mathcal{A}$ has an involutive or strict negation, then* $\mathrm{SAT}_{pos}(\mathcal{A})$ *is Π_1-complete.*

From these results, we can obtain some upper bounds for the arithmetical complexities with respect to the finite-chain semantics, when the class of finite chains is recursively enumerable:

Theorem 3.23. *Suppose that* L *is a (Δ)-core fuzzy logic such that there is a computable enumeration of all (up to isomorphism) finite* L*-chains. Then:*
(a) $\mathrm{finTAUT}_1(\mathrm{L}\forall)$ *and* $\mathrm{finTAUT}_{pos}(\mathrm{L}\forall)$ *are in Π_2.*
(b) $\mathrm{finSAT}_1(\mathrm{L}\forall)$ *and* $\mathrm{finSAT}_{pos}(\mathrm{L}\forall)$ *are in Σ_2.*

Proof. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n, \ldots$ be a computable enumeration of all finite L-chains. Then $\phi \in \mathrm{finTAUT}_1(\mathrm{L}\forall)$ iff $\forall n(\phi \in \mathrm{TAUT}_1(\mathcal{A}_n))$ and $\phi \in \mathrm{finTAUT}_{pos}(\mathrm{L}\forall)$ iff $\forall n(\phi \in \mathrm{TAUT}_{pos}(\mathcal{A}_n))$. Now since the sequence $\langle \mathcal{A}_n \mid n \in \omega \rangle$ is computable, by Theorem 3.21, the sets $\{\langle \phi, n\rangle \mid \phi \in \mathrm{TAUT}_1(\mathcal{A}_n)\}$ and $\{\langle \phi, n\rangle \mid \phi \in \mathrm{TAUT}_{pos}(\mathcal{A}_n)\}$ are in Σ_1, and claim (a) follows.

As regards to claim (b) we have that $\phi \in \mathrm{finSAT}_{pos}(\mathrm{L}\forall)$ iff $\exists n(\phi \in \mathrm{SAT}_{pos}(\mathcal{A}_n))$, and $\phi \in \mathrm{finSAT}_1(\mathrm{L}\forall)$ iff $\exists n(\phi \in \mathrm{SAT}_1(\mathcal{A}_n))$, and the claim follows from the computability of the sequence $\langle \mathcal{A}_n \mid n \in \omega\rangle$ and from Theorem 3.21 (note that if $R(n, x)$ is Π_1, then $\exists n R(n, x)$ is in turn Σ_2). □

Theorem 3.24. *If* L *is a finitely axiomatizable (Δ)-core fuzzy logic, then there is a computable enumeration of all (up to isomorphism) finite* L*-chains.*

Proof. We can obtain a computable enumeration of all finite L-chains as follows: clearly there is a computable enumeration of all the finite algebras of the signature of L (first put the trivial algebra in the list, then enumerate all the (finitely many) structures with two elements, 0 and 1, then all the (finitely many) structures with three elements, $0, 1$ and 2, etc.). Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n, \ldots$ be the computable list of structures obtained in this way, and assume without loss of generality that $\mathcal{C}_1$ is the trivial algebra. Now let $\mathcal{A}_1 = \mathcal{C}_1$ (note that the trivial algebra is a totally ordered algebraic model of any logic with that signature); then for every n, check if $\mathcal{C}_n$ is a chain and if it satisfies the finite axiomatization of L. This can be done with a finite computation. If so, let $\mathcal{A}_n = \mathcal{C}_n$; otherwise, let $\mathcal{A}_n = \mathcal{C}_1$. □

From the last two theorems, together with the general results in the previous subsection, we can obtain uniform bounds for the complexity of finite-chain semantics in recursively axiomatizable (Δ)-core fuzzy logics; see the results in Table 5. It applies, in particular, to all the prominent logics we have mentioned, i.e. for L $\in$ $\{\text{Ł}, \text{G}, \Pi, \text{BL}, \text{SBL}, \text{MTL}, \text{SMTL}, \text{IMTL}, \Pi\text{MTL}, \text{WCMTL}, \text{C}_n\text{MTL}, \text{C}_n\text{IMTL}, \text{WNM}, \text{NM}\}$ finTAUT$_1$(L$\forall$) is Π_2, etc. Note that this list may have repetitions. This is unavoidable if L has only finitely many totally ordered algebraic models, and this is the case for L $= \Pi$ or for L $= \Pi$MTL. In this case, finTAUT$_1$(L$\forall$) is not only Π_2, but even Σ_1: for instance finTAUT$_1$($\Pi\forall$) and finTAUT$_1$(ΠMTL$\forall$) coincide with the set of classical first-order tautologies, which is Σ_1-complete. Next will show that in some cases the upper bounds are reached as well.

	All logics	Logics with involutive or strict negation
finTAUT$_1$(L$\forall$)	Σ_1-hard, Π_2	Σ_1-hard, Π_2
finSAT$_1$(L$\forall$)	Π_1-hard, Σ_2	Π_1-hard, Σ_2
finTAUT$_{pos}$(L$\forall$)	Σ_1-hard, Π_2	Σ_1-hard, Π_2
finSAT$_{pos}$(L$\forall$)	Σ_2	Π_1-hard, Σ_2

Table 5: Arithmetical complexity bounds for the finite-chain semantics when L is recursively axiomatizable.

Theorem 3.25. *Let* L *be a (Δ)-core fuzzy logic such that the following conditions hold:*
(1) For every finite cardinal m, there is a finite L*-chain with at least m elements.*
(2) There is an L*-formula $\phi(p)$ such that for every* L*-chain $\mathcal{A}$ and for every $\mathcal{A}$-evaluation v, $v(\phi(p)) \in \{\overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\}$, and there are evaluations v_0 and v_1 such that $v_0(\phi(p)) = \overline{0}^{\mathcal{A}}$ and $v_1(\phi(p)) = \overline{1}^{\mathcal{A}}$.*
Then the set finTAUT$_1$(L$\forall$) *is Π_2-complete.*

Proof. Let PA^- be a finitely axiomatizable subtheory of arithmetics in which all recursive relations are representable. The numerals are inductively defined as usual: $\overline{0}$ is the constant for 0 and $\overline{n}$ denotes $S(\overline{n-1})$ for each $n > 0$, where S is the unary functional symbol for the successor. We also assume that PA^- has a binary predicate $\leq$ for the order, and that it proves all basic properties of order in the natural numbers. In particular, we assume that $PA^- \vdash (\forall x)(x \leq \overline{n} \to (x = \overline{0} \vee x = \overline{1} \vee \ldots \vee x = \overline{n}))$. Let Φ denote the conjunction of all axioms of PA^- and let for every formula γ, γ^+ be the result of replacing in γ every atomic formula δ by $\phi(\delta)$. Notice that for every model $\langle \mathbf{M}, \mathcal{A} \rangle$ the value $\|\gamma^+\|_{\mathbf{M},\mathcal{A}}$ is crisp.

