A basic fuzzy logic which is really basic and fuzzy

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Through the years, Mathematical Fuzzy Logic has undergone a process of increasing generality of its studied logical systems by progressively weakening their logical laws (described by Petr Hájek as *removing legs from the flea* [16]). The first works of the area focused on logics based on particular continuous t-norms: Łukasiewicz logic, Gödel–Dummett logic and Product logic. A first step in the generalization process was taken by Petr Hájek when he introduced BL as a first "basic fuzzy logic" [15], which turn out to be complete with respect to the semantics given by all continuous t-norms [4]. The three main fuzzy logics based on continuous t-norms we have just mentioned can all be seen as axiomatic extensions of BL. In a similar fashion, by dropping the divisibility law, Esteva and Godo introduced the system MTL, which was proved to be complete with respect to the semantics given by all left-continuous (i.e. residuated) t-norms [17]. This system was later further weakened in two different directions: (a) by dropping commutativity of conjunction Jenei and Montagna obtained a system, psMTL^r, complete with respect to the semantics on non-commutative residuated t-norms [18], and (b) by removing integrality (i.e. not requiring the neutral element of conjunction to be maximum of the order) Metcalfe and Montagna proposed the logic UL which is, in turn, complete with respect to left-continuous uninorms [20]. An alternative path in the search for weaker fuzzy systems has consisted in restricting the language by considering fragments of fuzzy logics (see e.g. [11, 5]). One can observe that the common feature of all the mentioned systems is that they enjoy a *standard completeness theorem*, i.e. completeness with respect to a semantics of algebras defined on the real unit interval $[0, 1]$, which is implicitly regarded by many authors (and sometimes even explicitly e.g. in [20]) as an essential requirement for fuzzy logics.

On the other hand, it is well-known that fuzzy logics are closely related to substructural logics (see e.g. [10]). Recall the full Lambek logic FL, a basic substructural logic which does not satisfy any of the usual three structural rules: exchange, weakening, and contraction. Although firstly presented by means of a Gentzen calculus, it can be given a Hilbert-style presentation and shown to be an algebraizable logic in the sense of [2] whose equivalent algebraic semantics is the variety of lattice-ordered residuated monoids (usually referred to as *residuated lattices* or FL-*algebras*). The main extensions of FL, obtained by adding some of the structural rules, correspond to subvarieties of residuated lattices satisfying corresponding extra algebraic properties (see e.g. [12]). Actually, many fuzzy logics have been shown to be axiomatic extensions of some of these prominent substructural logics by adding some axioms that enforce completeness with respect to some class of linearly ordered residuated lattices (or *chains*). For instance, Gödel–Dummett logic is the logic of linearly ordered Heyting algebras, MTL is the logic of $\mathrm{FL_{ew}}$-chains, UL is the logic of $\mathrm{FL_e}$-chains, and psMTL^r is the logic of $\mathrm{FL_w}$-chains. Interestingly enough, the logic FL^ℓ of FL-chains (a common generalization of UL and psMTL^r) does not enjoy standard completeness (see [21]), therefore, for many authors, it cannot be taken as a good candidate for a really basic *fuzzy* logic (even though for some it is *fuzzy enough* [1]). Moreover, one can also argue that FL^ℓ is still *not basic enough* because it satisfies a remaining structural rule: associativity. There have actually been several studies on non-associative substructural logics, starting with the original Lambek non-associative calculus [19] (without lattice connectives), and followed (in the full language) e.g. by Buszkowski and Farulewski in [3]. For our goals the most relevant publication is a recent paper by Galatos and Ono [14] where they have introduced a Gentzen-style and a Hilbert-style calculus for the non-associative version of the Full Lambek calculus and proved that it is an algebraizable logic with the equivalent algebraic semantics being the variety of lattice-ordered residuated unital groupoids.