Let $\mathbf{M}$ be a first-order structure over an L-chain $\mathcal{A}$ such that $\|\Phi^+\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$. We define a classical model $\mathbf{M}^{PA^-}$ for the language of PA^- as follows: the domain of $\mathbf{M}^{PA^-}$ is the domain M of $\mathbf{M}$ modulo the equivalence $\sim$ defined by $d \sim d'$ iff $\|\phi(d = d')\|^{\mathbf{M},\mathcal{A}} = 1$; for every n-ary function symbol f of PA^- and for every $d_1, \ldots, d_n \in M$, $f^{\mathbf{M}^{PA^-}}([d_1], \ldots, [d_n]) = [f^{\mathbf{M}}(d_1, \ldots, d_n)]$, where for each $d \in M$, $[d]$ denotes its equivalence class modulo $\sim$. Finally, for every n-ary predicate symbol P of PA^- and for every $d_1, \ldots, d_n \in M$, we stipulate that $\langle [d_1], \ldots, [d_n] \rangle \in P^{\mathbf{M}^{PA^-}}$ iff $\|\phi(P(d_1, \ldots, d_n))\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$. By induction on δ we can easily prove:

Claim 1: For every formula $\delta(x_1, \ldots, x_n)$ and for every $d_1, \ldots, d_n \in M$: $\|\delta(d_1, \ldots, d_n)^+\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$ iff $\mathbf{M}^{PA^-} \models \delta([d_1], \ldots, [d_n])$.

Conversely, given a model $\mathbf{N}$ of PA^- and an L-chain $\mathcal{A}$ we define a first-order structure $\mathbf{N}^{\mathcal{A}}$ on $\mathcal{A}$ (restricted to the language of PA^-) as follows: the domain of $\mathbf{N}^{\mathcal{A}}$ coincides with the domain N of $\mathbf{N}$ and the function symbols and the constants are interpreted as in $\mathbf{N}$; moreover, let z, o be elements of A such that for every evaluation v, we have $v(\phi(p)) = \overline{0}^{\mathcal{A}}$ if $v(p) = z$ and $v(\phi(p)) = \overline{1}^{\mathcal{A}}$ if $v(p) = o$. Then for every n-ary predicate symbol P and for every $d_1, \ldots, d_n \in N$, we define $\|\delta(d_1, \ldots, d_n)\|^{\mathbf{N}^{\mathcal{A}},\mathcal{A}} = o$ if $\mathbf{N} \models \delta(d_1, \ldots, d_n)$ and $\|\delta(d_1, \ldots, d_n)\|^{\mathbf{N}^{\mathcal{A}},\mathcal{A}} = z$ otherwise. Then, again by induction on δ, we can easily prove:

Claim 2: For every formula $\delta(x_1, \ldots, x_n)$ of PA^- and for every $d_1, \ldots, d_n \in N$ one has: $\|\delta(d_1, \ldots, d_n)^+\|^{\mathbf{N}^{\mathcal{A}},\mathcal{A}} = \overline{1}^{\mathcal{A}}$ iff $\mathbf{N} \models \delta(d_1, \ldots, d_n)$.

Now let $X = \{n \mid \forall m \exists k R(n, m, k)\}$, with R recursive, be a Π_2-complete set. Let $R'(x, y, z)$ be a formula of PA^- representing R in PA^-, that is, for all n, m, k, if $R(n, m, k)$ is true, then $R'(\overline{n}, \overline{m}, \overline{k})$ is provable in PA^- and if $R(n, m, k)$ is false, then $\neg R'(\overline{n}, \overline{m}, \overline{k})$ is provable in PA^-. Let R^+ be the formula obtained from R' by replacing every atomic subformula ψ by $\phi(\psi)$. Then R^+ behaves as a crisp formula. Finally, let P be a new unary predicate, and let $\Psi(x)$ be the formula

$$\Phi^+ \to \forall y(\exists u((u \leq y)^+ \wedge (P(S(u)) \to P(u))) \vee \exists z R^+(x, y, z)),$$

where S is the symbol of PA^- for the successor function.

We claim that for every n, $n \in X$ iff $\Psi(\overline{n})$ is true in every first-order model over a finite L-chain. Indeed, suppose $n \in X$. Let $\mathcal{A}$ be an L-chain with m elements, and let $\mathbf{M}$ be a first-order model over $\mathcal{A}$. If $\|\Phi^+\|^{\mathbf{M},\mathcal{A}} = \overline{0}^{\mathcal{A}}$, then $\|\Psi(\overline{n})\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$. Otherwise, $\|\Phi^+\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$, that is, the translation of every axiom of PA^- is true in $\langle \mathbf{M}, \mathcal{A} \rangle$.

Claim 3: If $\|\Phi^+\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$, then for every theorem ψ of PA^-, $\|\psi^+\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$.

Suppose not. Then, by Claim 1, $\mathbf{M}^{PA^-}$ would be a model of PA^- which does not satisfy ψ, a contradiction.

Now let y be an element of the universe of $\mathbf{M}$. If $\|(y \leq \overline{m})^+\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$, then by Claim 3, $\|(y = \overline{0} \vee y = \overline{1} \vee \ldots \vee y = \overline{m})^+\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$, because $PA^- \vdash \forall x(x \leq \overline{m} \leftrightarrow (x = \overline{0} \vee x = \overline{1} \vee \ldots \vee x = \overline{m}))$.

Moreover since $n \in X$, for $y = 0, 1, \ldots, m$, there is a k_y such that $R(n, y, k_y)$ is true. Then for such k_y, $R'(\overline{n}, \overline{y}, \overline{k_y})$ is provable in PA^- and $\|R^+(\overline{n}, \overline{y}, \overline{k_y})\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$, again by Claim 3. If $\|(y > \overline{m})^+\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$, then for some i such that $\|(\overline{i} \leq y)^+\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$, we must have $\|P(S(\overline{i})) \to P(\overline{i})\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$, otherwise $\|P(\overline{0})\|^{\mathbf{M},\mathcal{A}} < \|P(\overline{1})\|^{\mathbf{M},\mathcal{A}} < \ldots < \|P(\overline{m+1})\|^{\mathbf{M},\mathcal{A}}$ and A would have more than m elements. Thus in this case $\exists u((u \leq y)^+ \wedge (P(S(u)) \to P(u)))$. In any case, if $\|\Phi^+\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$ and $n \in X$ then $\|\forall y(\exists u((u \leq y)^+ \wedge (P(S(u)) \to P(u))) \vee \exists z R^+(x, y, z))\|^{\mathbf{M},\mathcal{A}} = \overline{1}^{\mathcal{A}}$. Thus $\Psi(\overline{n})$ has truth value $\overline{1}^{\mathcal{A}}$.

Now suppose that $n \notin X$. Then for some m there is no k such that $R(n, m, k)$. Let $\mathcal{A}$ be an L-chain with more than m elements. Let $\overline{0}^{\mathcal{A}} = a_0 < a_1 < \ldots < a_h = \overline{1}^{\mathcal{A}}$ with $h \geq m$, be the elements of A. Consider the first-order structure $\mathbf{N}^{\mathcal{A}}$ over $\mathcal{A}$ obtained from the standard model $\mathbf{N}$ of natural numbers according to Claim 2. Moreover, let us set, for $i = 0, \ldots, h$, $P^{\mathbf{N}^{\mathcal{A}}}(i) = a_i$ and for $i > h$, $P^{\mathbf{N}^{\mathcal{A}}}(i) = \overline{1}^{\mathcal{A}}$. Then by Claim 2, $\|\Phi^+\|^{\mathbf{N}^{\mathcal{A}},\mathcal{A}} = \overline{1}^{\mathcal{A}}$, $\|\exists u((u \leq \overline{m})^+ \wedge (P(S(u)) \to P(u)))\|^{\mathbf{N}^{\mathcal{A}},\mathcal{A}} = a_{h-1} < \overline{1}^{\mathcal{A}}$ and $\|\exists z R^+(\overline{n}, \overline{m}, z))\|^{\mathbf{N}^{\mathcal{A}},\mathcal{A}} = \overline{0}^{\mathcal{A}}$. It follows that $\|\Psi(\overline{n})\|^{\mathbf{N}^{\mathcal{A}},\mathcal{A}} = a_{h-1} < \overline{1}^{\mathcal{A}}$. □

Corollary 3.26. *Let* L *be a* (Δ)*-core fuzzy logic such that for every finite cardinal* m*, there is a finite* L*-chain with at least* m *elements. Then* $\mathrm{finTAUT}_1(\mathrm{L}\forall)$ *is* Π_2*-complete if one of the* *following sufficient conditions is satisfied:*

1. L *has a strict negation* $\sim$.
2. L *expands* WNM.
3. L *is a* Δ*-core fuzzy logic.*

Proof. By the hypothesis, all logics above satisfy condition (1) of Theorem 3.25. As regards to condition (2), for logics with a strict negation $\sim$, take $\phi(p) = \sim\sim p$, and note that for every evaluation v in an L-chain, if $v(p) = \overline{0}$, then $v(\phi(p)) = \overline{0}$, otherwise $v(\phi(p)) = \overline{1}$. For logics expanding WNM, take $\phi(p) = \neg((\neg((\neg\neg p)^2))^2)$ and note that for any evaluation v in an L-chain, if $v(\neg\neg p) \leq v(\neg p)$, then $v(\phi(p)) = \overline{0}$, otherwise $v(\phi(p)) = \overline{1}$. As regards to Δ-core fuzzy logics, it is clear that $\phi(p) = \Delta(p)$ satisfies condition (2) of Theorem 3.25. $\square$

Corollary 3.27. *For* $\mathrm{L} \in \{\mathrm{SMTL}, \mathrm{NM}, \mathrm{WNM}, \mathrm{SBL}, \mathrm{G}\}$*, we have that* $\mathrm{finTAUT}_1(\mathrm{L}\forall)$ *is* Π_2*-complete.*

Theorem 3.28. *Let* L *be a* Δ*-core fuzzy logic such that for every finite cardinal* m*, there is a finite* L*-chain with at least* m *elements. Then* $\mathrm{finSAT}_1(\mathrm{L}\forall)$ *and* $\mathrm{finSAT}_{pos}(\mathrm{L}\forall)$ *are* Σ_2*-complete.*

Proof. Let $X = \{x \in N \mid \exists y \forall z R(x, y, z)\}$ with R recursive, be a Σ_2-complete set. Let Φ be as in the proof of Theorem 3.25, and let for every formula γ, γ^+ be the result of replacing in γ every atomic formula P by $\Delta(P)$. Then γ^+ is a crisp formula. Now consider the formula $\Psi(x) = \Phi^+ \wedge \exists y[\forall u((u < y)^+ \to \neg\Delta(P(S(u)) \to P(u))) \wedge \forall z R^+(x, y, z)]$.

Claim A: for every n, $n \in X$ iff $\Psi(\overline{n})$ is 1 satisfiable in a finite L-chain iff $\Psi(\overline{n})$ is positively satisfiable in a finite L-chain.

Proof of Claim A. Since $\Psi(\overline{n})$ is crisp, it is 1-satisfiable in a finite L chain iff it is positively satisfiable there. Now suppose $n \in X$. Then there is $m \in N$ such that for all $z \in N$, $R(n, m, z)$ is true. Let $\mathbf{N}$ be the standard model of natural numbers, let $\mathcal{A}$ be a finite L-chain with $h + 1 > m$ elements $1 = a_0 > a_1 > \ldots > a_h = 0$, and let $\mathbf{N}^{\mathcal{A}}$ be as in Claim 2 of the proof of Theorem 3.25. Define $\|P(\bar{i})\|_{\mathcal{A},\mathbf{N}^{\mathcal{A}}} = a_i$ for $i = 0, \ldots, m$ and $\|P(\bar{i})\|_{\mathcal{A},\mathbf{N}^{\mathcal{A}}} = 1$ for $i > m$. Note that for $i = 0, \ldots, m$, $\|P(\bar{i}) \to P(S(\bar{i}))\|_{\mathcal{A},\mathbf{N}^{\mathcal{A}}} < 1$ and hence $\|\neg\Delta(P(\bar{i}) \to P(S(\bar{i})))\|_{\mathcal{A},\mathbf{N}^{\mathcal{A}}} = 1$. Moreover for every natural number k, $\|R^+(\overline{n}, \overline{m}, \overline{k})\|_{\mathcal{A},\mathbf{N}^{\mathcal{A}}} = 1$. It follows that $\|\Psi^+(\overline{n})\|_{\mathcal{A},\mathbf{N}^{\mathcal{A}}} = 1$, and $\Psi^+(\overline{n})$ is 1-satisfiable in a finite L-chain.