Recently, a general algebraic framework to study fuzzy logics as a subfamily of substructural logics with SL as the base logic has been developed in [8]. As a crucial tool, the notion of almost (MP)-based logic has been introduced: a logic with a Hilbert-style presentation where *modus ponens* is the only binary rule, there are no rules with more than two premises, and all unary rules are of the form $\varphi \vdash \delta(\varphi)$, for $\delta \in \mathrm{DT}$, where the set of terms DT satisfies some good properties; the logic is (MP)-based if $\mathrm{DT} = \emptyset$. It has been proved that every almost (MP)-based substructural logic enjoys a local deduction theorem and a certain form of proof by cases property (PCP), which can arguably be seen as the defining property of a reasonable generalized notion of disjunction (see [9, 6]). By using these disjunctions, given a substructural logic L, one can easily describe an axiomatization of the least logic L^ℓ extending L which is complete with respect to a semantics of chains. Such logics and their algebraic counterparts, following the terminology introduced in a previous paper [7], are called *semilinear*, because the subdirectly irreducible members of their corresponding classes of algebras are linearly ordered. Most fuzzy logics studied in the literature are actually semilinear substructural logics, including, of course, those which satisfy a standard completeness theorem in the sense we have mentioned above, and including FL^ℓ.

As shown in [8], the main associative substructural logics (FL, $\mathrm{FL_e}$, $\mathrm{FL_w}$, $\mathrm{FL_{ew}}$, etc.) are (almost) (MP)-based. Thus, in particular, we obtained axiomatizations for FL^ℓ, $\mathrm{FL}^\ell_\mathrm{e}$, $\mathrm{FL}^\ell_\mathrm{w}$, and $\mathrm{FL}^\ell_\mathrm{ew}$ (actually we obtained an alternative presentation, as some other axiomatizations of these logics were already known using different methods, see e.g. [13]). However the problems of axiomatization and standard completeness of SL^ℓ were left open. In this talk we show a solution for both of them. Actually we present the solution for all semilinear logics $\mathrm{SL}^\ell_\mathrm{S}$, where $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$ is a set of axioms corresponding to structural rules.

First we present an alternative Hilbert-style axiomatization of SL which allows to show that this logic, together with all its axiomatic extensions, is almost (MP)-based. Namely, besides *modus ponens*, it has the following rules:

$$
\begin{aligned}
(\mathrm{Adj_u}) &\quad \varphi \vdash \varphi \wedge \overline{1} \\
(\alpha) &\quad \varphi \vdash \delta \mathbin{\&} \varepsilon \to \delta \mathbin{\&} (\varepsilon \mathbin{\&} \varphi) \\
(\alpha') &\quad \varphi \vdash \delta \mathbin{\&} \varepsilon \to (\delta \mathbin{\&} \varphi) \mathbin{\&} \varepsilon \\
(\beta) &\quad \varphi \vdash \delta \to (\varepsilon \to (\varepsilon \mathbin{\&} \delta) \mathbin{\&} \varphi) \\
(\beta') &\quad \varphi \vdash \delta \to (\varepsilon \rightsquigarrow (\delta \mathbin{\&} \varepsilon) \mathbin{\&} \varphi)
\end{aligned}
$$

Thus we obtain an axiomatization of the logic SL^{ℓ} of linearly ordered residuated unital groupoids (and present simpler forms of this axiomatization in stronger logics $\mathrm{SL}^{\ell}_{\mathrm{S}}$). Furthermore this result entails several additional interesting logical and algebraic consequences for the logics $\mathrm{SL}^{\ell}_{\mathrm{S}}$: a form of the local deduction theorem, a description of intersection of filters and of the filter generated by a given set, an axiomatization of the intersection of two axiomatic extensions of a given logic, and equational bases of varieties of SL_{S}-algebras generated by positive universal classes of SL_{S}-algebras.

Secondly, by using purely algebraic constructions, we prove that every logic $\mathrm{SL}^{\ell}_{\mathrm{S}}$ is complete with respect to the class of all countably infinite dense SL_{S}-chains, and with respect to the class of all SL_{S}-chains on $[0,1]$, i.e. they enjoy the standard completeness theorem. Therefore, in particular, we have obtained a logic, SL^{ℓ}, which can reasonably be regarded as *really basic*, for it does not satisfy any structural rule (not even associativity), and *really fuzzy*, as it still enjoys a standard completeness theorem.

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