Conversely, suppose $n \notin X$. Let $\mathbf{M}$ be a model over a finite L-chain $\mathcal{A}$ and let k be its cardinality. We wish to prove that $\|\Psi^+(\overline{n})\|_{\mathcal{A},\mathbf{M}} < 1$, or equivalently that $\|\Psi^+(\overline{n})\|_{\mathcal{A},\mathbf{M}} = 0$, as Ψ is crisp. The claim is clear if $\|\Phi^+\|_{\mathcal{A},\mathbf{M}} < 1$. If $\|\Phi^+\|_{\mathcal{A},\mathbf{M}} = 1$, then we obtain a model $\mathbf{M}^{PA^-}$ as in Claim 2 of the proof of Theorem 3.25. Since $n \notin X$, for all $m \leq k$ there is $h_m \in N$ such that $R(n, m, h_m)$ is false. Thus, for all $m \leq k$ there is $h_m \in N$ such that $\|R^+(\overline{n}, \overline{m}, \overline{h}_m)\|_{\mathcal{A},\mathbf{M}} = 0$. It follows that $\|\forall z R^+(\overline{n}, \overline{m}, z)\|_{\mathcal{A},\mathbf{M}} = 0$. Finally, if $m > k$, then there must be $i < m$ such that $\|P(\bar{i})\|_{\mathcal{A},\mathbf{M}} \leq \|P(S(\bar{i}))\|_{\mathcal{A},\mathbf{M}}$. Hence, $\|\forall u((u < \overline{m})^+ \to \neg\Delta(P(S(u)) \to P(u)))\|_{\mathcal{A},\mathbf{M}} = 0$. In any case, $\|\Psi(\overline{n})\|_{\mathcal{A},\mathbf{M}} = 0$. $\square$

On the other hand, in [20, Th 5.4.30] it is proved that $\mathrm{finTAUT}_1(\text{Ł}\forall)$ is Π_2-complete (the proof of this fact and further consequences will be surveyed in the next subsection). This allows to prove the following result:

Proposition 3.29. $\mathrm{finTAUT}_1(\mathrm{BL}\forall)$ *is* Π_2*-complete.*

Proof. As in the proof of Proposition 3.10, for every sentence φ we consider the formula $\varphi^{\neg\neg}$ resulting from φ by adding double negation $\neg\neg$ to all atoms. Then for every $\varphi \in \mathrm{Sent}_\Gamma$ we have: $\varphi^{\neg\neg} \in \mathrm{finTAUT}_1(\mathrm{BL}\forall)$ iff $\varphi \in \mathrm{finTAUT}_1(\mathrm{Ł}\forall)$. Indeed, the left-to-right implication is obvious because the negation is involutive in Łukasiewicz logic; as for the converse one assume $\varphi \in \mathrm{finTAUT}_1(\mathrm{Ł}\forall)$ and consider any model $\mathbf{M}$ over a finite BL-chain $\mathcal{A}$. Taking into account the structure of BL-chains described in [20], it is enough to distinguish two cases: (1) Assume that $\mathcal{A}$ is an ordinal sum $\mathcal{C}_1 \oplus \mathcal{C}_2$ where $\mathcal{C}_1$ is a finite MV-chain. Then, we define a model $\mathbf{M}'$ over $\mathcal{C}_1$ from $\mathbf{M}$ in the following way: take the same domain, the same interpretation of constants and functionals, and for every n-ary predicate symbol P and elements $a_1, \ldots, a_n$ in the domain set $P_{\mathbf{M}'}(a_1, \ldots, a_n) = P_{\mathbf{M}}(a_1, \ldots, a_n)$ if $P_{\mathbf{M}}(a_1, \ldots, a_n) \in C_1$ and $P_{\mathbf{M}'}(a_1, \ldots, a_n) = \overline{1}^{\mathcal{A}}$ otherwise. Now it is easy to prove by induction that for every formula α and every evaluation v: $\|\alpha^{\neg\neg}\|^{\mathcal{A}}_{\mathbf{M},v} = \|\alpha\|^{\mathcal{C}_1}_{\mathbf{M}',v}$. Hence $\|\varphi^{\neg\neg}\|^{\mathcal{A}}_{\mathbf{M}} = \|\varphi\|^{\mathcal{C}_1}_{\mathbf{M}'} = \overline{1}^{\mathcal{A}}$. (2) Assume that $\mathcal{A}$ is an SBL-chain (i.e. its negation is strict). Then we define a model $\mathbf{M}'$ over $\mathcal{B}_2$ from $\mathbf{M}$ in the following way: take the same domain, the same interpretation of constants and functionals, and for every n-ary predicate symbol P and elements $a_1, \ldots, a_n$ in the domain set $P_{\mathbf{M}'}(a_1, \ldots, a_n) = 0$ if $P_{\mathbf{M}}(a_1, \ldots, a_n) = \overline{0}^{\mathcal{A}}$ and $P_{\mathbf{M}'}(a_1, \ldots, a_n) = 1$ otherwise. Now we have: $\|\varphi^{\neg\neg}\|^{\mathcal{A}}_{\mathbf{M}} = \|\varphi^{\neg\neg}\|^{\mathcal{B}_2}_{\mathbf{M}'} = \|\varphi\|^{\mathcal{B}_2}_{\mathbf{M}'} = 1$. Therefore, we have proved that $\mathrm{finTAUT}_1(\mathrm{BL}\forall)$ is Π_2-hard. The Π_2 containment follows from theorems 3.23 and 3.24. □

Some more results on complexity of finite-chain semantics will be obtained in next subsection when comparing such semantics with the standard and rational ones.

3.3 Complexity of standard and rational semantics and their relation to finite-chain semantics

Let us now consider the semantics given by all standard chains or all rational chains. The completeness properties with respect to those semantics have been studied in the literature. We extract Table 6 from [6] with some known results.

On the other hand, the arithmetical complexities w.r.t. the standard semantics of continuous t-norm based logics has been deeply studied in several papers. We summarize the known results in the following theorem.

Theorem 3.30. *Let us denote by $\mathcal{R}$ the set of all standard* BL*-chains, and given $\mathcal{A} \in \{[0,1]_{\mathrm{Ł}}, [0,1]_\Pi, [0,1]_{\mathrm{G}}\}$ let us denote by $\mathcal{A}\oplus$ the subset of $\mathcal{R}$ of those chains whose ordinal sum decomposition starts with $\mathcal{A}$.*

- $\mathrm{TAUT}_1([0,1]_{\mathrm{Ł}})$ *is Π_2-complete [43];* $\mathrm{TAUT}_1([0,1]_{\mathrm{G}})$ *is Σ_1-complete [20]; for every $\mathbb{K} \subseteq \mathcal{R} \setminus \{[0,1]_{\mathrm{G}}\}$* $\mathrm{TAUT}_1(\mathbb{K})$ *is Π_2-hard [37]; if $\mathbb{K} \subseteq \mathcal{R}$ contains at least one algebra non-isomorphic to any of $[0,1]_{\mathrm{G}}$, $[0,1]_{\mathrm{Ł}}$, $[0,1]_{\mathrm{Ł}} \oplus [0,1]_{\mathrm{G}}$, $[0,1]_{\mathrm{G}} \oplus [0,1]_{\mathrm{Ł}}$, $[0,1]_{\mathrm{Ł}} \oplus [0,1]_{\mathrm{Ł}}$ and $[0,1]_{\mathrm{Ł}} \oplus [0,1]_{\mathrm{G}} \oplus [0,1]_{\mathrm{Ł}}$, then* $\mathrm{TAUT}_1(\mathbb{K})$ *is not arithmetical [37].*

- $\mathrm{SAT}_1([0,1]_{\mathrm{Ł}}) = \mathrm{SAT}_1([0,1]_{\mathrm{Ł}}\oplus)$ *and* $\mathrm{SAT}_1([0,1]_{\mathrm{G}}) = \mathrm{SAT}_1([0,1]_{\mathrm{G}}\oplus)$*, and they are Π_1-complete; if $\mathbb{K} \subseteq \mathcal{R}$ contains some chain from $[0,1]_\Pi\oplus$, then* $\mathrm{SAT}_1(\mathbb{K})$ *is not arithmetical [21, 24].*

- $\mathrm{TAUT}_{pos}([0,1]_{\mathrm{Ł}}) = \mathrm{TAUT}_{pos}([0,1]_{\mathrm{Ł}}\oplus)$ *and* $\mathrm{TAUT}_{pos}([0,1]_{\mathrm{G}}) = \mathrm{TAUT}_{pos}([0,1]_{\mathrm{G}}\oplus)$*, and they are Σ_1-complete; if $\mathbb{K} \subseteq \mathcal{R}$ contains some chain from $[0,1]_\Pi\oplus$, then* $\mathrm{TAUT}_{pos}(\mathbb{K})$ *is not arithmetical [21, 24].*

Logic	$\mathcal{R}$C, FS$\mathcal{R}$C	$\mathcal{Q}$C, FS$\mathcal{Q}$C
MTL∀	Yes	Yes
IMTL∀	Yes	Yes
SMTL∀	Yes	Yes
WCMTL∀	?	?
ΠMTL∀	?	?
BL∀	No	No
SBL∀	No	No
Ł∀	No	Yes
Π∀	No	Yes
G∀	Yes	Yes
C_nMTL∀	Yes	Yes
C_nIMTL∀	Yes	Yes
WNM∀	Yes	Yes
NM∀	Yes	Yes

Table 6: Completeness properties w.r.t. distinguished semantics for some prominent first-order fuzzy logics.

- $\mathrm{SAT}_{pos}([0,1]_{\text{Ł}}) = \mathrm{SAT}_{pos}([0,1]_{\text{Ł}}\oplus)$ *and it is* Σ_2*-complete;* $\mathrm{SAT}_{pos}([0,1]_{\mathrm{G}}) = \mathrm{SAT}_{pos}([0,1]_{\mathrm{G}}\oplus)$ *and it is* Π_1*-complete; if* $\mathbb{K} \subseteq \mathcal{R}$ *contains some chain from* $[0,1]_{\Pi}\oplus$*, then* $\mathrm{SAT}_{pos}(\mathbb{K})$ *is not arithmetical [21, 24].*

Combining this knowlegde with our new general results in Subsection 3.1 we can obtain many arithmetical complexity results w.r.t. the standard and rational semantics for prominent logics as collected in tables 7 and 8. In the case of BL∀ and SBL∀ we need an additional result:

Proposition 3.31. *The sets* $\mathrm{ratTAUT}_1(\mathrm{BL}\forall)$ *and* $\mathrm{ratTAUT}_1(\mathrm{SBL}\forall)$ *are in* Σ_1*, and hence* $\mathrm{ratSAT}_{pos}(\mathrm{BL}\forall)$ *and* $\mathrm{ratSAT}_{pos}(\mathrm{SBL}\forall)$ *are in* Π_1*.*

Proof. Following [6] consider the extensions of BL∀ and SBL∀ by adding the schema $\Phi = (\forall x)(\chi \& \varphi) \to (\chi \& (\forall x)\varphi)$, where x is not free in χ. Call them $\mathrm{BL}\forall^+$ and $\mathrm{SBL}\forall^+$, respectively. Φ is known to be satisfied by every model on a densely ordered BL-chain [20, page 102] but it is not a 1-tautology for all BL-chains [12]. In [6, Th. 5.34] it is proved that $\mathrm{BL}\forall^+$ (resp. $\mathrm{SBL}\forall^+$) enjoys strong completeness with respect to models over rational BL-chains (resp. SBL-chains), and it is not necessary to require that those models satisfy the additional schema because their chains are densely ordered. Therefore, $\mathrm{ratTAUT}_1(\mathrm{BL}\forall)$ turns out to be the set of theorems of the logic $\mathrm{BL}\forall^+$, and analogously for $\mathrm{ratTAUT}_1(\mathrm{SBL}\forall)$; this proves the result. □

Observe that the completeness properties imply that for many prominent logics we have $\mathrm{genTAUT}_1(\mathrm{L}\forall) = \mathrm{stTAUT}_1(\mathrm{L}\forall) = \mathrm{ratTAUT}_1(\mathrm{L}\forall)$. On the other hand, in [20, Theorem 5.4.30] Hájek proved that the standard 1-tautologies of first-order Łukasiewicz logic coincide with the 1-tautologies over the finite-chain semantics, i.e. $\mathrm{stTAUT}_1(\text{Ł}\forall) = \mathrm{finTAUT}_1(\text{Ł}\forall)$. Now, in addition, we will consider the semantics given by canonical rational chains. Of course, this can only be done for those logics where it makes sense to have an intended semantics over the rational unit interval, i.e. logics L_* given by a left-continuous t-norm $*$ such that

Logic	stTAUT_1	stSAT_1	stTAUT_{pos}	stSAT_{pos}
MTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1
IMTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
SMTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
WCMTL∀	Σ_1-hard	Π_1-hard	Σ_1-hard	?
ΠMTL∀	Σ_1-hard	Π_1-hard	Σ_1-hard	Π_1-hard
BL∀	Non-arithmetical	Non-arithmetical	Non-arithmetical	Non-arithmetical
SBL∀	Non-arithmetical	Non-arithmetical	Non-arithmetical	Non-arithmetical
Ł∀	Π_2-complete	Π_1-complete	Σ_1-complete	Σ_2-complete
Π∀	Non-arithmetical	Non-arithmetical	Non-arithmetical	Non-arithmetical
G∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
C_nMTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1
C_nIMTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
WNM∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1
NM∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete

Table 7: Complexity results for the standard semantics.

its restriction to $[0,1]^{\mathbb{Q}}$ is well-defined. We denote the corresponding algebra as $[0,1]^{\mathbb{Q}}_*$. This can be done, for instance, for a special kind of t-norms corresponding to extensions of WNM, namely those which are determined by a finite partition of the unit interval in subintervals where the negation function is either constant or involutive, provided that the extremal points of such subintervals are rational numbers; denote the class of those t-norms by $\mathbf{WNM\text{-}fin}^{\mathbb{Q}}$ (see [41] for further details).

Proposition 3.32. *Let $* \in \mathbf{WNM\text{-}fin}^{\mathbb{Q}}$ and let I_a be its maximum constant interval (with possibly $a = 1$) and let $\mathcal{A}$ be a countable L_*-chain. Then:*

- *If $I^{\mathcal{A}}_{\bar{1}^{\mathcal{A}}} = \{\bar{1}^{\mathcal{A}}\}$, then there exists an embedding from $\mathcal{A}$ into $[0,1]^{\mathbb{Q}}_*$.*
- *If $I^{\mathcal{A}}_{\bar{1}^{\mathcal{A}}} \neq \{\bar{1}^{\mathcal{A}}\}$, then there exists an embedding from $\mathcal{A}$ into $[0,1]^{\mathbb{Q}}_*/F^a$ (where F^a denotes the filter generated by a).*

Proof. It suffices to inspect the proof of the analogous fact for real chains in [41] and realize that the embedding can be in fact defined into the rationals. □

Corollary 3.33. *Let $* \in \mathbf{WNM\text{-}fin}^{\mathbb{Q}}$ and let I_a be its maximum positive constant interval, if it exists. Then:*

- *If $a = 1$ or $*$ has no positive constant intervals, then the logic L_* is strongly complete with respect to $[0,1]^{\mathbb{Q}}_*$, and hence $\mathrm{TAUT}_1([0,1]^{\mathbb{Q}}_*)$ is Σ_1-complete.*
- *If $a \neq 1$, then the logic L_* is strongly complete with respect to $\{[0,1]^{\mathbb{Q}}_*, [0,1]^{\mathbb{Q}}_*/F^a\}$, and hence $\mathrm{TAUT}_1([0,1]^{\mathbb{Q}}_*, [0,1]^{\mathbb{Q}}_*/F^a)$ is Σ_1-complete.*

Notice, in particular, that the logic NM∀ is under the scope of the first point in the last corollary, and so we have $\mathrm{genTAUT}_1(\mathrm{NM}\forall) = \mathrm{stTAUT}_1(\mathrm{NM}\forall) = \mathrm{canratTAUT}_1(\mathrm{NM}\forall)$ and they are Σ_1-complete.

Logic	ratTAUT$_1$	ratSAT$_1$	ratTAUT$_{pos}$	ratSAT$_{pos}$
MTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1
IMTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
SMTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
WCMTL∀	Σ_1-hard	Π_1-hard	Σ_1-hard	?
ΠMTL∀	Σ_1-hard	Π_1-hard	Σ_1-hard	Π_1-hard
BL∀	Σ_1-complete	Π_1-hard	Σ_1-hard	Π_1
SBL∀	Σ_1-complete	Π_1-hard	Σ_1-hard	Π_1-complete
Ł∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
Π∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
G∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
C$_n$MTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1
C$_n$IMTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
WNM∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1
NM∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete

Table 8: Complexity results for the rational semantics.

Observe now that the three main continuous t-norms satisfy the required property as well, i.e. we have well-defined algebras over the rationals $[0,1]_{\text{Ł}}^{\mathbb{Q}}$, $[0,1]_{\Pi}^{\mathbb{Q}}$ and $[0,1]_{\text{G}}^{\mathbb{Q}}$; the same goes for their ordinal sums. Let $\mathcal{Q}$ be the set of ordinal sums of these three rational BL-chains. Given $\mathbb{K} \subseteq \mathcal{Q}$, $\bar{\mathbb{K}}$ will denote the subset of $\mathcal{R}$ given by the substitution in the elements of $\mathbb{K}$ of each component for its corresponding basic real chain. We start with the case of $[0,1]_{\text{Ł}}^{\mathbb{Q}}$.

Lemma 3.34. *Let* $\mathbf{M}$ *and* $\mathbf{M}'$ *be two first-order structures with the same domain* M *over* $[0,1]_{\text{Ł}}$*, and let* ϕ, ψ *be first-order sentences with parameters from* M *and* $\delta(x)$ *be a first-order formula with parameters from* M *and with* x *as its only free variable. Let for any two real numbers* α, β*,* $d(\alpha,\beta)$ *denote the distance between* α *and* β*, that is,* $d(\alpha,\beta) = \max\{\alpha-\beta, \beta-\alpha\}$*. Then for every positive real number* γ *we have:*
(i) If $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq \gamma$ *then* $d(\|\neg\phi\|_{\mathbf{M}}, \|\neg\phi\|_{\mathbf{M}'}) \leq \gamma$.
(ii) If $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq \gamma$ *and* $d(\|\psi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}'}) \leq \gamma$ *then* $d(\|\phi\&\psi\|_{\mathbf{M}}, \|\phi\&\psi\|_{\mathbf{M}'}) \leq 2\gamma$.
(iii) If for all $a \in M$*,* $d(\|\delta(a)\|_{\mathbf{M}}, \|\delta(a)\|_{\mathbf{M}'}) \leq \gamma$ *then* $d(\|\forall x\delta(x)\|_{\mathbf{M}}, \|\forall x\delta(x)\|_{\mathbf{M}'}) \leq \gamma$.

Proof. Almost trivial. □

Corollary 3.35. *If* ϕ *is a sentence of complexity* k *and* $\mathbf{M}$ *and* $\mathbf{M}'$ *are first-order structures with the same domain* M *over* $[0,1]_{\text{Ł}}$ *such that for every closed atomic subformula* ψ *of* ϕ*,* $d(\|\psi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}'}) \leq \gamma$*, then* $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq 2^k\gamma$.

Lemma 3.36. *Let* ϕ *be a first-order sentence of complexity* k *and let for every* n*,* $\mathbf{Ł}_n$ *denote the finite* MV*-chain with* $n+1$ *elements. Let* $\mathbf{M}$ *be a first-order structure over* $[0,1]_{\text{Ł}}$ *with domain* M *such that* $\|\phi\|_{\mathbf{M}} < 1$ *and let* n *be such that* $2^{-n} < 1 - \|\phi\|_{\mathbf{M}}$*. Then there is a first-order structure* $\mathbf{M}'$ *over* $\mathbf{Ł}_{2^{k+n}}$ *such* $\|\phi\|_{\mathbf{M}'} < 1$.

Proof. Let for every atomic formula ψ with parameters in M, $m(\psi)$ denote the maximum natural number such that $\frac{m(\psi)}{2^{n+k}} \leq \|\psi\|_{\mathbf{M}}$. Define a new first-order structure $\mathbf{M}'$ with domain M letting for every atomic formula ψ with parameters in M, $\|\psi\|_{\mathbf{M}'} = \frac{m(\psi)}{2^{n+k}}$. Then

$d(\|\psi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}'}) < \frac{1}{2^{n+k}}$ and by Corollary 3.35, $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq 2^k \frac{1}{2^{n+k}} = \frac{1}{2^n} < 1 - \|\phi\|_{\mathbf{M}}$. It follows that $\|\phi\|_{\mathbf{M}'} < 1$. Now $\mathbf{M}'$ has been defined as a first-order structure over $[0,1]_{\text{Ł}}$, but since for every sentence δ, $\|\delta\|_{\mathbf{M}'} \in \mathbf{Ł}_{2^{k+n}}$, $\mathbf{M}'$ can be also regarded as a first-order structure over $\mathbf{Ł}_{2^{k+n}}$. This ends the proof. □

Theorem 3.37. $\text{stTAUT}_1(\text{Ł}\forall) = \text{canratTAUT}_1(\text{Ł}\forall) = \text{finTAUT}_1(\text{Ł}\forall)$ *and they are* Π_2-*complete.*

Proof. Inclusions $\subseteq$ follow from Lemma 3.3, therefore it suffices to prove that $\text{finTAUT}_1(\text{Ł}\forall) \subseteq \text{stTAUT}_1(\text{Ł}\forall)$. But this is immediate from Lemma 3.36. □

Now recall that if $\mathbb{K}$ is a class of MV-chains then $\text{SAT}_{pos}(\mathbb{K}) = \{\phi \mid \neg\phi \notin \text{TAUT}_1(\mathbb{K})\}$. Thus we obtain:

Theorem 3.38. $\text{stSAT}_{pos}(\text{Ł}\forall) = \text{canratSAT}_{pos}(\text{Ł}\forall) = \text{finSAT}_{pos}(\text{Ł}\forall)$ *and they are* Σ_2-*complete.*

Open problem: Is it true that $\text{finTAUT}_{pos}(\text{Ł}\forall) \subseteq \text{stTAUT}_{pos}(\text{Ł}\forall)$? This would imply: $\text{stTAUT}_{pos}(\text{Ł}\forall) = \text{canratTAUT}_{pos}(\text{Ł}\forall) = \text{finTAUT}_{pos}(\text{Ł}\forall)$ and $\text{stSAT}_1(\text{Ł}\forall) = \text{canratSAT}_1(\text{Ł}\forall) = \text{finSAT}_1(\text{Ł}\forall)$.

Theorems 3.37 and 3.38 do not extend to finite consequence relation as shown in the next theorem.

Theorem 3.39. *There are formulae* ϕ *and* ψ *of Łukasiewicz logic such that* $\phi \in \text{finCons}(\text{Ł}, \psi)$ *and* $\phi \notin \text{stCons}(\text{Ł}, \psi)$.

Proof. Let S be a unary function symbol, P be a unary predicate symbol and 0 be a constant symbol. Let $\psi = \forall x(P(x) \leftrightarrow ((P(S(x)) \oplus P(S(x))))$ and $\phi = P(0) \vee \neg P(0)$. We claim that $\phi \in \text{finCons}(\psi)$ but $\phi \notin \text{stCons}(\psi)$. Let $\mathbf{M}$ be a first-order structure over a finite chain $\mathbf{Ł}_n$ such that $\|\psi\|_{\mathbf{M}} = 1$, let m be such that $2^m > n$ and let for every k, $k_{\mathbf{M}} = S(S \ldots (S(0)) \ldots)$ (k times). If $\|P(0)\|_{\mathbf{M}} \in \{0,1\}$ we have $\|\phi\|_{\mathbf{M}} = 1$. Otherwise, by the definition of ψ we have $\|P(m_{\mathbf{M}})\| = \frac{\|P(0)\|_{\mathbf{M}}}{2^m}$. Now if for some x with $0 < x < 1$, $\frac{x}{2^m}$ belongs to a finite MV-chain $\mathbf{C}$, then $\mathbf{C}$ must have $2^m + 1$ elements at least. Since $n < 2^m$, $\frac{\|P(0)\|_{\mathbf{M}}}{2^m}$ cannot belong to $\mathbf{Ł}_n$ and a contradiction is reached. To prove that $\phi \notin \text{stCons}(\text{Ł}, \psi)$, define a structure $\mathbf{M}$ on $[0,1]_{\text{Ł}}$ letting the domain M of $\mathbf{M}$ be the set ω of natural numbers, 0 the minimum of ω, S the successor function and $\|P(n)\| = \frac{1}{2^{n+1}}$. Then we have $\|\psi\|_{\mathbf{M}} = 1$, but $\|P(0)\|_{\mathbf{M}} = \|\neg P(0)\|_{\mathbf{M}} = \|\phi\|_{\mathbf{M}} = \frac{1}{2} < 1$. □

We consider now other canonical rational chains.

Theorem 3.40. *Let* $\mathbb{K} \subseteq \mathcal{Q}$ *such that there exists* $\mathcal{A} \in \mathbb{K}$ *whose first component is product. Then* $\text{TAUT}_1(\mathbb{K})$, $\text{SAT}_1(\mathbb{K})$, $\text{TAUT}_{pos}(\mathbb{K})$ *and* $\text{SAT}_{pos}(\mathbb{K})$ *are non-arithmetical.*

Proof. In [36] the author defines a formula Ψ obtained as the conjunction of the following formulae:

(a) $\gamma^{\neg\neg}$, for each axiom γ of PA^-.

(b) $\forall x \neg\neg U(x)$, $\neg\forall x U(x)$ and $\forall x(U(S(x)) \to U(x)^3)$.

Then one can prove the following claims for every Φ:

1. $\mathbf{N} \models \Phi$ iff $\Psi \to \Phi^{\neg\neg} \in \text{TAUT}_1(\mathbb{K})$ iff $\Psi \to \Phi^{\neg\neg} \in \text{TAUT}_{pos}(\mathbb{K})$.

2. $\mathbf{N} \models \Phi$ iff $\Psi \& \Phi^{\neg\neg} \in \mathrm{SAT}_1(\mathbb{K})$ iff $\Psi \& \Phi^{\neg\neg} \in \mathrm{SAT}_{pos}(\mathbb{K})$.

Assume that $\mathbf{N} \models \Phi$. As proved in [36] this implies that $\Psi \to \Phi^{\neg\neg} \in \mathrm{TAUT}_1(\bar{\mathbb{K}})$, and hence $\Psi \to \Phi^{\neg\neg} \in \mathrm{TAUT}_1(\mathbb{K})$ and $\Psi \to \Phi^{\neg\neg} \in \mathrm{TAUT}_{pos}(\mathbb{K})$. Assume now that $\mathbf{N} \not\models \Phi$. Let Then we construct a countermodel $\mathbf{M}$ over $\mathcal{A}$ as follows:

(a) the domain of $\mathbf{M}$ is the set of natural numbers, and the constant 0 and the function symbols of PA^- are interpreted as in the standard model $\mathbf{N}$ of natural numbers.

(b) if P is an n-ary predicate symbol of PA^- and $k_1, \dots, k_n$ are natural numbers, then $P^M(k_1, \dots, k_n) = 1$ if $P(k_1, \dots, k_n)$ is true in $\mathbf{N}$ and $P^M(k_1, \dots, k_n) = 0$ otherwise.

(c) Let f be the affine bijective transformation from $[0,1]^{\mathbb{Q}}$ to the interval where the first component of $\mathcal{A}$ is defined. For every natural number n, $U_{\mathbf{M}}(n) = f(2^{-3n})$.

It is readily seen that $\|\Psi\|^{\mathbf{M},\mathcal{A}} = 1$ and $\|\Phi^{\neg\neg}\|^{\mathbf{M},\mathcal{A}} = 0$. Therefore, $\|\Phi \to \Phi^{\neg\neg}\|^{\mathbf{M},\mathcal{A}} = 0$ and hence $\Psi \to \Phi^{\neg\neg} \notin \mathrm{TAUT}_1(\mathbb{K})$ and $\Psi \to \Phi^{\neg\neg} \notin \mathrm{TAUT}_{pos}(\mathbb{K})$.

The second claim is proved analogously. □

Theorem 3.41. *1. If $\mathcal{C}$ is the first component of a chain $\mathcal{A} \in \mathcal{Q}$, then* $\mathrm{TAUT}_{pos}(\mathcal{A}) = \mathrm{TAUT}_{pos}(\mathcal{C})$ *and* $\mathrm{SAT}_{pos}(\mathcal{A}) = \mathrm{SAT}_{pos}(\mathcal{C})$.

2. $\mathrm{SAT}_{pos}([0,1]^{\mathbb{Q}}_{\mathrm{G}}) = \mathrm{SAT}_1([0,1]^{\mathbb{Q}}_{\mathrm{G}})$ *and it is* Π_1*-complete.*

3. If $\mathcal{A} \in \mathcal{Q}$ begins with a Gödel component, then $\mathrm{SAT}_1(\mathcal{A}) = \mathrm{SAT}_1([0,1]^{\mathbb{Q}}_{\mathrm{G}})$.

4. $\mathrm{TAUT}_{pos}([0,1]^{\mathbb{Q}}_{\mathrm{G}})$ *is* Σ_1*-complete.*

5. If $\mathbb{K} \subseteq \mathcal{Q}$ contains some component non-isomorphic to $[0,1]^{\mathbb{Q}}_{\mathrm{G}}$, then $\mathrm{TAUT}_1(\mathbb{K})$ *is* Π_2*-hard.*

6. if $\mathbb{K} \subseteq \mathcal{Q}$ contains at least one algebra non-isomorphic to any of $[0,1]^{\mathbb{Q}}_{\mathrm{G}}$, $[0,1]^{\mathbb{Q}}_{\mathrm{Ł}}$, $[0,1]^{\mathbb{Q}}_{\mathrm{Ł}} \oplus [0,1]^{\mathbb{Q}}_{\mathrm{G}}$, $[0,1]^{\mathbb{Q}}_{\mathrm{G}} \oplus [0,1]^{\mathbb{Q}}_{\mathrm{Ł}}$, $[0,1]^{\mathbb{Q}}_{\mathrm{Ł}} \oplus [0,1]^{\mathbb{Q}}_{\mathrm{Ł}}$ and $[0,1]^{\mathbb{Q}}_{\mathrm{Ł}} \oplus [0,1]^{\mathbb{Q}}_{\mathrm{G}} \oplus [0,1]^{\mathbb{Q}}_{\mathrm{Ł}}$, then $\mathrm{TAUT}_1(\mathbb{K})$ *is not arithmetical*

Proof. The first four claims are proved by checking that the proofs of the corresponding results for standard semantics in [24] actually work as well for canonical rational semantics. Point 5. is shown by reducing the problem to $\mathrm{TAUT}_1([0,1]^{\mathbb{Q}}_{\mathrm{Ł}})$, which we know is Π_2-hard, as in [37]. Similarly, for the last point we use the fact, proved in the previous theorem, that $\mathrm{TAUT}_1([0,1]^{\mathbb{Q}}_{\Pi})$ is non-arithmetical and perform the analogous reduction to that problem as in [37]. □

As for the relation between 1-tautologies over real and finite chains, the issue has been already considered in [17] where it has been proved that for every left-continuous t-norm $*$, $\mathrm{stTAUT}_1(\mathrm{L}_*\forall) = \mathrm{finTAUT}_1(\mathrm{L}_*\forall)$ iff $[0,1]_* \cong [0,1]_{\mathrm{Ł}}$. That result is achieved by showing for every $*$ not isomorphic to Łukasiewicz t-norm a sentence φ such that every model over a finite L_*-chain validates φ but there is a $[0,1]_*$-model $\mathbf{M}$ such that $\langle \mathbf{M}, [0,1]_* \rangle \not\models \varphi$. That allows for the following generalization to our framework:

Theorem 3.42. *Let* L *be a consistent (Δ-)core fuzzy logic. If there exist* L*-chains over* $[0,1]$ *whose t-norm is not isomorphic to Łukasiewicz, then* $\mathrm{stTAUT}_1(\mathrm{L}\forall) \neq \mathrm{finTAUT}_1(\mathrm{L}\forall)$ *and* $\mathrm{genTAUT}_1(\mathrm{L}\forall) \neq \mathrm{finTAUT}_1(\mathrm{L}\forall)$.

Proof. It suffices to take the aforementioned counterexamples from [17]. □

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